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EXTENDING THE ALIAS MONTE CARLO SAMPLING METHOD TO GENERAL DISTRIBUTIONS

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ABSTRACT

The alias method is a Monte Carlo sampling technique that offers significant advantages over more traditional methods. It equals the accuracy of table lookup and the speed of equal probable bins. The original formulation of this method sampled from discrete distributions and was easily extended to histogram distributions. We have extended the method further to applications more germane to Monte Carlo particle transport codes: continuous distributions. This paper presents the alias method as originally derived and our extensions to simple continuous distributions represented by piecewise linear functions. We also present a method to interpolate accurately between distributions tabulated at points other than the point of interest. We present timing studies that demonstrate the method's increased efficiency over table lookup and show further speedup achieved through vectorization.

INTRODUCTION

The alias method, a Monte Carlo sampling technique that offers significant advantages over more traditional methods, was originally developed nearly 15 years ago for sampling from discrete distributions.[1] Six years later it was independently rediscovered[2] and applied to discrete sampling in a vectorized Monte Carlo particle transport code. Since then, although it has not seen widespread use in the Monte Carlo community, the method has been extended to continuous distributions (see, for example, reference 3). The alias method equals the accuracy of table lookup and nearly equals the speed of equal probable bins. Further, unlike table lookup, the alias method can be effectively vectorized. Perhaps a reason for its limited use is the alias method's heretofore inability to interpolate between tabulated distributions. This paper remedies this situation and presents a statistical method of interpolation that reproduces to within statistical errors an accepted interpolation technique.

This paper reviews the traditional sampling techniques: equal probable bins and table lookup. It reviews the alias method as applied to discrete distributions and presents a new alias-like method for sampling from a linearly varying continuous distribution. Together with the discrete alias method, this technique can be used to sample from piecewise-linear distributions. The statistical alias interpolation technique is presented. Comparisons of Cray-XMP execution times of the various methods sampling from a typical distribution are given.

TRADITIONAL SAMPLING TECHNIQUES

Sampling techniques randomly sample values of a parameter, x, according to a probability distribution $y(x)$, which is usually tabulated at discrete values of the parameter, $y_j=y(x_j)$, between which a specified interpolation procedure applies, usually linear-linear.

EQUAL PROBABLE BIN METHOD

The equal probable bin method divides the parameter space into N intervals, or bins, defined by the set $\{\eta\}$, such that the probability of all bins are equal,

$$\int_{\eta_{j-1}}^{\eta_j} y(x)dx = \frac{1}{N} \ .$$

A value of x is sampled by first selecting a bin randomly

$$j = N\xi_1 + 1$$

and then randomly selecting a value within the bin with a uniform probability

$$x = (1 - \xi_2)\eta_{j+1} + \xi_2\eta_j \ ,$$

where ξ_1 and ξ_2 are random numbers uniformly distributed between 0 and 1.

This method is extremely fast but at the expense of accuracy. Much of the detail of the original distribution is lost by the necessary assumption of uniform likelihood within each bin.

TABLE LOOKUP METHOD

The table lookup method searches a table of increasing cumulative probabilities until the interval, j, is found within which a given random number, ξ_1, falls,

$$P_{j-1} \le \xi_1 < P_j \ ,$$

where

$$P_0 = 0$$

$$P_j = P_{j-1} + \int_{x_{j-1}}^{x_j} y(x)dx \ .$$

The sets $\{x\}$ and $\{P\}$ define the table.

The value of x is found by sampling over the interval

$$\left(x_{j-1}, x_j\right)$$

according to a uniform likelihood. This method retains almost all the detail of the original distribution. *All* is retained if a linearly varying, rather than uniform, probability is used in the last step. The speed of this method is dependent on the size and shape of the distribution. Because the random number may fall at various positions in the table, this method cannot be vectorized.

THE DISCRETE ALIAS METHOD

Consider a distribution, $\{p\}$, describing the likelihood of M discrete outcomes. The alias method recasts this distribution into M equal probable events, each with likelihood $1/M$. Each event, i, consists of a non-alias outcome, i, an alias outcome, Λ_i, and the probability of the non-alias outcome, Π_i. The alias probability is the complement of the non-alias probability. The new distribution is formed by adding to the probability of each outcome with less than average likelihood enough probability from an outcome with more than average likelihood such that the total is equal to $1/M$. The donor outcome's probability is then reduced accordingly. The original outcome is called the non-alias outcome; the donor is the alias outcome. The non-alias probability is equal to its original probability multiplied by M. During the course of the generation of the new distribution, a donor is allowed to become a recipient as its probability falls below the average $1/M$.

Table 1 presents an example of generating an alias representation for this simple, 6-element probability distribution:

$$\{p\} = \{.24,.08,.28,.12,.12,.16\}.$$

Note that event 1 is a donor event during step 1 and, after dropping in liklihood below the average, 1/6, becomes a recipeint event in step 2. The resulting alias distribution,

$$\{\Pi/\Lambda\} = \{.92/3,.48/1,1.00/3,.72/1,.72/1,.96/3\},$$

is not unique. Others are equally valid, for example,

$$\{\Pi/\Lambda\} = \{.88/3,.48/1,1.00/3,.72/1,.72/1,.96/3\}.$$

The outcome is sampled by first randomly selecting a equal probable non-alias/alias set,

$$j = M\xi_1 + 1 .$$

The method then compares a second random number against the non-alias probability to select either the non-alias or alias outcome:

$$
\begin{aligned}
&\mathit{if}\ \xi_2 \le \Pi_j, \\
&\qquad \mathit{choose}\ j \quad (\text{non-alias}), \\
&\mathit{else}, \\
&\qquad \mathit{choose}\ \Lambda_j\ (\text{alias}).
\end{aligned}
$$

The alias method is easily vectorized because each sample undergoes the same operations.

THE GENERAL ALIAS METHOD

The general alias method constructs from a piece-wise linear tabulated distribution a series of discrete probabilities corresponding to the likelihood of x lying in the each tabulated interval,

$$p_j = \int_{x_{j-1}}^{x_j} y(x)dx \ .$$

From the set $\{p\}$ a discrete alias distribution is calculated as described above. Sampling with the discrete alias method results in an interval from which the value of x is then sampled. The accuracy of this method is identical to that of the table lookup method.

SAMPLING FROM A LINEARLY VARYING CONTINUOUS DISTRIBUTION

The accuracy of both the table lookup and continuous alias methods can be improved by sampling over the sampled interval in x according to a linearly varying continuous distribution instead of a uniform distribution. The value x' is found from a uniform distribution between x_l and x_r, as shown in Figure 1, by

$$x' = (1-\xi)x_l + \xi x_r$$

where ξ is a random number uniformly distributed between 0 and 1. Various methods of sampling from a linearly varying distribution exist, such as choosing the greater (or lessor) of two random numbers or solving a quadratic equation. Presented here is a method[4] that converts a non-uniform distribution into a uniform distribution analogous to the alias method coercing non-equal probable bins into equal probable bins.

Figure 2 shows a distribution varying from y_l on the left to y_r on the right. The method randomly chooses a value x' a distance $\xi\Delta x$ from the left and x'' an equal distance from the right,

$$x' = (1-\xi)x_l + \xi x_r$$
$$x'' = \xi x_l + (1-\xi)x_r \ .$$

If a second random number, selected between 0 and y_l+y_r, i.e. $\xi'(y_l+y_r)$, falls below the linear function evaluated at x', the value x' is retained, otherwise x'' is chosen:

$$\begin{aligned}&\textit{if}\quad \xi'(y_l+y_r) \le (1-\xi)y_l + \xi y_r,\\ &\qquad\qquad \textit{choose } x = x',\\ &\textit{else},\\ &\qquad\qquad \textit{choose } x = x''.\end{aligned}$$

The non-alias outcome can be thought of as x', the alias outcome as x'', and the linear function as the non-alias probability.

INTERPOLATION BETWEEN TABULATED DISTRIBUTIONS

Often in Monte Carlo particle transport codes a value must be sampled from a distribution that is not available. An example helps illuminate this apparent paradox: a neutron with energy 7.32 MeV may undergo a reaction resulting in its absorption and the emission of a second neutron. Spectra for this second neutron are tabulated only at, say, integral MeV neutron energies. Spectra exist at 7 MeV and 8 MeV but not at 7.32 MeV. One could derive a spectrum at 7.32 MeV using an interpolation procedure and then sample from that distribution with one of the techniques described above. This procedure would result in a different spectra for each possible neutron energy and is absurdly impractical. Instead, means to sample a value at an intermediate energy from distributions tabulated at specific energies have been developed.

Generalizing the example of the previous paragraph, a value T is desired from a nonexistent distribution at E which lies between the distributions tabulated at E^{lo} and E^{hi}. The parameter α specifies the fractional position of E in the interval,

$$\alpha = \frac{E - E^{lo}}{E^{hi} - E^{lo}}$$
$$E = (1-\alpha)E^{lo} + \alpha E^{hi} \ .$$

EQUAL PROBABLE BIN METHOD

Again the simplest method is equal probable bins. The corresponding bins are selected from both the 'lo' and 'hi' bin sets

$$j = N\xi_1 + 1 .$$

A value for T is found from each set

$$T^{lo} = (1-\xi_2)\eta_{j-1}^{lo} + \xi_2\eta_j^{lo}$$
$$T^{hi} = (1-\xi_2)\eta_{j-1}^{hi} + \xi_2\eta_j^{hi}$$

and the final value found through linear interpolation

$$T = (1-\alpha)T^{lo} + \alpha T^{hi} .$$

Although this method is fast and effectively vectorized, it suffers from the same liabilities of the equal probable bin method mentioned earlier. Moreover, the final interpolation to find T has no theoretical basis and may introduce errors.

TABLE LOOKUP METHOD

The traditional technique for interpolation with the table lookup method is very similar to the equal probable bin method. Again, with the same random number, values T^{lo} and T^{hi} are found and then used in the final interpolation. Observations of the table lookup method made earlier apply here, too.

STATISTICAL INTERPOLATION

The alias method cannot take advantage of the simple linear interpolation used as the last step in the other interpolation methods because there is no correlation between the values selected with the same random number from the 'lo' and 'hi' alias distributions. This is a consequence of the rearrangement of probabilities inherent to the alias method. Instead, a statistical method of interpolation must be used. This technique presented here can also be applied to the table lookup method.

Instead of sampling from both the 'lo' and 'hi' distributions for each event, the statistical interpolation method samples from *either* 'lo' or 'hi' with a probability governed by the parameter α. The result, T^{hilo}, is transposed to the interval (T_{min}, T_{max}) determined again by α. The 'lo' distribution is sampled if

$$\xi \le (1-\alpha) ,$$

otherwise, the 'hi' distribution is sampled. The final value T is given by

$$T = \frac{T_{min}\left(T_{max}^{hilo} - T^{hilo}\right) + T_{max}\left(T^{hilo} - T_{min}^{hilo}\right)}{T_{max}^{hilo} - T_{min}^{hilo}},$$

where

$$T_{min} = (1-\alpha)T_{min}^{lo} + \alpha T_{min}^{hi}$$
$$T_{max} = (1-\alpha)T_{max}^{lo} + \alpha T_{max}^{hi}$$

and 'hilo' refers to either 'hi' or 'lo' depending on which distribution was sampled. The 'max' and 'min' values of T are the largest and smallest values of T tabulated.

This method reproduces the distribution found with the unit base transformation interpolation scheme[5] to within statistical errors if a linear function is sampled between tabulated points. The unit base scheme is used to generate ENDF multi-group transfer matrices.

NUMERICAL RESULTS

Execution times on the Cray-XMP for the various methods were found by sampling from a typical distribution: the neutron spectra arising from the (n,n') reaction on vanadium-51. The tabulated spectra contain 105 values at the incident neutron energy of 5 MeV and 103 values at 7 MeV. The interpolation schemes used both distributions while others used only the lower energy spectra. The equal probable bin method used 32 bins. All methods sampled 1000 events.

Table 2 gives the times required for each sample for the various methods, both in scalar and vector modes.

Table lookup times vary with different tables; the other times are insensitive to table length or makeup. A time is given for a vectorized table lookup. As mentioned early, table lookup cannot be vectorized. The speedup given in the table accounts for vectorized generation of random numbers. The hand assembly coded routine LUF[6] was used to perform the table lookup. This routine uses a combined binary and linear search and has been optimized for vector architectures. All other routines are coded in FORTRAN.

The equal probable bin method is the fastest of all the methods and profits from vectorization. The alias method is 30% faster than table lookup in scalar mode but the vectorized version is nine times faster. The vectorized alias method is 50% slower than equal probable bin. Recall, however, that equal probable bin gives up accuracy for its speed; alias is nearly as fast as equal probable bin and is as accurate as table lookup. Adding the increased accuracy of linear sampling between tabulated points increases run time by 40%.

Interpolation between tabulated distributions adds only 20% to the time required by the equal probable bin method. Interpolation nearly doubles the compute time required by alias. The transformation from the sampled basis to the interpolated basis requires substantial arithmetic. Alias takes almost four times the compute time of equal probable bin but avoids the problem of losing detail in the distribution. As is always the case, users of these methods must evaluate the trade-off between speed and accuracy. In contrast to table lookup, this trade-off is not offensive.

ACKNOWLEDGMENTS

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Table 1. Generating an Example Alias Representation.

		step 1	step 2	step 3	step 4	step 5	step 6
recipient event		2	1	4	5	6	3
donor event		1	3	3	3	3	3
event	original prob.	updated probability distribution after step					
1	.24	.1533	0.0	0.0	0.0	0.0	0.0
2	.08	0.0	0.0	0.0	0.0	0.0	0.0
3	.28	.28	.2667	.22	.1733	.1667	0.0
4	.12	.12	.12	0.0	0.0	0.0	0.0
5	.12	.12	.12	.12	0.0	0.0	0.0
6	.16	.16	.16	.16	.16	0.0	0.0
	non-alias event (j)	evolving alias representation (Π_j / Λ_j)					
	1		.92/3	.92/3	.92/3	.92/3	.92/3
	2	.48/1	.48/1	.48/1	.48/1	.48/1	.48/1
	3						1.00/3
	4			.72/3	.72/3	.72/3	.72/3
	5				.72/3	.72/3	.72/3
	6					.96/3	.96/3

Table 2. Cray-XMP execution times.

method	µsec/sample scalar	µsec/sample vectorized	speedup
without interpolation			
equal probable bin	2.280	0.294	7.76
table lookup (uniform)	6.245	3.790	1.65
alias (uniform)	4.290	0.428	10.02
alias (linear)	5.831	0.642	9.08
with interpolation			
equal probable bin	3.227	0.340	9.49
alias (linear)	8.471	1.160	7.30

Figure 1. Sampling From a Uniform Distribution.

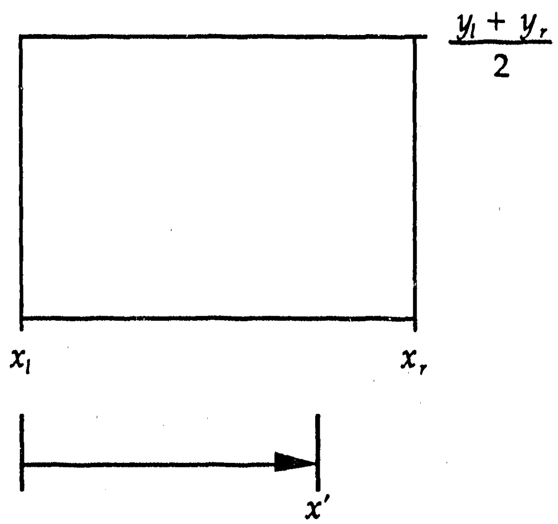
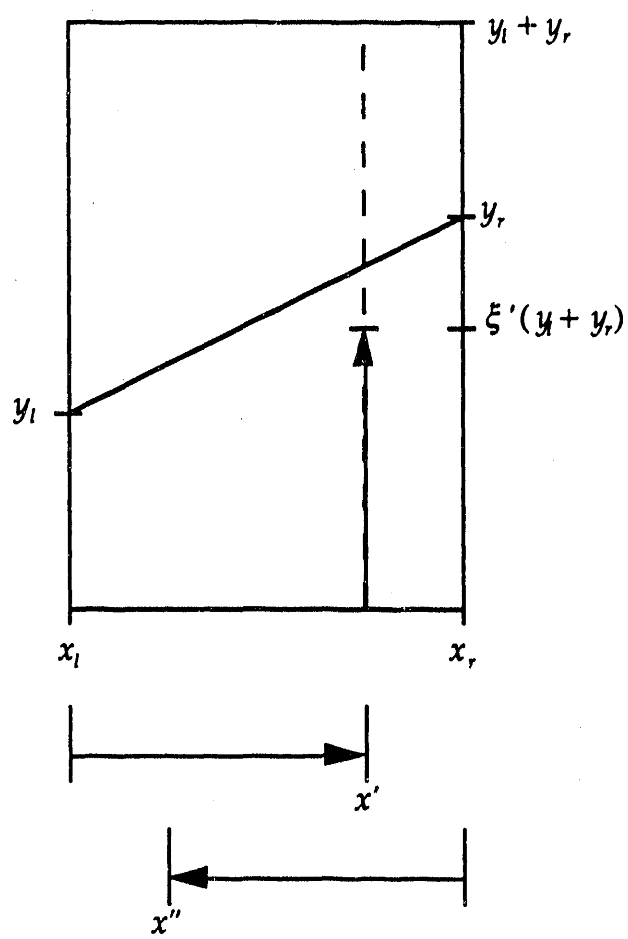


Figure 2. Sampling From a Linearly Varying Distribution.



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