DEPARTMENT OF COMPUTER SCIENCE

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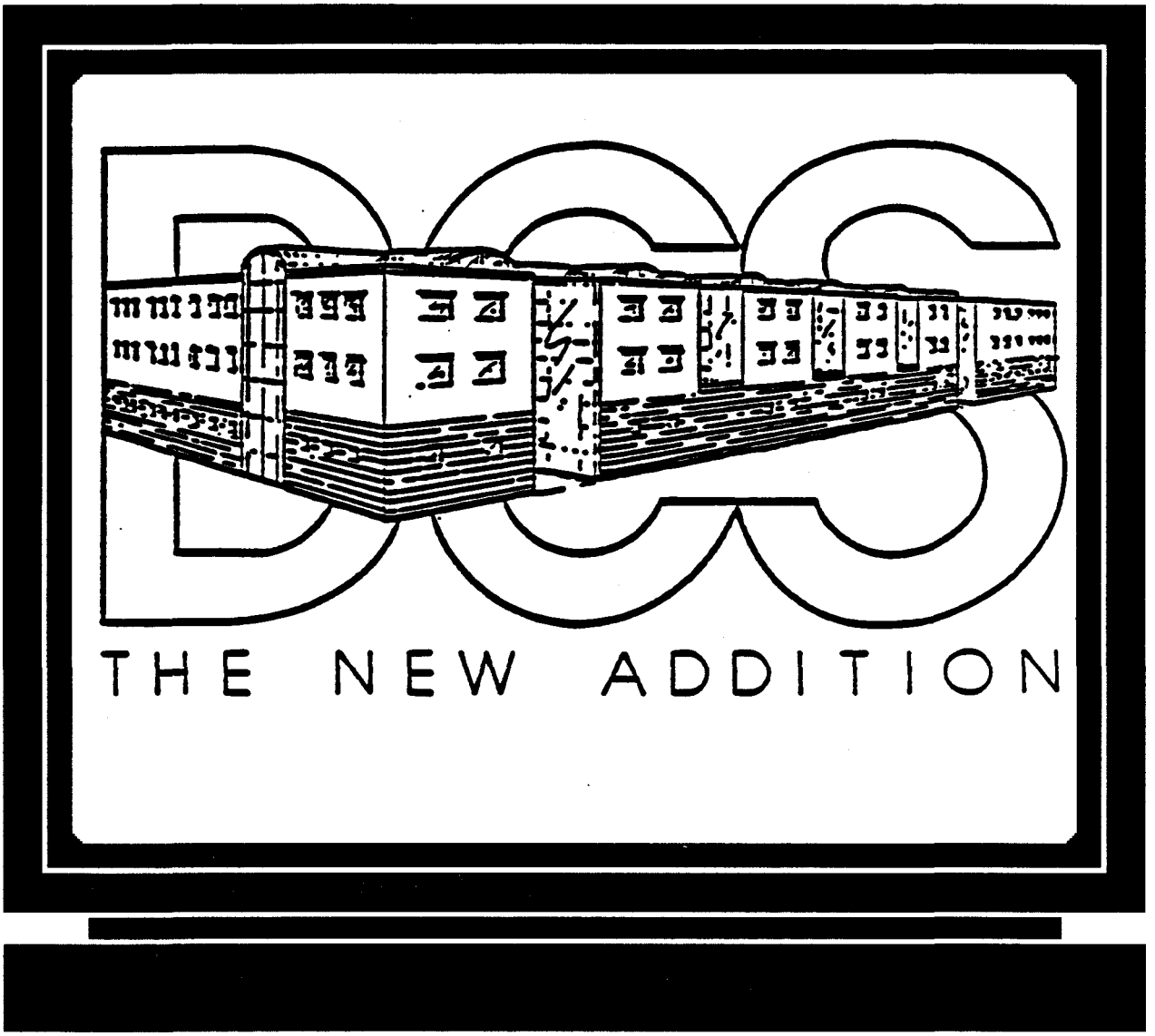

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ACCURACY INCREASE IN WAVEFORM GAUSS SEIDEL

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1 Introduction

An iterative approach, the waveform relaxation method, for solving large systems of initial value problems has been shown to converge superlinearly in [1]. In [2] we define the *order of accuracy* of approximate solutions generated by the waveform relaxation method and have shown that the accuracy after one relaxation step is at least one order higher than before the step starts. In this paper we will discuss the accuracy increase (by which we mean increase in the order of accuracy) of a special scheme, the waveform Gauss-Seidel method, and show that the average accuracy increase for the waveform Gauss-Seidel method is equal to the minimum value C/d among all cycles in the dependency graph, where C is the length of a cycle and d is the number of times the numbering of succesive nodes around the cycle decreases. Note that the value, C, depends on the coupling relation after a system is partitioned and the value, d, depends on the numbering of nodes in the system's dependency graph that is imposed by the Gauss-Seidel scheme. So after a systems's partitioning, we should order the nodes to maximize this minimum.

2 Basic Results

Consider the following system of ordinary differential equations

$$\dot{\mathbf{u}} = F(\mathbf{u}), \qquad \mathbf{u}(0) = \mathbf{u}_0 \tag{1}$$

where $\mathbf{u} \in R^n$, and $F : R^n \to R^n$. The iteration formula of the waveform relaxation method is

$$\dot{\mathbf{u}}^{[k+1]} - G(\mathbf{u}^{[k+1]}, \mathbf{u}^{[k]}) = F(\mathbf{u}^{[k]}) - G(\mathbf{u}^{[k]}, \mathbf{u}^{[k]}), \qquad \mathbf{u}^{[k+1]}(0) = \mathbf{u}_0 \tag{2}$$

where $\mathbf{u}^{[k]}$ is the k^{th} iterate of waveform relaxation. Suppose the system is partitioned into m coupled subsystems

$$\begin{aligned} \dot{u}_1 &= f_1(u_1, u_2, \ldots u_m), & u_1(0) &= u_{1,0} \\ &\vdots & & \\ \dot{u}_m &= f_m(u_1, u_2, \ldots u_m), & u_m(0) &= u_{m,0} \end{aligned}$$

where $u_i \in R^{n_i}$, $\mathbf{u} = (u_1^T, u_2^T, \ldots u_m^T)^T$, $f_i : R^n \to R^{n_i}$, $F = (f_1^T, f_2^T, \ldots f_m^T)^T$, $1 \le i \le m$, and $\sum_{i=1}^m n_i = n$, then the iteration formula of the Gauss-Seidel scheme becomes

$$\dot{\mathbf{u}}^{[k+1]} - G(\mathbf{u}^{[k+1]}, \mathbf{u}^{[k]}) = 0, \qquad \mathbf{u}^{[k+1]}(0) = \mathbf{u}_0,$$

where $G = (g_1, g_2, \ldots, g_m)^T$ and for $i = 1, \ldots, m$

$$g_i(\mathbf{u}^{[k+1]}, \mathbf{u}^{[k]}) = f_i(u_1^{[k+1]}, \ldots, u_{i-1}^{[k+1]}, u_i^{[k+1]}, u_{i+1}^{[k]}, \ldots, u_m^{[k]}).$$

That is, when a subsystem is being integrated in the waveform Gauss-Seidel method it always uses the most recently computed values of other subsystems. Before proceeding we redefine the *order of accuracy* of an approximate solution. (This order of accuracy is one less than that defined in [2].)

Let $u_i(t)$ be the i^{th} component of the exact solution and $z_i(t)$ be the i^{th} component of an approximate solution to Equation (1).

Definition 2.1 *If $z_i(t) - u_i(t) = O(t)^{M_i+1}$ over a fixed, finite interval $[0, T]$, then the order of accuracy of $z_i(t)$ is defined as: $N(z_i) \doteq M_i$ for $1 \le i \le n$. The order of accuracy of $\mathbf{z}(t)$ is defined as: $N(\mathbf{z}) \doteq \min_{1 \le i \le n} N(z_i)$.*

Since, in waveform Gauss-Seidel iteration, we solve each subsystem sequentially and independently, as one subsystem is integrated during a Gauss-Seidel iteration the remaining subsystems are specified approximants to the exact solutions. By definition of the order of accuracy we then have the following result:

Theorem 2.2 *Consider the equation for the i^{th} component after partitioning,*

$$\dot{u}_i = f_i(u_1, \ldots, u_i, \ldots, u_m), \qquad u_i(0) = u_{i,0}. \tag{3}$$

The equation to be solved after applying the Gauss-Seidel scheme is

$$\dot{u}_i^{[k+1]} - f_i(u_1^{[k+1]}, \ldots, u_{i-1}^{[k+1]}, u_i^{[k+1]}, u_{i+1}^{[k]}, \ldots, u_m^{[k]}) = 0, \qquad u_i^{[k+1]}(0) = u_{i,0}. \tag{4}$$

Assume that

$$\left.\begin{aligned} E_j^{[k+1]} &\doteq u_j^{[k+1]} - u_j = O(t)^{N_j^{[k+1]}+1} \qquad \text{for } j \le i, \\ E_j^{[k]} &\doteq u_j^{[k]} - u_j = O(t)^{N_j^{[k]}+1} \qquad \text{for } j > i. \end{aligned}\right\} \tag{5}$$

and all the $E_j^{[k]}$'s and $E_j^{[k+1]}$'s are sufficiently smooth. Then

$$N_i^{[k+1]} \geq min(N_1^{[k+1]}, \ldots, N_{i-1}^{[k+1]}, N_{i+1}^{[k]}, \ldots, N_m^{[k]}) + 1. \tag{6}$$

(For detailed proof see [2].)

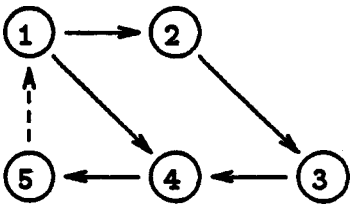
In the Gauss-Seidel scheme the numbering of the subsystems is important since it determines the order of their sequential solution. One subsystem affects another if any of its variables appear on the right-hand side of the differential equations describing the other. This coupling is an oriented relationship and an adjacency matrix can be used to describe the coupling relations among all subsystems. A directed graph that is built from the adjacency matrix is called the dependency graph of a system. If a system's dependency graph is acyclic we could get the exact solution with only one waveform Gauss-Seidel iteration when each subsystem is sequentially integrated in a proper order; otherwise iterations are needed until a sufficiently accurate solution has been computed. (From now on a subsytem is referred as a node in a dependency graph.)

From Theorem 2.2, we know that the order of accuracy at one node after one waveform Gauss-Seidel iteration is at least one greater than the minimum order of its incoming nodes, and possibly more if there is fortuitous cancellation. But the fortuitous cancellation can only occur under very special coupling, so it will be ignored in general. Hence we assume equality in that theorem and investigate some examples to study the accuracy increase of the waveform Gauss-Seidel method. From these examples we will see that the accuracy increase in the waveform Gauss-Seidel method is related to the coupling and the numbering on a given system's dependency graph.

3 Ascending Chains and Accuracy Increase

First let us consider:

Example 1 Consider a system with the following dependency graph after partitioning and ordering. This graph has only two cycles and all the nodes inside each cycle are sequentially ordered, i.e. there is only one decrease in the numbering of all nodes around every cycle.



The two cycles are $\tilde{A}_1 = \{(1,2,3,4,5)\}$ of length $C_1 = 5$ and $\tilde{A}_2 = \{(1,4,5)\}$ of length $C_2 = 3$. The sole numbering decrease is the branch $(5,1)$ shown dashed. So $d_1 = d_2 = 1$. Hence $C_1/d_1 = 5$ and $C_2/d_2 = 3$. In the following tables we list the order of accuracy and accuracy increase after each waveform Gauss-Seidel iteration assuming we start with $\mathbf{u}^{(0)}(t) = \mathbf{u}_0$.

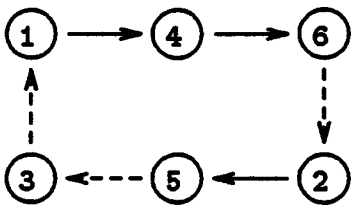
Order of Accuracy												
Node	Iteration No											
No	0	1	2	3	4	5	6	7	8	9	10	...
1	0	1	4	7	10	13	16	19	22	25	28	...
2	0	2	5	8	11	14	17	20	23	26	29	...
3	0	3	6	9	12	15	18	21	24	27	30	...
4	0	2	5	8	11	14	17	20	23	26	29	...
5	0	3	6	9	12	15	18	21	24	27	30	...

Accuracy Increase												
Node	Iteration No											
No	0	1	2	3	4	5	6	7	8	9	10	...
1		1	3	3	3	3	3	3	3	3	3	...
2		2	3	3	3	3	3	3	3	3	3	...
3		3	3	3	3	3	3	3	3	3	3	...
4		2	3	3	3	3	3	3	3	3	3	...
5		3	3	3	3	3	3	3	3	3	3	...

From Example 1 we see that, after the waveform Gauss-Seidel iteration stabilizes, the accuracy increase after one iteration is equal to the minimum cycle length in the dependency graph. Note, however, that in this example the internal node of the cycle with minimum length have been ordered sequentially around this cycle. In general, the internal node of a cycle may not be ordered sequentially; in this case we can not achieve an accuracy increase equal to the cycle length in one waveform Gauss-Seidel iteration. However, we will show that, in the case of a single cycle, an order increase equal to the length of the cycle will occur in some number

of iterations. Let us consider another example in which the nodes are not sequentially ordered around a cycle.

Example 2: Consider a dependency graph which contains only one cycle and nodes inside the cycle are not sequentially ordered,



$$\tilde{A} = \{(1,4,6),(2,5),(3)\}; \qquad C/d = 6/3$$

The tables for order of accuracy and accuracy increase are given below.

Order of Accuracy												
Node No	Iteration No											
	0	1	2	3	4	5	6	7	8	9	10	...
1	0	1	2	4	7	8	10	13	14	16	19	...
2	0	1	4	5	7	10	11	13	16	17	19	...
3	0	1	3	6	7	9	12	13	15	18	19	...
4	0	2	3	5	8	9	11	14	15	17	20	...
5	0	2	5	6	8	11	12	14	17	18	20	...
6	0	3	4	6	9	10	12	15	16	18	21	...

Accuracy Increase												
Node No	Iteration No											
	0	1	2	3	4	5	6	7	8	9	10	...
1		1	1	2	3	1	2	3	1	2	3	...
2		1	3	1	2	3	1	2	3	1	2	...
4		2	1	2	3	1	2	3	1	2	3	...
5		2	3	1	2	3	1	2	3	1	2	...
6		3	1	2	3	1	2	3	1	2	3	...

We see in Example 2 that the order increase of 6 , the cycle length, is achieved in 3 iterations. The reader might notice that in this example, three is the number of times the numbering of succesive nodes around the cycle decreases. This will be seen to be a general result, for which we need to define the concept of an *ascending chain* in a cycle.

Suppose an ordering for the Gauss-Seidel method applied on a graph containing a cycle $\tilde{A}$ of length C has been chosen. Number the nodes of the graph by the Gauss-Seidel ordering.

Definition 3.1 *An ascending chain of length l in a cycle $\tilde{A}$ is a sequence of nodes with numerical ordering $j_0, j_1, \ldots, j_{l-1}$, such that (1) $j_0 < j_1 < \cdots < j_{l-1}$, (2) there exists an edge from node j_i to node j_{i+1} for $i = 0, 1, \ldots, l-2$ in the cycle, and (3) no ascending chain in cycle $\tilde{A}$ contains $\{j_0, j_1, \ldots, j_{l-1}\}$ as a subsequence (in other words, it is as long as possible).*

It then follows from the definition that any cycle can be decomposed into a mutually exclusive set of ascending chains and the number of ascending chains in a cycle equals the number of times the numbering of successive nodes around the cycle decreases. By the equality assumption in Theorem 2.2, we know that after one waveform Gauss-Seidel iteration, each node, j_i, in an ascending chain can not have order of accuracy more than one greater than the order of accuracy of its predecessor node, $j_i - 1$, in the chain at this iteration, while the first node in an ascending chain can not have order of accuracy greater than one plus the order of accuracy, prior to the iteration, of its predecessor, the last node in the chain precedes it. (If no other node except its predecessor in the cycle is connected to a node k, it will achieve exactly this order increase.) Now we introduce some simple notations to express these ideas.

Let $\tilde{A}$ be a cycle of length C with d ascending chains. Let $l_1, l_2, \ldots, l_d$ be the lengths of the d ascending chains that follow the orientation of $\tilde{A}$ and W_{n,i,k_i} be the order of accuracy of the k_i^{th} node of the i^{th} ascending chain at the n^{th} waveform Gauss-Seidel iteration. For convenience, define $l_{i+d} = l_i$ and $W_{n,i+d,k} = W_{n,i,k}$ for all i. Then by assuming equality in Theorem 2.2, we have

$$\begin{aligned} W_{n,i,k_i+1} &\leq W_{n,i,k_i} + 1 \\ W_{n+1,i+1,0} &\leq W_{n,i,l_i-1} + 1 \end{aligned}$$

for $k_i = 0, 1, \ldots, l_i - 2$,and $n \geq 1$. For ease of derivation later, we now define $W^T_{n,i} \equiv W_{n,i,0}$ and $W^H_{n,i} \equiv W_{n,i,l_i-1}$, i.e. let $W^T_{n,i}$ ($W^H_{n,i}$) denote the order of accuracy at the tail (head) node of the i^{th} ascending chain at the n^{th} waveform Gauss-Seidel iteration. And it is easy to see that

$$W_{n,i,k_i} \leq W^T_{n,i} + k_i \tag{7}$$

$$W^H_{n+1,i} \leq W^H_{n,i-1} + l_i \tag{8}$$

$$W^T_{n+1,i} \leq W^T_{n,i-1} + l_{i-1}. \tag{9}$$

Based on these relations, we then have the following result.

Theorem 3.2 *If a cycle $\tilde{A}$ of length C consists of d ascending chains, then, after the first iteration, the accuracy increase at the internal nodes of $\tilde{A}$ due to d waveform Gauss-Seidel iterations is bounded by C.*

Proof: By Eqn (9) for $n \geq 1$

$$\begin{aligned}
W^T_{n+d,i} &= W^T_{n+d,i+d} \\
&\leq W^T_{n+d-1,i+d-1} + l_{i+d-1} \\
&\leq W^T_{n+d-2,i+d-2} + l_{i+d-2} + l_{i+d-1} \\
&\vdots \\
&\leq W^T_{n+1,i+1} + l_{i+1} + \cdots + l_{i+d-1} \\
&\leq W^T_{n,i} + l_i + l_{i+1} + \cdots + l_{i+d-1} \\
&\leq W^T_{n,i} + C.
\end{aligned}$$

The proof for the remaining nodes in an ascending chain is similar:

$$\begin{aligned}
W_{n+d,i,k_i} &= W_{n+d,i+d,k_i} \\
&\leq W^T_{n+d,i+d} + k_i \\
&\leq W^H_{n+d-1,i+d-1} + 1 + k_i \\
&\leq W^H_{n+d-2,i+d-2} + l_{i+d-1} + 1 + k_i \\
&\leq W^H_{n+d-3,i+d-3} + l_{i+d-2} + l_{i+d-1} + 1 + k_i \\
&\vdots \\
&\leq W^H_{n+1,i+1} + l_{i+2} + \cdots + l_{i+d-2} + l_{i+d-1} + 1 + k_i \\
&\leq W^H_{n,i} + l_{i+1} + l_{i+2} + \cdots + l_{i+d-2} + l_{i+d-1} + 1 + k_i \\
&\leq W_{n,i,k_i} + l_i - 1 - k_1 + l_{i+1} + l_{i+2} + \cdots + l_{i+d-2} + l_{i+d-1} + 1 + k_i \\
&= W_{n,i,k_i} + \sum_{j=1}^{d} l_j \\
&= W_{n,i,k_i} + C.
\end{aligned}$$

Q. E. D.

When $d = 1$, i.e. when all the internal nodes of a cycle are solved in cyclic order, the accuracy increase in one waveform Gauss-Seidel iteration is then bounded by the cycle length. That is the result we saw in Example 1.

If we assume that each node of a cycle has no other nodes connected to it except its predecessor in the cycle, then its order of accuracy after each waveform Gauss-Seidel iteration is exactly one over its predecessor's in the cycle, i.e.

$$W_{n,i,k_i+1} = W_{n,i,k_i} + 1 \tag{10}$$

$$W^T_{n+1,i+1} = W^H_{n,i} + 1 \tag{11}$$

for all i and k_i's. Thus

$$W_{n,i,k_i} = W^T_{n,i} + k_i \tag{12}$$

$$W^H_{n+1,i} = W^H_{n,i-1} + l_i \tag{13}$$

$$W^T_{n+1,i} = W^T_{n,i-1} + l_{i-1} \tag{14}$$

for $k_i = 0, 1, \ldots, l_i - 1$, $i = 1, 2, \ldots, d$ and $n \geq 1$. We then see that for a single cycle, the average accuracy increase in each iteration is C/d. Ignoring fortuitous cancellation, a cycle can not have a greater average accuracy increase, so it is clear that a bound on the average accuracy increase for a graph is given by $\min_i(C_i/d_i)$ where i indexes all the cycles in the graph. We will show that this bound is realized by all the graphs, so that we should order the nodes to maximize this minimum.

4 Accuracy Increase in a Single Cycle

If we examine Example 2 again we see that the accuracy increase at a given node at successive iteration followed a repetitive pattern after some initial irregularity. For some it was (1,2,3), for others it was (2,3,1), and for the remainder it was (3,1,2). The important property of these patterns is that they are *circular shifts* of the *partition* of the cycle length $C = 6$, where there are $d = 3$ members in the partition. In general we say that a set of strictly positive integers $\{q_1, q_2, \ldots, q_d\}$ is an *integer partition* of C, if $\sum_{i=1}^{d} q_i = C$. We now show that given a

cycle of length C with d ascending chains of lengths $l_1, l_2, \ldots$, and l_d, and an integer partition, $\{q_1, q_2, \ldots, q_d\}$, of C, if the initial orders of accuracy at all nodes of the cycle are chosen carefully, then the accuracy increase at each node at every d successive waveform Gauss-Seidel iterations is $\{q_d, q_{d-1}, \ldots, q_2, q_1\}$ or its circular shifts.

Theorem 4.1 *Let $\tilde{A}$ be a cycle of length C with d ascending chains of lengths, $l_1, l_2, \ldots, l_d$, and let $\{q_1, q_2, \ldots, q_d\}$ be an integer partition of C. If the initial order of accuracies are chosen such that*

$$\begin{aligned} W^T_{0,i} &= \textstyle\sum_{j=i}^{d}(q_j - l_j) && \text{for } i = 1, 2, \ldots, d \\ W_{0,i,k_i} &= W^T_{0,i} + k_i && \text{for } k_i = 0, 1, \ldots, l_i - 1, \end{aligned}$$

then

$$W_{n,i,k_i} = W_{n-1,i,k_i} + q_{i-n} \tag{15}$$

for $k_i = 0, 1, 2, \ldots, l_i - 1$, $i = 1, 2, \ldots, d$, and $n \geq 1$, where $q_n \equiv q_{n\%d}$ and $n\%d = n \bmod (d)$ for any integer n.

Proof: By Eqn (12), $W_{n,i,k_i} - W_{n-1,i,k_i} = W^T_{n,i} - W^T_{n-1,i}$, i.e. all the nodes in an ascending chain have the same accuracy increase after each waveform Gauss-Seidel iteration, so we only need to show

$$W^T_{n,i} = W^T_{n-1,i} + q_{i-n} \tag{16}$$

for $i = 1, 2, \ldots, d$ and $n \geq 1$.

When $n = 1$ and $i = 1, 2, \ldots, d$, from Eqn (14) and the choice of initial orders, we have

$$\begin{aligned} W^T_{1,i} &= W^T_{0,i-1} + l_{i-1} \\ &= \sum_{j=i-1}^{d} (q_j - l_j) + l_{i-1} \\ &= \sum_{j=i}^{d} (q_j - l_j) + q_{i-1} \\ &= W^T_{0,i} + q_{i-1}. \end{aligned}$$

Hence, (16) holds for $n = 1$. Let us use induction and assume the statement is true when $n \leq m$ and $i = 1, 2, \ldots, d$, i.e.

$$W^T_{m,i} = W^T_{m-1,i} + q_{i-m}. \tag{17}$$

Then when $n = m + 1$ and $i = 1, 2, \ldots, d$, from Eqns (13), (14), and (17) we have

$$\begin{aligned} W^T_{m+1,i} &= W^T_{m,i-1} + l_{i-1} \\ &= W^T_{m-1,i-1} + q_{(i-1)-m} + l_{i-1} \\ &= W^T_{m,i} + q_{(i-1)-m} \\ &= W^T_{m,i} + q_{i-(m+1)} \end{aligned}$$

Hence, by induction, Eqn (16) holds for all $n \geq 1$.

Q. E. D.

From Eqn (15) and the periodic behavior of q_j's we have

$$\begin{aligned} W_{n+d,i,k_i} &= W_{n,i,k_i} + \sum_{j=n+1}^{n+d} q_{i-j} \\ &= W_{n,i,k_i} + \sum_{j=1}^{d} q_j \\ &= W_{n,i,k_i} + C, \end{aligned}$$

which are exactly the results we saw in previous examples. Since all the nodes in an ascending chain have the same accuracy increase after each waveform Gauss-Seidel iteration, from now on we will use the head or tail node at each ascending chain to discuss the accuracy increase property. In Table 1 we list the accuracy increase at the tail node of each ascending chain after each waveform Gauss-Seidel iteration using the result in Theorem 4.1. From this table it is easy to see that the accuracy increase at any node in every d successive iterations is $\{q_d, q_{d-1}, \ldots, q_1\}$ or its circular shifts.

To avoid the decreasing subscripts in q_j's as iteration proceeds, we let $p_j \doteq q_{d-j}$ for $j = 0, 1, \ldots, d-1$ and rewrite Table 1 to obtain Table 2. With some manipulation we have the following formula for the order of accuracy at the n^{th} waveform Gauss-Seidel iteration:

$$W^T_{n,i} = \sum_{j=i}^{d} (p_{d-j} - l_j) + \lfloor \frac{n}{d} \rfloor \times C + \sum_{j=1}^{n\%d} p_{d-i+j}, \tag{18}$$

for $i = 1, 2, \ldots, d$ and $n \geq 1$.

In particular, if we choose a specific integer partition of the cycle length C, we will not only have a nice formula for the order of accuracy at each node, but will have an accuracy increase

	Iteration Index							
	0	1	2	$\cdots$	d	d+1	d+2	$\cdots$
l_1	$0 = \sum_{j=1}^{d}(q_j - l_j)$	$+q_d$	$+q_{d-1}$	$+\cdots$	$+q_1$	$+q_d$	$+q_{d-1}$	$+\cdots$
l_2	$l_1 - q_1 = \sum_{j=2}^{d}(q_j - l_j)$	$+q_1$	$+q_d$	$+\cdots$	$+q_2$	$+q_1$	$+q_d$	$+\cdots$
l_3	$l_1 - q_1 + l_2 - q_2 = \sum_{j=3}^{d}(q_j - l_j)$	$+q_2$	$+q_1$	$+\cdots$	$+q_3$	$+q_2$	$+q_1$	$+\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l_i	$\sum_{1}^{i-1}(l_j - q_j) = \sum_{j=i}^{d}(q_j - l_j)$	$+q_{i-1}$	$+q_{i-2}$	$+\cdots$	$+q_i$	$+q_{i-1}$	$+q_{i-2}$	$+\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l_{d-1}	$q_d - l_d + q_{d-1} - l_{d-1}$	$+q_{d-2}$	$+q_{d-3}$	$+\cdots$	$+q_{d-1}$	$+q_{d-2}$	$+q_{d-3}$	$+\cdots$
l_d	$q_d - l_d$	$+q_{d-1}$	$+q_{d-2}$	$+\cdots$	$+q_d$	$+q_{d-1}$	$+q_{d-2}$	$+\cdots$

Table 1: Accuracy increase at tail node of each ascending chain

	Iteration Index							
	0	1	2	$\cdots$	d	d+1	d+2	$\cdots$
l_1	$0 = \sum_{j=1}^{d}(p_{d-j} - l_j)$	$+p_0$	$+p_1$	$+\cdots$	$+p_{d-1}$	$+p_0$	$+p_1$	$+\cdots$
l_2	$\sum_{j=2}^{d}(p_{d-j} - l_j)$	$+p_{d-1}$	$+p_0$	$+\cdots$	$+p_{d-2}$	$+p_{d-1}$	$+p_0$	$+\cdots$
l_3	$\sum_{j=3}^{d}(p_{d-j} - l_j)$	$+p_{d-2}$	$+p_{d-1}$	$+\cdots$	$+p_{d-3}$	$+p_{d-2}$	$+p_{d-1}$	$+\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l_i	$\sum_{j=i}^{d}(p_{d-j} - l_j)$	$+p_{d-i+1}$	$+p_{d-i+2}$	$+\cdots$	$+p_{d-i}$	$+p_{d-i+1}$	$+p_{d-i+2}$	$+\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
l_{d-1}	$p_0 - l_d + p_1 - l_{d-1}$	$+p_2$	$+p_3$	$+\cdots$	$+p_1$	$+p_2$	$+p_3$	$+\cdots$
l_d	$p_0 - l_d$	$+p_1$	$+p_2$	$+\cdots$	$+p_0$	$+p_1$	$+p_2$	$+\cdots$

Table 2: Accuracy increase at tail node of each ascending chain

pattern that can not be destroyed by other cycles in the same graph that do not have a smaller C/d.

Lemma 4.2 *Given a cycle of length C with d ascending chains of lengths $l_1, l_2, \ldots, l_d$, respectively. If*

$$p_j = \lfloor (j+1)\frac{C}{d} \rfloor - \lfloor j\frac{C}{d} \rfloor \qquad \text{for } j = 0, 1, \ldots, d-1, \tag{19}$$

then $\{p_0, p_1, \ldots, p_{d-1}\}$ is an integer partition of C. Further, if

$$W^T_{0,i} = C - \sum_{j=i}^{d} l_j + \lfloor (1-i)\frac{C}{d} \rfloor \tag{20}$$

then

$$W^T_{n,i} = C - \sum_{j=i}^{d} l_j + \lfloor (n+1-i)\frac{C}{d} \rfloor \tag{21}$$

for $i = 1, 2, \ldots, d$ and $n \geq 1$.

Proof: Since $\frac{C}{d} = \lfloor \frac{C}{d} \rfloor + \epsilon$, for some $0 \leq \epsilon < 1$, by assumption

$$\begin{aligned} p_j &= \lfloor (j+1)\frac{C}{d} \rfloor - \lfloor j\frac{C}{d} \rfloor \\ &= \lfloor j\frac{C}{d} + \lfloor \frac{C}{d} \rfloor + \epsilon \rfloor - \lfloor j\frac{C}{d} \rfloor \\ &= \lfloor \frac{C}{d} \rfloor + \lfloor j\frac{C}{d} + \epsilon \rfloor - \lfloor j\frac{C}{d} \rfloor \\ &\geq \lfloor \frac{C}{d} \rfloor \\ &\geq 1 \end{aligned}$$

and

$$\begin{aligned} \sum_{0}^{d-1} p_j &= \sum_{0}^{d-1} \lfloor (j+1)\frac{C}{d} \rfloor - \lfloor j\frac{C}{d} \rfloor \\ &= \lfloor d\frac{C}{d} \rfloor - \lfloor 0\frac{C}{d} \rfloor \\ &= C, \end{aligned}$$

therefore $\{p_0, p_1, \ldots, p_{d-1}\}$ is an integer partition of C. Moreover

$$\begin{aligned} \sum_{j=1}^{n\%d} p_{d-i+j} &= \lfloor (d-i+n\%d+1)\frac{C}{d} \rfloor - \lfloor (d-i+1)\frac{C}{d} \rfloor \\ &= \lfloor (n\%d+1-i)\frac{C}{d} \rfloor - \lfloor (1-i)\frac{C}{d} \rfloor. \end{aligned} \tag{22}$$

Substituting (22) into (18), we have

$$
\begin{aligned}
W^T_{n,i} &= \sum_{j=i}^{d}(p_{d-j}-l_j)+\lfloor\frac{n}{d}\rfloor\times C+\lfloor(n\%d+1-i)\frac{C}{d}\rfloor-\lfloor(1-i)\frac{C}{d}\rfloor \\
&= \sum_{j=i}^{d}(p_{d-j}-l_j)+\lfloor\lfloor\frac{n}{d}\rfloor\times C+(n\%d+1-i)\frac{C}{d}\rfloor-\lfloor(1-i)\frac{C}{d}\rfloor \\
&= \sum_{j=i}^{d}(p_{d-j}-l_j)+\lfloor(\lfloor\frac{n}{d}\rfloor\times d+n\%d+1-i)\frac{C}{d}\rfloor-\lfloor(1-i)\frac{C}{d}\rfloor \\
&= \sum_{j=i}^{d}(p_{d-j}-l_j)+\lfloor(n+1-i)\frac{C}{d}\rfloor-\lfloor(1-i)\frac{C}{d}\rfloor \\
&= \sum_{j=i}^{d}(p_{d-j}-l_j)+\lfloor(n+1-i)\frac{C}{d}\rfloor+\sum_{j=1}^{i-1}p_{d-j} \\
&= \sum_{j=1}^{d}p_{d-j}-\sum_{j=i}^{d}l_j+\lfloor(n+1-i)\frac{C}{d}\rfloor \\
&= C-\sum_{j=i}^{d}l_j+\lfloor(n+1-i)\frac{C}{d}\rfloor
\end{aligned}
$$

Q. E. D.

For a cycle of length C with d ascending chains, the set of integers, $\{p_0, p_1, \ldots, p_{d-1}\}$, defined in (19) will be called the *natural partition* of C. Let us use Example 2 again but choose the initial orders specified by (20), that is, based on the natural partition {2,2,2} of the cycle of length 6. The accuracy increase pattern is:

Order of Accuracy							
Node No	Iteration No						
	0	1	2	3	4	5	···
1	0	2	4	6	8	10	···
2	1	3	5	7	9	11	···
3	1	3	5	7	9	11	···
4	1	3	5	7	9	11	···
5	2	4	6	8	10	12	···
6	2	4	6	8	10	12	···

Accuracy Increase							
Node No	Iteration No						
	0	1	2	3	4	5	···
1		2	2	2	2	2	···
2		2	2	2	2	2	···
3		2	2	2	2	2	···
4		2	2	2	2	2	···
5		2	2	2	2	2	···
6		2	2	2	2	2	···

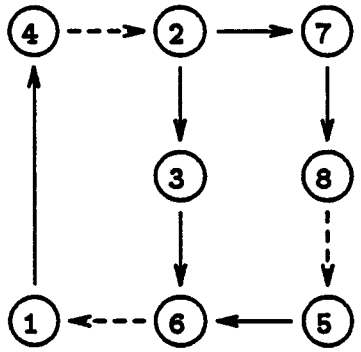
From the table we see that the phenomena described in Theorem 4.1 and Lemma 4.2 are satisfied. Next we examine how one cycle interacts with the remainder of a graph.

5 Accuracy Increase in General Graphs

We now want to discuss general graphs. Our analysis technique will be to analyze part of the graph and consider *driving terms* from other parts of the graph. These driving terms are the branches entering the part of the graph selected for analysis. The orders on the nodes at the start of these branches may, or may not, reduce the order of subsequent iterations of nodes in the selected part of the graph.

From Lemma 4.2 we see that, for a cycle of length C with d ascending chains, if the initial orders at all nodes of the cycle are chosen properly and if a driving term, if there is any, does not interfere with the order of accuracy in this cycle, then at the n^{th} waveform Gauss-Seidel iteration, the order of accuracy at each node of the cycle can be expressed as $\beta + \lfloor (n+\gamma)\frac{C}{d} \rfloor$ for some constant integers β and γ. If such a cycle is the only cycle in a system's dependency graph, then all the nodes in the dependency graph that are reachable from this cycle will have a similar pattern for their accuracy increase. (A node U is reachable from a cycle if there exists a directed path from any node in this cycle to U.) Now let's look at an example first, which shows that result in Lemma 4.2 is satisfied not only by the nodes in the cycle with minimum C/d but also by any node that is reachable from that cycle.

Example 3: Consider a dependency graph that has two cycles and nodes are ordered as shown.



$\tilde{A}_1 = \{(1,4),(2,7,8),(5,6)\}; \quad C_1/d_1 = 7/3$
$\tilde{A}_2 = \{(1,4),(2,3,6)\}; \quad C_2/d_2 = 5/2$

Since cycle $\tilde{A}_1$ has smaller C/d ratio, we choose the initial orders inside $\tilde{A}_1$ according to (20) in Lemma 4.2. That is the *natural partition* {2,2,3} of 7, length of $\tilde{A}_1$, is considered. Below we list the order of accuracy and accuracy increase at each node.

Order of Accuracy												
Node	Iteration No											
No	0	1	2	3	4	5	6	7	8	9	10	...
1	0	3	5	7	10	12	14	17	19	21	24	...
2	0	2	5	7	9	12	14	16	19	21	23	...
3	4	3	6	8	10	13	15	17	20	22	24	...
4	1	4	6	8	11	13	15	18	20	22	25	...
5	1	3	5	8	10	12	15	17	19	22	24	...
6	2	4	6	9	11	13	16	18	20	23	25	...
7	1	3	6	8	10	13	15	17	20	22	24	...
8	2	4	7	9	11	14	16	18	21	23	25	...

Accuracy Increase												
Node	Iteration No											
No	0	1	2	3	4	5	6	7	8	9	10	...
1		3	2	2	3	2	2	3	2	2	3	...
2		2	3	2	2	3	2	2	3	2	2	...
3		-1	3	2	2	3	2	2	3	2	2	...
4		3	2	2	3	2	2	3	2	2	3	...
5		2	2	3	2	2	3	2	2	3	2	...
6		2	2	3	2	2	3	2	2	3	2	...
7		2	3	2	2	3	2	2	3	2	2	...
8		2	3	2	2	3	2	2	3	2	2	...

From the previous accuracy increase table, we see that the result in Lemma 4.2 is satisfied by all the other nodes, besides the nodes in cycle $\tilde{A}_1$, in the graph. Next we want to see what will occur if the initial accuracies are not specified with the natural partition of 7. Let us use the partition {1,2,4} of 7 to specify the initial accuracies and list the order of accuracy and accuracy increase at all nodes.

Order of Accuracy												
Node No	Iteration No											
	0	1	2	3	4	5	6	7	8	9	10	...
1	0	4	5	7	10	12	14	17	19	21	24	...
2	1	2	6	7	9	12	14	16	19	21	23	...
3	4	3	7	8	10	13	15	17	20	22	24	...
4	1	5	6	8	11	13	15	18	20	22	25	...
5	2	4	5	9	10	12	15	17	19	22	24	...
6	3	4	6	9	11	13	16	18	20	23	25	...
7	2	3	7	8	10	13	15	17	20	22	24	...
8	3	4	8	9	11	14	16	18	21	23	25	...

Accuracy Increase												
Node No	Iteration No											
	0	1	2	3	4	5	6	7	8	9	10	...
1		4	1	2	3	2	2	3	2	2	3	...
2		1	4	1	2	3	2	2	3	2	2	...
3		-1	4	1	2	3	2	2	3	2	2	...
4		4	1	2	3	2	2	3	2	2	3	...
5		2	1	4	1	2	3	2	2	3	2	...
6		1	2	3	2	2	3	2	2	3	2	...
7		1	4	1	2	3	2	2	3	2	2	...
8		1	4	1	2	3	2	2	3	2	2	...

In this case we see that the partition {1,2,4} does not appear in the acuuracy increase table, whereas the natural partition {2,2,3} of 7 does. Let us use another partition {1,1,5} of 7 to specify the initial orders and see how it affects the accuracy increase pattern.

Order of Accuracy												
Node No	Iteration No											
	0	1	2	3	4	5	6	7	8	9	10	...
1	0	5	5	7	10	12	14	17	19	21	24	...
2	1	2	7	7	9	12	14	16	19	21	23	...
3	4	3	8	8	10	13	15	17	20	22	24	...
4	1	6	6	8	11	13	15	18	20	22	25	...
5	3	4	5	10	10	12	15	17	19	22	24	...
6	4	4	6	9	11	13	16	18	20	23	25	...
7	2	3	8	8	10	13	15	17	20	22	24	...
8	3	4	9	9	11	14	16	18	21	23	25	...

Accuracy Increase												
Node	Iteration No											
No	0	1	2	3	4	5	6	7	8	9	10	...
1		5	0	2	3	2	2	3	2	2	3	...
2		1	5	0	2	3	2	2	3	2	2	...
3		-1	5	0	2	3	2	2	3	2	2	...
4		5	0	2	3	2	2	3	2	2	3	...
5		1	1	5	0	2	3	2	2	3	2	...
6		0	2	3	2	2	3	2	2	3	2	...
7		1	5	0	2	3	2	2	3	2	2	...
8		1	5	0	2	3	2	2	3	2	2	...

From the table we see that this partition {1,1,5} of 7 is also not preserved in the accuracy increase pattern. From this example we see that when a dependency graph contains more than one cycle, only the natural partition of the length of the cycle with minimum C/d in the graph may be preserved in the accuracy increase pattern when it is used to specify the initial orders.

Now we select for analysis any cycle with a minimum value of C/d. We will call this the *minimum cycle.* Let us initialize the orders in the graph such that the nodes in this cycle follow the pattern specified in (20) and all other nodes in the graph are infinitely accurate. (This is not possible in practice, but is used to show that the minimum cycle determines the average accuracy increase.) We know from Lemma 4.2 that the minimum cycle will maintain an average order increase of C/d unless a driving term restricts the order of some node in the cycle. Since all other nodes were initially set to order infinity, the only way for this to happen is for the orders of a chain of nodes starting from some point on the minimum cycle and ending on the minimum cycle (called the nodes on a *sidetrack path* of the minimum cycle) to be lowered by the minimum cycle so as to reduce the order of the minimum cycle. Suppose the chain is as shown in Figure 1 by the sidetrack path from node U to node V on cycle $\tilde{A}$, where cycle $\tilde{A}$ is a *minimum cycle.* We want to show that the order in cycle $\tilde{A}$ will not be lowered by any sidetrack path.

Let $l_1, l_2, \ldots, l_d$ be the lengths of the d ascending chains of cycle $\tilde{A}$. Then $\sum_{j=1}^{d} l_j = C$. Let $\tilde{B}$ be the cycle consisting of the sidetrack path of $\tilde{A}$ from U to V and the path from V to U on $\tilde{A}$. Suppose $\tilde{B}$ has $\hat{d}$ ascending chains of length $m_1, m_2, \ldots, m_{\hat{d}}$ and $\sum_{j=1}^{\hat{d}} m_j = \hat{C}$. By assumption $C/d \leq \hat{C}/\hat{d}$. We now number the ascending chains on both cycles such that node

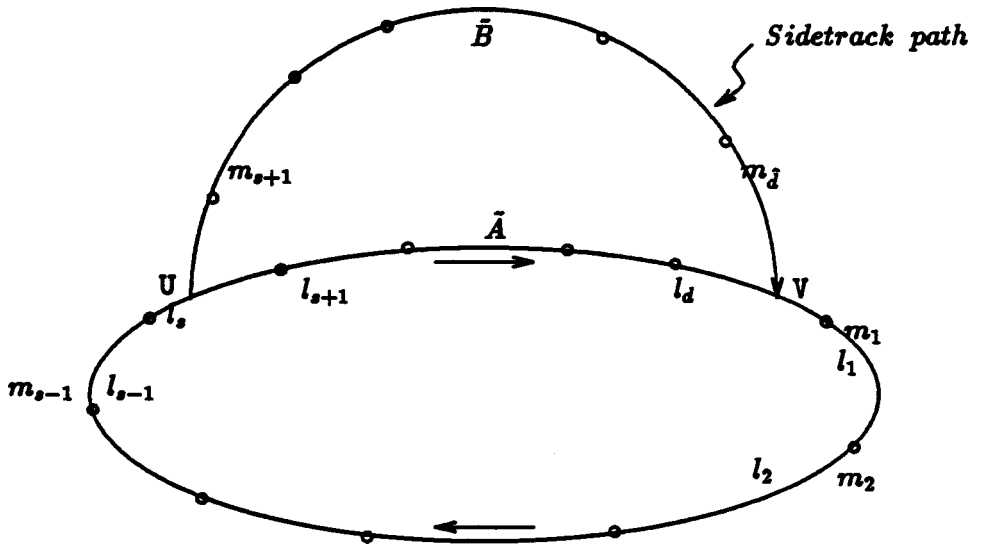


Figure 1: A minimum cycle with a sidetrack path

U is in the s^{th} chain of both cycles and node V is in the d^{th} chain of cycle $\tilde{A}$ and in the $\hat{d}^{th}$ chain of cycle $\tilde{B}$. This means $m_j = l_j$ for $j = 1, 2, \ldots, s-1$. Furthermore we assume node U is the k_s^{th} node in chain s and node V is the k_d^{th} node in chain d of cycle $\tilde{A}$. (Note the indexing of a node in a chain starts with 0.) So node V will be the $(m_{\hat{d}} - l_d + k_d)^{th}$ node in chain $\hat{d}$ of cycle $\tilde{B}$. Now we want to show that the order coming into node V from the path on $\tilde{A}$ is no greater than the order coming from the sidetrack path.

Lemma 5.1 *Let W_{n,i,k_i} ($\hat{W}_{n,i,k_i}$) denote the order of the k_i^{th} node in chain i of cycle $\tilde{A}$ ($\tilde{B}$) at the n^{th} iteration. If*

$$W_{0,i,k_i} = C - \sum_{j=i}^{d} l_j + \lfloor (1-i)\frac{C}{d} \rfloor + k_i \qquad \text{for nodes on cycle} \tilde{A}$$

and

$$\hat{W}_{0,i,k_i} = W_{0,i,k_i} \qquad \text{for nodes on path V to U}$$
$$\hat{W}_{0,i,k_i} = \infty \qquad \text{for nodes on the sidetrack path,}$$

Then

$$W_{n,d,k_d-1} \leq \hat{W}_{n,\hat{d},(m_{\hat{d}}-l_d+k_d)-1}. \tag{23}$$

Note that the $(k_d - 1)^{th}$ node on chain d of cycle $\tilde{A}$ and the $((m_{\hat{d}} - l_d + k_d) - 1)^{th}$ node on chain $\hat{d}$ of cycle $\tilde{B}$ are the only predecessors of node V.

Proof: From Figure 1 we can see that the order of node U propagating through the sidetrack path will not affect node V until $\hat{d} - s + 1$ iterations later. So the first possible lowering of order at node V by the sidetrack path will happen at the $(\hat{d} - s + 1)^{th}$ iteration. Thus we first show that

$$W_{\hat{d}-s+1,d,k_d-1} \leq \hat{W}_{\hat{d}-s+1,\hat{d},(m_{\hat{d}}-l_d+k_d)-1}. \tag{24}$$

Since

$$\begin{aligned}
\hat{W}_{\hat{d}-s+1,\hat{d},(m_{\hat{d}}-l_d+k_d)-1} &= \hat{W}^T_{\hat{d}-s+1,\hat{d}} + (m_{\hat{d}} - l_d + k_d) - 1 \\
&= \hat{W}^H_{\hat{d}-s,\hat{d}-1} + (m_{\hat{d}} - l_d + k_d) \\
&= \hat{W}^H_{1,s} + \sum_{j=s+1}^{\hat{d}-1} m_j + (m_{\hat{d}} - l_d + k_d) \\
&= \hat{W}^T_{1,s} + m_s - 1 + \sum_{j=s+1}^{\hat{d}-1} m_j + (m_{\hat{d}} - l_d + k_d) \\
&= W^T_{1,s} + \sum_{j=s}^{\hat{d}} m_j - l_d + k_d - 1 \\
&= C - \sum_{j=s}^{d} l_j + \lfloor (2-s)\frac{C}{d} \rfloor + \sum_{j=i}^{\hat{d}} m_j - l_d + k_d - 1 \\
&= \sum_{j=1}^{s-1} l_j + \lfloor (2-s)\frac{C}{d} \rfloor + \sum_{j=s}^{\hat{d}} m_j - l_d + k_d - 1 \\
&= \sum_{j=1}^{s-1} m_j + \lfloor (2-s)\frac{C}{d} \rfloor + \sum_{j=s}^{\hat{d}} m_j - l_d + k_d - 1 \\
&= \hat{C} + \lfloor (2-s)\frac{C}{d} \rfloor - l_d + k_d - 1
\end{aligned}$$

and

$$\begin{aligned}
W_{\hat{d}-s+1,d,k_d-1} &= W^T_{\hat{d}-s+1,d} + k_d - 1 \\
&= C - l_d + \lfloor (\hat{d} - s + 1 + 1 - d)\frac{C}{d} \rfloor + k_d - 1 \\
&= C - l_d + \lfloor (\hat{d} - s + 2)\frac{C}{d} - C \rfloor + k_d - 1
\end{aligned}$$

$$= \lfloor (\hat{d} - s + 2)\frac{C}{d} \rfloor - l_d + k_d - 1,$$

to prove (24) we only need to show that

$$\lfloor (\hat{d} - s + 2)\frac{C}{d} \rfloor \leq \hat{C} + \lfloor (2 - s)\frac{C}{d} \rfloor.$$

This is easy, since $\frac{C}{d} \leq \frac{\hat{C}}{\hat{d}} \Rightarrow \hat{d}\frac{C}{d} \leq \hat{C}$, and hence

$$\lfloor (\hat{d} - s + 2)\frac{C}{d} \rfloor = \lfloor \hat{d}\frac{C}{d} + (2 - s)\frac{C}{d} \rfloor \leq \lfloor \hat{C} + (2 - s)\frac{C}{d} \rfloor = \hat{C} + \lfloor (2 - s)\frac{C}{d} \rfloor.$$

In subsequent iterations we have

$$\begin{aligned} \hat{W}_{n+\hat{d}-s+1,\hat{d},(m_{\hat{d}}-l_d+k_d)-1} &= W^T_{n+1,s} + \sum_{j=s}^{\hat{d}} m_j - l_d + k_d - 1 \\ &= \hat{C} + \lfloor (n + 2 - s)\frac{C}{d} \rfloor - l_d + k_d - 1 \end{aligned}$$

and

$$W_{n+\hat{d}-s+1,d,k_d-1} = \lfloor (n + \hat{d} - s + 2)\frac{C}{d} \rfloor - l_d + k_d - 1$$

from which (23) follows directly.

Q. E. D.

6 Average Accuracy Increase in Waveform Gauss-Seidel

We define the *average accuracy increase* of a node to be the limit of p_n/n as $n \to \infty$ where p_n is the order of accuracy of a node and n is the iteration number. The average accuracy increase of a numbered graph is the minimum average accuracy increase over all nodes. Theorem 3.2 shows that the average accuracy increase in a waveform Gauss-Seidel method can not exceed the C/d of the minimum cycle. Lemma 5.1 shows that if the minimum cycle is initialized to the natural accuracy (accuracy specified by the natural partition of its length) and the remaining nodes are infinitely accurate, no sidetrack paths destroy the natural accuracy. These results can now be combined to show that the average accuracy increase for the waveform Gauss-Seidel method is exactly the C/d of the minimum cycle in the dependency graph of a given system after partitioning and ordering.

Theorem 6.1 *Suppose a minimum cycle in the dependency graph of a given system is of length C and has d ascending chains. Then the waveform Gauss-Seidel method applied to this system has average accuracy increase C/d.*

Proof: Consider two identical dependency graphs, $\hat{G}_1$ and $\hat{G}_2$, with N nodes and identical numberings. If we start with all initial orders of accuracy on both graphs set to zero and run the waveform Gauss-Seidel iterations on both synchronously, the order of accuracy of corresponding nodes on the two graphs will be the same at all steps.

After M iterations, we will perturb the iteration on $\hat{G}_2$ in the following way: lower the order of accuracy at the nodes in the minimum cycle, $\tilde{A}_2$, of $\hat{G}_2$ following (20), i.e. using the natural partition of C. Then resume waveform Gauss-Seidel iterations on both graphs. The accuracy at any node in $\hat{G}_2$ will never surpass the accuracy of its corresponding node in $\hat{G}_1$. This is easy to see inductively: the new accuracy at a node being integrated is equal to one plus the mininum accuracy of its predecessors on a graph. If the accuracy of every node in $\hat{G}_1$ is at least as large as the accuracy of its corresponding node in $\hat{G}_2$, then the same condition holds after the integration and hence before the next integration. Therefore, the average accuraacy increase of $\hat{G}_2$ is a lower bound for the average accuracy increase of the unperturbed problem on $\hat{G}_1$.

It remains to show that the average accuracy increase for $\hat{G}_2$ is C/d. Since M is fixed, the first M iterations can be ignored in computing the average. The important step is to choose M large enough that the orders of accuracy of the nodes not in the minimum cycle $\tilde{A}_2$ of $\hat{G}_2$ are effectively infinite at the perturbation, so that the result in Lemma 5.1 applies. Note that (i) after M iterations starting from accuracy 0, all nodes have accuracy $\geq M$, and (ii) when a cycle is set to the accuracy specified by its natural partition as in (20), the accuracies assigned are $\leq N$. Hence the nodes not in $\tilde{A}_2$ will have an order of accuracy at least $M - N$ greater than those in $\tilde{A}_2$ after the perturbation.

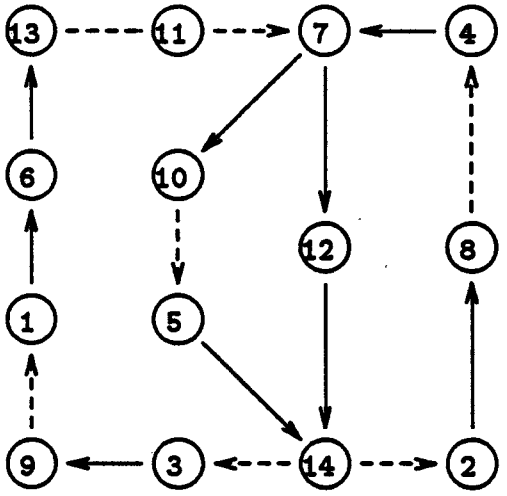
Now consider iterations on $\hat{G}_2$ after the perturbation. If the average accuracy increase is less than C/d, the graph $\hat{G}_2 - \tilde{A}_2$ must be lowering the order of $\tilde{A}_2$ (Lemma 5.1 implies that the propagation of an accuracy from $\tilde{A}_2$ into $\hat{G}_2 - \tilde{A}_2$ and back to $\tilde{A}_2$ can not be responsible for lowering the accuracy). Since the order of accuracy of $\hat{G}_2 - \tilde{A}_2$ can be made arbitrarily higher

than that of $\tilde{A}_2$ at the perturbation ($M - N$ higher), the average accuracy of $\hat{G}_2 - \tilde{A}_2$ must be less than C/d. The argument can now be completed by induction on the size of the graph: It is certainly true for $N = 2$. Assume it is true for $2 < N \leq K - 1$. If $|\hat{G}_2| = K$ than either $\hat{G}_2$ has no cycles or $|\hat{G}_2 - \tilde{A}_2| \leq K - 2$ where $\tilde{A}_2$ is a minimum cycle. Hence the average accuracy increase of $\hat{G}_2 - \tilde{A}_2$ is at least C/d so it can not lower the order of $\tilde{A}_2$.

Q. E. D.

In the following we give some examples to illustrate this result.

Example 4: Consider a dependency graph that has four cycles and the nodes are ordered as shown.



$\tilde{A}_1 = \{(1, 6, 13), (11), (7, 10), (5, 14), (3.9)\}$
$\tilde{A}_2 = \{(1, 6, 13), (11), (7, 12, 14), (3.9)\}$
$\tilde{A}_3 = \{(2, 8), (4, 7, 10), (5, 14))\}$
$\tilde{A}_4 = \{(2, 8), (4, 7, 12, 14)\}$

$C_1/d_1 = 10/5$
$C_2/d_2 = 9/4$
$C_3/d_3 = 7/3$
$C_4/d_4 = 6/2$

From the graph we see that cycle $\tilde{A}_1$ is the minimum cycle. We would like to see how the minimum cycle affects the remaining cycles in the graph. First we use the same initial orders for all nodes in the graph and list the order of accuracy and accuracy increase at all nodes.

Order of Accuracy																	
Node No	Iteration No																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1	0	1	3	5	7	8	11	13	15	17	18	21	23	25	27	28	...
2	0	1	3	5	6	9	11	13	15	16	19	21	23	25	26	29	...
3	0	1	3	5	6	9	11	13	15	16	19	21	23	25	26	29	...
4	0	1	3	5	7	8	11	13	15	17	18	21	23	25	27	28	...
5	0	1	3	4	7	9	11	13	14	17	19	21	23	24	27	29	...
6	0	2	4	6	8	9	12	14	16	18	19	22	24	26	28	29	...
7	0	1	2	5	7	9	11	12	15	17	19	21	22	25	27	29	...
8	0	2	4	6	7	10	12	14	16	17	20	22	24	26	27	30	...
9	0	2	4	6	7	10	12	14	16	17	20	22	24	26	27	30	...
10	0	2	3	6	8	10	12	13	16	18	20	22	23	26	28	30	...
11	0	1	4	6	8	10	11	14	16	18	20	21	24	26	28	30	...
12	0	2	3	6	8	10	12	13	16	18	20	22	23	26	28	30	...
13	0	3	5	7	9	10	13	15	17	19	20	23	25	27	29	30	...
14	0	2	4	5	8	10	12	14	15	18	20	22	24	25	28	30	...

Accuracy Increase																	
Node No	Iteration No																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1		1	2	2	2	1	3	2	2	2	1	3	2	2	2	1	...
2		1	2	2	1	3	2	2	2	1	3	2	2	2	1	3	...
3		1	2	2	1	3	2	2	2	1	3	2	2	2	1	3	...
4		1	2	2	2	1	3	2	2	2	1	3	2	2	2	1	...
5		1	2	1	3	2	2	2	1	3	2	2	2	1	3	2	...
6		2	2	2	2	1	3	2	2	2	1	3	2	2	2	1	...
7		1	1	3	2	2	2	1	3	2	2	2	1	3	2	2	...
8		2	2	2	1	3	2	2	2	1	3	2	2	2	1	3	...
9		2	2	2	1	3	2	2	2	1	3	2	2	2	1	3	...
10		2	1	3	2	2	2	1	3	2	2	2	1	3	2	2	...
11		1	3	2	2	2	1	3	2	2	2	1	3	2	2	2	...
12		2	1	3	2	2	2	1	3	2	2	2	1	3	2	2	...
13		3	2	2	2	1	3	2	2	2	1	3	2	2	2	1	...
14		2	2	1	3	2	2	2	1	3	2	2	2	1	3	2	...

From the table we see that after every 5 iterations the accuracy at each node increases by 10, the cycle length of $\tilde{A}_1$. And the increase pattern at each node is {2,2,2,1,3}, or its circular shifts, which happens to be the ascending chain lengths of $\tilde{A}_1$ in reverse orientation. Next we choose the initial orders inside $\tilde{A}_1$ such that the natural partition {2,2,2,2,2} of C_1 is

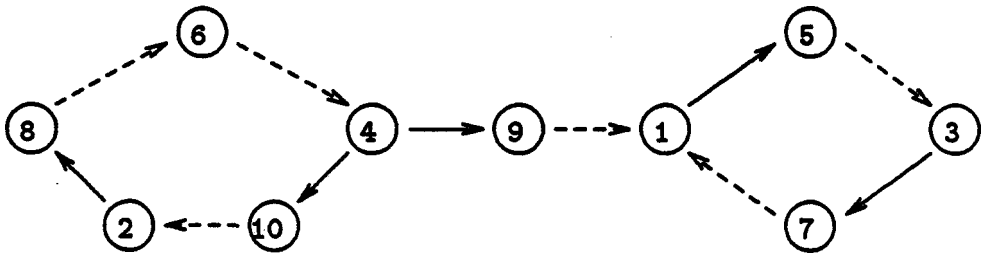
considered.

Order of Accuracy																	
Node	Iteration No																
No	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
2	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
3	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
4	1	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
5	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
6	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...
7	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	...
8	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...
9	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...
10	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...
11	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...
12	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...
13	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	...
14	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	...

Accuracy Increase																	
Node	Iteration No																
No	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
1		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
2		1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
3		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
4		1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
5		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
6		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
7		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
8		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
9		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
10		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
11		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
12		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
13		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...
14		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	...

From the table we see that all the nodes in the graph have the same accuracy increase pattern, the natural partition of C_1.

Example 5 Consider a dependency graph that has two cycles and one of the cycle is reachable by the other.



$$\tilde{A}_1 = \{(2,8),(6),(4,10)\}; \qquad C_1/d_1 = 5/3$$
$$\tilde{A}_2 = \{(1,5),(3,7)\}; \qquad C_2/d_2 = 4/2$$

Again we list the order of accuracy and accuracy increase at all nodes after each waveform Gauss-Seidel iteration. From the table we can see that the minimum cycle dominates the accuracy increase for the entire graph.

Order of Accuracy												
Node No	Iteration No											
	0	1	2	3	4	5	6	7	8	9	10	...
2	0	1	3	4	6	8	9	11	13	14	16	...
3	0	1	3	5	6	8	10	11	13	15	16	...
4	0	1	2	4	6	7	9	11	12	14	16	...
5	0	2	4	5	7	9	10	12	14	15	17	...
6	0	1	3	5	6	8	10	11	13	15	16	...
7	0	2	4	6	7	9	11	12	14	16	17	...
8	0	2	4	5	7	9	10	12	14	15	17	...
9	0	2	3	5	7	8	10	12	13	15	17	...
10	0	2	3	5	7	8	10	12	13	15	17	...

Accuracy Increase												
Node	Iteration No											
No	0	1	2	3	4	5	6	7	8	9	10	...
1		1	2	1	2	2	1	2	2	1	2	...
2		1	2	1	2	2	1	2	2	1	2	...
3		1	2	2	1	2	2	1	2	2	1	...
4		1	1	2	2	1	2	2	1	2	2	...
5		2	2	1	2	2	1	2	2	1	2	...
6		1	2	2	1	2	2	1	2	2	1	...
7		2	2	2	1	2	2	1	2	2	1	...
8		2	2	1	2	2	1	2	2	1	2	...
9		2	1	2	2	1	2	2	1	2	2	...
10		2	1	2	2	1	2	2	1	2	2	...

7 Conclusion

Effective use of the Waveform Gauss-Seidel method depends on the ordering of the nodes. The result in this paper provides a basis for selecting the ordering. It is probable that the problem of finding an ordering that maximizes the average order increase is NP hard so not computationally feasible. However, the result suggests heuristics that could be used to maximize the lengths of ascending chain in short cycles. Future work will test some heuristics.

Although the waveform Gauss-Seidel is usually thought of as a serial method, there are several ways in which it can be used for parallel computation. In one approach, the integration of later nodes can be staggered in time. In another, several nodes can be integrated in parallel. In this approach, for a p processor system, up to p nodes can be numbered with each number in the sequence so long as there are no branches between nodes with the same number. At each step of a single waveform Gauss-Seidel sweep, all nodes with the same number can be integrated in parallel. Since they are mutually independent, the order result of Theroem 2.2 applies, and hence all results in this paper apply.

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16. Abstracts

The traditional approach for solving large dynamical systems is time consuming. Waveform relaxation, an iterative technique for solving systems of differential equations, can be used to reduce the processing time. It has been shown to converge superlinearly on finite intervals. In this paper, the order of accuracy of solutions generated by a relaxation approach, the waveform Gauss-Seidel method, is discussed.

In this approach, a directed graph, called a dependency graph, is used to indicate the coupling relations among all components. The relation between the accuracy increase after each Gauss-Seidel iteration and the lengths of ascending chains (simple directed paths) in cycles in the dependency graph is discussed. It is proved that the cycle in the dependency graph which has the minimum ratio of its length to its number of ascending chains, determines the average accuracy increase. Effective use of the waveform Gauss-Seidel method depends on the ordering of the components. The result in this paper provides a basis for selecting the ordering.

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