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# MODIFIED LOCAL SIMILARITY FOR NATURAL CONVECTION ALONG A NONISOTHERMAL VERTICAL FLAT PLATE INCLUDING STRATIFICATION*

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**ABSTRACT**

An approximate method has been developed to analyze natural convection along a vertical flat plate with variable surface conditions and temperature stratification. This method uses the boundary layer velocity and temperature profiles from the local similarity method and imposes explicit conservation of energy along the plate resulting in required relationships for the similarity parameters for energy conservation. The results from this Modified Local Similarity (MLS) method are compared to those from other methods for a number of nonsimilar natural convection problems. Based on these comparisons, the MLS method is a significant improvement to the local similarity approach and is a useful approximate tool for analyzing natural convection on vertical surfaces for nonsimilar conditions.

**NOMENCLATURE**

- A area
- $c_p$ specific heat
- $f'$ velocity similarity variable
- g gravitational constant
- $Gr_x$ Grashof number based on x
- J stratification similarity parameter
- LS local similarity
- $\dot{m}$ mass flow rate per unit width
- n temperature difference similarity parameter
- N temperature difference constant
- PL Power Law Distribution
- Pr Prandtl number
- $q''$ heat flux
- Q integrated heat flux per unit width
- $\Delta T$ temperature difference, $T_w - T_f$
- T temperature
- u x-direction velocity
- v y-direction velocity
- W width of plate
- $\Delta x$ difference in x, $x_2 - x_1$
- x distance along plate surface
- y distance normal to plate surface

Greek

- $\alpha$ thermal diffusivity
- $\beta$ coefficient of thermal expansion
- $\delta$ boundary layer thickness
- $\eta$ dimensionless coordinate
- $\mu$ viscosity
- $\nu$ kinematic viscosity
- $\rho$ density
- $\theta$ dimensionless temperature

Subscripts

- 1 value at position $x_1$
- 2 value at position $x_2$
- 12 value between $x_1$ and $x_2$
- f fluid
- r reference
- w wall

Superscripts

- $-$ average value
- $'$ derivative with respect to $\eta$
- * entrainment or ejected value

**INTRODUCTION**

Natural convection along vertical surfaces occurs in the more than 50 oil-filled caverns in the Strategic Petroleum Reserve (SPR). These caverns are located in a number of large salt domes where the geothermal temperature difference over the cavern height of up to 600 m can be 15°C or more. The hotter salt is located at the bottom of the cavern; this configuration causes natural convection in the enclosed fluids as a result of buoyancy forces. Due to the large length scale, highly turbulent boundary layer conditions will be encountered with Rayleigh numbers up to approximately

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$10^{16}$. Since the heat transfer between the salt and the fluids in the cavern is coupled, heat transfer to the oil and the resulting natural convection can occur during the entire anticipated storage period of up to 30 years. SPR cavern wall conditions are nonuniform due to the geothermal temperature difference. In addition, the fluid temperature is nonuniform owing to the thermal stratification of the oil. Thus, the wall conditions and the ambient fluid temperature are both variable. In order to efficiently evaluate the natural convection boundary layer behavior in each cavern, a rapid analysis technique is needed.

The methods in general use for the analysis of natural convection are the integral (Sparrow, 1955), similarity, local similarity, local nonsimilarity (Sparrow, et al., 1970, 1971 and Minkowycz and Sparrow, 1974), and finite difference approaches (Cebeci and Bradshaw, 1984). In addition, approximate methods have been developed by Raithby, et al. (1975, 1977, 1978), Kao, et al. (1977), Yang, et al. (1982), and Lee and Yovanovich (1987, 1988).

The integral method could be used, although the assumed profiles are a problem for turbulent flow conditions. The wall and fluid temperature variations preclude direct use of the similarity solutions. The local similarity method, which applies the similarity solutions based only on the local boundary conditions, would be appropriate for SPR since the boundary layer results can be tabulated for use at each time step; therefore, the resulting calculations would be fast. However, the method does not consider the history of the boundary layer, and errors in the heat transfer rate can be significant even for simple cases. Local nonsimilarity and finite difference methods are impractical due to long estimated computing times for the 30 year transient involved.

Approximate methods have been proposed by a number of authors. However, the methods developed by Raithby, et al. (1975, 1977, 1978) and Lee and Yovanovich (1987, 1988) were not considered for use in SPR since neither method reduces to the similarity solutions for similar boundary conditions. Differences of up to 20% have been noted. The methods developed by Kao, et al. (1977) and by Yang, et al. (1982), approach the local nonsimilarity method in complexity and were therefore not considered.

If the heat transfer rate errors noted for the local similarity approach can be significantly reduced, the method would be ideal for SPR. The present study attempts to minimize this problem by modifying the local similarity approach to explicitly conserve energy as the boundary layer develops along the surface. This Modified Local Similarity (MLS) approach is developed and compared to results from other methods in this paper. This method is used in the SPR velocity model developed by Webb (1988a).

## FORMULATION

Consider natural convection boundary layer flow along a flat plate as depicted in Figure 1. The boundary layer energy equation can be integrated along the plate using the local boundary layer velocity and temperature profiles. This equation must be satisfied for global energy conservation. In the present study, the boundary layer profiles used in this equation are calculated by the local similarity method. The local similarity method has two parameters which are mathematical descriptions of the temperature variation along the plate and in the surrounding fluid. In addition to being mathematical parameters, these variables have physical significance with regard to conservation of energy. The global energy conservation equation, which must be satisfied, can be written in terms of the local similarity parameters. Relationships for the two local similarity parameters can then be developed to explicitly satisfy energy conservation.

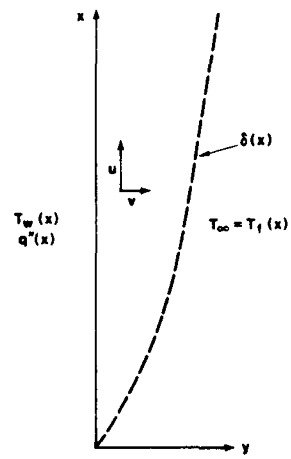


Fig. 1. Boundary layer coordinates.

For example, if the wall heat flux and fluid temperature variation are known, the use of the local similarity profiles in the global energy conservation equation results in a required variation of the two similarity parameters. If these relationships are satisfied, global energy conservation is achieved. This approach is called the MLS method and is detailed below.

### Global Energy Equation

Conservation of energy along the plate per unit width can be written as

$$\dot{m}_1 \; c_p \; (\bar{T}_1 - T_f^*) \; + \; \bar{q}''_{12} \; \Delta x = \dot{m}_2 \; c_p \; (\bar{T}_2 - T_f^*) \qquad (1)$$

where

$$\bar{q}''_{12} = \frac{1}{\Delta x} \int_{x_1}^{x_2} q'' \; dx \qquad (2)$$

and $T_f^*$ is the average temperature of the fluid entrained into or ejected from the boundary layer between $x_1$ and $x_2$. The value of $T_f^*$ is then equal to $T_f$ at location $x^*$. In the present analysis, the MLS approach results in a relationship for $x^*$ based on energy conservation considerations. Note that the fluid specific heat, $c_p$, is assumed to be constant.

The average temperature of the entrained or ejected fluid will be assumed equal to the local environmental fluid temperature for this analysis. For Prandtl number fluids of order 1.0 and higher, such as air, water, and oil, the velocity boundary layer thickness is larger than the the thermal boundary layer, so any fluid exchange will be at the environmental temperature. This assumption breaks down for low Prandtl number fluids such as liquid metals where the similarity solution gives a larger thermal boundary layer than velocity boundary layer (Gebhart, 1985).

The average boundary layer temperature, $\bar{T}$, to be used in equation (1) is simply the bulk fluid temperature at that location, or

$$\bar{T} = \frac{\int u\ T\ dy}{\int u\ dy} \tag{3}$$

which for the present analysis is rewritten as

$$\bar{T} = T_f(x) + \frac{\int u\ (T - T_f(x))\ dy}{\int u\ dy}. \tag{4}$$

Combining the equations (1) and (4) results in

$$\dot{m}_1\ c_p\ (\bar{T}_1 - T_f^*) + \bar{q}''_{12}\ \Delta x$$
$$= \dot{m}_2\ c_p\ (T_{f_2} - T_f^* + \frac{\int u\ (T - T_f(x))\ dy}{\int u\ dy}). \tag{5}$$

where the integrals in equation (5) are evaluated at $x_2$. The above equation is general; any restrictions as to the orientation, etc. are from evaluation of the boundary layer velocity and temperature profiles. These profiles will be based on local similarity.

<u>MLS Method</u>

The boundary layer profiles in this study are for laminar natural convection over a nonisothermal vertical flat plate in a variable temperature fluid medium. Invoking the Boussinesq approximation with otherwise constant properties and neglecting viscous dissipation and the pressure-work term, the steady-state conservation equations are (Jaluria, 1980)

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6}$$

x-Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\ \beta\ (T - T_f(x)) + \nu\frac{\partial^2 u}{\partial y^2} \tag{7}$$

Energy

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \tag{8}$$

The above conservation equations can be integrated across the boundary layer resulting in

Momentum

$$\frac{d}{dx}\int u^2\ dy = g\ \beta\int (T - T_f(x))\ dy - \nu\left.\frac{\partial u}{\partial y}\right|_w \tag{9}$$

Energy

$$\frac{d}{dx}\int u\ (T - T_f(x))\ dy + \frac{dT_f}{dx}\int u\ dy$$
$$= -\ \alpha\left.\frac{\partial T}{\partial y}\right|_w \tag{10}$$

where the second term on the LHS of the energy equation accounts for temperature stratification.

Two energy equations are considered in the present analysis. The global energy equation (5) is concerned with the energy in the boundary layer as it develops along the plate. The local energy equation (10) is related to the energy in the boundary layer at location x only. Both equations must be satisfied. Similarity variables will be used to rewrite the energy equations. These equations will then be combined to lead to relationships for the similarity variables that must be satisfied for global and local energy conservation.

According to Sparrow and Gregg (1958) and Yang (1960), similarity exists for two temperature distributions: the power-law and the exponential distributions. The power-law distribution is the more useful case and is discussed in this paper. Results for the exponential distribution are given by Webb (1988b).

For the power-law distribution, the temperature difference between the wall and fluid is a function of the distance x to a power, or

$$\Delta T(x) = T_w(x) - T_f(x) = N\ x^n. \tag{11}$$

For similarity, the fluid temperature variation must be of the same form, or (Jaluria, 1980)

$$T_f(x) - T_r = \frac{J\ N}{4\ n}\ x^n = \frac{J}{4\ n}\ \Delta T(x) \tag{12}$$

where the reference temperature, $T_r$, is the fluid temperature at x = 0. If the fluid temperature is constant, J is equal to 0.

The similarity variables for this case are (Gebhart and Mollendorf, 1969)

$$f' = \frac{x}{2\ \nu}\ Gr_x^{-1/2}\ u. \tag{13}$$

$$\eta = \frac{y}{x}\left(\frac{Gr_x}{4}\right)^{1/4} \tag{14}$$

$$\theta(\eta) = \frac{T(x) - T_f(x)}{T_w(x) - T_f(x)} \tag{15}$$

where

$$Gr_x = \frac{g\ \beta\ x^3\ (T_w(x) - T_f(x))}{\nu^2} \tag{16}$$

Using the similarity variables and assuming local similarity, the boundary layer partial differential equations (6)-(8) reduce to a set of coupled ordinary differential equations which are (Jaluria, 1980)

$$f''' + (n + 3)\ f\ f'' - 2\ (n + 1)\ f'^2 + \theta = 0 \tag{17}$$

$$\frac{\theta''}{Pr} + (n + 3)\, f\, \theta' - 4\, n\, f'\, \theta - J\, f' = 0 \qquad (18)$$

so the boundary layer velocity and temperature profiles are a function of the similarity parameters n and J. The above equations have been solved by a finite difference method as summarized by Webb (1989).

Expressing the fluid temperature difference in terms of the similarity parameter J gives

$$T_{f_2} - T_f^* = \frac{J\, \Delta T_2}{4\, n} \left(1 - \left(\frac{x^*}{x_2}\right)^n\right). \qquad (19)$$

Using the similarity parameters along with equation (19), the global energy equation (5) becomes

$$\dot{m}_1\, c_p\, (\bar{T}_1 - T_f^*) + \bar{q}''_{12}\, \Delta x = \dot{m}_2\, c_p \left(\frac{J}{4\, n}\left(1 - \left(\frac{x^*}{x_2}\right)^n\right) + \frac{\int f'\, \theta\, d\eta}{\int f'\, d\eta}\right) \Delta T_2. \qquad (20)$$

The mass flow rate per unit width can be expressed in terms of the local similarity variables as

$$\dot{m} = \rho\, \bar{u}\, A / W = \rho\, \bar{u}\, \delta = 4\, \mu \int f'\, d\eta \left(\frac{g\, \beta}{4\nu^2}\right)^{1/4} x^{3/4}\, \Delta T^{1/4}. \qquad (21)$$

The heat flux relationship

$$q'' = - k \left.\frac{\partial T}{\partial y}\right|_w = - k\, \theta'_w \left(\frac{g\, \beta}{4\nu^2}\right)^{1/4} \Delta T^{5/4}\, x^{-1/4} \qquad (22)$$

can be used to get the temperature difference as a function of x, and the mass flow rate per unit width is

$$\dot{m} = 4\, \mu \int f' d\eta \left(\frac{g\, \beta}{4\nu^2}\right)^{0.2} x^{0.8} \left(\frac{q''}{- k\, \theta'_w}\right)^{0.2} \qquad (23)$$

The global energy equation (20) then becomes

$$\dot{m}_1\, c_p\, (\bar{T}_1 - T_f^*) + \bar{q}''_{12}\, \Delta x = 4\, Pr \int f' d\eta \left(\frac{q''_2}{- \theta'_w}\right) \left(\frac{J}{4\, n}\left(1 - \left(\frac{x^*}{x_2}\right)^n\right) + \frac{\int f'\, \theta\, d\eta}{\int f'\, d\eta}\right) x_2. \qquad (24)$$

Using the similarity variables in the integrated local boundary layer equations (9) and (10) results in the following equations for the boundary layer quantities under the assumption of local similarity

Momentum

$$(5 + 3n) \int f'^2\, d\eta = \int \theta\, d\eta - f''_w \qquad (25)$$

Energy

$$(5n + 3) \int f'\, \theta\, d\eta = - \frac{\theta'_w}{Pr} - J \int f'\, d\eta. \qquad (26)$$

Rearranging the local boundary layer energy equation (26) and substituting it into the global energy equation (24) results in

$$(\dot{m}_1\, c_p\, (\bar{T}_1 - T_f^*) + \bar{q}''_{12}\, \Delta x) / (q''_2\, x_2) = \frac{\frac{J}{n}\left(1 - \left(\frac{x^*}{x_2}\right)^n\right) + 4 \int f'\, \theta\, d\eta / \int f'\, d\eta}{J + (5n + 3) \int f'\, \theta\, d\eta / \int f'\, d\eta}. \qquad (27)$$

The similarity parameter n is independent of J. Therefore, from equation (27), the expression for n is

$$n = \frac{1}{5}\, [4\, q''_2\, x_2 / (\dot{m}_1\, c_p\, (\bar{T}_1 - T_f^*) + \bar{q}''_{12}\, \Delta x) - 3]. \qquad (28)$$

Taking $x_1$ at the leading edge of the plate ($x_1$=0.) with no initial mass flow rate, which is usually the case, then $\Delta x = x_2$, and the equation for n simplifies to

$$n = \frac{1}{5}\, [4\, q''_2 / \bar{q}''_{12} - 3]. \qquad (29)$$

The value of the similarity parameter n is just a function of the ratio of the local to the average heat flux up to that point.

Similarly, from equations (27) and (28),

$$x^* = x_2 \left[1 - \frac{4n}{(5n + 3)}\right]^{1/n}. \qquad (30)$$

The fluid temperature evaluated at $x^*$ is that required for global energy conservation.

Surprisingly, the stratification parameter, J, is independent of global conservation of energy. Instead, the value of J is determined by the local value of the heat flux, $q''_2$. Equating equations (1) and (28) gives

$$\dot{m}_2\, c_p\, (\bar{T}_2 - T_f^*) = \frac{4}{(5n + 3)}\, q''_2\, x_2 \qquad (31)$$

where the values on the RHS are known. This equation includes the effect of temperature stratification on the local energy balance. Using the relationships developed above for $\dot{m}$ and $\Delta T$, the equation can be written as

$$A_1 \frac{\int f' d\eta}{(-\theta'_w)^{0.2}} \left(A_2 \frac{\int f'\, \theta\, d\eta}{\int f' d\eta\, (-\theta'_w)^{0.8}} + T_{f_2} - T_f^*\right) = A_3 \qquad (32)$$

where

$$A_1 = 4\, c_p\, \mu \left(\frac{g\, \beta}{4\nu^2}\right)^{0.2} x_2^{0.8} \left(\frac{q''_2}{k}\right)^{0.2} \qquad (33)$$

$$A_2 = \left(\frac{g\, \beta}{4\nu^2}\right)^{-0.2} x_2^{0.2} \left(\frac{q''_2}{k}\right)^{0.8} \qquad (34)$$

$$A_3 = \frac{4}{(5n + 3)} q_2'' x_2 . \quad (35)$$

For uniform fluid conditions, $T_{f2}$ is equal to $T_f^*$, and the above expression reduces to the local integrated energy equation with J equal to zero. As discussed earlier, the boundary layer parameters in the above expression are dependent on the similarity parameters n and J. Since the value of n is determined by equation (28) or (29), the only undefined parameter is J.

Solution of the equation (32) for J initially looks difficult. In practice, however, solution is straightforward and, for the present investigation, has been accomplished by iterating on the form

$$\frac{\int f' \theta \, d\eta}{- \theta_w'} = \frac{A_3 - \dot{m}_2^{i-1} c_p (T_{f_2} - T_f^*)}{4 \Pr q_2'' x_2} \quad (36)$$

where $\dot{m}_2^{i-1}$ is the value of $\dot{m}_2$ from the previous iteration and $T_f^*$ is evaluated from conservation of energy. The ratio of the LHS of the equation is a strong function of J for a given value of n, and convergence has not been a problem.

In summary, for a specified heat flux problem, the similarity parameter n is determined directly from equation (28) or (29). For a uniform environmental fluid temperature, the similarity parameter J is equal to 0. Otherwise, the value of J is determined by iterating on equation (36). All the boundary layer parameters are uniquely determined by these values of n and J. For situations where similarity conditions are imposed, the similarity solutions are obtained. This is not the case for the approximate methods developed by Raithby, et al. (1975, 1977, 1978) and by Lee and Yovanovich (1987, 1988). For variable conditions where an exact similarity solution does not exist, the MLS method provides an estimate of "equivalent" similarity conditions including velocity and temperature profiles by requiring global conservation of energy and the same local heat flux at position $x_2$.

In the above development, the heat flux variation is assumed to be specified. This situation is not always the case, as the temperature distribution is sometimes given. In order to calculate the similarity parameters, energy consistency between the specified problem and the MLS method is required. The integrated heat flux per unit width for constant properties is proportional to the following integral

$$Q = \int q'' \, dx \propto \int \theta_w' \, \Delta T^{5/4} \, x^{-1/4} \, dx \quad (37)$$

and the expression for n becomes

$$n = \frac{1}{5} \left[ 4 \, q_2'' \, x_2 \, / \, (\dot{m}_1 \, c_p \, (\bar{T}_1 - T_f^*) + Q) - 3 \right] \quad (38)$$

which, for $x_1$ and $\dot{m}_1$ equal to 0. can be written as

$$n = \frac{1}{5} \left[ \frac{4 \, \theta_w' \, \Delta T_2^{5/4} \, x_2^{3/4}}{\int \theta_w' \, \Delta T^{5/4} x^{-1/4} dx} - 3 \right] . \quad (39)$$

The temperature gradient for a given Prandtl number is only a function of n and J, so iteration is required on this equation and, when necessary, equation (39) for J.

For specified surface temperatures, the MLS method is not a local similarity approach since the answer at x depends on the results at the upstream locations. Iteration is required for the variation of the similarity parameters with x. However, this iteration is easily accomplished since the only term that depends on n and J is $\theta_w'$, and convergence is rapid for the cases analyzed in this report.

While specified temperatures are a convenient analytical case, the wall temperature and wall heat fluxes are usually coupled to each other through heat conduction, and either the wall temperature or the heat flux can be used in the solution scheme. For the MLS method, heat fluxes are considerably more convenient than temperatures since no iteration is involved.

## EVALUATION

The Modified Local Similarity (MLS) method derived above has been applied to a number of nonsimilar wall temperature and heat flux cases with uniform fluid temperature and to an isothermal plate in a stratified fluid environment. The results in this section compare the predictions from the MLS method with those from other approaches and, for the case of an isothermal plate in a stratified fluid, to experimental data. The results from another possible implementation of the local similarity approach in addition to the MLS method are also given. While the MLS method is based on conservation of energy as the boundary layer develops and matching the local heat flux, another reasonable approach would be matching the local value of the specified parameter (temperature difference or heat flux) as well as the local slope of that parameter. The predictions from this method will be referred to as the LS* approach.

### Uniform Fluid Temperature

For uniform fluid temperature conditions, the MLS and LS* methods have been applied to specified wall temperature and specified heat flux cases. Results of these cases for a number of other methods are summarized by Yang, et al. (1982) for a Prandtl number of 0.7 where the property term is assumed equal to 1.0, or

$$\left( \frac{g \, \beta}{4 \nu^2} \right) = 1.0. \quad (40)$$

In all these cases, the stratification parameter, J, is equal to 0. since the fluid temperature is uniform.

The results from the MLS method and the LS* approach will be compared to the following predictions.

1. Numerical - as given by Kao, et al. (1977).
2. Kao LS - Kao, et al. (1977) local similarity.
3. Kao method - The method of Kao, et al. (1977) which is basically a perturbation approach.
4. Yang method - The method of Yang, et al. (1982) which is a series expansion approach.

Predictions from other methods, such as the integral approach, will also be included where available.

Specified Surface Temperature. The comparisons are based on the temperature gradient at the surface which is related to the local heat transfer coefficient. In addition, the approximation of the temperature difference behavior is presented.

1) $\Delta T = e^x$. Figure 2a shows the desired temperature difference as well as the variation predicted by the MLS and LS* approaches. The predictions depend on n which itself is a function of x. Therefore, in Figure 2a, two curves for the appropriate value of n corresponding to the two x values of 0.5 and 2.0 are shown for each approach. In general, the variation of the temperature difference is reasonably close to the desired behavior. The temperature difference variation is well represented by both methods. The surface temperature gradient as a function of x is depicted in Figure 2b. The gradient is underpredicted by the MLS method by approximately 5%. While the error is larger than the other methods, the magnitude is still relatively small. For the LS* approach, a slight overprediction of the gradient, especially near the front of the plate, is noted. This behavior is also seen for the Kao LS method. Predictions for the Kao and Yang methods are not shown in this figure since both approaches yield predictions indistinguishable from the numerical results.

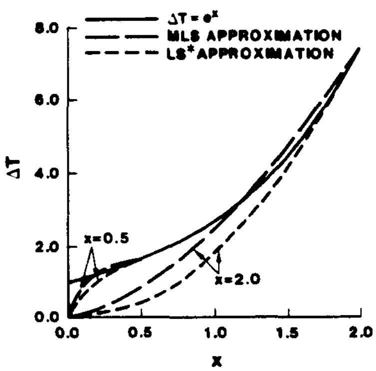


Fig. 2a. Approximation of $\Delta T$ for $\Delta T = e^x$.

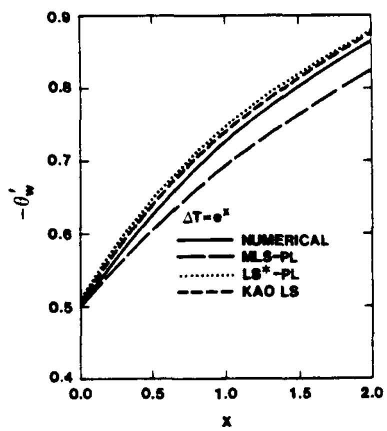


Fig. 2b. Surface temperature gradient for $\Delta T = e^x$.

2) $\Delta T = \sin x$. The predicted surface temperature gradient as a function of x for a number of other methods is depicted in Figure 3. The Yang method gives excellent results up to an x value of 2.2 after which the method has convergence problems. The Kao method also gives good results out to an x value of 2.3; after this point, the Kao method also no longer converges. The Kao local nonsimilarity (LNS) results are surprisingly poor; only results to an x value of 2.0 are given by Kao (1976). The Kao LS results show reasonable agreement for the entire problem, although a systematic underprediction is evident.

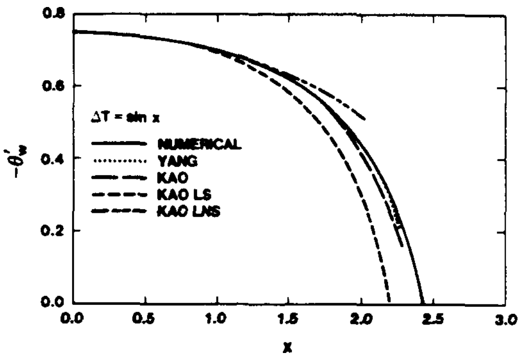


Fig. 3. Surface temperature gradient for $\Delta T = \sin x$. Results from other methods.

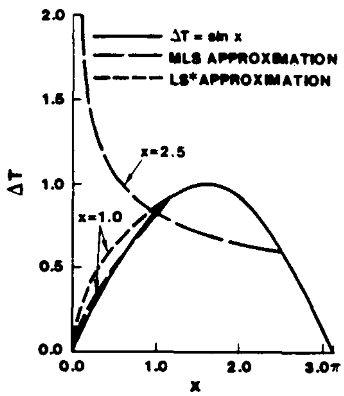


Fig. 4a. Approximation of $\Delta T$ for $\Delta T = \sin x$.

Results for the MLS and LS* power-law approaches are shown in Figure 4. The temperature difference is given in Figure 4a for n corresponding to x values of 1.0 and 2.5. For small x values ($<\sim\pi/2$), both methods give a good approximation of the sinusoidal temperature difference. For larger x values, the MLS distribution becomes increasingly poor since n becomes negative for x values greater than about 2.16. When n changes sign, the shape changes dramatically as shown in Figure 4a. No temperature difference variation is presented for the LS* method at an x value of 2.5 since the zero heat flux location is $x \sim 1.88$ for this approach. From Figure 4b, the temperature gradient is well predicted for small x values. For large x values, the MLS method overpredicts the temperature gradient and the location of zero heat flux, while the LS* approach underpredicts the results. While not as good as some of the other methods, the MLS approach is better than the LS* method and about the same as the Kao LS approach. This discrepancy is not unexpected due to the poor approximation of the temperature difference behavior by the MLS method at large x values.

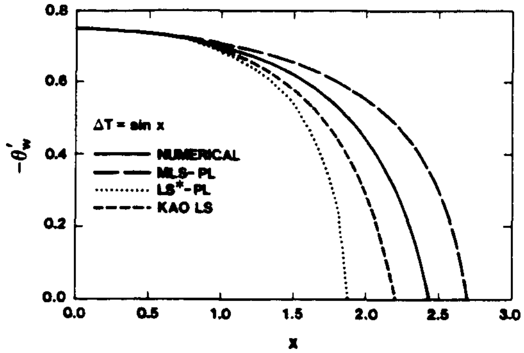


Fig. 4b. Surface temperature gradient for ΔT = sin x.

Specified Wall Heat Flux. The comparisons between the various methods are based on the wall to fluid temperature difference. In addition, the approximation of the heat flux behavior is presented.

1) $q''/k=e^x$. The heat flux distribution for the MLS and LS* approaches is given in Figure 5a for n corresponding to x values of 0.5 and 2.0. In general, the heat flux variation is well represented by both methods. These conclusions are similar to those for the exponential temperature difference case discussed earlier. The surface temperature as a function of x is depicted in Figure 5b. The MLS and LS* methods both successfully predict the surface temperature variation with x. The predictions of the Kao and the Yang methods are not shown since they are indistinguishable from the numerical results. All the methods perform well for this case.

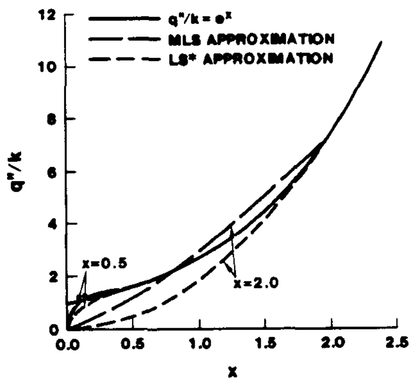


Fig. 5a. Approximation of $q''/k$ for $q''/k = e^x$.

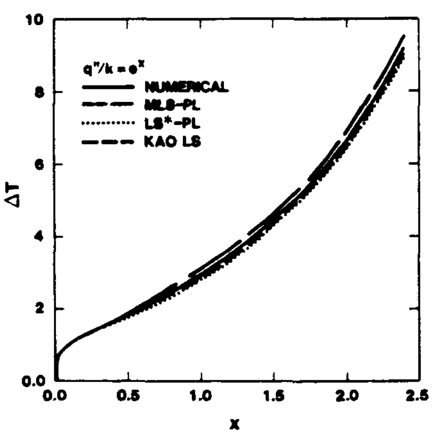


Fig. 5b. Temperature difference for $q''/k = e^x$.

2) $q''/k=1+x$. Figure 6a compares the heat flux variation for the power-law distribution to the desired variation for n corresponding to x values of 0.5 and 3.0. As with a number of the previous cases, the heat flux behavior is reasonably well represented by both approaches. Figure 6b shows the temperature difference variation along the plate. The answers from the Kao method and the Yang method are not given since they essentially coincide with the numerical results. The results from the integral analysis as given by Sparrow (1955) are also shown. All methods give good predictions for this case including the integral method.

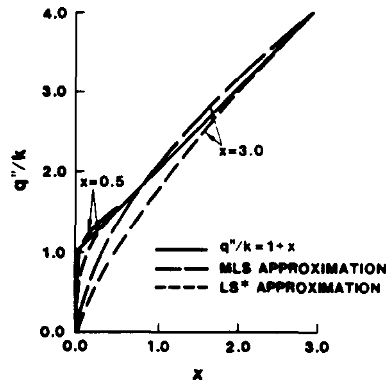


Fig. 6a. Approximation of $q''/k$ for $q''/k = 1 + x$.

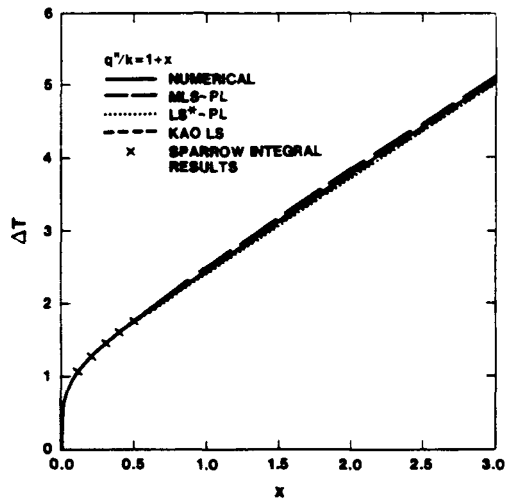


Fig. 6b. Temperature difference for $q''/k = 1 + x$.

3) $q''/k=1-x$. The predicted temperature difference variation along the plate for a number of different methods is given in Figure 7. The Kao and Yang methods diverge from the numerical solution for x values greater than about 0.5. The Kao LS method gives widely different results. The integral results from Sparrow (1955) seem to be well behaved, although the results are only provided out to an x value of 0.5 due to the limited information presented by Sparrow.

Figure 8a gives the heat flux predictions for the MLS and LS* methods for n corresponding to x values of 0.1 and 0.5. The behavior of both methods is not unreasonable, although significant differences can be seen between the approximation and the desired variation. Figure 8b shows the temperature difference results. The MLS method provides a reasonable prediction for the surface temperature behavior; the results are superior to all the other methods based on

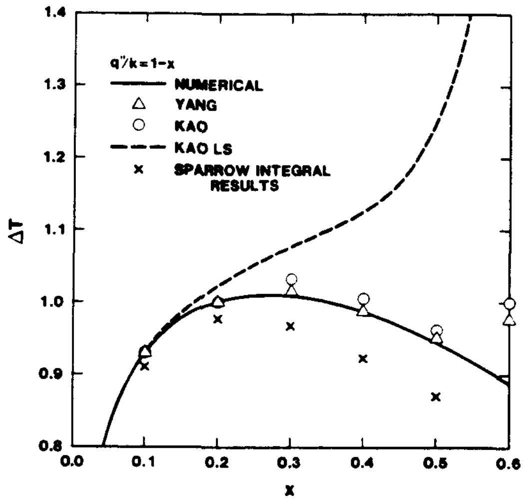


Fig. 7. Temperature difference for q"/k = 1 - x. Results from other methods.

comparison to the numerical predictions. The LS* predictions diverge like the Kao LS results.

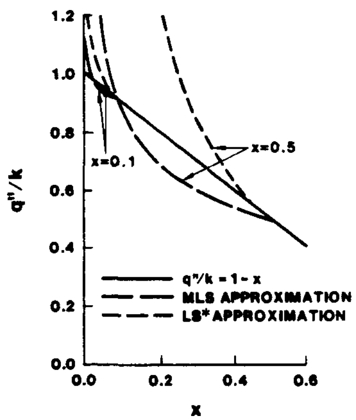


Fig. 8a. Approximation of q"/k for q"/k = 1 - x.

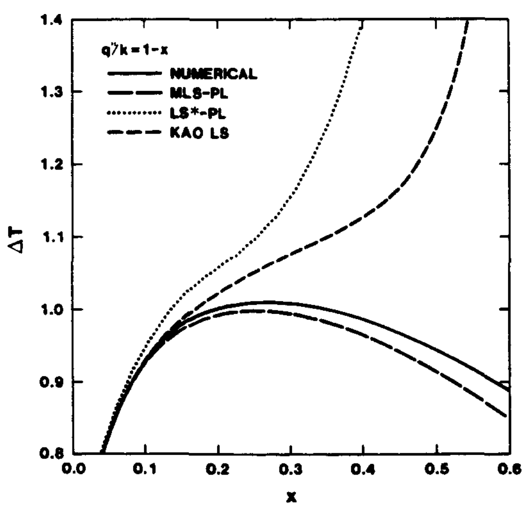


Fig. 8b. Temperature difference for q"/k = 1 - x.

Stratified Fluid Temperature

The stratified fluid temperature problem is an isothermal plate in a linearly stratified fluid as shown in Figure 9. The temperature difference between the plate and the fluid decreases linearly up the plate. A similar solution is not available for this problem. Chen and Eichhorn (1976) present an analysis of this problem using local similarity and local non-similarity methods for their coordinate transformation. In addition, Chen and Eichhorn (1976) acquired heat transfer data for water with a nominal Prandtl number of 6.0. Raithby and Hollands (1978) have applied their approximate technique (Raithby, et al. (1975, 1977)) to this problem with good results.

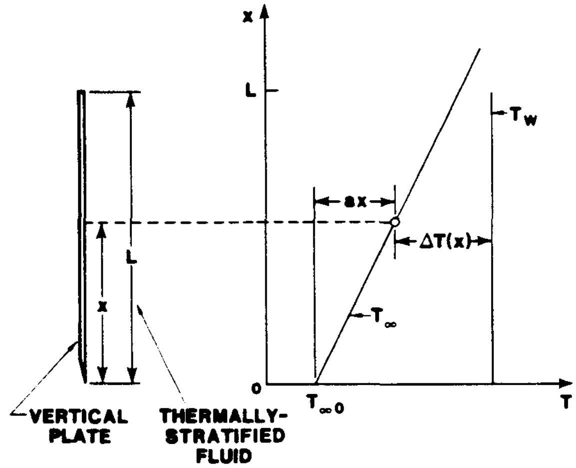


Fig. 9. Stratified fluid temperature problem.

Results for this problem are given in terms of the ratio of Nusselt numbers for the stratified fluid to that for an isothermal fluid as a function of the stratification parameter S, which is

$$S = \frac{L}{\Delta T} \frac{dT_f}{dx}. \tag{41}$$

When $S \leq 2$, the entire plate is hotter than the fluid. For $S > 2$, the bottom portion of the plate is hotter than the fluid while the top is colder.

The MLS and LS* approaches have been used to analyze this case for Prandtl numbers of 0.7 and 6.0. Only the results for a Prandtl number of 6.0 are included in this paper since this is the only case where data are available. The results for a Prandtl number of 0.7 are given in Webb (1988b). For the LS* approach, the value of n is determined by matching the local temperature difference value and the local slope; the value of J is calculated by the appropriate fluid temperature variation equation.

Figure 10 gives the variation of the temperatures for n corresponding to an x value of 0.5. For both methods, the reference value of the fluid temperature, $T_r$, is calculated by matching the temperatures at an x value of 0.5. The temperature variation for the MLS method is more reasonable than the LS* approach as the temperature difference is closer to the desired behavior. Both methods give an adequate, though certainly not perfect, prediction of the temperature variation.

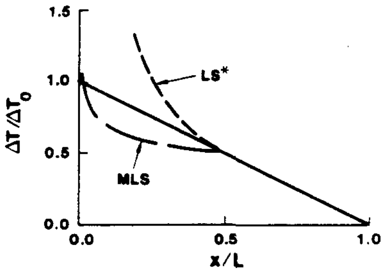


Fig. 10a. Approximation of $\Delta T$ for Stratified Fluid Case. (S=2).

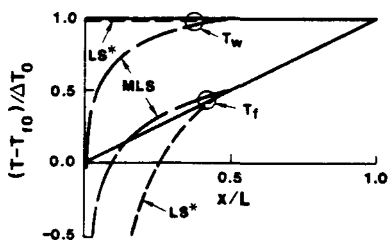


Fig. 10b. Approximation of $T_w$ and $T_f$ for Stratified Fluid Case. (S=2).

Figure 11 shows the predicted value of the average Nusselt number for a stratified fluid over that for an isothermal fluid with the same average temperature difference for a Prandtl number of 6.0. The MLS predictions are shown on this figure; results from the LS* method are not included as discussed below. The local similarity and local non-similarity (LNS) results are shown as well as the experimental data from Chen and Eichhorn (1976). The predictions by Raithby and Hollands (1978) are also included in the figure.

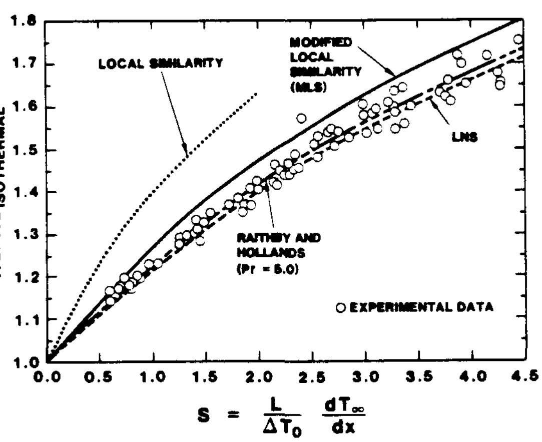


Fig. 11. Variation of Nusselt Number with Stratification for Pr=6.0.

The MLS results were calculated for a number of discrete S values out to 2.0. For S>2, the predictions are based on an $S^{1/4}$ dependence as used by Chen and Eichhorn (1976) and Raithby and Hollands (1978). The MLS predictions show reasonable agreement with the data with a consistent overprediction of about 4%. The local similarity results by Chen and Eichhorn (1976) are much higher than the data with an error of about 16%. The Raithby and Hollands predictions go right through the data, although their results are for a Prandtl number of 5.0, not 6.0. The LNS results show good agreement with the data with a small consistent underprediction. Overall, the MLS, Raithby and Hollands, and LNS results are in good agreement with the data. The maximum difference between these methods is about 5%, while the uncertainty in the data is of this order, or ±3.2% for Nu and ±3.5% for S (Chen and Eichhorn, 1976).

The LS* method performs poorly for this case. For a Prandtl number of 6.0 and an S value of 2.0, the wall is always as hot or hotter than the fluid. The LS* method predicts that the wall temperature gradient will change sign about 1/4 up the plate. For the first 1/4 of the plate, heat is transferred from the hotter wall to the fluid. However, for the last 3/4 of the plate, heat is predicted to flow from the colder fluid to the hotter plate, which is unreasonable. Therefore, the LS* predictions are not shown on the figure.

## SUMMARY AND CONCLUSIONS

The MLS method has been developed and evaluated for a number of nonsimilar temperature and heat flux cases for the power-law similarity distribution. For variable conditions where an exact similarity solution does not exist, the MLS method provides an estimate of "equivalent" similarity conditions including velocity and temperature profiles. This estimate is achieved by requiring global conservation of energy and the same local heat flux at position $x_2$. In addition, another possible application of the local similarity approach, has been evaluated. This method, designated the LS* approach, matches the local value and local slope of the prescribed parameter whether it be temperature difference or heat flux. The MLS and LS* results have been compared to those from a number of other methods, including a numerical approach.

The predictions from the LS* approach vary from reasonable to absurd, so the LS* method is not a reliable technique. The MLS method is not the most accurate approach as expected but is superior to the traditional local similarity approach. In addition, many of the other more complex approaches, such as the methods of Kao, et al. (1977) and of Yang, et al. (1982), have problems with certain cases such as the linearly decreasing heat flux situation and have not been applied to a nonuniform fluid temperature case. In contrast, all the MLS predictions are reasonable even where the more complex methods fail or no longer apply. Through the introduction of global conservation of energy, the MLS method has significantly improved the predictive capability of the local similarity approach and may be superior to more complex methods.

The MLS method is not without its problems. For specified temperature cases, iteration is required which violates the local similarity assumption. However, in most practical cases, temperatures and heat fluxes are related through heat conduction in the wall, and the more convenient variable can be used. Use of heat flux information permits use of the MLS method on a local basis consistent with the local similarity approach.

Where computing times are a major constraint, as in the analysis of natural convection in SPR caverns (Webb (1988a)), standard techniques such as finite differences and local nonsimilarity are impractical. In this case, the approximate results provided by the MLS method should usually provide reasonable results with minimal computing time. Thus, the MLS method is a useful approximate tool for natural convection analysis for analyzing vertical plates for nonsimilar conditions.

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