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NONLINEAR ERGODIC THEORY IN BANACH SPACES

by

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TABLE OF CONTENTS

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ABSTRACT

We prove the mean ergodic theorem for nonlinear nonexpansive mappings in Banach spaces, extend it by a different argument, deduce several consequences, and point out open problems.

Introduction

The purpose of this report is to present a proof of the mean ergodic theorem for nonlinear nonexpansive mappings T in Banach spaces, to extend it by a different argument, and to point out several consequences and open problems. Theorem 1, announced in [10], establishes the weak almost convergence of the iterates $\{T^n x\}$ in uniformly convex Banach spaces with a Fréchet differentiable norm. Its proof uses the ideas of Baillon [1] who proved the weak convergence of the Cesàro means of $\{T^n x\}$ in L^p, $1<p<\infty$. Although a simpler proof is available (Bruck [4]), we feel that Baillon's ideas, as well as our modifications of his arguments, are of independent interest and may be applied to other problems. Theorem 2 is an extension of Theorem 1. It is concerned with the sequence $\{x_n\}$ defined inductively by $x_{n+1} = c_n T x_n + (1-c_n) x_n$, $0 \le c_n \le 1$, and is proved by a different, shorter argument. The Proposition in [10] is also used. The importance of establishing almost convergence is illustrated by the fact that both parts of Theorem 3 are immediate consequences of Theorem 1. They deal with the equivalence between the weak asymptotic regularity of T and the weak convergence of $\{T^n x\}$, and with the summability of

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$\{T^n x\}$ by strongly regular matrices. The method of [7] yields analogous results for nonlinear nonexpansive semigroups (Theorem 4). Finally, we mention two open problems. They are concerned with the possibility of a "blow-up" result, and with strong convergence for odd operators.

1. The sequence $\{T^n x\}$

Let C be a closed convex subset of a Banach space E. Recall that a mapping $T\colon C \to C$ is said to be nonexpansive of $|Tx-Ty| \leq |x-y|$ for all x and y in C, and that a sequence $\{x_n\} \subset E$ is weakly almost convergent (cf. [6]) to $y \in E$ if the weak

$$\lim_{n\to\infty} \left(\sum_{i=0}^{n-1} x_{i+k}\right) \Big/ n = y$$

uniformly in $k \geq 0$. In this section we present a proof of the following nonlinear mean ergodic theorem, announced in [10]. The proof uses the ideas of Baillon [1], who showed that if $E = L^p$, $1<p<\infty$, then the Cesàro means of the iterates $\{T^n x\}$ converge weakly to a fixed point of T. Theorem 1 improves upon this result of Baillon.

<u>Theorem 1</u>. Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, and let $T\colon C \to C$ be nonexpansive. If T has a fixed point, then for each x in C, $\{T^n x\}$ is weakly almost convergent to a fixed point of T.

<u>Proof</u>. Let δ be the modulus of convexity of E. If $\max(|u|,|v|) \leq M_1$, then

$$|(1-c)u+cv| \leq M_1 \left(1 - 2\min(1-c,c)\,\delta\left(\frac{|u-v|}{M_1}\right)\right)$$

for all $0 \leq c \leq 1$. Taking $u = cT(cx+(1-c)y)-cTx$ and $v = (1-c)Ty-(1-c)T(cx+(1-c)y)$ where $x,y \in C$ and $|x-y| \leq M$, we obtain

$$|u|, |v| \le c(1-c)|x-y| = M_1 \le M/4$$

and

$$2M_1 \min(1-c, c)\,\delta\left(\frac{|u-v|}{M_1}\right) \le c(1-c)\{|x-y| - |Tx-Ty|\}.$$

Since $\delta(\varepsilon)/\varepsilon$ is a nondecreasing function of ε, it follows that

$$(M/2)\,\delta\left(\frac{4|u-v|}{M}\right) \le |x-y| - |Tx-Ty| \ .$$

Therefore for each M there is a nondecreasing $f\colon [0, M/2] \to [0, M/2]$, with $f(r) > 0$ for $r > 0$, such that

$$f(|T(cx+(1-c)y)-(cTx+(1-c)Ty)|) \le |x-y| - |Tx-Ty|$$

for all $|x-y| \le M$. Defining a function $g\colon [0, M/2] \to [0, M/2]$ by

$$g(r) = \frac{2}{M}\int_0^r f(t)\,dt \ , \qquad 0 \le r \le M/2 \ ,$$

we obtain

$$g(|T(cx+(1-c)y)-(cTx+(1-c)Ty)|) \le |x-y| - |Tx-Ty| , \tag{1}$$

where g is convex, strictly increasing and continuous. (δ is not a convex function in general.) Now let f_1 and f_2 be two fixed points of T, and let $J\colon E \to E^*$ be the (normalized) duality mapping. Since $L = \lim_{n\to\infty}(T^n x, J(f_1-f_2))$ exists (see the Proposition in [10]), it follows that if $\{k(n)\}$ is an arbitrary sequence of natural numbers and s is a weak subsequential limit of

$$\left(\sum_{i=0}^{n-1} T^{k(n)+i}\, x\right)\Big/ n \ ,$$

then

$$(s, J(f_1-f_2)) = L \ .$$

Consequently, the theorem will be established if we can prove that such an s must be a fixed point of T.

To this end, we first note that since T has a fixed point, we may assume that C is bounded, with diameter d. Let

$$v(n_1,n_2,\dots,n_q) = \frac{1}{q}\left(T^{n_1}u_1 + T^{n_1+n_2}u_2 + \dots + T^{n_1+\dots+n_q}u_q\right)$$

with $u_i \in C$, $1 \le i \le q$. We now show, by induction on q, that there is, for each q and N_1, a constant $I(q,N_1)$ such that

$$\frac{1}{N_1}\frac{1}{N_2}\cdots\frac{1}{N_q}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\cdots\sum_{n_q=0}^{N_q-1}|Tv(n_1,\dots,n_q)-v(n_1+1,n_2,\dots,n_q)|$$

$$\le I(q,N_1) \ ,$$

and

$$\lim_{N_1\to\infty} I(q,N_1) = 0 \ .$$

Denote $v(n_1,\dots,n_q)$ by $w(n_1)$ and $T^{n_1+\dots n_q+n_{q+1}}u_{q+1}$ by $z(n_1)$. We have

$$|Tv(n_1,\dots,n_{q+1})-v(n_1+1,\dots,n_{q+1})| \le$$

$$\left|T\left(\frac{q}{q+1}\,w(n_1) + \frac{1}{q+1}\,z(n_1)\right) - \left(\frac{q}{q+1}\,Tw(n_1) + \frac{1}{q+1}\,Tz(n_1)\right)\right|$$

$$+ \frac{q}{q+1}\,|Tw(n_1)-w(n_1+1)| \ ,$$

and by (1),

$$g\left(\left|T\left(\frac{q}{q+1}\,w(n_1) + \frac{1}{q+1}\,z(n_1)\right) - \left(\frac{q}{q+1}\,Tw(n_1) + \frac{1}{q+1}\,Tz(n_1)\right)\right|\right)$$

$$\le |w(n_1)-z(n_1)| - |Tw(n_1)-Tz(n_1)| \le$$

$$|w(n_1)-z(n_1)| - |w(n_1+1)-z(n_1+1)| + |Tw(n_1)-w(n_1+1)| \ .$$

Summing, we see that since g is convex, we may take

$$I(q+1,N_1) = \frac{q}{q+1} I(q,N_1) + g^{-1}(d/N_1 + I(q,N_1)) .$$

Denoting

$$\left(\sum_{i=0}^{n-1} x_{k+i}\right)\Big/ n$$

by $S_n(x_k)$, and letting

$$s_m(n_1,n_2,\ldots,n_p) = \frac{1}{p}\Big(S_m(T^{n_1} u_1) + S_m(T^{n_1+n_2} u_2) + \ldots + S_m(T^{n_1+n_2+\ldots+n_p} u_p)\Big) ,$$

we also obtain

$$\frac{1}{N_1}\frac{1}{N_2}\cdots\frac{1}{N_p}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\cdots\sum_{n_p=0}^{N_p-1}|Ts_m(n_1,n_2,\ldots,n_p) - s_m(n_1+1,n_2,\ldots,n_p)| \le I(mp,N_1) . \tag{2}$$

Note that $I(q,N_1)$ is independent of $\{u_i\}$.

Now let $x \in C$, $x_n = T^n x$, and assume that the weak $\lim_{j\to\infty} S_{n_j}(x_{k(n_j)}) = z$. We wish to prove that z is a fixed point of T. Let $K = \{n_j\}$ and

$$s_m(n_1,n_2,\ldots,n_p) = \frac{1}{p}\Big(S_m(x_{n_1+k(N_1)}) + \ldots + S_m(x_{n_1+\ldots+n_p+k(N_p)})\Big)$$

(thus $u_i = T^{k(N_i)}x$). We claim that

$$\lim_{p\to\infty}\ \limsup_{\substack{N_1\to\infty\\ N_1\in K}} \cdots \limsup_{\substack{N_p\to\infty\\ N_p\in K}} \frac{1}{N_1}\cdots\frac{1}{N_p}\sum_{n_1=0}^{N_1-1}\cdots\sum_{n_p=0}^{N_p-1} |s_m(n_1,n_2,\ldots,n_p)-z| = 0 . \tag{3}$$

To prove (3), we first repalce the norm of E by an equivalent norm for which E* is also uniformly convex. This is possible by [5]. Thus there is [9, p. 89] a continuous nondecreasing function $b: [0,\infty) \to [0,\infty)$ such that $b(0) = 0$, $b(ct) \le cb(t)$ for $c \ge 1$, and

$$|x+y|^2 \le |x|^2 + 2(y,Jx) + \max(|x|,1)|y|b(|y|)$$

for all x and y in E. Let

$$r_p = s_m(n_1,n_2,\ldots,n_p)-z .$$

Since

$$r_p = \frac{p-1}{p} r_{p-1} + \frac{1}{p}\left(S_m(x_{n_1+\ldots+n_p+k(N_p)})-z\right) ,$$

we obtain

$$\frac{1}{N_p}\sum_{n_p=0}^{N_p-1}|r_p|^2 \le \left|\frac{p-1}{p} r_{p-1}\right|^2 +$$

$$\frac{2}{p}\left(\frac{1}{N_p}\sum_{n_p=0}^{N_p-1} S_m(x_{n_1+\ldots+n_p+k(N_p)})-z,\ J\left(\frac{p-1}{p} r_{p-1}\right)\right)$$

$$+ M\frac{1}{p} b\left(\frac{1}{p}\right) \quad \text{for some } M \ge 1 .$$

It is not difficult to check that for a fixed i,

$$S_{N_p}\left(x_{i+k(N_p)}\right) \xrightarrow[N_p\in K]{N_p\to\infty} z .$$

Since

$$\left|\frac{1}{N_p}\sum_{n_p=0}^{N_p-1} S_m(x_{i+n_p}) - S_{N_p}(x_i)\right| \le \frac{(m-1)d}{2N_p} ,$$

it follows that

$$\limsup_{\substack{N_p\to\infty \\ N_p\in K}} \frac{1}{N_p}\sum_{n_p=0}^{N_p-1} |r_p|^2 \le \left(\frac{p-1}{p}\right)^2 |r_{p-1}|^2 + M\,\frac{1}{p}\,b\left(\frac{1}{p}\right) .$$

Hence,

$$f(p) = \limsup_{\substack{N_1\to\infty \\ N_1\in K}} \frac{1}{N_1}\sum_{n_1=0}^{N_1-1} \limsup_{\substack{N_2\to\infty \\ N_2\in K}} \frac{1}{N_2}\sum_{n_2=0}^{N_1-1} \cdots \limsup_{\substack{N_p\to\infty \\ N_p\in K}} \frac{1}{N_p}\sum_{n_p=0}^{N_p-1} |r_p|^2$$

$$\le \left(\frac{p-1}{p}\right)^2 f(p-1) + M\,\frac{1}{p}\,b\left(\frac{1}{p}\right) .$$

Since $\lim_{p\to\infty} b\left(\frac{1}{p}\right) = 0$, it follows that

$$\lim_{p\to\infty} f(p) = 0 \quad \left(\sum_{p=1}^{\infty} \frac{1}{p} = \infty\right) .$$

Since

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2 \le \frac{1}{n}\sum_{i=1}^{n} a^2 ,$$

(3) follows too.

Now let $y \in C$, denote $s_m(n_1,n_2,\ldots,n_p)$ by s_m, and consider

$$(y-Ty,\ J(s_m-y)) = (y-s_m + Ts_m-Ty + s_m-Ts_m,\ J(s_m-y))$$

$$\le (s_m-Ts_m,\ J(s_m-y)) =$$

$$(s_m-s_m(n_1+1,n_2,\ldots,n_p) + s_m(n_1+1,n_2,\ldots,n_p)-Ts_m,$$

$$J(s_m-y)) .$$

By (2),

$$\frac{1}{N_1}\frac{1}{N_2}\cdots\frac{1}{N_p}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\cdots\sum_{n_p=0}^{N_p-1} (y-Ty,J(s_m-y))$$

$$\le d^2/m + dI(mp,N_1) .$$

(3) implies that

$$\lim_{p\to\infty} \limsup_{\substack{N_1\to\infty\\ N_1\in K}} \cdots \limsup_{\substack{N_p\to\infty\\ N_p\in K}} \frac{1}{N_1}\cdots\frac{1}{N_p} \sum_{n_1=0}^{N_1-1} \cdots \sum_{n_p=0}^{N_p-1} |(y-Ty, J(s_m-y)-J(z-y))|$$
$$= 0 \ .$$

It follows that for each m, $(y-Ty, J(z-y)) \le d^2/m$. Therefore $(y-Ty, J(z-y)) \le 0$ for all y in C. Taking $y = (1-t)z+tTz$, dividing by t, and letting $t \to 0+$, we obtain $z = Tz$. This completes the proof.

2. The sequence $\{c_n Tx_n+(1-c_n)x_n\}$

In this section we establish an extension of Theorem 1 by a different argument. In the setting of the previous section, let $\{c_n : n=0,1,2,\ldots\}$ be a real sequence with $0 \le c_n \le 1$ for all n. For x_0 in C we define a sequence $\{x_n\} \subset C$ by

$$x_{n+1} = c_n Tx_n + (1-c_n)x_n \ , \qquad n \ge 0 \ . \tag{4}$$

It is known [10, Theorem 2] that if

$$\sum_{n=0}^{\infty} c_n(1-c_n) = \infty \ ,$$

then $\{x_n\}$ converges weakly to a fixed point of T. This happens, of course, if $c_{n_k} \to p$ for some $0<p<1$, or if $c_n \to 0$ and

$$\sum_{n=0}^{\infty} c_n = \infty \ .$$

(If

$$\sum_{n=0}^{\infty} c_n < \infty \ ,$$

the limit is not a fixed point in general.) The remaining case, $c_n \to 1$, is covered by the following result, announced in [11].

Theorem 2. Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, and let $T: C \to C$ be a nonexpansive mapping with a fixed point. Let $\{x_n\}$ be defined by (4). If $\lim_{n\to\infty} c_n = 1$, then $\{x_n\}$ is weakly almost convergent to a fixed point of T.

Proof. Denote

$$\left(\sum_{i=0}^{n-1} x_{k+i}\right)\Big/ n$$

by $S_n(x_k)$. If the weak $\lim_{j\to\infty} S_{n_j}(x_{k(n_j)}) = z$, then the Proposition in [10] implies that

$$(z, J(f_1-f_2)) = \lim_{n\to\infty}(x_n, J(f_1-f_2))$$

for any two fixed points f_1 and f_2 of T. Therefore all we have to prove is that z is a fixed point of T.

By [10, Theorem 2] we may assume that

$$\sum_{n=0}^{\infty} (1-c_n) < \infty .$$

We may also assume that C is bounded with diameter d. Induction on m shows that

$$|x_{n+m}-T^m x_n| \le \sum_{k=n}^{n+m-1} (1-c_k)|x_0-Tx_0| .$$

Thus

$$\lim_{n\to\infty} |x_{n+m}-T^m x_n| = 0$$

uniformly in $m \ge 0$. Since each T^m satisfies (1), it follows that

$$\lim_{n\to\infty} |S_q(x_{n+m})-T^m S_q(x_n)| = 0$$

for each fixed $q \ge 1$. Noting that

$$|S_q(x_{n+1}) - S_q(x_n)| \leq d/q \ ,$$

we obtain

$$|S_q(x_n) - TS_q(x_n)| \leq d/q + \varepsilon(q,n) \ ,$$

where $\lim_{n\to\infty} \varepsilon(q,n) = 0$ for each q.

Let p be a seminorm for the weak topology of E, $d_p(x,D) = \inf\{p(x-y) : y \in D\}$, and let F(T) be the fixed point set of T. It follows from [3, Theorem 3] that for each $\varepsilon > 0$ there is $\delta > 0$ such that if $x \in C$ and $|x-Tx| < \delta$, then $d_p(x,F(T)) < \varepsilon$. Given $\varepsilon > 0$, we first choose q and then n_0 such that

$$|S_q(x_n) - TS_q(x_n)| < \delta$$

and

$$d_p(S_q(x_n), F(T)) < \varepsilon$$

for all $n \geq n_0$.

We also have

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} S_q(x_{k+i}) - S_n(x_k) \right| \leq \frac{(q-1)}{2n} d \ .$$

Hence

$$\begin{aligned} d_p(z,F(T)) &= \lim_{j\to\infty} d_p(S_{n_j}(x_{k(n_j)}), F(T)) \\ &= \lim_{j\to\infty} d_p\left(\frac{1}{n_j} \sum_{i=0}^{n_j-1} S_q(X_{i+k(n_j)}), F(T)\right) \\ &\leq \limsup_{j\to\infty} \frac{1}{n_j} \sum_{i=0}^{n_j-1} d_p(S_q(x_{i+k(n_j)}), F(T)) \leq \varepsilon \ . \end{aligned}$$

Consequently, $z \in F(T)$ and the proof is complete.

Theorem 2 can also be established by first showing that the set of sequences $\{x_n\}$ for which $\{|x_{n+1} - Tx_n|\}$ is almost convergent to 0 is convex. To prove this fact, note that by (1),

$$|(1-a)x_{i+1} + ay_{i+1} - T((1-a)x_i+ay_i)|$$

$$\leq |(1-a)(x_{i+1}-Tx_i) + a(y_{i+1}-Ty_i)|$$

$$+ |(1-a)Tx_i + aTy_i - T((1-a)x_i+ay_i)|$$

$$\leq (1-a)|x_{i+1}-Tx_i| + a|y_{i+1}-Ty_i|$$

$$+ g^{-1}(|x_i-y_i|-|Tx_i-Ty_i|)$$

$$\leq (1-a)|x_{i+1}-Tx_i| + a|y_{i+1}-Ty_i|$$

$$+ g^{-1}(|x_i-y_i| - |x_{i+1}-y_{i+1}| + |x_{i+1}-Tx_i| + |y_{i+1}-Ty_i|) ,$$

and that g^{-1} is concave. Now Theorem 2 can be proved by the method of [4].

3. Applications and open problems

In this section we present some applications, as well as open problems. The following two results are immediate consequences of the almost convergence of $\{T^n x\}$ established in the previous sections (cf. [8]).

Theorem 3. Let C be a closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, $T: C \to C$ a nonexpansive mapping with a fixed point, and $x \in C$.

(a) $\{T^n x\}$ converges weakly to a fixed point of T if and only if the weak $\lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$.

(b) If the matrix $\{a_{n,k}\}$ is strongly regular and

$$y_n = \sum_{k=0}^{\infty} a_{n,k} T^k x ,$$

then $\{y_n\}$ converges weakly to a fixed point of T.

The method of [7] yields analogous results for nonexpansive semigroups on C. (They can also be obtained directly by the method used in the proof of Theorem 2.)

Theorem 4. Let C be a closed convex subset of a uniformly convex Banach space with a Fréchet differentiable norm, $S: [0,\infty)\times C \to C$ a nonexpansive semigroup with a fixed point, and $x \in C$.

(a) The weak

$$\lim_{T\to\infty} \frac{1}{T}\int_c^{T+c} S(t)x\,dt$$

exists uniformly in $c \geq 0$ and is a fixed point of S.

(b) $S(t)x$ converges weakly to a fixed point of S if and only if the weak $\lim_{t\to\infty}(S(t+h)x-S(t)x) = 0$ for all positive h.

(c) If the kernel K is strongly regular and

$$R(s)x = \int_0^\infty K(s,t)S(t)x\,dt ,$$

then $R(s)x$ converges weakly to a fixed point of S.

In the setting of Theorem 1, it is not known if

$$\left|\left(\sum_{i=0}^{n-1} T^i x\right)\Big/ n\right| \xrightarrow[n\to\infty]{} \infty$$

when T is fixed point free. This is true in Hilbert space. Also, in the setting of Theorem 2, if $\{x_n\}$ is defined by (4) and

$$\sum_{n=0}^{\infty} c_n(1-c_n) = \infty ,$$

then $|x_n| \xrightarrow[n\to\infty]{} \infty$ if T is fixed point free.

Another open problem is whether strong almost convergence occurs in the setting of Theorem 1 when T is odd. Again this is known to be the case in Hilbert space. Although the same question applies to part (b) of Theorem 3, it is known [2], that if T is odd, then the strong $\lim_{n\to\infty}(T^n x - T^{n+1}x) = 0$ if and only if $\{T^n x\}$ converges strongly to a fixed point of T.

It would also be of interest to determine whether a pointwise ergodic theorem is possible, and whether similar results hold for other Banach spaces.

Remark: We have already noted that Theorem 4 can also be established by the method used in the proof of Theorem 2. It follows that Theorem 4 remains true even if S is assumed to be only strongly measurable. (A strongly measurable S is continuous on $(0,\infty)$, but not necessarily on $[0,\infty)$.)

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