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A Multilevel Preconditioner for Domain Decomposition Boundary Systems

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Abstract. In this note, we consider multilevel preconditioning of the reduced boundary systems which arise in non-overlapping domain decomposition methods. It will be shown that the resulting preconditioned systems have condition numbers which are bounded in the case of multilevel spaces on the whole domain and grow at most proportional to the number of levels in the case of multilevel boundary spaces without multilevel extensions into the interior.

1. INTRODUCTION.

This paper deals with the analysis of multilevel preconditioners for boundary systems which arise in non-overlapping domain decomposition. Domain decomposition algorithms are important in that they represent a basic tool for the development of effective algorithms for solving the discrete systems which arise from the numerical approximation of elliptic boundary value problems on computers with modern parallel computing environments.

The effective preconditioning of the boundary system is often the most critical part of a domain decomposition algorithm. In order to minimize the amount of interprocessor communication, domains are chosen of quasi-uniform size and shape. Some fundamental work on the development of preconditioners for the boundary systems for problems in two and three dimensions was given in the series of papers [4]–[7]. In [4] and [7], preconditioners were developed such that the resulting preconditioned system had condition numbers which increased at most like $O(1+\ln(d/h)^2)$. Here d is the subdomain size and h is the mesh size. In [6] a technique was provided which gave rise to a uniform preconditioner (independent of d and h) with only a moderate increase in computational effort.

The possibility of using a multilevel technique to develop preconditioners for the domain decomposition boundary systems has been known for a long time [9],[11]. In this note, we show that the condition number of the resulting preconditioned boundary system is bounded independently of the number of levels J for the canonical application defined from multilevel spaces on the whole domain. We also consider the case when multilevel extensions to the full domain are not available, e.g. tetrahedral mesh in three spatial dimensions. In this case, we show that the condition number grows at most by $O(J)$. These results improve the existing conditioning bound of $O(J^2)$ (here, J is the number of levels) which follows easily from [9] (see also, [11]).

The reduced boundary system iteration can be developed from more than one point of view. One popular approach is to describe the reduction in terms of block matrices.

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The reduced boundary system then corresponds to the Schur Complement. Unfortunately, the matrix approach does not provide much analytical information. A second approach is based on the fact that the reduced system is equivalent to solving a variational problem on the trace (with respect to the union of the boundaries of the subdomains) of the finite element space. One then can apply multilevel techniques ([1],[9]) directly to this problem. A third approach is to consider the reduced system as an operator equation on a subspace of the original finite element space. We use this approach in this note, in particular, in the case when multilevel spaces on the whole domain are available. For this application, multilevel theory [1] on the whole domain gives rise to corresponding uniform results for the related reduced systems.

The outline of the remainder of the paper is as follows. In Section 2, the model problem and the assumptions on the full space and subdomain boundary meshes are described. The reduced system, formulated as an operator on a lower dimensional subspace of the original approximation space, is described in Section 3. Finally, condition number bounds for preconditioned reduced systems are given in Section 4.

2. Model problem and mesh assumptions.

Let Ω be a bounded domain in R^d, for $d = 2$ or $d = 3$, with polygonal or polyhedral boundary $\partial\Omega$. We consider the Dirichlet problem

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where

$$Lv = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial v}{\partial x_j}\right).$$

We shall place some very weak assumptions on the coefficients above. We first assume that a_{ij} is in $L^\infty(\Omega)$. We further assume that the matrix $\{a_{ij}(x)\}$ is uniformly positive definite almost everywhere.

Let $A(\cdot,\cdot)$ denote the generalized Dirichlet form corresponding to (2.1), i.e.,

$$A(v,w) = \sum_{i,j=1}^{d} \int_\Omega a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}\,dx, \tag{2.2}$$

This is defined for all v and w in the Sobolev space $H^1(\Omega)$ (the space of distributions with square-integrable first derivatives).

We shall consider the case of quasi-uniform finite element approximation of the solution of (2.1). To define the approximation spaces, we will first define the underlying mesh partitioning. We assume that the domain is first partitioned into the union of triangles or rectangles ($\Omega = \cup\tau_H^i$) of quasi-uniform size H (in the case of three dimensions, this partitioning is in terms of tetrahedra or bricks). Along with this partitioning, there is a fine partitioning of the region into similar structures ($\Omega = \cup\tau_h^i$) of size h. We assume that the fine partitioning **aligns** with the coarser in the sense that the faces of the coarser can be written as a union of faces of the finer. For convenience, we shall subsequently refer to all of the above mesh elements generically as triangles.

Let Γ denote the union of the boundaries of the coarse triangles, $\Gamma = \cup_i \partial\tau_H^i$. Our goal is to analyze multilevel preconditioners for the reduced systems resulting from subdomain solves in a domain decomposition algorithm. Thus, we shall always assume that the trace of the full approximation space Γ has a multilevel structure. We provide two types of results depending on whether or not the fine mesh approximation space (on Ω) has a multilevel structure. The case where the fine grid does not come from a multilevel sequence is encountered in three dimensional domains with tetrahedral partitioning where sequences of multilevel partitionings are not available. In contrast, it is easy to define multilevel partitioning of three dimensional regions based on "brick-like" elements.

The boundary Γ inherits a partition from $\{\tau_h^i\}$ which consists of the faces on Γ and will be denoted $\{\tau_\Gamma^i\}$. We make the further restriction on the fine grid mesh and assume that $\{\tau_\Gamma^i\}$ results from a multilevel sequence of partitioning $\{\tau_{\Gamma,k}^i\}$ with elements of size h_k for $k = 1, \ldots, J$ with

$$c_0 2^{-k} \leq h_k \leq c_1 2^{-k}.$$

Here c_0 and c_1 are positive constants, independent of J. We assume that this restriction corresponds to the restriction of a mesh of quasi-uniform size $\{\tau_{h,k}^i\}$ defined on Ω. We can take $\{\tau_{h,J}^i\}$ to be $\{\tau_h^i\}$. We finally assume that the meshes $\{\tau_{\Gamma,k}^i\}$ are nested in the sense that each triangle in $\{\tau_{\Gamma,k}^i\}$ can be written as a union of triangles of $\{\tau_{\Gamma,k+1}^i\}$. Our two types of results will depend on whether or not it is assumed that the triangles $\{\tau_{h,k}^i\}$, $k = 1, \ldots, J$, form a nested sequence.

We assume that nodal finite element spaces M_k are defined with respect to the mesh partitioning $\{\tau_{h,k}^i\}$. For example, continuous piecewise linear functions can be used when the meshes are defined in terms of triangles and tetrahedra in two and three spatial dimensions respectively. The multilevel boundary spaces V_k are defined to be the trace of M_k with respect to Γ, i.e. V_k is the space of functions defined on Γ which are the restrictions of those in M_k. The nested assumption imposed on the triangles $\{\tau_{\Gamma,k}^i\}$ implies that the sequence of spaces $\{V_k\}$ are nested, i.e.,

$$V_1 \subset V_2 \cdots \subset V_J.$$

The Galerkin approximation to solution of (2.1) is defined to be the unique function $u_J \in M_J$ satisfying

$$A(u_J, \psi) = (f, \psi) \qquad \text{for all } \psi \in M_J. \tag{2.3}$$

Here $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. Without loss of generality, we may assume that $f \in M_J$.

3. Reduced boundary systems.

In this paper, we shall be concerned with developing well conditioned iterative methods involving reduced systems resulting from subspace solves. In the domain decomposition application in the next section, this reduction will be associated with subdomain solves.

However, in this section, we shall consider this technique in terms of an arbitrary subspace $M^0 \subset M_J$.

We shall consider two techniques. One comes from preconditioning a reduced system and the other from the direct development of a preconditioned reduced system. In terms of block matrices, preconditioning the reduced system corresponds to the preconditioning the so-called Schur Complement. Alternatively, the second technique uses a preconditioner for the full problem directly before reduction.

Equation (2.3) can be cast in terms of operators by defining the operator $A : M_J \mapsto M_J$ by

$$(A\phi, \theta) = A(\phi, \theta) \qquad \text{for all } \phi, \theta \in M_J. \tag{3.1}$$

Note that (2.3) can be restated as $Au_J = f$.

The reduced systems are defined in terms of a subspace $M^0 \subset M_J$. Let $M^\perp$ be the orthogonal complement of M^0 with respect to the $A(\cdot,\cdot)$ inner product, i.e.,

$$M^\perp = \{\phi \in M_J \mid A(\phi, \theta) = 0 \text{ for all } \theta \in M^0\}.$$

The solution u_J of (2.3) can be written $u_J = u^\perp + u^0$ where u^0 satisfies

$$A(u^0, \theta) = (f, \theta) \qquad \text{for all } \theta \in M^0.$$

Note that $u^0 = P^0 u$ where $P^0 : M_J \mapsto M^0$ denotes the $A(\cdot,\cdot)$ orthogonal projector onto M^0. Moreover, the remainder $u^\perp$ is in $M^\perp$ and satisfies the equation

$$Au^\perp = f - Au^0. \tag{3.2}$$

Let $Q^\perp$ denote the $L^2(\Omega)$ orthogonal projection onto the subspace $M^\perp$. The first reduced system is defined by considering the operator $Q^\perp A$. Note that $Q^\perp A$ restricted to $M^\perp$ is a symmetric and positive definite operator (with respect to $(\cdot,\cdot)$). Hence (3.2) can be reduced to the equation (on $M^\perp$)

$$A^\perp u^\perp = Q^\perp(f - Au^0). \tag{3.3}$$

Here $A^\perp \equiv Q^\perp A$. The preconditioning problem for (3.3) is to define an operator $B^\perp : M^\perp \mapsto M^\perp$ so that the action of $B^\perp$ is efficient to compute and the condition number $K(B^\perp A^\perp)$ is relatively small.

The second reduced system is defined in terms of a preconditioner B for the full operator A. Here we assume that B is a symmetric (with respect to $(\cdot,\cdot)$) positive definite operator on M_J and that

$$\lambda_0 A(v,v) \le A(BAv, v) \le \lambda_1 A(v,v) \qquad \text{for all } v \in M_J, \tag{3.4}$$

with λ_1/λ_0 not very large. Let $P^\perp = (I - P^0)$. We consider the alternative reduced system

$$\tilde{R}u^\perp \equiv P^\perp BAu^\perp = P^\perp B(f - Au^0). \tag{3.5}$$

For $v, w \in M^\perp$,

$$A(\tilde{R}v, w) = A(BAv, w)$$

and hence the reduced system is symmetric on $M^\perp$ with respect to the $A(\cdot,\cdot)$ inner product. Moreover,

$$\lambda_0 A(v,v) \le A(\tilde{R}v, v) \le \lambda_1 A(v,v) \qquad \text{for all } v \in M^\perp, \tag{3.6}$$

where λ_1, λ_0 are the constants appearing in (3.4). The following proposition follows from (3.4) and (3.6).

PROPOSITION 3.1. *The reduced system $\tilde{R}$ is symmetric on $M^\perp$ and positive definite with respect to the $A(\cdot,\cdot)$ inner product. Moreover $K(\tilde{R}) \leq K(BA)$.*

4. MULTILEVEL PRECONDITIONING FOR DOMAIN DECOMPOSITION BOUNDARY SYSTEMS.

In this section, we consider the reduced system which results from subdomain solves in a domain decomposition algorithm. Let the mesh structure be as discussed in Section 2. The subspace M^0 is defined to be the functions $\phi \in M_J$ which vanish on Γ. Note that functions in $M^\perp$ are completely determined by their values on Γ. For $v \in V_J$, we define v_H to be the unique function in $M^\perp$ which coincides with v on Γ.

Let $\{\psi_k^i\}_i$ denote the usual nodal basis for the finite element boundary space V_k and set $\tilde{\psi}_k^i = (\psi_k^i)_H$. Following [9], the first multilevel preconditioner for the reduced system $A^\perp$ on $M^\perp$ is defined by

$$B^\perp v = \sum_{k=1}^{J} h_k^{2-d} \sum_i (v, \tilde{\psi}_k^i)\, \tilde{\psi}_k^i \qquad \text{for all } v \in M^\perp. \tag{4.1}$$

The sum over i is taken over all boundary nodes of V_k on Γ.

The above preconditioner can be effectively used in a preconditioned iteration for solving (3.3). Actual implementation avoids the computation of $Q^\perp$ as well as computation of the values of $\tilde{\psi}_k^i$ off Γ. However, it does require the solution of problems on the subdomains on each step of the iteration. Before analyzing this preconditioner, we shall discuss its implementation in more detail.

Since the value of a function in $M^\perp$ is completely determined by its boundary trace, one implements the solution to (3.3) as a boundary iteration. A typical preconditioned iteration for (3.3) with preconditioner $B^\perp$ requires computation of $B^\perp A^\perp v$ for vectors $v \in M^\perp$ as well as $B^\perp$ applied to the right hand side of (3.3). We assume that we are given a computational basis $\{\theta^i{}_J\}$ for M_J. The data for $f - Au^0$ is represented by a vector of values

$$(f, \theta^i{}_J) - A(u^0, \theta^i{}_J), \quad i = 1, 2, \ldots \tag{4.2}$$

which are assumed known at the start of the iteration. The boundary values of $B^\perp Q^\perp(f - Au^0)$ are trivially computed from the quantities

$$F_k^i = (Q^\perp(f - Au^0), \tilde{\psi}_k^i) = (f, \tilde{\psi}_k^i) - A(u^0, \tilde{\psi}_k^i) = (f, \bar{\psi}_k^i) - A(u^0, \bar{\psi}_k^i). \tag{4.3}$$

Here $\bar{\psi}_k^i$ denotes the function in M_J which equals $\tilde{\psi}_k^i$ on Γ and vanishes at all nodes of M_J not on Γ. Note that for $k = J$, $\bar{\psi}^i{}_J$ coincides with $\theta^l{}_J$ for some l and thus the quantities in (4.3) are provided by (4.2). The quantities in (4.3) for $k < J$ are calculated recursively using the fact that F_k^i can be written as a simple (local) linear combination of the values in $\{F_{k+1}^l\}$. The evaluation of the action of $B^\perp$ applied to $A^\perp v$ is similar. Given the value of v on Γ, we first compute v everywhere by discrete harmonic extension. This involves the solution of subdomain problems. Next, the quantities $\{A(v, \theta^i{}_J)\}$ are computed by applying the "stiffness matrix" for M_J. The boundary values of $B^\perp A^\perp v$ are computed from these quantities as discussed above in the case of $B^\perp Q^\perp(f - Au^0)$.

We now proceed with the analysis of (4.1) and first consider the case when the subspaces M_k form a nested sequence, i.e.,

$$M_1 \subset M_2 \subset \ldots \subset M_J. \tag{4.4}$$

In this case, we will apply the theory in [1] and Proposition 3.1. To apply this theory, we introduce the following regularity estimate for the domain Ω.

CONDITION C.1: We will assume that there is an α in $(0, 1]$ such that solutions u of (2.1) with $L = -\Delta$ satisfy the following regularity estimate:

$$\|u\|_{1+\alpha} \leq C \|f\|_{-1+\alpha}. \tag{4.5}$$

Here $\|\cdot\|_{-1+\alpha}$ is the is the interpolated norm between $L^2(\Omega)$ and $H^{-1}(\Omega)$ (the dual of $H_0^1(\Omega)$). Thus, we assume that the domain results in some elliptic regularity for smooth coefficient problems (but not necessarily full elliptic regularity). In two dimensions, (4.5) holds for any polygonal domain including a domain with a crack. This assumption is weak since (4.5) may not hold for any $\alpha > 0$ for the original equation with possibly bad coefficients.

When Condition C.1 holds, we can apply Theorem 4.1 of [1] to get that

$$Bv = \sum_{k=1}^{J} h_k^{2-d} \sum_i (v, \theta_k^i)\, \theta_k^i \tag{4.6}$$

provides a preconditioner for A with resulting condition number $K(BA)$ bounded independently of J. Here $\{\theta_k^i\}_i$ denotes the nodal basis for the space M_k and the sum over i in (4.6) is taken over all interior and boundary nodes (on Γ) of the k'th mesh. By Proposition 3.1, $K(\tilde{R}) \leq K(BA)$.

We next show that

$$B^\perp A^\perp w = \tilde{R} w \equiv P^\perp BAw \qquad \text{for all } w \in M^\perp. \tag{4.7}$$

Since the images of $B^\perp$ and $P^\perp$ are in $M^\perp$, it suffices to show that $B^\perp A^\perp w = BAw$ on Γ. Note that the expression (4.1) defining $B^\perp$ defines an extension $B^\perp : M_J \mapsto M^\perp$. Clearly, $B^\perp Q^\perp = B^\perp$. By rearrangement, we may assume that the function ψ_k^i is the trace of θ_k^i. Then, on Γ,

$$\begin{aligned} B^\perp A^\perp w = B^\perp A w &= \sum_{k=1}^{J} h_k^{2-d} \sum_i (Aw, \tilde{\psi}_k^i)\, \tilde{\psi}_k^i \\ &= \sum_{k=1}^{J} h_k^{2-d} \sum_i (Aw, \theta_k^i)\, \theta_k^i = BAw. \end{aligned}$$

Both sums above are taken only on the nodes of Γ since the additional terms (4.6) vanish on Γ. Thus, we have proved the following theorem.

THEOREM 4.1. *Assume that the spaces $\{M_k\}$ are nested and that Condition C.1 holds. Then $K(B^{\perp}A^{\perp})$ is bounded independently of the number of levels J.*

Before investigating the case when spaces satisfying (4.4) are not available, we consider the following remark.

REMARK 4.1: There are applications when $H = h_j$ for $j > 1$ and no obvious set of coarser spaces $\{M_k\}$ for $k = 1, \ldots, j-1$ are available. Let $M_0^{\perp}$ denote the functions in M which are orthogonal with respect to $(\cdot,\cdot)$ to M^0. Note that in the preconditioned algorithm , one only evaluates the action of $B^{\perp}$ on functions of the form $Q^{\perp}w$ for $w \in M_0^{\perp}$. Since $A : M^{\perp} \mapsto M_0^{\perp}$ and $A^{\perp}$ is invertible, $Q^{\perp}$ restricted to $M_0^{\perp}$ has an inverse $(Q^{\perp})^{-1}$. We replace the lower terms in (4.1) by a solve on the j'th level and define

$$B^{\perp}v = P^{\perp}A_j^{-1}Q_j(Q^{\perp})^{-1}v + \sum_{k=j+1}^{J} h_k^{2-d}\sum_i (v, \tilde{\psi}_k^i)\,\tilde{\psi}_k^i \qquad \text{for all } v \in M^{\perp}. \tag{4.8}$$

Here Q_j denotes the $(\cdot,\cdot)$ orthogonal projection onto M_j and $A_j : M_j \mapsto M_j$ is defined by

$$(A_j v, w) = A(v, w) \qquad \text{for all } v, w \in M_j.$$

The analysis of (4.8) is similar to the case considered above. One shows that $P^{\perp}BA = B^{\perp}A^{\perp}$ on $M^{\perp}$ for B given by

$$Bv = A_j^{-1}Q_j v + \sum_{k=j+1}^{J} h_k^{2-d}\sum_i (v, \theta_k^i)\,\theta_k^i \tag{4.9}$$

Note that this immediately implies that $B^{\perp}A^{\perp}$ is symmetric with respect to $A(\cdot,\cdot)$ and hence $B^{\perp}$ is symmetric with respect to $(\cdot,\cdot)$. When Condition C.1 holds, we can apply Theorem 4.1 of [1] to get that $K(BA)$ is bounded independently of J. Hence, Proposition 3.1 shows that Theorem 4.1 holds for $B^{\perp}$ given by (4.8).

The above definition of $B^{\perp}$ is computable. Indeed, as we have already observed, it is only necessary to compute $B^{\perp}v$ where $v = Q^{\perp}w$, $w \in M_0^{\perp}$ and the data $\{(w, \tilde{\psi}_J^i)\}_i$ are given. For such a v, $A_j^{-1}Q_j(Q^{\perp})^{-1}v = \chi$ where $\chi \in M_j$ satisfies

$$A(\chi, \theta) = (w, \theta) \qquad \text{for all } \theta \in M_j.$$

The application of $P^{\perp}$ is avoided since we only need the boundary values of $B^{\perp}v$.

We now turn to the case when nested spaces on the full domain are not available, i.e., (4.4) fails to hold. For the purpose of analysis, we consider operators defined on Γ. The bilinear form on V_J corresponding to $A^{\perp}$ is

$$s(v, w) = (A^{\perp}v_H, w_H) = A(v_H, w_H) \qquad \text{for all } v, w \in V_J.$$

By (3.3), the boundary values U of u^0 satisfy the equation

$$s(U, w) = \left\langle \tilde{f}, w \right\rangle \qquad \text{for all } w \in V_J. \tag{4.10}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product on Γ and $\tilde{f}$ is the unique function in V_J satisfying

$$\left\langle \tilde{f},\theta\right\rangle = (f - Au^0,\bar{\theta}) \qquad \text{for all } \theta \in V_J.$$

As usual, $\bar{\theta}$ denotes any extension of θ into M_J. The value of $\tilde{f}$ does not depend on the choice of extension since $f - Au^0$ is in $M_0^{\perp}$. Define the operator $s_J : V_J \mapsto V_J$ by

$$\langle s_J v, w\rangle = s(v,w) \qquad \text{for all } v,w \in V_J.$$

Then (4.10) can be rewritten $s_J v = \tilde{f}$. We first consider the case when H is of unit size and $B^{\perp}$ is defined by (4.1). In this case, we define $b : V_J \mapsto V_J$ by

$$(4.11) \qquad bv = \sum_{k=1}^{J} h_k^{2-d} \sum_i \left\langle v,\psi_k^i\right\rangle \psi_k^i \qquad \text{for all } v \in V_J.$$

It is easy to check that $bs_J\, v$ is equal to the trace of $B^{\perp}A^{\perp}v_H$ and thus, $K(bs_J) = K(B^{\perp}A^{\perp})$.

Let $A_i(\cdot,\cdot)$ denote the form which results from (2.2) but with integration only over the subregion τ_H^i. It is well known (cf. [3], [4]) that $A_i(v_H,v_H)$ is uniformly (independent of H and J) equivalent to the square of the half Sobolev semi-norm ($|v|_{1/2,\partial\tau_H^i}$). Thus, $s(v,v)$ is equivalent to

$$\|v\|_{1/2,\Gamma}^2 = \sum_i |v|_{1/2,\partial\tau_H^i}^2 .$$

We now prove the following theorem.

THEOREM 4.2. *The condition number $K(bs_J) = K(B^{\perp}A^{\perp})$ is less than or equal to CJ where C is independent of J.*

PROOF: We will apply Theorem 3.1 of [1] to the form $s(v,v)$ with the spaces V_i, $i = 1,\ldots,J$. By Corollary 3.1 of [1], it suffices to prove the theorem in the case of constant coefficient L.

Following [1], we define operators

(1) $s_k : V_k \mapsto V_k$ by

$$\langle s_k\theta,\eta\rangle = s(\theta,\eta) \qquad \text{for all } \theta,\eta \in V_k.$$

(2) $q_k : V_J \mapsto V_k$ by

$$\langle q_k\theta,\eta\rangle = \langle\theta,\eta\rangle .$$

(3) $r_k : V_k \mapsto V_k$ by

$$r_k v = h_k^{2-d}\sum_i \left\langle v,\psi_k^i\right\rangle \psi_k^i.$$

It is easy to show that the largest eigenvalue λ_k of s_k is bounded by a constant times h_k^{-1}. We next show that for $v \in V_l$ with $l \leq k$,

$$\lambda_k^{-1} \|s_k v\|_\Gamma^2 \leq C(h_k/h_l)^{2\epsilon} s(v,v) \tag{4.12}$$

holds for some $\epsilon > 0$. Here $\|\cdot\|_\Gamma$ denotes the L^2 norm on Γ. We give this argument in the case when the subdomains are of unit size for convenience. The more general case is similar but uses norms which depend H (related to the norms on unit size domains by scaling).

First, we note that

$$\|s_k v\|_\Gamma^2 = \sup_{\phi \in V_k} \left(\frac{s(v,\phi)}{\|\phi\|_\Gamma} \right)^2 . \tag{4.13}$$

Let $\bar{v}$ denote the A-harmonic extension of v, i.e., $\bar{v}$ is equal to v on Γ and satisfies the homogeneous equation

$$A(\bar{v}, \theta) = 0$$

for all functions in $H_0^1(\Omega)$ which vanish on Γ. Then,

$$|s(v,\phi)| = |A(v_H, \phi_H)| \leq |A(v_H - \bar{v}, \phi_H)| + |A(\bar{v}, \phi_H)|. \tag{4.14}$$

We note that

$$|A_i(v_H - \bar{v}, \phi_H)| \leq C A_i(\tilde{W} - \bar{v}, \tilde{W} - \bar{v})^{1/2} \|\phi_H\|_{1,\tau_H^i}$$

holds for any function $\tilde{W} \in M_J$ which equals v_H on Γ. Choosing $\tilde{W}$ as in Section 3 of [7], it follows that

$$|A_i(v_H - \bar{v}, \phi_H)| \leq C h_J^\epsilon \|\bar{v}\|_{1+\epsilon,\tau_H^i} \|\phi_H\|_{1,\tau_H^i} .$$

Without loss of generality, we may assume that v_H has zero mean value on τ_H^i. Let $\tilde{v}$ denote the discrete harmonic extension of v in the space M_l. It is well known (cf. [7], [10]) that

$$\|\tilde{v}\|_{1,\tau_H^i} \leq C \|v\|_{1/2,\partial\tau_H^i} .$$

Writing $\bar{v} = \theta + \tilde{v}$ and using the regularity theory (cf. [14]) and Theorem 1.4.4.6 of [12], it is easy to see that

$$\|\bar{v}\|_{1+\epsilon,\tau_H^i} \leq C \|\tilde{v}\|_{1+\epsilon,\tau_H^i} ,$$

holds for some $0 < \epsilon < 1/2$. Applying the inverse property of [8] gives

$$\|\bar{v}\|_{1+\epsilon,\tau_H^i} \leq C h_l^{-\epsilon} \|\tilde{v}\|_{1,\tau_H^i} \leq C h_l^{-\epsilon} \|v\|_{1/2,\partial\tau_H^i} .$$

Combining the above estimates and using a standard inverse property gives

$$\begin{aligned} |A_i(v_H - \bar{v}, \phi_H)| &\leq C(h_J/h_l)^\epsilon h_k^{-1/2} \|v\|_{1/2,\partial\tau_H^i} \|\phi\|_{\partial\tau_H^i} \\ &\leq C(h_J/h_l)^\epsilon h_k^{-1/2} |v_H|_{1,\tau_H^i} \|\phi\|_{\partial\tau_H^i} . \end{aligned} \tag{4.15}$$

We have used the Poincaré inequality for the last estimate above to replace the norm by the semi-norm.

We next estimate the second term on the right hand side of (4.14). The subdomains have Lipschitz continuous boundaries and hence it follows from [13] that

$$\left\| \frac{\partial \bar{v}}{\partial n} \right\|_{-1/2+\epsilon, \partial\tau^i_{H}} \le c \|v\|_{1/2+\epsilon, \partial\tau^i_{H}}$$

holds for any ϵ in $[0, 1/2]$. Here $\partial/\partial n$ denotes the outward pointing normal derivative on $\partial\tau^i_{H}$. Thus,

$$\begin{aligned} |A_i(\bar{v}, \phi_H)| &= \left| \left\langle \frac{\partial \bar{v}}{\partial n}, \phi \right\rangle_{\partial\tau^i_{H}} \right| \le C \|v\|_{1/2+\epsilon, \partial\tau^i_{H}} \|\phi\|_{1/2-\epsilon, \partial\tau^i_{H}} \\ &\le C (h_k/h_l)^{\epsilon} h_k^{-1/2} \|v\|_{1/2, \partial\tau^i_{H}} \|\phi\|_{\partial\tau^i_{H}} \\ &\le C (h_k/h_l)^{\epsilon} h_k^{-1/2} |v_H|_{1, \tau^i_{H}} \|\phi\|_{\partial\tau^i_{H}} . \end{aligned} \tag{4.16}$$

Combining (4.14), (4.15), (4.16), summing and applying the Schwarz inequality gives

$$|s(v, \phi)| \le C (h_k/h_l)^{\epsilon} h_k^{-1/2} s(v, v)^{1/2} \|\phi\|_{\Gamma} .$$

This completes the proof of (4.12) which implies (3.5) of [1].

We next show that (3.1) of [1] is satisfied. Let Q_k denote the $L^2(\Omega)$ orthogonal projection onto M_k. Then,

$$\|(I - Q_k)w\|^2_{\Omega} + h_k^2 \|(I - Q_k)w\|^2_{1,\Omega} \le h_k^2 \|w\|^2_{1,\Omega} . \tag{4.17}$$

For $v \in V_J$,

$$\begin{aligned} \|(I - q_k)v\|^2_{\Gamma} &\le \|(I - Q_k)v_H\|^2_{\Gamma} \\ &\le C (h_k^{-1} \|(I - Q_k)v_H\|^2_{\Omega} + h_k \|(I - Q_k)v_H\|^2_{1,\Omega}) \\ &\le C h_k A(v_H, v_H) = C h_k s(v, v). \end{aligned} \tag{4.18}$$

By using an inverse inequality and (4.18), we see that

$$\begin{aligned} \|q_k v - Q_k v_H\|^2_{1/2,\Gamma} &\le C h_k^{-1} \|q_k v - Q_k v_H\|^2_{\Gamma} \\ &\le C h_k^{-1} (\|(q_k - I)v\|^2_{\Gamma} + \|(I - Q_k)v_H\|^2_{\Gamma}) \le C s(v, v). \end{aligned}$$

But, (4.17) implies that

$$\|Q_k v_H\|^2_{1/2,\Gamma} \le C A(v_H, v_H) = C s(v, v)$$

which implies that

$$s(q_k v, q_k v) \le C s(v, v) \qquad \text{for all } v \in V_J. \tag{4.19}$$

Combining (4.18), (4.19) and Theorem 1 of [9] gives that (3.1) of [1] holds with constant $C_0 \le CJ$ where C is independent of J.

The final condition which must be verified before applying Theorem 3.1 of [1] is that r_k satisfies

$$c\frac{\|v\|_\Gamma^2}{\lambda_k} \le \langle r_k v, v\rangle \le C\frac{\|v\|_\Gamma^2}{\lambda_k} \qquad \text{for all } v \in V_k. \tag{4.20}$$

The inequalities in (4.20) follow immediately from Theorem 3.1 of [2]. Theorem 3.1 of [1] implies that the condition number of the operator $(\sum_{k=1}^{J} r_k)s_J = bs_J$ is bounded by a constant times J. This completes the proof of the theorem.

REMARK 4.2: Analogous to Remark 4.1, there are applications when $H = h_j$ for $j > 1$ and no obvious set of coarser spaces $\{V_k\}$ for $k = 1, \dots, j-1$ are available. The previous theorem still remains valid if we define b by

$$bv = s_j^{-1} q_j v + \sum_{k=j+1}^{J} h_k^{2-d} \sum_i \langle v, \psi_k^i \rangle \ \psi_k^i \qquad \text{for all } v \in V_J. \tag{4.21}$$

Computationally, one builds the sparse stiffness matrix corresponding to s_j. This involves doing fine grid extensions of coarse grid boundary data. The resulting system has relatively few unknowns and can be solved by direct methods. We have not been able to show that it suffices to replace the first term in (4.21) by the analogous term coming from (4.8). Thus, without multilevel spaces on the full domain, it may not suffice to use the coarse solution A_j in the multilevel boundary iteration.

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