KRALL ASSOCIATES

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MICROSTABILITY THEORY FOR THE FIELD REVERSED CONFIGURATION

FINAL REPORT

Prepared by

Nicholas A. Krall

Krall Associates
Del Mar, CA 92014

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INTRODUCTION

This document is the Final Grant Report required by the U.S. Department of Energy at the conclusion of a Research Grant. It summarizes the work done in the last contract period. Previous work has been described in Annual Performance Reports.

The work carried on under this Research Grant and not included in previous progress and annual reports includes two distinct items:

One work is a study of the nonlocal high beta microstability of the FRC (Field Reversed Configuration), which we began sometime ago. This study identified the limiting beta ($=4\pi nT/B^2$) for the mode to remain unstable. The study found that as beta increases, the wavenumbers (k_y, K_z) for maximum growth changes, so that the limiting beta is not the one found by fixing (k_y, K_z) and increasing beta. It also appears that the criterion for nonlocal terms to influence the result, as beta increases, is substantially weaker than might have been thought. We identify the parameter that determines this effect. This study is presented as Appendix 1 of this report.

The second study is of the effect of collisions on the lower hybrid drift instability. The result is that the effect of collisions is substantially more important than might have been expected. It might have been expected that since in the absence of collisions the growth rate $\omega_i \simeq \omega_r \simeq \omega_{pi}/(1 + 4\pi nmc^2/B^2)^{1/2}$, collisions would damp the wave when $\nu \gtrsim \omega_i$, with ν a collision frequency. However, the result we get is that

$$\omega_i \sim \frac{\Omega - \nu}{D} \quad ,$$

where $(\Omega/D)_{max} = \omega_{pi}/(1 + 4\pi nmc^2/B^2)^{1/2}$; in the collisionless case, the largest growth Ω/D is achieved by minimizing D, with the appropriate choice of wave number k. But now minimizing D not only maximizes Ω/D, but also maximizes ν/D, and stability is reached when

$$\nu > \Omega \quad ,$$

rather than $\nu > \Omega/D$. The results of this study, which calculates Ω, are derived in Appendix 2.

These two studies are in different stages of completion. The second is in fact complete, and could be published virtually as is, although it would benefit from a small amount of numerical analysis. The first study is far richer than the second, in that it includes a variety of regimes and effects. The formulation presented in it could be used as the basis for a series of papers, although in its present stage it is not ready for publication. It is unfortunate, but the level of the research Grant, and its untimely end, did not permit further progress on that study.

APPENDIX 1

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NONLOCAL AND HIGH β DRIFT WAVE THEORY

Nicholas A. Krall

Krall Associates
Del Mar, CA 92014

1. INTRODUCTION

Drift waves are an important part of transport theory. They include, among others,

- Eta-i modes ($d \ln T_i / d \ln n_i$)
- Trapped particle instabilities
- Collisionless and dissipative drift instabilities.

Collisionless and dissipative drift waves, driven by gradients in the plasma density and/or temperatures, are believed to dominate or at least influence the transport properties of a variety of plasma confinement devices.[1,2] In the present work, with particular reference to the Field Reversed Configuration (FRC),[3-5] we develop a nonlocal theory of these waves in an arbitrary β ($\equiv$ plasma pressure/magnetic pressure) plasma, including the effect of perturbed flow in the direction of the plasma density.

It is well known that finite β has a strong effect on the structure and stability of drift waves; "finite" is here defined by $\beta \gtrsim \beta_c$ with β_c the value of β at which the $\beta = 0$ results are significantly altered. Previous studies have shown that β_c is strongly mode-specific. For modes already studied, the variety of β_c included

- MHD interchange: $\beta_c \sim 1$
- Electrostatic universal instability: $\beta_c \sim (r_p/L_p)^2$
- Ion cyclotron drift wave: $\beta_c \sim (m_e/m_i)^{1/2}$
- Lower hybrid drift wave: $\beta_c \sim (a_{Li}/r_p)$

where r_p = plasma radius, L_p = plasma length, and a_{Li} = ion gyroradius.

The present work is a natural extension of previous calculations which were limited to $\beta \simeq 1$ in a local approximation;[6,7] the high β nature of the FRC makes a proper treatment of high β effects vital to an understanding of that device. The major result of this study is a comprehensive dispersion equation for the low frequency stability FRC, which shows clearly how the universal and dissipative drift wave instabilities evolve in wavenumber as β increases. A major finding from this is that the effect of finite β begins to dominate long before $\beta \to 1$; the expansion parameter is $\beta f(k, a_i, K, \omega, L_n)$ where f is a function which can be substantially greater than 1, depending on the wavenumber of the wave parallel to the magnetic field (K), the wavenumber parallel to the particle drifts (k), the wave frequency (ω), the strength of the density gradient (L_n), and the ion gyroradius (a_i). The fact that finite β effects can onset for quite small β make this study also applicable to confinement schemes such as tokamak in which $\beta \sim$ 1-10%, in addition to the natural application to the FRC.

A second result from the study was the surprising finding that including finite β could result in a compressional flow in the direction of the density gradient, and could also generate a perturbed electric field in that direction, which changes the perturbed orbits. These finite β drift effects prove to be lower order in ka_i than the $\beta = 0$ drift effects. Thus, finite β effects set in for $\beta \ll 1$ for modes with $ka_i \ll 1$.

In this report we derive and quantify these results. Section 2 discusses the elements which parameterize drift waves, and cause deviation from the results obtained in the original rather primitive calculations of drift instabilities.[8] In Section 3 we develop the nonlocal, finite β dispersion equation for drift waves including arbitrary polarization, identifying a new critical parameter involving $E_i \nabla n_0$, and in Appendix A we give details of the derivation. In

Section 4 we solve the equations developed in Section 3 in the limit $k_x = E_x = 0$ and β arbitrary, extending and connecting previous specialized results.[6-9] We leave the $k_x \neq 0$, $E_x \neq 0$ exploration of the formulation documented in Section 3 for other researchers.

2. ELEMENTS WHICH PARAMETRIZE DRIFT WAVE BEHAVIOR

There are a number of plasma parameters and phenomena which can drive or alter drift wave instabilities. Despite the extensive literature, not all of these parameters and phenomena have been explored. The list of effects includes the following:

<u>Plasma Gradient Drifts</u>: Drifts proportional to ∇nT are responsible for virtually all drift wave activity and are included in all theories.

<u>Magnetic Gradient Drifts</u>: These include magnetic curvature effects. They have been modelled in a limited number of examples as a pseudogravity.

<u>Finite Larmor Radius Effects</u>: In many cases, drift wave growth is of order $(ka_i)^2$. FLR effects are routinely included in drift wave calculations.

<u>Finite Collisionality</u>: Particle collisions allow cross field transport, but also provide a dissipation which can drive negative energy waves unstable; they are included in calculations of dissipative drift waves.

<u>Finite Beta Effects</u>: These have been largely ignored, with the notable exception of Ref. 6, which included the electromagnetic component of the drift wave introduced by finite β. A subsequent calculation[7] has questioned the existence of drift instability in the finite β regime.

The present study obtains a complete description of the transition from $\beta = 0$ drift instability to higher β instability.

Nonlocal Effects: In this category we combine effects which operate in the direction of the plasma gradients. With few exceptions,[10] previous drift wave theories have been local, in the sense that variations of the perturbed quantities with x, where $n = n(x)$, were neglected, along with perturbed fields E_x. In the local zero β treatments, the perturbed fields were $E_y e^{iky} e^{iKz}$ and $E_z e^{iky} e^{iKz}$. However, finite β requires that perturbed magnetic fields be retained. Specifically, finite β can introduce an E_x and B_z, through the electromagnetic equations $dB_z/dt = (dE_x/dy - dE_y/dx)c$. In the present study we keep these effects, and show that there is a parameter range in which they can be significant.

3. NONLOCAL, FINITE β, ARBITRARY POLARIZATION DRIFT WAVES

In this section we derive a general expression for drift waves which retains the effects of finite β, variations in x, where $n_p = n_p(x)$, and electric fields also in the x-direction. The details of the derivation of these expressions are given in Appendix A.

We consider the slab plasma as shown in Figure 1, where the plasma is infinite and uniform in the z-y plane, $B = B_z\hat{z}$ is the magnetic field, and n is the plasma density. The plasma can be described by the distribution function

$$f_o = f_M\ (v_z,\ v_\perp^2)\ g(\eta)$$

$$\eta \equiv v_y - \int B_z dx \quad .$$

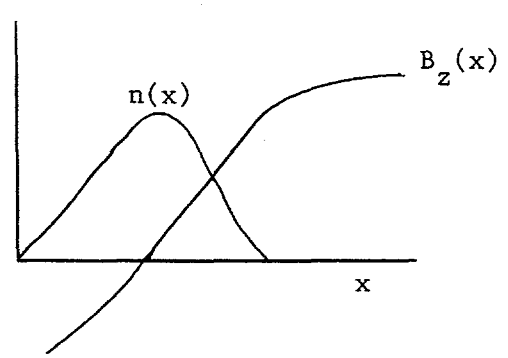

Figure 1. Plasma geometry for drift waves.

A perturbation $\delta E = E(x)e^{iky}e^{iKz}e^{i\omega t}$, $(i\omega/c)\delta B = -\nabla x \delta E$ is applied to the plasma and the response calculated from the Vlasov and Maxwell equations,

$$f_{1\alpha} = -\frac{q}{m}\int dt\, e^{ik\cdot r'}\left(E + \frac{v' \times \delta B}{c}\right)\cdot \nabla_v f_o \quad , \tag{1}$$

$$\nabla \cdot E = 4\pi \sum_\alpha q_\alpha \int f_{1\alpha}\, dv \quad , \tag{2}$$

$$\nabla \times \delta B = \frac{4\pi}{c}\sum_\alpha q_\alpha \int v f_1 dv \quad , \quad \text{(x and z components)} \tag{3}$$

$$\frac{i\omega}{c}\delta B = -\nabla \times E \quad ,$$

where the sums are over particle species α, and r′, v′ are the particle orbits in the unperturbed magnetic field. Making a small Larmor radius approximation ($a_i < L_n = (d\ln n/dx)^{-1}$) the integrand of (1) can be expanded

$$E(x') = E(x) + \frac{\partial E}{\partial x}(x' - x) + \frac{1}{2}\frac{\partial^2 E}{\partial x^2}(x' - x)^2 \quad , \tag{4}$$

and the orbit integrals carried out in the usual way. The technique for obtaining a dispersion relation $D(\omega,k) = 0$ from Eqs. (1)-(4) has been used for so many years[8] that its details may have been forgotten. For completeness,

Appendix A derives $D(\omega,k) = 0$ for this problem in sufficient detail to allow future researchers to reproduce and expand our results.

The result, after much algebra, is

$$a_i^2 \left(1 + \frac{\omega^X}{\omega}\right) \frac{\partial^2 E_y}{\partial x^2} - \frac{k^2 a_i^2 \omega_{ci}}{X\,\omega} \left[W_e - \frac{\omega^X}{\omega}(2 + W_e)\right] \frac{1}{k}\frac{dE_y}{dx} -$$

$$\left\{k^2\lambda_D^2 + k^2 a_i^2 \left(1 + \frac{\omega^X}{\omega}\right) - \frac{\omega^X}{\omega}\left[1 + \frac{T_e}{T}\right] \frac{\beta}{X} \left[W_e - \frac{\omega^X}{\omega}(2 + W_e)\right]\right\} E_y =$$

$$\left\{K^2\lambda_D^2 - \left[1 - \frac{\omega^X}{\omega} - \frac{i\nu}{\omega}(1 + W_e)\left(1 - \frac{\omega^X}{\omega}\right)\right] W_e + \right.$$

$$\left.\left[W_e - \frac{\omega^X}{\omega}(2 + W_e)\right] \left(1 - \frac{\omega^X}{\omega}\right) \frac{\beta}{X} W_e\right\} E_z \frac{k}{K} \qquad , \qquad (5)$$

$$\frac{k^2 a_i^2 \omega_{ci}}{X\,\omega} W_e\left(1 - \frac{\omega^X}{\omega}\right) \frac{1}{k}\frac{dE_y}{dx} + \left[\frac{k^2\lambda_D^2 K^2 c^2}{\omega^2} - \frac{\beta W_e}{X}\frac{\omega^X}{\omega}\left(1 + \frac{T_e}{T_i}\right)\left(1 - \frac{\omega^X}{\omega}\right)\right] E_y =$$

$$\left\{\frac{k^2\lambda_D^2 K^2 c^2}{\omega^2} - \frac{k^2\lambda_D^2 K^2 c^2}{\omega^2}\frac{1}{k^2}\frac{d^2}{dx^2} + W_e\left(1 - \frac{\omega^X}{\omega}\right)\left[1 - \frac{i\nu}{\omega}(1 + W_e)\right] + \right.$$

$$\left. W_i\left(1 + \frac{\omega^X}{\omega}\right) - \frac{\beta W_e}{X}\left(1 - \frac{\omega^X}{\omega}\right)^2 \left[1 + \frac{i\nu}{\omega}(1 + W_e)\right]\right\} \frac{k}{K} E_z \qquad , \qquad (6)$$

where $\omega^X \equiv \frac{kT}{M\omega_{ci}} \frac{1}{f_o} \frac{\partial f_o}{\partial x}$, W_α is defined below in Eqs. (9)-(10), and $X \equiv 1 + \Sigma_\alpha \beta_\alpha (1 + \frac{\omega^X}{\omega})_\alpha (1 + W_\alpha)$.

The perturbed field E_x has been expressed in terms of E_y, E_z by

$$E_x \simeq \frac{1}{X} \frac{\omega}{\omega^X} \frac{\beta}{ikL_n} \left\{ \left[\frac{2\omega^X}{\omega} + \frac{\omega^X}{\omega} (1 + \frac{\omega^X}{\omega}) L_n \frac{\partial}{\partial x} \right] E_y - W_e (1 - \frac{\omega^X}{\omega}) \frac{k}{K} E_z \right\} \quad . \tag{7}$$

Clearly, E_x is not necessarily negligible for finite small β because the parameter which determines the generation of E_x from finite β is not β itself, but

$$\frac{\beta}{kL_n}$$

where $kL_n = ka_i(L_n/a_i)$ can be a small parameter even when $L_n > a_i$.

There is a variety of information contained in Eqs. (5)-(7). One possibility is to solve the differential equation (4th order) as an eigenvalue problem for ω. This requires a specific profile n(x), B(x). We do not attempt this solution, on the grounds that the result would be specific to the FRC and probably not worth the time such a device-specific calculation would require. Two more modest efforts are to:

- Delete all $\partial/\partial x$ and E_x effects and find $\omega(\omega^X, \beta, \nu, ka_i, KL_n)$.
- Write $\partial/\partial x = ik_x$ and find the effect of E_x and harmonic spatial structure in the x-direction.

We discuss these in the next two sections.

4. DRIFT WAVES FOR ARBITRARY β; $E_x = k_x = 0$

Setting $E_x = k_x = 0$, Eqs. (5)-(6) reduce to (ν is the electron collision frequency),

$$k^2 a_i^2 \frac{\omega}{\omega^x}\left(\frac{\omega}{\omega^x}+1\right)\left[1+\frac{\beta W_e}{K^2 L_n{}^2}\left(\frac{\omega}{\omega^x}-1\right)\left(\frac{\omega}{\omega^x}-\frac{i\nu}{\omega^x}-\frac{i\nu}{\omega^x}W_e\right)+\right.$$

$$\left.\frac{\beta}{K^2 L_n{}^2}\left(\frac{\omega}{\omega^x}+1\right)\frac{\omega}{\omega^x}W_i\right]=\left(\frac{\omega}{\omega^x}-1\right)\left(\frac{\omega}{\omega^x}-\frac{i\nu}{\omega^x}-\frac{i\nu}{\omega^x}W_e\right)W_e+$$

$$\frac{\omega}{\omega^x}\left(\frac{\omega}{\omega^x}+1\right)W_i \quad , \tag{8}$$

where ω^x is the ion drift frequency and the W's are the limit as $\epsilon \rightarrow 0$ of

$$W_e = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{ve^{-v^2}\,dv}{v+[(\omega-i\nu)/Kv_e]-i\epsilon} \quad , \tag{9}$$

$$W_i = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{ve^{-v^2}\,dv}{v+(\omega/Kv_i)-i\epsilon} \quad , \tag{10}$$

with v_e and v_i the thermal velocities.

In the limit $\beta = 0$, $\nu = 0$, and $Kv_i < \omega < Kv_e$, Eq. (8) gives

$$\omega = \omega^X + (2k^2a_i^2/W_e)\ \omega^X$$

$$= \omega^X\ (1 - 2k^2a_i^2 - i\ \sqrt{\pi}\ 2k^2 a_i^2 \omega^X/Kv_e) \qquad , \qquad (11)$$

which is the familiar collisionless branch of the drift wave instability spectrum. When $\beta = 0$, $\nu \neq 0$, $\omega > Kv_e$, Eq. (8) gives

$$\omega = \omega^X + 4k^2a_i^2\ (\frac{\omega^X}{Kv_e})^2\ (\omega^X - i\nu) \qquad , \qquad (12)$$

which is the drift dissipative branch of the drift wave spectrum. So Eq. (8) extends both the collisionless and dissipative drift waves into the finite β regime.

We first consider the collisionless drift instability. Rewriting Eq. (8) gives

$$\omega = \omega^X + \frac{k^2a_i^2\ (\omega + \omega^X)\ \frac{\omega}{\omega^X}}{W_e\left[1 - \frac{k^2a_i^2\beta}{K^2L_n^2}\frac{\omega}{\omega^X}\ (\frac{\omega}{\omega^X} + 1)\right]\left[\frac{\omega - i\nu}{\omega^X} - \frac{i\nu W_e}{\omega^X}\right]} -$$

$$\frac{W_i}{W_e}\ (\frac{\omega}{\omega^X} + 1)\ \frac{\omega}{\left[\frac{\omega - i\nu}{\omega^X} - \frac{i\nu}{\omega^X}W_e\right]} \qquad . \qquad (13)$$

This shows that the appropriate "finite-β" parameter is $k^2a_i^2\beta/K^2L_n{}^2$. When this parameter is small, Eq. (13) becomes

$$\omega \equiv \omega^X \left[1 - \frac{2k^2a_i^2 + i\sqrt{\pi}\, 2k^2a_i^2\omega^X/Kv_e}{1 - \dfrac{2k^2a_i^2}{K^2L_n{}^2}\beta} \right] \quad . \tag{14}$$

This shows the path that the drift instability follows in $k^2a_i^2$ and $K^2L_n{}^2$ parameter space as β increases. From Eq. (13) we see further that as $k^2a_i^2\beta/K^2L_n{}^2$ increases, $\omega/\omega^X < 1$ extends the β range of the collisionless drift instability.

Using a numerical method developed by N. T. Gladd,[10] we have solved Eq. (13) directly for increasing values of β. Figure 2 shows the development of the collisionless drift instability with β. Equation (13) essentially gives ω/ω^X in terms of three parameters, $k^2a_i^2$, $K^2L_n{}^2$, β. In reducing the result to $\mathrm{Im}\,\omega(\beta)$, we varied ka_i and KL_n as well as β in a manner consistent with the idea that $\omega/Kv_i > 1$ and $W_i < 2k^2a_i^2$ would constrain these parameters. Figure 2 is a qualitative representation of the maximum value of $\mathrm{Im}\,\omega_i$ for a given β. When $ka_i > 0.7$ or $KL_n > 0.233$, the collisionless drift instability disappeared for all β.

Next we turn to the dissipative drift wave (DDW) branch. Here

$$\frac{\omega}{\omega^X} \simeq 1 + 4k^2a_i^2 \left[\frac{\omega^X}{Kv_e}\right]^2 \left(\frac{\omega}{\omega^X} - \frac{i\nu}{\omega^X}\right) \left[1 - \frac{k^2a_i^2}{K^2L_n^2}\beta\,\frac{\omega}{\omega^X}\left(\frac{\omega}{\omega^X} + 1\right)\right]^{-1} , \tag{15}$$

where $(\omega^X - i\nu)/Kv_e \gtrsim 1$. This constraint is a severe limit on $k^2a_i^2\beta/K^2L_n{}^2 = (\omega^X/Kv_e)(M/m)^{1/2}\beta$, and finite β quickly forces the mode to $\omega << \omega^X$ or to the branch $\omega/\omega^X \cong -1$, both of which are stable. As a practical matter, this means that the DDW would appear unstable only for $\beta < (m_e/m_i)$. However, as β increases the frequency ω/ω^X decreases, until $(\omega - i\nu)/Kv_e < 1$. This mode remains unstable, with

$$\frac{\omega}{\omega^X} \simeq \frac{K^2 L_n^{\ 2}}{\beta k^2 a_i^2} - i \left[\frac{K^2 L_n^{\ 2}}{k^2 a_i^2 \beta}\right]^2 \frac{M}{m} \frac{K^2 L_n^{\ 2} \omega^{X2}}{m\nu^2} \left[\frac{ka_i}{KL_n} \sqrt{\frac{\bar{m}}{M}} \frac{\nu^2}{\omega^{X2}}\right] \quad . \tag{16}$$

Figure 3 shows the evolution of the DDW from the $\beta = 0$ limit to larger β, as given by Eq. (16). This drift wave branch is discussed in Ref. 8.

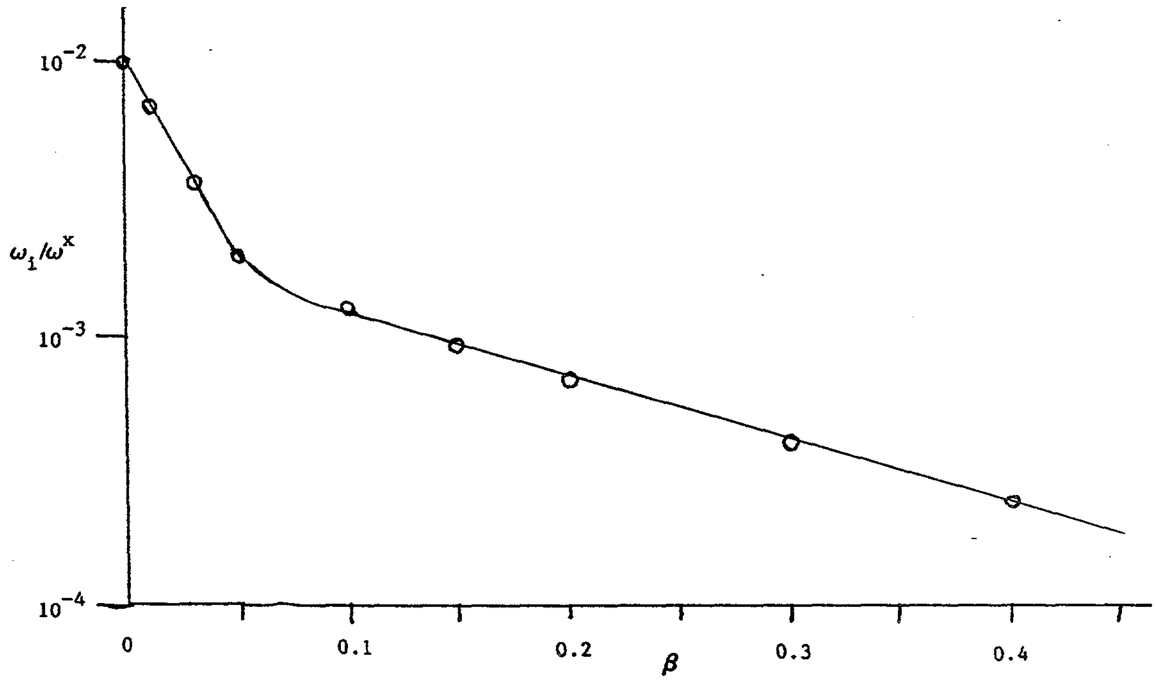

Figure 2. ω_i/ω^X vs. β for the collisionless drift instability.

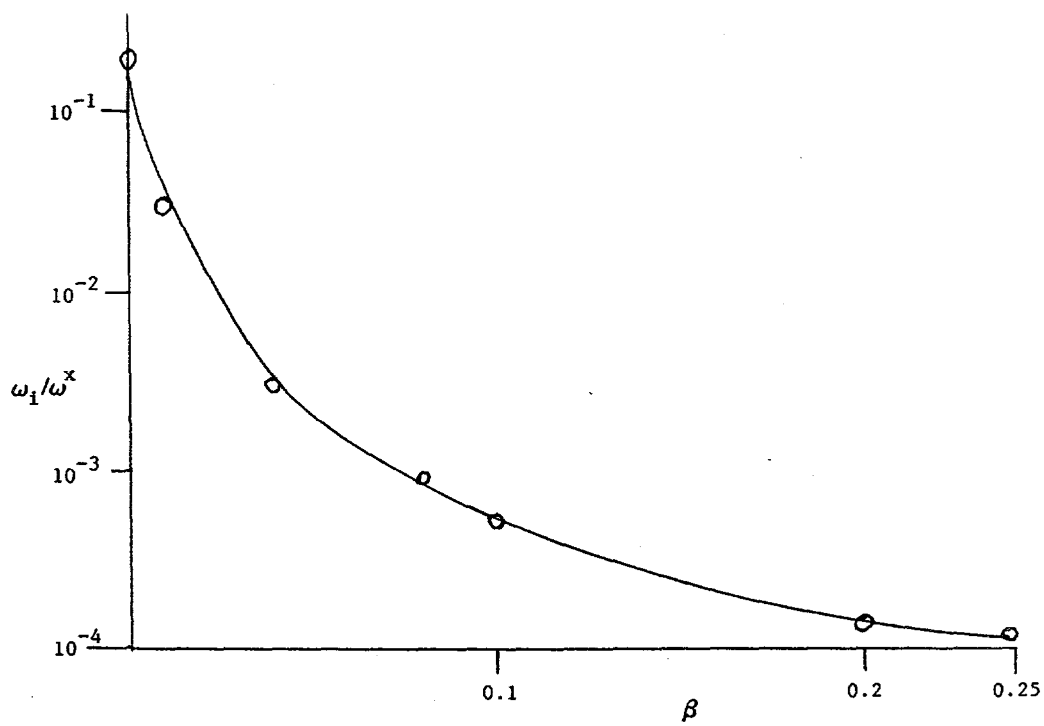

Figure 3. ω/ω^X vs. β for the DDW.

5. NONLOCAL EFFECTS AT $\beta \neq 0$

When the E_x and k_x terms are retained in the general derivation of high β drift waves, an interesting feature is apparent from Eqs. (15)-(16). A new parameter,

$$\frac{k_x}{k_y}\frac{\omega_{ci}}{\omega} \quad , \tag{17}$$

competes with terms of order 1. While we have not yet explored the consequence of this new parameter, the physics of its appearance is clear, as follows.

The perturbed charge density in the drift wave is determined by the perturbed velocities,

$$i(\omega + kV_D)\, n_1 = -\, V_{1x}\frac{dn_o}{dx} - n_o \nabla \cdot V_1 \quad . \tag{18}$$

Because for low frequency waves V_{1x} is the same for electrons and ions to order $k^2a_i^2$, and $dn_{oe}/dx = dn_{oi}/dx$, the RHS of Eq. (17) is $o(k^2a_i^2)$ when only E_y perturbations are included. The k_x and E_x terms produce a δB_z, which gives a $V_D \times \delta B_z$ contribution to δV_x which is opposite for electrons and ions. This leads to

$$\sum_{i,e} n_o \nabla \cdot V_1 = \frac{\omega}{\omega_{ci}} kE_y + \frac{kV_d}{\omega}\left(\frac{d}{dx} E_y - ikE_x\right) \quad ,$$

$$\sum_{i,e} V_x \cdot \frac{dn_o}{dx} = \frac{k^2a_i^2E_yc}{B} + \frac{i\omega}{\omega_c}\left[\frac{E_xc}{B} + \frac{kc}{\omega}\frac{E_xV_d}{B_o}\right] + \frac{ik_xV_d}{\omega_c}\frac{cE_y}{B} \quad ,$$

where $E_x \sim (\beta/kL_n)E_y = \beta(a_i/L_n)(1/ka_i)E_y$ shows that the contributions from k_x, E_x can be substantial even for $\beta \ll 1$.

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APPENDIX A. DERIVATION OF THE DISPERSION EQUATION

For a highly elongated FRC, we assume that the stability problem can be calculated in slab geometry. Therefore, we consider an equilibrium in which an equilibrium distribution of plasma particles is described by a function of space and velocity $f_o(\mathbf{v},x)$ confined by electric and magnetic fields $B_z(x)$, $E_x(x)$. If this distribution is perturbed by a field $\mathbf{E}_1(x)\exp(iky + iKz + i\omega t)$, $\mathbf{B}_1(x)\exp(iky + iKz + i\omega t)$ the linear plasma response $f_1(\mathbf{v}, x, y, z)$ for a plasma species of charge q and mass m is determined by the Vlasov equation,

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \nabla f_1 + \frac{q}{m}\left(\mathbf{E}_o + \frac{\mathbf{v} \times \mathbf{B}_o}{c}\right) \cdot \nabla_v f_1$$

$$= -\frac{q}{m}\left(\mathbf{E}_1 + \frac{\mathbf{v} \times \mathbf{B}_1}{c}\right) \cdot \nabla_v f_o - \nu f_1 + \nu f_m \frac{n_1}{n_o} \quad , \qquad \text{(A1)}$$

where the effect of collisions has been modelled by a collision operator

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} = -\nu f_1 + \nu f_m \frac{n_1}{n_o} \quad ,$$

$$n_1 = \int d^3v\, f_1 \quad ,$$

$$n_o = \int d^3v\, f_o = \int d^3v\, f_m \quad ,$$

with f_m a Maxwellian distribution. This simple collision term at least conserves density. The plasma response sets up currents and charges which must be self-consistent with $\mathbf{E}_1$ and $\mathbf{B}_1$ through Maxwell's equations

$$\nabla \cdot \mathbf{E}_1 = 4\pi e \; (n_{1i} - n_{1e}) \quad , \tag{A2}$$

$$\nabla \times \mathbf{B}_1 = \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t} + \frac{4\pi}{c} (ne\mathbf{V}_i - ne\mathbf{V}_e)_1 \quad , \tag{A3}$$

$$\nabla \times \mathbf{E}_1 = - \frac{1}{c} \frac{\partial \mathbf{B}_1}{\partial t} \quad , \tag{A4}$$

where i and e refer to ions and electrons. In this Appendix we give the details of the derivation of the dispersion equation. These details are all contained in earlier papers, and are repeated here simply for the convenience of the reader.

A1. CHOICE OF EQUILIBRIUM

The equilibrium distribution function can be made up of any function of the constants of the motion,

$$H = v^2 - \frac{2q}{m} \int E_o \, dx \quad , \qquad p = v_y + \int \omega_c \, dx \quad , \tag{A5}$$

where $\omega_c = qB/mc$ and E_o is the equilibrium electric field. We assume that plasma wavelengths and gyroradius are small compared with the scale length of the equilibrium, and so expand $\int \omega_c \, dx = \omega_c(x_o)(x - x_o)$, $\int E_o \, dx = E_o(x_o)(x - x_o)$. For example, a local equilibrium which includes a density gradient, a diamagnetic drift due to the density gradient, and a drift due to the electric field would be

$$f_o = \bar{n}_o \left(\frac{m}{2\pi T}\right)^{3/2} e^{-\frac{mv^2}{2T} + \frac{qE_o x}{2T}} \left[1 + \epsilon' \left(x + \frac{v_y}{\omega_c}\right)\right] \quad , \tag{A6}$$

with

$$n_o(x_o) = \bar{n}_o \quad , \tag{A7}$$

$$\frac{dn_o}{dx} = n_o \left(\epsilon' + \frac{qE_o}{2T}\right) \quad , \tag{A8}$$

$$V_y = \frac{1}{n_o} \int v_y f_o d^3v = \frac{\epsilon'}{\omega_c} \frac{T}{m} = \frac{T}{m\omega_c} \frac{1}{n} \frac{dn}{dx} - \frac{E_o c}{B} \quad . \tag{A9}$$

Most of the analysis in this Appendix does not require the simplified form (A6)-(A9), but will use an arbitrary $f_o = f_o(H,p)$.

A2. PERTURBED WAVE VECTORS

For low β, the low frequency stability problem is dominated by electrostatic waves, $\phi = - \nabla\phi$, with wave vectors E_y, E_z in the direction of the drift and the B-field. The FRC has high β, so we keep magnetic perturbations. If the study is nonlocal, i.e., $d\phi/dx \neq 0$, electric fields E_x in the direction of the density gradient also contribute. Therefore, we keep a full set of field vectors

$$E_1 = E_y\hat{y} + E_z\hat{z} + E_x\hat{x} \quad , \tag{A10}$$

which we write, in order to stay in contact with the low β work, as

$$E_y = ik\phi \quad , \tag{A11}$$

$$E_z = iK\phi + \lambda_z \quad , \tag{A12}$$

$$E_x = \frac{d\phi}{dx} + \lambda_x \quad . \tag{A13}$$

The associated perturbed magnetic fields are obtained from Faraday's law,

$$\frac{i\omega}{c} B_{1x} = - ik\lambda_z \quad , \tag{A14}$$

$$\frac{i\omega}{c} B_{1y} = \frac{d}{dx} \lambda_z - iK\lambda_x \quad , \tag{A15}$$

$$\frac{i\omega}{c} B_{1z} = ik\lambda_x \quad . \tag{A16}$$

A3. CHOICE OF MAXWELL'S EQUATIONS

Using Eq. (A4) to eliminate B_1 in terms of E_1, Eqs. (A2) and (A3) give four equations for the three wave vectors E_x, E_y, E_z. This gives us a choice as to which set of three we use in the calculation. A good choice would be one in which there is a minimum number of large electron terms which cancel large ion terms, so that the j_e and j_i don't have to be calculated to higher order to take account of the cancellation. The set we choose is

$$\nabla \cdot \mathbf{E} = 4\pi e \ (n_i - n_e)_1 \quad , \tag{A17}$$

$$(\nabla \times \mathbf{B})_{1z} = \frac{\partial E_z}{\partial t} + \frac{4\pi e}{c} [(n_i V_{zi})_1 - (n_e V_{ze})_1] \quad , \tag{A18}$$

$$(\nabla \times \mathbf{B})_{1x} = \frac{\partial E_x}{\partial t} + \frac{4\pi e}{c} [(n_i V_{xi})_1 - (n_e V_{xe})_1] \quad . \tag{A19}$$

The reasoning is as follows: Equation (A17) is Poisson's equation, which contains the electrostatic part of the problem. Equation (A18) involves V_z, which is flow along the magnetic field due to the perturbation E_z. Since $V_z \sim qE_z/\omega m$, the electron term is large and dominant. For the third equation the choice was between Eq. (A19) which involves j_x and the component of Eq. (A3) which involves j_y. We note that in the low β calculations, the j_{1y} equation combines with the j_{1z} equation to give Poisson's equation. To avoid a redundancy in the leading E_y, E_z terms, we use Eq. (A19) to find E_x in terms of E_y and E_z.

A4. THE PERTURBED DISTRIBUTION FUNCTION

Equation (A1) is solved by defining time dependent variables x′,v′ which are orbits of particles in the equilibrium fields E_o, B_o, with the boundary condition x′(t = 0) = x, v′(t = 0) = v, where x, v are the phase space variables in (A1). In terms of x′, v′, Eq. (A1) can then be integrated to give

$$f_1(\mathbf{r},\,\mathbf{v}) = \int_{-\infty}^{0} dt\; e^{ik(y'-y)+iK(z'-z)+(i\omega+\nu)t}$$

$$\times\left[-\frac{q}{m}\left(\mathbf{E}_1(x') + \frac{\mathbf{v}'\times\mathbf{B}_1}{c}\right)\cdot\nabla_{v'}\, f_o(x',v') + \nu f_m\,\frac{n_1(x')}{n_o}\right] \quad , \qquad \text{(A20)}$$

$$\frac{d\mathbf{r}'}{dt} = \mathbf{v}' \quad ,$$

$$\frac{d\mathbf{v}'}{dt} = \frac{q}{m}\left(\mathbf{E}_o + \frac{\mathbf{v}'\times\mathbf{B}_o}{c}\right) \quad .$$

Now we use

$$\nabla_v f_o = 2\mathbf{v} \frac{\partial f_o}{\partial H} + \hat{y} \frac{\partial f_o}{\partial p} \quad , \tag{A21}$$

and

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0 \quad , \tag{A22}$$

to write

$$f_1 = \int_{-\infty}^{0} dt\ e^{iky' + iKz' (i\omega + \nu)t}$$

$$\times \left\{ -\frac{q}{m} 2\mathbf{E}_1 \cdot \mathbf{v}' \frac{\partial f_o}{\partial H} - \frac{q}{m} \left[E_y + \left(\frac{\mathbf{v}' \times \mathbf{B}_1}{c} \right)_y \right] \frac{\partial f_o}{\partial p} + \nu f_m \frac{n_1(x')}{n_o} \right\}$$

$$= \int_{-\infty}^{0} dt\ e^{iky' + iKz' (i\omega + \nu)t}$$

$$\times \left[-\frac{2q}{m} \nabla\phi \cdot \mathbf{v}' \frac{\partial f_o}{\partial H} - \frac{2q}{m} (\lambda_x v_x' + \lambda_z v_z') \frac{\partial f_o}{\partial H} \right.$$

$$\left. - \frac{q}{m} \left(ik\phi - \frac{kv_z'}{\omega} \lambda_z - \frac{kv_x'}{\omega} \lambda_x \right) \frac{\partial f_o}{\partial p} + \nu f_m \frac{n_1(x')}{n_o} \right] \quad . \tag{A23}$$

Now some simplification follows from using (A11)-(A13) for E_1, and noting that

$$\int_{-\infty}^{0} dt\ e^{iky'+iKz'+(i\omega+\nu)t} \left[(ikv_y' + iKv_z')\phi + \frac{d\phi}{dx} v_x' \right]$$

$$= \phi - (i\omega + \nu) \int_{-\infty}^{0} dt\ e^{iky'+iKz'(i\omega+\nu)t}\ \phi(x') \quad . \tag{A24}$$

This leads at once to

$$f_1 = -\frac{2q}{m}\frac{\partial f_o}{\partial H}\phi + \left(\frac{2q}{m}\frac{\partial f_o}{\partial H} - \frac{q}{m}\frac{k}{\omega}\frac{\partial f_o}{\partial p}\right)$$

$$\times \int_{-\infty}^{0} dt \left(i\omega\phi - \lambda_x v_x' - \lambda_z v_z'\right) e^{iky+iKz+(i\omega+\nu)t}$$

$$+ \nu \int_{-\infty}^{0} dt \left(\frac{2q}{m}\frac{\partial f_o}{\partial H}\phi + \frac{n_1}{n_o} f_m\right) e^{iky'+iK'(i\omega+\nu)t} \quad . \tag{A25}$$

A5. THE ORBITS

The orbits **r**′ and **v**′ satisfy

$$v_z' = v_z \quad , \qquad z' = v_z t \quad ,$$

$$\frac{dv_x'}{dt} = \frac{q}{m}\left(E_o + \frac{v_y B_o}{c}\right) \quad , \tag{A26}$$

$$\frac{dv_y'}{dt} = -\frac{q}{m}\frac{v_x B_o}{c} \quad .$$

Now expanding about an arbitrary x = 0, y = 0 point, $B_o = B_o(0) + x'(dB_o/dx)$ and $E_o = E_o(0) + x'(dE_o/dx)$, we can solve Eq. (A26) by iteration, e.g., $v_x' = v_x^o + v_x^1$, $x' = x^o + x^1$, where $B_o(0)$ is zeroth order and $x(dB_o/dx)$ is first order. The equation set to first order is

$$\frac{dv_x^o}{dt} = \frac{q}{m}\left(E_o + \frac{v_y^o B_o}{c}\right) \quad , \qquad \frac{dv_y^o}{dt} = -\frac{q}{m}\frac{v_x^o B_o}{c} \quad ,$$

$$\frac{dv_x^1}{dt} = \frac{q}{m}\frac{dE_o}{dx}x^o + \frac{q}{mc}\frac{dB_o}{dx}v_y^o x^o + v_y^1\frac{qB_o}{mc} \quad ,$$

$$\frac{dv_y^1}{dt} = -\frac{q}{mc}v_x^1 B_o - \frac{q}{mc}\frac{dB_o}{dx}v_x^o x^o \quad .$$

Neglecting the d/dx terms gives the solution to zeroth order:

$$v_x^o = V\sin(\theta + \omega_c t) \quad , \qquad x^o = -\frac{V}{\omega_c}\cos(\theta + \omega_c t) + \frac{V}{\omega_c}\cos\theta \quad ,$$

$$v_y^o = V\cos(\theta + \omega_c t) - \frac{E_o c}{B} \quad , \qquad y^o = \frac{V}{\omega_c}\sin(\theta + \omega_c t) - \frac{V}{\omega_c}\sin\theta - \frac{E_o c}{B} t \quad ,$$

where

$$v_x = V\sin\theta \quad , \qquad v_y = V\cos\theta \quad ,$$

define V and θ in terms of the phase space variables v_x, v_y, and $\omega_c = qB_o(0)/mc$. Next we time average the first order equations, keeping only the secular (non-oscillating in time) first order terms v^1 and $dx^1/dt = v^1 t$. This gives to first order (neglecting oscillating terms such as $(E_o c/B)\cos\omega_c t$)

$$v_y' = V\cos(\theta + \omega_c t) - \frac{E_o c}{B} + \frac{1}{2}\frac{V^2}{\omega_c}\frac{1}{B}\frac{dB}{dx} - \frac{V}{\omega_c}\frac{E_o c}{B_o}\left(\frac{B_o}{E_o}\right)\frac{d}{dx}\left(\frac{E_o}{B_o}\right)\cos\theta \quad , \tag{A27}$$

$$v_x' = V\sin(\theta + \omega_c t) \quad , \tag{A28}$$

$$x' = -\frac{V}{\omega_c}\cos(\theta + \omega_c t) + \frac{V}{\omega_c}\cos\theta \quad , \tag{A29}$$

$$y' = \frac{V}{\omega_c}\sin(\theta + \omega_c t) - \frac{V}{\omega_c}\sin\theta - \alpha_o t - \alpha_1 t\cos\theta \quad , \tag{A30}$$

$$\alpha_o \equiv \frac{E_o c}{B} - \frac{1}{2}\frac{V^2}{\omega_c}\frac{1}{B}\frac{dB}{dx} \quad , \tag{A31}$$

$$\alpha_1 \equiv \frac{V}{\omega_c}\frac{d}{dx}\frac{E_o c}{B_o} \quad . \tag{A32}$$

A6. TABLE OF INTEGRALS

The orbits listed above allow the time integrals in Eq. (A25) to be done explicitly. If the calculation is nonlocal, $\phi(x') = \phi(x) + (x' - x)\, d\phi/dx + (d^2\phi/dx^2)(x' - x)^2/2$ changes (A25) into an explicit set of integrals, of the form

$$f_1 \sim \int dt\; e^{iky'+iKz'+(i\omega+\nu)t}$$

$$\times \left[a_1 + a_x(x' - x) + a_{xx}(x' - x)^2 + a_v v_x + a_{vx} v_x(x' - x)\right] \quad . \tag{A33}$$

We define

$$I_\alpha = \int dt\; e^{iky'+iKz'+(i\omega+\nu)t}\, \alpha \quad ,$$

and calculate I_1, I_x, etc. Next we note that Eqs. (A17) and (A18) require $\int d^3v f_1$ and $\int d^3v v_z f_1$, respectively, while Eq. (A19) requires $\int d^3v v_x f_1$. In this section we collect the integrals required to assemble the dispersion relation. We make repetitive use of the following relations:

$$e^{i(kV/\omega_c)\sin(\theta+\omega_c t)} = \sum_{\ell=-\infty}^{\infty} e^{i\ell(\theta+\omega_c t)} J_\ell\left(\frac{kV}{\omega_c}\right) \quad .$$

Further, in the Σ, only leading terms will be retained, ordering $\omega/\omega_c << 1$, $k^2a_L^2 << 1$, so $\ell = 0, \pm 1$. It is tedious but straightforward to obtain (the argument of every Bessel function is kV/ω_c), and $\sum_{m=-\infty}^{\infty}$ is implied

$$I_1 = \frac{J_m J_o e^{-im\theta}}{i(\omega + Kv_z - k\alpha_o - k\alpha_1\cos\theta - i\nu)} + \frac{2J_m J_1 e^{-im\theta}\cos\theta}{i\omega_c} \quad , \tag{A34}$$

$$I_x = \frac{V}{i\omega_c} \frac{J_m J_o e^{-im\theta}\cos\theta}{(\omega + Kv_z - k\alpha_o - k\alpha_1\cos\theta - i\nu)} - \frac{V}{\omega_c^2} J_m J_o e^{-im\theta}\sin\theta \quad , \tag{A35}$$

$$I_{xx} = \frac{V^2}{\omega_c^2} \frac{J_m J_o\, e^{-im\theta}}{i(\omega + Kv_z - k\alpha_o - k\alpha_1\cos\theta - i\nu)} \quad , \tag{A36}$$

$$I_{v_x} = \frac{VJ_m J_1 e^{-im\theta}}{\omega + Kv_z - k\alpha_o - k\alpha_1\cos\theta - i\nu} - \frac{VJ_m J_o e^{-im\theta}\cos\theta}{\omega_c} \quad , \tag{A37}$$

$$I_{xv_x} = \frac{V^2}{\omega_c} \frac{\cos\theta J_m J_1 e^{-im\theta}}{\omega + Kv_z - k\alpha_o - k\alpha_1\cos\theta - i\nu} \quad . \tag{A38}$$

Note that v_z = constant makes $\int dt\, v_z = v_z \int dt$, so only the five integrals listed are significant. We have determined that the sixth integral, I_{x2v_x} is negligible, and haven't listed it.

Next we examine the velocity integrals, writing

$$\int d^3v = \int_0^\infty VdV \int_{-\infty}^{\infty} dv_z \int_0^{2\pi} d\theta \quad ,$$

with $v_y = V\cos\theta$ and $v_x = V\sin\theta$. Integrating over the velocity angle θ and the perpendicular velocity V, we obtain the moments of f_1 needed to calculate the charge density $\int f_1 d^3v$:

$$\int d\theta\, I_1 = \frac{2\pi J_o^2}{i(\omega - i\nu + Kv_z - k\alpha_o)} \quad ,$$

$$\int d\theta \cos\theta I_1 = \frac{2\pi J_o J_1}{i\omega_c} + \frac{J_o^2}{2} \frac{\frac{Vk}{\omega_c}\frac{d}{dx}\frac{Ec}{B}}{i(\omega - i\nu + Kv_z - k\alpha_o)^2} \quad , \qquad \text{(A39)}$$

$$\int d\theta I_x = -\frac{2\pi V J_o J_1}{i\omega_c^2} + \frac{2\pi V^2 J_o^2}{\omega_c^2} \frac{\left(k \frac{d}{dx}\frac{Ec}{B}\right)}{2i(\omega - i\nu + Kv_z - k\alpha_o)^2} \quad ,$$

$$\int d\theta \cos\theta I_x = \frac{\pi J_o^2 V}{i\omega_c(\omega - i\nu + Kv_z - k\alpha_o)} \quad , \qquad \text{(A40)}$$

$$\int d\theta I_{x^2} = \frac{2\pi V^2}{\omega_c^2} \frac{J_o^2}{i(\omega - i\nu + Kv_z - k\alpha_o)} \quad , \qquad \int d\theta \cos\theta I_{x^2} = 0 \quad , \qquad \text{(A41)}$$

$$\int d\theta I_{v_x} = \frac{2\pi V J_1 J_o}{(\omega - i\nu + K v_z - k\alpha_o)} \quad , \quad \int d\theta \cos\theta I_{v_x} = -\frac{\pi V}{\omega_c} J_o^2 \quad , \tag{A42}$$

$$\int d\theta I_{v_x x} = 0 \quad , \quad \int d\theta \cos\theta I_{v_x} x = \frac{\pi V^2 J_o J_1}{\omega_c(\omega - i\nu + K v_z - k\alpha_o)} \quad , \tag{A43}$$

$$\alpha_o \equiv \frac{E_y c}{B} - \frac{V^2}{2\omega_c} \frac{1}{B} \frac{dB}{dx} \quad , \quad J_\ell = J_\ell\left(\frac{kV}{\omega_c}\right) \quad .$$

The integrals required to compute $j_{1x} = \int V\sin\theta f_1 d^3v$ are

$$\int d\theta \sin\theta \ (1) \ e^{iky+iKz-k\alpha_o+(i\omega+\nu)t} = -\frac{J_o J_1}{(\omega + K v_z - k\alpha_o - i\nu)} + 0\left(\frac{\omega}{\omega_c} k^3 a_L^3\right) \quad , \tag{A44}$$

$$\int d\theta \sin\theta \ (x' - x) \ e^{iky+iKz-k\alpha_o+(i\omega+\nu)t} = \frac{V J_o^2}{2\omega_c^2} \quad , \tag{A45}$$

$$\int d\theta \sin\theta \ (x' - x)^2 \ e^{iky+iKz-k\alpha_o+(i\omega+\nu)t} = -\frac{V^2}{\omega_c^2} \frac{J_o J_1}{(\omega + K v_z - k\alpha_o - i\nu)} \quad , \tag{A46}$$

$$\int d\theta \sin\theta \ v_x' \ e^{iky+iKz-k\alpha_o+(i\omega+\nu)t} = \frac{V J_1^2}{i(\omega + K v_z - k\alpha_o - i\nu)} \quad , \tag{A47}$$

$$\int d\theta \sin\theta\ v_x'(x' - x)\ e^{iky+iKz-k\alpha_o+(i\omega+v)t}$$

$$= \frac{V^2}{\omega_c} \frac{kV}{2\omega_c} \left(\frac{d}{dx} \frac{Ec}{B}\right) \frac{J_1^2}{(\omega + Kv_z - k\alpha_o - iv)^2} \sim 0(ka_L)^4 = 0 \quad . \tag{A48}$$

A7. PERTURBED CHARGE AND CURRENT DENSITIES

We can use the integral table from the previous section to calculate the charge and current densities needed for Maxwell's equations. Using Eq. (A25) for f_1, expanding J_ℓ in kV/ω_c

$$n_{1i} = \int VdVdv_z \left\{ - \frac{4\pi q}{m} \frac{\partial f_o}{\partial H} \phi(x) \right.$$

$$+ \left(\frac{2q}{m} \frac{\partial f_o}{\partial H} - \frac{q}{m} \frac{k}{\omega} \frac{\partial f_o}{\partial p}\right) \left[(i\omega\phi - \lambda_z v_z) \int I_1 d\theta \right.$$

$$+ \frac{d}{dx} (i\omega\phi - \lambda_z v_z) \int I_x d\theta + \frac{d^2}{dx^2} (i\omega\phi$$

$$\left.\left. - \lambda_z v_z) \int I_{x^2} d\theta - \lambda_x \int I_{v_x} d\theta - \int I_{v_x x} d\theta \frac{d\lambda_x}{dx}\right]\right\} \quad , \tag{A49}$$

$$(nV_x)_{1i} = \int V^2 dV dv_z \left(\frac{2q}{m} \frac{\partial f_o}{\partial H} - \frac{q}{m} \frac{k}{\omega} \frac{\partial f_o}{\partial p} \right) \left[(i\omega\phi - \lambda_z v_z) \int I_1 \sin\theta d\theta \right.$$

$$+ \frac{d}{dx} (i\omega\phi - \lambda_z v_z) \int I_x \sin\theta d\theta + \frac{d^2}{dx^2} (i\omega\phi - \lambda_z v_z) \int I_{x^2} \sin\theta d\theta$$

$$\left. - \lambda_x \int I_{v_x} \sin\theta d\theta - \frac{d\lambda_x}{dx} \int I_{v_x x} \sin\theta d\theta \right] \quad , \tag{A50}$$

where the $\int I$ and $\int I\sin\theta$ are given explicitly in the previous section.

$$n_{1e} = \int V dV dv_z \left\{ - \frac{4\pi q}{m} \frac{\partial f_o}{\partial H} \phi(x) + \left(\frac{2q}{m} \frac{\partial f_o}{\partial H} - \frac{q}{m} \frac{k}{\omega} \frac{\partial f_o}{\partial p} \right) \right.$$

$$\times \left[(i\omega\phi - \lambda_z v_z) \int I_1 d\theta - \lambda_x \int I_{v_x} d\theta \right]$$

$$\left. + \nu \left(\frac{2q}{m} \frac{\partial f_o}{\partial H} \phi + \frac{n_{1e}}{n} f_m \right) \int I_1 d\theta \right\} \quad , \tag{A51}$$

$$(nV_x)_{1e} = \int V^2 dV dv_z \left\{ \left(\frac{2q}{m} \frac{\partial f_o}{\partial H} - \frac{q}{m} \frac{k}{\omega} \frac{\partial f_o}{\partial p} \right) \right.$$

$$\times \left[(i\omega\phi - \lambda_z v_z) \int I_1 \sin\theta d\theta - \lambda_x \int I_{v_x} \sin\theta d\theta \right]$$

$$\left. + \nu \left(\frac{2q}{m} \frac{\partial f_o}{\partial H} \phi + \frac{n_{1e}}{n} f_m \right) \int I_1 \sin\theta \right\} \quad , \qquad \text{(A52)}$$

where we have kept ka_{Li} terms, dropped ka_{Le} terms, and retained electron collisions only.

In the electron charge density, Eq. (A51), the perturbed density n_{1e} appears on both the left and right hand sides, so Eq. (A51) can be explicitly solved for n_1,

$$n_1 = [\text{All Eq. (A51) RHS terms not multiplied by } n_1]/D \quad ,$$

where

$$D = 1 - \int \frac{\nu F_m dv_z}{i(\omega + Kv_z - k\alpha_o)} \quad ,$$

and F_m is f_m integrated over $dv_x dv_y / n_o$. This n_1 can be used in $(nv_x)_1$, etc.

The only further perturbations needed are $(nv_z)_1$. These can be obtained from Eqs. (A49) and (A51) by simply replacing dv_z by $v_z dv_z$, which immediately transforms n_1 into $(nv_z)_1$.

A8. PERTURBED CHARGE DENSITY AND CURRENT → DISPERSION RELATION

First we use the x-component of Ampere's law to express λ_x in terms of ϕ and λ_z. Equation (A19) gives

$$\frac{ik^2c}{\omega}\lambda_x = \frac{Kc}{\omega}\frac{d}{dx}\lambda_z + \frac{i\omega}{c}\left(\frac{d\phi}{dx} + \lambda_x\right)$$

$$- \frac{4\pi e}{c}\int V^2 dV dv_z \left[\left(\frac{2e}{m}\frac{\partial f_{oe}}{\partial H} - \frac{e}{m}\frac{k}{\omega}\frac{\partial f_{oe}}{\partial p}\right)(i\omega\phi - \lambda_z v_z)\right.$$

$$\left. + \nu\left(\frac{2e}{m}\frac{\partial f_{oe}}{\partial H}\phi - \frac{n_{1e}}{n}f_m\right)\right]\frac{2\pi J_o J_1}{\omega + Kv_z - k\alpha_o - i\nu}$$

$$- \frac{4\pi e}{c}\int V^2 dV dv_z \left\{\left(\frac{2e}{m}\frac{\partial f_{oi}}{\partial H} - \frac{e}{m}\frac{k}{\omega}\frac{\partial f_{oi}}{\partial p}\right)\left[(i\omega\phi - \lambda_z v_z)\frac{2\pi J_o J_1}{\omega + Kv_z - k\alpha_o}\right.\right.$$

$$+ \frac{d}{dx}(i\omega\phi - \lambda_z v_z)\frac{VJ_o^2 2\pi}{2\omega_c^2} + \frac{d^2}{dx^2}(i\omega\phi - \lambda_z v_z)\frac{2\pi V^2 J_o J_1}{\omega_c^2(\omega + Kv_z - k\alpha_o)}$$

$$\left.\left. + \lambda_x\frac{VJ_1^2}{i(\omega + Kv_z - k\alpha_o)}\right]\right\} \quad . \qquad \text{(A53)}$$

It is useful to define some recurring quantities, z(λ), ω_j* and v_j as

$$\frac{\omega_j^x}{\omega} \equiv \frac{kcT}{q_j B \omega}\left(\frac{1}{n_o}\frac{dn_o}{dx} - \frac{\omega_{cj}}{v_j^2}\frac{E_o c}{B}\right) \quad ,$$

$$Z(\lambda) \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{e^{-x^2}dx}{x - \lambda + i\epsilon} = -\frac{1}{\lambda} \qquad \lambda \gg 1 \quad ,$$

$$= -2\lambda - \sqrt{\pi}\, i\, e^{-\lambda^2} \qquad \lambda < 1 \quad ,$$

$$v_j \equiv \left(\frac{T}{m}\right)_j^{1/2} \quad ,$$

In terms of these quantities and doing $\int d^3v$, Eq. (A53) becomes

$$i\lambda_x \frac{k^2 c}{\omega}\left[1 - \sum_j \beta_j\left(\frac{\omega}{Kv_j}\right)\left(1 + \frac{\omega_j^x}{\omega}\right) Z\left(\frac{\omega - k\alpha_o}{Kv_j}\right)\right] = \frac{Kc}{\omega}\frac{d\lambda_z}{dx} + \frac{i\omega}{c}\frac{d\phi}{dx} -$$

$$\frac{k\omega_{pj}^2}{\omega_{cj}c}\left[\left(1 + \frac{\omega_j^x}{\omega}\right)\left(i\phi + \frac{\lambda_z}{K} - \frac{(k\alpha_o + i\nu)}{\omega}\frac{\lambda_z}{K}\right)\frac{\omega}{Kv_j} Z\left(\frac{\omega - k\alpha_o - i\nu}{Kv_j}\right) + \left(1 + \frac{\omega_j^x}{\omega}\right)\right.$$

$$- \frac{k\omega_{pe}^2}{\omega_{ce}c}\frac{\nu}{Kv_e}\left(\phi + \frac{n_{1e}}{n_o}\frac{T}{e}\right) Z\left(\frac{\omega - k\alpha_o}{Kv_e}\right) \quad , \qquad \text{(A54)}$$

where we have used $\partial f_o/\partial H = -(m/2T)f_o$ and $\partial f_o/\partial p = (1/\omega_c)[(1/n_o)(\partial n_o/\partial x) - (eE_o/T)]$. From (A53) and (A54) we note that

$$\lambda_x \sim \frac{\omega_p \omega}{k^2 c^2} k\phi + \frac{\omega_p \omega}{k^2 c^2} \frac{\omega}{\omega_c} \frac{d\phi}{dx} + \frac{\omega \omega_p}{k^2 c^2} \frac{V^2}{\omega_c^2} \frac{d^2\phi}{dx^2} \quad . \tag{A55}$$

Even the leading term in E_x is down from E_y by $\omega\omega_p/k^2c^2$, which for $k \sim 1/a_i \sim \omega/v_d$ is less than $T_i/mc^2 << 1$. This justifies the fact that we have neglected the $d\lambda_x/dx$ terms in ρ_i and j_{1z}. We have also used $(a_L{}^2/\lambda_D{}^2)(T/mc^2) = 4\pi nT/B^2 \equiv \beta$.

Keeping the leading terms in λ_x then gives

$$\frac{i\lambda x}{k}\left[1 - \sum \beta \left(1 + \frac{\omega^x}{\omega}\right)\frac{\omega Z}{K\bar{v}}\right] = \frac{K}{k^3}\frac{d\lambda_z}{dx} - \sum \frac{\omega}{\omega_c}\frac{\beta}{k^2 a_L^2}\left(1 + \frac{\omega^x}{\omega}\right)$$

$$\left[\left(i\phi + \frac{\lambda_z}{K}\left(\omega - \frac{k\alpha_o}{\omega}\right)\frac{\omega Z}{K\bar{v}}\right) + \frac{\lambda_z}{K}\right] \quad . \tag{A56}$$

APPENDIX 2

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THE COLLISIONAL LOWER HYBRID DRIFT INSTABILITY

Prepared by

Nicholas A. Krall

Krall Associates
Del Mar, CA 92014

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1. INTRODUCTION

Since it was first discussed[1] and explored with reference to theta pinch experiments,[2] there has been extensive study and application of the lower hybrid drift (lhd) instability.[3-6] The reason for this interest is that there are a variety of experiments and experimental conditions for which the lhd mode produces the turbulence that dominates the observed transport and heating behavior. The dominance of the lhd mode in so many contexts is due to the fact that it does not require a close resonance with any particular particle drift or orbit feature, and because it is driven by plasma currents in a plasma edge or sheath, and this is a universal feature of plasma experiments.

One effect which has not been treated is the effect of collisions on this instability. This may be a significant feature in existing experiments, particularly near the magnetic separatrix, where the temperature is low compared with the interior. This is also the region where high β (plasma pressure/magnetic pressure) fails to stabilize the mode; further, this is the region where FRC experiments[7] have looked for this instability and have failed to find it. Although there are other effects[8] which have been proposed as supression mechanisms for this instability in the FRC, it must be recognized that collisional effects may also play a role. In this study we investigate this effect.

One important result of the study is that the effect of collisions is substantially more important than might have been expected. It might have been expected that since in the absence of collisions the growth rate $\omega_i \simeq \omega_r \simeq \omega_{pi}/(1 + 4\pi nmc^2/B^2)^{1/2}$, collisions would damp the wave when $\nu \gtrsim \omega_i$, with ν a collision frequency. However, the result we get is that

$$\omega_i \sim \frac{\Omega - \nu}{D} \quad ,$$

where $(\Omega/D)_{max} = \omega_{pi}/(1 + 4\pi nmc^2/B^2)^{1/2}$; in the collisionless case, the largest growth Ω/D is achieved by minimizing D, with the appropriate choice of wave number k. But now minimizing D not only maximizes Ω/D, but also maximizes ν/D, and stability is reached when

$$\frac{\Omega}{D} \gg \nu > \Omega \quad ,$$

rather than $\nu > \Omega/D$.

In this report we derive the results stated above and explicitly calculate the form of Ω. In subsequent work we will develop a numerical method to analyze the dispersion relation, and later extend the work by using the numerical method to plot ω_i vs. k for a variety of plasma conditions and collisionality.

2. THE EQUILIBRIUM

As in all calculations of lhd stability, we work in slab geometry, with x representing radial gradients, y the axis of the cylinder, and z the magnetic field direction. The equilibrium ion and electron velocity distributions must satisfy

$$v \cdot \nabla f_o + \frac{e}{m}\left(E_{xo} + \frac{vxB_{zo}}{c}\right) \cdot \nabla_v f_o = 0 \quad , \tag{1}$$

which implies a functional form for f_o,

$$f_o = f_o\left[\frac{1}{2}\, m\left(v_x^2 + v_y^2\right) - \frac{eE_o x}{T} \, , \quad x + \frac{v_y mc}{eB} \, , \quad v_z\right] \quad . \tag{2}$$

Expanding f_o gives the usual form for ions and electrons,

$$f_{oi} = n_{oi}\left[1 + \epsilon_i\left(x + \frac{v_y mc}{eB}\right) + \frac{eE_o x}{T}\right] e^{-mv^2/2T}\left(\frac{m}{2\pi T}\right)^{3/2} \quad , \tag{3}$$

$$f_{oe} = n_{oe}\left[1 + \epsilon_e\left(x - \frac{v_y mc}{eB}\right) - \frac{eE_o x}{T}\right] e^{-mv^2/2T}\left(\frac{m}{2\pi T}\right)^{3/2} \quad , \tag{4}$$

with the constraints

$$n_{oe}\left(\epsilon_e - \frac{eE_o}{T}\right) = n_{oi}\left(\epsilon_i + \frac{eE_o}{T}\right) \quad , \tag{5}$$

$$eE_o = \frac{1}{n_i}\frac{dn_iT_i}{dx} \quad . \tag{6}$$

The first constraint is required for quasi-charge neutrality, the second gives ion pressure balance. In this equilibrium, the electrons have single particle drifts $v_e = -E_oc/B$, and diamagnetic drifts $V_{De} = (cT/eB)(d \ln n/dx)$, which are clearly present in Eq. (4), since

$$\epsilon_e = \frac{eE_o}{T} + \frac{1}{n_e}\frac{dn_e}{dx} \quad . \tag{7}$$

Also, Eq. (6) requires that $\epsilon_i = 0$, since $\epsilon_i = (1/n)(dn/dx) - eE_o/T$. Since in the FRC, B (x=0) = 0, we will keep the electromagnetic terms in the plasma response to lhd waves.

3. COLLISION MODEL

The effect of binary collisions can be included in the lhd calculation by adding a term $-\nu(f - f_o)$ to the Vlasov equation, e.g.,

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m}\left(E + \frac{vxB}{c}\right) \cdot \nabla_v f = -\nu\ (f - f_R) \quad . \tag{8}$$

The problem is in choosing what collision frequency ν to use, and what relaxation distribution f_R to choose. For f_R, we note that although collisions eventually relax the system to thermal equilibrium, this takes a very long time. On the lhd time scale, the effect of ν will be to restore the metaequilibrium state described by Eqs. (3) and (4). In deciding what collision frequency to use, we first calculate a table of useful parameters:[9]

Electron Gyrofrequency

$$\omega_{ce} = 2 \times 10^{11}\ B\ (\text{tesla})\ s^{-1}$$

Ion Gyrofrequency

$$\omega_{ci} = \frac{B(\text{tesla})}{A}\ 10^8\ s^{-1}$$

Ion Thermal Speed

$$V_{Ti} = \left(\frac{T(eV)}{A}\right)^{1/2} 10^6\ \text{cm/s}$$

Electron Thermal Speed

$$V_{Te} = 4.5\ T(eV)^{1/2}\ 10^{7}\ \text{cm/s}$$

Cross Section for 90° Electron Scattering by Singly Charged Ions

$$\sigma_{ei} = \frac{2.6}{T^2\ (eV)}\ 10^{-12}\ \text{cm}^2$$

Cross Section for 90° Ion Scattering by Electrons

$$\sigma_{ie} = \frac{2.6}{T^2(eV)} \left(\frac{m_e}{m_i}\right) 10^{-12}\ \text{cm}^2$$

Collision Frequencies $\equiv n\ \sigma\ v$

$$\nu_{e,i} = 3 \times 10^{9}\ s^{-1} \qquad \text{for } 10\ \text{eV},\ n = 10^{15}\ \text{cm}^{-3}$$

$$\nu_{i,e} = 2 \times 10^{6}\ \frac{1}{A}\ s^{-1} \qquad \text{for } 10\ \text{eV},\ n = 10^{15}\ \text{cm}^{-3}$$

For the electrons, the lower hybrid frequency of the wave means that the drifts induced by the wave will give a relative speed of electrons relative to the ions, primarily due to the drift induced by the waves electric field,

$$\delta v_e = \frac{\delta E c}{B_o}\ e^{iky - i\omega t} \quad .$$

Reducing this relative drift requires shorting out the charge separation which supports δE, which is equivalent to the electrons diffusing a distance k^{-1}, the lhd wavelength. Since like-particle collisions do not give spatial diffusion, electron-ion collisions will dominate. The ions could also diffuse. The fastest process for this would be collisions between different ion species. However, for the lhd wave, the ions are essentially unmagnetized, and are moving to short out the fields as rapidly as their heavier mass m_i will allow. Collisions would impede this process, giving the ions a greater effective mass. However, from the calculations above, by the time ion collisions affect the wave, the electrons will have diffused many electron gyroradii. Since the lhd wavelength is comparable to the electron gyroradius, electron collisions will dominate. Thus we use $\nu_{e,i}$ (90°) as the relaxation frequency.

4. COLLISIONAL LOWER HYBRID WAVES

Vlasov theory[10] gives the response of the plasma to a wave

$$\delta E = E_1 \, e^{iky+iKz-i\omega t} \quad ,$$

$$\delta B = B_1 \, e^{iky+iKz-i\omega t} \quad ,$$

by expressing the velocity distributions as $f = f_o + f_1$ and finding that

$$f_{1\alpha}(y,z,v) = -\frac{q_\alpha}{m} \int_{-\infty}^{0} dt' \left(E_1 + \frac{v' x B_1}{c} \right) \cdot \nabla_v \cdot f_{o\alpha} \, e^{iky'+iKz'-i\omega t'+v_\alpha t'} \quad , \qquad (9)$$

where y′, z′, v′ are the orbits of particles of species α, in the equilibrium fields E_o, B_o, with the constraint that at t′ = 0, y′ = y, z′ = z, and v′ = v, which are the phase space points for which f_1 is calculated. In the present case, the time scale is taken to be the lower hybrid drift scale, so the ion orbits are straight line,

$$v_i' = v \quad .$$

The electrons are magnetized, so we use the orbits

$$v_{ye}' = v \cos(\theta - \omega_c t) + v_e \quad ,$$

$$v_{xe}' = v \sin(\theta - \omega_c t) \quad ,$$

$$v'_{ze} = v_{zz} \quad ,$$

$$y' = \frac{v}{\omega_c} \sin(\theta - \omega_c t) - \frac{v}{\omega_c} \sin\theta + y \quad ,$$

$$z' = v_z t + z \quad ,$$

$$v_e = -\frac{E_x c}{B} + \frac{1}{B}\frac{dB}{dx}\frac{(v_x^2 + v_y^2)}{eB/mc} \quad .$$

The techniques for solving Eq. (9) are published in many articles and books[9,10], and need not be reproduced here. This is not to say that they are trivial. After a dozen or so pages of algebra we find forms for the dispersion tensor in terms of selected components of the wave fields; the representation of the wave fields, explicitly defined below, were chosen, from past experience, to simplify the algebra

$$E_1 = -\nabla\phi + iK\lambda_z\hat{z} \quad ,$$

$$B_1 = \frac{ikK\lambda_z c}{\omega}\hat{x} \quad .$$

In terms of these fields, the dispersion tensor is

$$(k^2 + K^2)\,\phi - K^2\lambda_z = -4\pi e \int f_{1e}\, d^3v + 4\pi e \int f_{1i} d^3v \quad , \qquad (10)$$

$$\frac{k^2Kc}{\omega}\lambda_z = -\frac{4\pi e}{c}\int v_z f_{1e} d^3v + \frac{4\pi e}{c}\int v_z f_i d^3v \quad , \tag{11}$$

where

$$f_{1e} = \frac{ef_o}{T}\phi - J_o^2 \frac{\left[-\frac{e(\omega + i\nu)}{T}\left(1 + \frac{\epsilon_e kcT}{eB(\omega + i\nu)}\right)\phi + \frac{v_z Ke}{T}\lambda_z\right] f_o}{Kv_z + kv_e - \omega - i\nu}$$

$$- \frac{J_o^2 \epsilon_e v_z kKc}{B_o \omega \ (Kv_z + kv_e - \omega - i\nu)}\lambda_z \quad , \tag{12}$$

$$f_{1i} = \frac{-\frac{e}{T}(\vec{k}\cdot\vec{v})\phi + \frac{eKv_z}{T}\lambda_z}{Kv_z + kv_y - \omega} f_o \quad . \tag{13}$$

and $J_o \equiv J_o(kv/\omega_c)$ is the Bessel function of order 0.

The fact that in f_{1e} ω sometimes appears alone and sometimes in the combination $\omega + i\nu$ is because ν appears explicitly in the Vlasov equation for f_1 but not in the Maxwell equation $dB/dt = -\nabla xE$ that relates E_1 and B_1.

In the ion term of course $\nu_i = 0$ from arguments in Section 3. Continuing, we define

$$Z\left(\frac{\omega}{kV_i}\right) = \frac{1}{n}\int \frac{kV_i f_o d^3v}{k\cdot v - \omega} \quad , \tag{14}$$

which has limits (V_i is the ion thermal speed)

$$Z = -\frac{kV_i}{\omega} + i\sqrt{\pi}\, e^{-\omega^2/k^2 v_i^2} \quad , \qquad \omega \gg kV_i \quad , \tag{15}$$

$$Z = \frac{2\omega}{kV_i} + i\sqrt{\pi} \quad , \qquad \omega \ll kV_i \quad . \tag{16}$$

We also use K << k, because long wavelength along B_o (which means small K) is required to prevent electron flow along B_o from shorting out the wave fields.

In calculating the resistivity we will need the fluctuation densities δn_i, δn_e, since[2]

$$m\left(\frac{n\partial V_y}{\partial t}\right)_{lhd} = \langle \delta n \delta E \rangle \; q \quad . \tag{17}$$

In a low β, ion-electron plasma, this gives

$$\left(\frac{n_e m_e \partial V_{ye}}{\partial t}\right)_{lhd} = -\, e \; \langle \delta n_e \delta E \rangle \quad , \tag{18}$$

$$\left(\frac{n_i m_i \partial V_{yi}}{\partial t}\right)_{lhd} = e \; \langle \delta n_i \delta E \rangle \quad , \tag{19}$$

and momentum conservation then requires

$$\langle \delta n_i \delta E \rangle = \langle \delta n_e \delta E \rangle \quad . \tag{20}$$

Explicitly, the perturbed densities are

$$\delta n_e = \frac{n_o e}{T} \left\{ \phi - I_o\, e^{-b} \left(1 + \frac{\omega^x}{\omega}\right) \right.$$

$$\cdot \left[\lambda_z - \frac{(\omega\phi - \omega\lambda_z - i\nu\lambda_z + kv_e\lambda_y)}{KV_e} Z\left(\frac{\omega + i\nu - kv_e}{KV_e}\right) \right] \tag{22}$$

$$\left. + \frac{i\nu}{KV_e} I_o e^{-b} Z\left(\frac{\omega + i\nu - kv_e}{KV_e}\right) \phi \right\} \quad ,$$

$$\delta n_{ii} = \frac{n_i 1_i e}{T_i} \left[\phi + \frac{\omega}{kV_i} Z\left(\frac{\omega}{kV_i}\right) \phi \right] \quad , \tag{23}$$

where we have defined

$$\omega_e^x = k_y \frac{cT\epsilon_e}{eB} = 2k_y \frac{cT}{eB} \frac{1}{n_e} \frac{dn_e}{dx} \quad ,$$

and I_o is the Bessel function of pure imaginary argument, of argument $b = k^2 a_e^2$. In addition to the perturbed densities, which go into Eq. (10), we must calculate explicitly the currents J_z needed in Eq. (11), the second half of the dispersion matrix. Again, the technique is straightforward and well documented, though in practice it is really tedious. The result is

$$\frac{4\pi q_e}{c} \int v_z f_{1e} d^3v = - \frac{4\pi n_e e^2 I_{oe}^{-b}}{Tc} \left[\left(1 + \frac{\omega^x}{\omega}\right) \left(\frac{\omega}{K} \phi - \frac{\omega + i\nu - kv_e}{K} \lambda_z \right) + \frac{i\nu}{K} \phi \right]$$

(24)

$$\cdot \left[1 + \frac{\omega + i\nu - kv_e}{KV_e} Z \left(\frac{\omega + i\nu - kv_e}{KV_e} \right) \right] \quad ,$$

$$\frac{4\pi q_i}{c} \int v_z f_{1i} d^3v = \frac{4\pi e^2 n_i}{m_i c} \frac{K}{kV_i}$$

(25)

$$\cdot \left\{ \lambda_z Z \left(\frac{\omega}{kV_i} \right) - \frac{\omega \phi}{kV_i} \left[1 + \frac{\omega}{kV_i} Z \left(\frac{\omega}{kV_i} \right) \right] \right\} \quad .$$

Using the perturbed densities and currents found in Eqs. (22)-(25), we can evaluate Eqs. (10) and (11) to get the dispersion matrix

$$\left\{ k^2 + K^2 + \frac{4\pi n_e e^2}{T} \left[1 + \left(\frac{i\nu}{KV_e} + \frac{\omega}{KV_e} + \frac{\omega^x}{KV_e} \right) I_o e^{-b} Z \left(\frac{\omega + i\nu - kv_e}{KV_e} \right) \right] \right.$$

$$\left. + \frac{4\pi n_i e^2}{T_i} \left[1 + \frac{\omega}{kV_i} Z \left(\frac{\omega}{kV_i} \right) \right] \right\} \phi \qquad (26)$$

$$= \left[K^2 + \frac{4\pi n_e e^2}{T} \left(1 + \frac{\omega^x}{\omega} \right) \left(1 + \frac{\omega + i\nu - kv_e}{KV_e} \right) I_o e^{-b} Z \left(\frac{\omega + i\nu - kv_e}{KV_e} \right) \right] \lambda_z \quad ,$$

and

$$\left\{ \frac{4\pi n_e e^2}{T} \left(\omega + \omega_e^x + i\nu\right) I_o e^{-b} \left[1 + \frac{\omega + i\nu - kv_e}{KV_e} Z\left(\frac{\omega + i\nu - kv_e}{KV_e}\right)\right]\right.$$

$$\left. + \frac{4\pi n_i e^2}{m_i} \frac{\omega K^2}{k^2 V_i^2} \left[1 + \frac{\omega}{kV_i} Z\left(\frac{\omega}{kV_i}\right)\right] \right\} \phi$$

$$= \left\{ - \frac{k^2 K^2 c^2}{\omega} + \frac{4\pi n_e e^2}{T} \left[I_o e^{-b_e} \left(1 + \frac{\omega^x}{\omega}\right)(\omega + i\nu - kv_e)\left(1 + \frac{\omega + i\nu - kv_e}{KV_e} Z\left(\frac{\omega + i\nu - kv_e}{KV_e}\right)\right)\right]\right. \tag{27}$$

$$\left. + \frac{4\pi n_i e^2}{m_i} \frac{K^2}{kV_i} Z\left(\frac{\omega}{kV_i}\right) \right\} \lambda_z \quad .$$

To the best of my knowledge, the matrix Eq. (26) and (27) is an original result, including for the first time electron collisions in an electromagnetic calculation of the lower hybrid drift wave.

5. ANALYSIS OF THE DISPERSION MATRIX

To analyze the implications of Eqs. (26) and (27), a reasonable program would be to assume that finite β will stabilize the interior, and so to look at the electrostatic limit, $\lambda_z = 0$, making the usual approximations[1,2] that allow an analytic solution to the problem. This brings into focus the differences between the well analyzed case of an electron-ion collisionless plasma, and the case in which collisional effects can contribute. Eventually however a numerical integration of Eqs. (26) and (27) would be necessary to complete the study. We do not anticipate attempting that step, due to the limited resources in the contract.

In the electrostatic (low β) limit, $\lambda_z = 0$, and the dispersion relation is (from Eq. (26)),

$$k^2 + K^2 + \frac{\omega_{pe}^2}{V_e^2}\left[1 + I_o(b)\, e^{-b}\left(\frac{\omega + \omega^x + i\nu}{KV_e}\right) Z\left(\frac{\omega + i\nu - kv_e}{KV_e}\right)\right]$$

$$+ \frac{\omega_{pi}^2}{V_i^2}\left[1 + \left(\frac{\omega}{kV_i}\right) Z\left(\frac{\omega}{kV_i}\right)\right] = 0 \quad , \tag{28}$$

where $\omega^2_{pi} = 4\pi n_i e^2/m_i$, $b = k^2 a^2_e$, I_o is the Bessel function of order zero, of pure imaginary argument, and a_e is the electron gyroradius.

The limit which recovers the lower hybrid wave is that of a nearly flute-like mode with $k_y a_e$ of order 1 and $(K/k) < (m_e/m_i)^{1/2}$. In that limit $(\omega - kv_e)/KV_e > 1$, and $Z = - KV_e/(\omega + i\nu - kv_e)$, so that Eq. (28) becomes

$$k^2 + \frac{\omega_{pe}^2}{V_e^2}\left(1 - I_o e^{-b}\right) - \frac{\omega_{pe}^2}{V_e^2}\,\frac{I_o e^{-b}\,(\omega^x + kv_e)}{\omega + i\nu - kv_e}$$

$$+ \frac{\omega_{pi}^2}{V_i^2}\left[1 + \left(\frac{\omega}{kV_i}\right) Z\left(\frac{\omega}{kV_i}\right)\right] = 0 \quad . \tag{29}$$

To do a really accurate job of analyzing Eq. (29) requires some numerical work, since the dimensionless constants are of order unity. However, a good indication of the physics can be had by taking $\omega/kV_i >> 1$, since that is the limit in which lower hybrid waves are found; this limit becomes increasingly good if $T_i < T_e$. With this assumption, along with $k^2a_e^2 < 1$, Eq. (29) becomes

$$k^2\left(\lambda_D^2 + a_e^2\right) - \frac{\omega^x + kv_e}{\omega + i\nu - kv_e} - \frac{k^2C_s^2}{\omega^2}$$

$$+ i\sqrt{\pi}\,\frac{T_e}{T_i}\,\frac{\omega}{kV_i}\,\exp\left(-\frac{\omega^2}{k^2V_i^2}\right) = 0 \quad , \tag{30}$$

where C_s is the ion-acoustic speed, $C_s^2 = T_e/m_i$.

An adjunct to the fact (see after Eq. (6)) that the ions are electrostatically confined in the sheath is that

$$\omega^x = \frac{kcT\epsilon}{eB} = -2kv_e \quad , \tag{31}$$

which further simplifies Eq. (30), reducing it to

$$k^2\lambda_D^2 + \left(1 - I_o e^{-b}\right) + I_o e^{-b} \frac{kv_e}{\omega - kv_e + i\nu} - \frac{k^2 C_s^2}{\omega^2}$$

$$+ i\sqrt{\pi}\,\frac{\omega}{kV_i}\exp\left(-\frac{\omega^2}{k^2 V_i^2}\right) = 0 \quad . \tag{32}$$

Even with this simplification the dispersion relation is cubic in ω, if the exponential factor $\exp(-\omega^2/k^2V^2)$ is set equal to 1, and a numerical analysis, albeit a fairly easy one, would be needed for truly quantitative answers. However, the idea is still to get analytic formulas to show trends, and we can do this by treating the growth rate of the lhd wave as smaller than the wave frequency, with the results given in the next few subsections.

5.1 WAVE FREQUENCY AND WAVELENGTH

Taking $\Omega_r = \mathrm{Re}\,\omega/kv_e$ and $\Omega_i = \mathrm{Im}\,\omega/kv_e$, the real part of Eq. (32) becomes

$$k^2\lambda_D^2 + \left(1 - I_o e^{-b}\right) + \frac{I_o e^{-b}(\Omega_r - 1)}{(\Omega_r - 1)^2 + \left(\Omega_i + \frac{\nu}{kv_e}\right)^2} - \frac{C_s^2}{v_e^2}\left[\frac{\Omega_r^2 - \Omega_i^2}{(\Omega_r^2 + \Omega_i^2)^2}\right] = 0 \quad , \tag{33}$$

and the imaginary part becomes

$$-\frac{I_o e^{-b}\left(\Omega_i + \frac{\nu}{kv_e}\right)}{(\Omega_r - 1)^2 + \left(\Omega_i + \frac{\nu}{kv_e}\right)^2} + \left(\frac{C_s}{v_e}\right)^2\left[\frac{2\Omega_r\Omega_i}{\left(\Omega_r^2 + \Omega_i^2\right)^2}\right] + i\sqrt{\pi}\,\frac{\Omega v_e}{V_i}\, e^{-\frac{\Omega^2 v_e^2}{V_i^2}} = 0 \quad , \tag{34}$$

and $\Omega v_e/V_i > 1$ has been assumed in estimating $Z(\omega/kV_i)$.

Taking $\Omega < 1$, from the previous work in Ref. 2, and taking ω_i and ν smaller than ω_r, Eq. (33) gives

$$\omega_r^2 = \frac{\dfrac{e^2B^2}{m_e m_i c^2}}{\dfrac{\left(1 - I_o e^{-b}\right)}{b} + \dfrac{\omega_c^2}{\omega_p^2} - \dfrac{I_o e^{-b}}{b}} < k^2 v_e^2 \quad . \tag{35}$$

Now in order for ω to be real the denominator of Eq. (35) must be positive, which requires $b > 1$, since otherwise the denominator $\doteq - 1 + \omega_c^2/\omega_p^2 < 0$. This restricts the wave number k compared to the inverse of the electron gyroradius,

$$k^2 a_e^2 > 1 \quad . \tag{36}$$

Using this fact gives

$$\omega^2 = \frac{\omega_{ce}\omega_{ci}}{\dfrac{\omega_{ce}^2}{\omega_{pe}^2} + \dfrac{1}{b} - \dfrac{2}{b^2}} \sim \frac{k^2 V_i^2 \dfrac{T_e}{T_i}}{1 + k^2\lambda_D^2 - \dfrac{2}{k^2 a_e^2}} \quad , \tag{37}$$

where V_i is the ion thermal velocity. The limits $\omega^2 > 0$, $\omega < kv_e$, $\omega > kV_i$ bound $k^2a_e^2$ to be of order greater than 2, the actual value depending on T_e/T_i.

5.2 MAXIMUM GROWTH RATES AND INSTABILITY CONDITIONS

Now we can use ω and k from Eqs. (36)-(37) to find the instability conditions and growth rate. From Eq. (34) we have

$$\omega_i = \frac{\sqrt{\pi}\,\dfrac{\omega v_e}{V_i}\, e^{-\omega^2/k^2 V_i^2} - \nu\, I_o e^{-b}}{\dfrac{1}{k^2 a_e^2} - \dfrac{2C_s^2}{v_e^2}\left(\dfrac{kv_e}{\omega_r}\right)^3} . \tag{38}$$

When collisions are neglected, instability requires $\omega_i > 0$, which means

$$\frac{2}{1 + k^2\lambda_D^2} < k^2 a_e^2 < \frac{v^2}{2C_s^2}\,\Omega_r^3 = \frac{C_s}{2v_e}\left(1 + k^2\lambda_D^2 - \frac{2}{k^2 a_e^2}\right)^{-3/2} , \tag{39}$$

where the left-hand inequality follows from the requirement that the denominator of Eq. (37) be positive, while the right-hand comes from the need to make the denominator of Eq. (38) positive.

To understand the implication of the <u>form</u> of Eq. (38), we first examine the limit of no collisions, $\nu = 0$. Because of the exponential in the numerator, which gets small for $\omega/kV_i >> 1$, it might be expected that ω_i^{max} would come when the numerator is maximum. After a little algebra we find that the numerator is maximum at

$$\frac{2}{k^2 a_e^2} = \left(1 + k^2\lambda_D^2\right)\left(1 - \frac{2T_e}{T_i}\right) ; \tag{40}$$

this means that for $T_e/T_i > 1/2$, there is no maximum and the numerator increases monotonically from 0, $k^2a_e^2 = 2/(1 + k^2\lambda_D^2)$ to

$$\left(\frac{\pi \frac{T_e}{T_i}}{1 + k^2\lambda_D^2}\right)^{1/2} \exp - \frac{\frac{T_e}{T_i}}{1 + k^2\lambda_D^2} \quad , \quad k^2a_e^2 \gg \frac{2}{1 + k^2\lambda_D^2} \quad .$$

However, even if there is a maximum in the numerator ($T_e/T_i < 1/2$ and Eq. (40) holds), the inequality of Eq. (39) is violated. So maximizing the numerator is not an option, except to make $k^2a_e^2$ as large as possible while still satisfying the right-hand inequality Eq. (39). This means that the largest growth occurs when the denominator is small, namely at

$$k^2a^2 = \frac{C_s}{2v_e}\left(1 + k^2\lambda_D^2 - \frac{2}{k^2a_e^2}\right)^{-3/2} = \frac{C_s}{2v_e}\frac{\left(k^2a_e^2\right)^{3/2}}{\left[\left(1 + k^2\lambda_D^2\right)k^2a_e^2 - 2\right]^{3/2}} \quad . \tag{41}$$

Equation (41) can in fact be solved for $k^2a_e^2$. Defining

$$Z = \frac{\left(1 + k^2\lambda_D^2\right)k^2a_e^2}{2} \quad ,$$

Eq. (41) can be written

$$(Z - 1)^3 = \frac{C_s^2}{2^5v_e^2}\frac{2}{1 + k^2\lambda_D^2} Z \quad ,$$

which has the solution

$$\frac{\left(1 + k^2\lambda_D^2\right) k_{max}^2 a_e^2}{2} = 1 + \left(\frac{C_s^2}{2^4 v_e^2} \frac{1}{1 + k^2\lambda_D^2}\right)^{1/3} \quad , \tag{42}$$

where k_{max} means the k that gives maximum growth, not the largest value of k. This value of $k^2a_e^2$ will make the denominator of Eq. (38) vanish, so in practice $k^2a_e^2$ will be at the slightly larger (than $k^2_{max}a_e^2$) value which makes $\omega_i \approx \omega_r$, whereupon the approximation $\omega_i << \omega_r$ breaks down. Near k_{max} we have

$$\omega_r^2 = \frac{k^2C_s^2}{1 + k^2\lambda_D^2} \left(\frac{2^4 v_e^2 \left(1 + k^2\lambda_D^2\right)}{C_s^2}\right)^{1/3} \quad , \tag{43}$$

which for $v_e > C_s$ satisfies $\omega > kV_i$ as well as $\omega < kv_e$

$$\frac{\omega}{kV_i} \sim \sqrt{\frac{T_e}{T_i}\left(\frac{v_e}{C_s}\right)^{1/3}} \frac{2^{2/3}}{\left(1 + k^2\lambda_D^2\right)^{2/3}} > 1 \quad \text{for } v_e > C_S \quad . \tag{44}$$

$$\frac{\omega}{kv_e} \sim \left(\frac{C_s}{v_e}\right)^{2/3} \frac{2^{2/3}}{\left(1 + k^2\lambda_D^2\right)^{1/3}} < 1 \quad \text{for } v_e > C_s \quad . \tag{45}$$

Thus for $k \simeq k_{max}$, ω_r satisfies the two approximations used in the derivation.

Near k_{max}, ω_i becomes

$$\omega_i = \frac{\sqrt{\pi}\, kv_e \left(\frac{T_e}{T_i}\right)^{1/2} \left(\frac{v_e}{C_s}\right)^{1/3} \frac{2^{2/3}}{(1+k^2\lambda_D^2)^{2/3}}\, e^{-\left[\frac{T_e}{T_i}\left(\frac{v_e}{C_s}\right)^{2/3} \frac{2^{4/3}}{(1+k^2\lambda_D^2)^{2/3}}\right]} - \frac{\nu\,(1+k^2\lambda_D^2)}{2}}{\frac{1}{k^2 a_e^2} - \frac{1}{k_{max}^2 a_e^2}} \quad . \tag{46}$$

The point of Eq. (36) is that although $k = k_{max}$ maximizes the instability driving term, it also maximizes the collisional damping term. To provide the strongest resonance, $2/k^2a_e^2$ should be as small as possible. But if $2/k^2a_e^2$ is too small, the driving resonance becomes a damping term, and this limit on $2/k^2a_e^2$ comes when ω_r/kV_i is still fairly large and $\exp(-\omega^2/k^2V_i^2)$ is very small (see Eq. (46)). This means that the collisional term damps the lower hybrid wave not when $\nu \sim \omega_i \sim \omega_r$, but when

$$\nu > \sqrt{\pi}\, \frac{2}{(1 + k^2\lambda_D^2)}\, \frac{\omega_r k v_e}{kV_i}\, e^{-\omega_r^2/k^2V_i^2} =$$

$$4\sqrt{\pi}\, \omega_{LH} \left(\frac{v_e}{C_s}\right)^{2/3} \left(\frac{T_e}{T_i}\right)^{1/2} e^{-\frac{T_e}{T_i}\left(\frac{v_e}{C_s}\right)^{2/3} 2^{4/3}} \ll \omega_{LH} \quad . \tag{47}$$

It's not easy to get around the limit Eq. (47) because to minimize the collisional part of the numerator one would increase $k^2a_e^2$, and in Eqs. (41)-(47) the value of $k^2a_e^2$ has been taken as large as is consistent with the lower hybrid drift instability.

6. OTHER MODES AND NEEDED WORK

Although the lhd instability has been the dominant source of turbulence in the bulk of plasma implosion studies to date, it is possible that collisions might introduce a dissipative branch to the lower hybid drift instability.

Examining Eq. (38) shows that in the usual case where the denominator is positive the effect of ion Landau resonance is wave growth, and the effect of collisions is wave damping. However, when the denominator is negative, collisions can drive instability if ν is large enough to dominate the ion resonance term. The reason for this behavior is that the electron drift splits the lower hybrid wave into a positive and a negative energy branch, in the same way that counterstreaming electrons split the plasma wave ω_p into a fast and a slow wave which have opposite energies. The negative energy branch is then destabilized by any dissipation mechanism, in this case electron-ion collisions. Of course, the collision term must dominate for this to be possible. In previous applications collisions were neglected, so this dissipative instability was overlooked. It might be worthwhile to explore this branch in some future study.

There are a number of extensions of this work that would further help application to understanding the turbulent structure of the FRC. For example,

- The dissipative lhd mode could be examined to determine its linear and nonlinear properties.
- There could be some numerical work to make our analytic estimate of lhd properties fully reliable. At the minimum, the electrostatic limit Eq. (29), or at

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