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PROCEEDINGS OF THE 1984 WORKSHOP SPECTRAL THEORY OF STURM-LIOUVILLE DIFFERENTIAL OPERATORS*

Held at Argonne National Laboratory

May 14 - June 15, 1984

Hans G. Kaper and Anton Zettl, editors

Gail W. Pieper, technical editor

Mathematics and Computer Science Division

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Abstract

This report contains the proceedings of the workshop on "Spectral Theory of Sturm-Liouville Differential Operators," which was held at Argonne during the period May 14 through June 15, 1984. The report contains 22 articles, authored or co-authored by the participants in the workshop.

Topics covered at the workshop included the asymptotics of eigenvalues and eigenfunctions; qualitative and quantitative aspects of Sturm-Liouville eigenvalue problems with discrete and continuous spectra; polar, indefinite, and non-selfadjoint Sturm-Liouville eigenvalue problems; and systems of differential equations of Sturm-Liouville type.

Preface

This report contains the proceedings of the workshop "Spectral Theory of Sturm-Liouville Differential Operators," which was held at Argonne National Laboratory during the period May 14 through June 15, 1984. The workshop was organized by the Mathematics and Computer Science Division (MCS) as part of its activities in applied analysis. Twenty-six mathematicians, nine of them from abroad, took part in the scientific activities. These proceedings are the permanent record of the research done at or stimulated by the workshop.

The objectives of the workshop were (1) to encourage the transfer of theoretical research results to the domain of computational mathematics, and (2) to identify open research problems in the area of Sturm-Liouville eigenvalue problems. The format of the workshop was chosen to emphasize communication and cooperation. Each participant was given the opportunity to present a summary of his or her past or current work, but most of the time was spent on discussions and joint research.

Each week the workshop focused on a particular theme. The five themes were (1) asymptotics of eigenvalues and eigenfunctions; (2) qualitative and quantitative aspects of Sturm-Liouville eigenvalue problems with discrete spectra; (3) qualitative and quantitative aspects of Sturm-Liouville eigenvalue problems with continuous spectra; (4) polar, indefinite, and non-selfadjoint Sturm-Liouville problems; and (5) systems of second-order Sturm- Liouville equations. For each week one participant was invited to act as program coordinator.

The main financial support came from the University of Chicago Fund for Argonne Activities. The MCS Division generously supported the activities of the workshop.

Following this preface is a list of names and addresses of all colleagues who took part in the workshop. We express our gratitude to these colleagues and especially to those who contributed manuscripts to the proceedings.

Special thanks are due to Gail Pieper, technical editor of the MCS Division, who so ably handled all the work involved in the production of these proceedings from rough draft manuscripts to the end product that we have before us. We also thank our colleague Jim Cody for his help in making the necessary physical arrangements for the workshop. Finally, we offer our sincere appreciation to Paul Messina, MCS division director, for encouraging us to organize this workshop and for generally stimulating the proper environment for creative research within the division.

The organizing committee:

F.V. Atkinson (Toronto)
Hans G. Kaper (Argonne)
Man Kam Kwong (DeKalb)
Tony Zettl (DeKalb)

December 1984

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Schedule of Lectures

Monday, May 14

2:00pm	B.J. Harris (Northern Illinois U., DeKalb) "Asymptotics of Sturm-Liouville Eigenvalue Problems"

Tuesday, May 15

10:00am	J. Hejtmanek (U. of Vienna, Austria) "Asymptotics of Linear Semigroups"
2:00pm	H.G. Kaper (ANL) "Asymptotics of Eigenvalues and Eigenfunctions of Indefinite Sturm-Liouville Problems"

Wednesday, May 16

10:00am	H. Benzinger (U. of Illinois, Urbana) "Well-Bounded Operators and Semigroups"
2:00pm	A.B. Mingarelli (U. of Ottawa, Canada) "Asymptotics of Sturm-Liouville Eigenvalue Problems with Indefinite Weights"

Thursday, May 16

2:00pm	J. Fleckinger (U. of Toulouse, France) "Asymptotics of Eigenvalues of Differential Operators with Indefinite Weights"

Tuesday, May 22

10:00am	A. Zettl (Northern Illinois U., DeKalb) "The Essential Spectrum of Non-Selfadjoint Sturm-Liouville Operators"
1:30pm	Man Kam Kwong (Northern Illinois U., DeKalb) "Oscillation of Sturm-Liouville Equations"
2:15pm	A.B. Mingarelli (U. of Ottawa, Canada) "Lower Bounds for the Spectrum of Sturm-Liouville Operators with an Almost Periodic Potential"

Wednesday, May 23

9:30am	J. Fleckinger (U. of Toulouse, France) "Eigenvalues of Schroedinger Operators"
2:00pm	M. Ashbaugh (U. of Missouri, Columbia) "Stability of Resonances"

Thursday, May 24

10:00am	G. Halvorsen (U. of Trondheim, Norway) "Limit Circle Criteria for Sturm-Liouville Operators"
2:00pm	R.C. Brown (U. of Alabama, University) "Sum and Product Inequalities with Some Applications"

Friday, May 25

10:00am	J. Hooker (Southern Illinois U., Carbondale) "Principal Solutions of Discrete Linear Systems and Riccati Transformations"
2:00pm	C. Ahlbrandt (U. of Missouri, Columbia) "Principal Solutions of Discrete Linear Systems and Riccati Transformations, cont'd"

Monday, May 28

2:30pm	A.B. Mingarelli (U. of Ottawa, Canada) "Lower Bounds for the Spectra of Differential and Difference Operators"

Tuesday, May 29

2:00pm	A. Zettl (Northern Illinois U., DeKalb) "Bounds on the Infimum of the Essential Spectrum for Ordinary Differential Equations"

Wednesday, May 30

2:00pm	D. Sorensen (ANL) "Numerical Solution of Riccati Equations"
3:30pm	C. Ahlbrandt (U. of Missouri, Columbia) "Riccati Differential Equations (Periodic and Constant Coefficients)"

Thursday, May 31

9:30am	G. Halvorsen (U. of Trondheim, Norway) "Sharp Bounds for Solutions of Sturm-Liouville Equations"
11:00am	Excursion to CP-5
2:00pm	M. Ashbaugh (U. of Missouri, Columbia) "Potentials Having Extremal Eigenvalues Subjected to p-Norm Constraints"

Friday, June 1

10:00am	M. Kon (Boston U.) "Sturm-Liouville Semigroups"
2:00pm	L. Raphael (Howard U., Washington D.C.) "Equisummability of Eigenfunction Expansions"

Monday, June 4

10:30am	A.B. Mingarelli (U. of Ottawa, Canada) "A Survey of the Regular Weighted Sturm-Liouville Problem: The Non-Definite Case"

Tuesday, June 5

9:00am	B. Schultze (U. of Essen, W.Germany) "The Essential Spectrum of a Class of Ordinary Differential Equations"
10:30am	R.C. Brown (U. of Alabama, University) "Some Non-Standard Inequalities Associated with Second-Order Sturm-Liouville Problems (and the Indefinite Methods Needed to Prove Them)"

Wednesday, June 6

9:00am	Man Kam Kwong (Northern Illinois U., DeKalb) "An Inequality for Hermitian Operators and Applications"
10:30am	H. de Snoo (U. of Groningen, Netherlands) "Self-Adjoint Relations in Indefinite Inner Product Spaces"

Thursday, June 7

9:00am	R.J. Hangelbroek (Western Illinois U., Macomb) "Some Problems of Transport Theory"
11:00am	Excursion to IPNS
2:00pm	M. Lapidus (U. of Southern California) "Eigenvalues and Eigenfunctions of Elliptic Boundary Value Problems with an Indefinite Weight Function"

Friday, June 8

9:00am	H.-D. Niessen (U. of Essen, W.Germany) "J-Symmetric Differential Systems"
10:30am	H.G. Kaper (ANL) "A Regularizing Transformation for a Class of Singular Sturm-Liouville Problems"

Monday, June 11

10:30am — Man Kam Kwong (Northern Illinois U., DeKalb)
"Oscillation Theory for Systems of
Sturm-Liouville Differential Equations"

Tuesday, June 12

9:00am — R.C. Brown (U. of Alabama, University)
"A Survey of Some Results Concerning the
Dirichlet Index"

10:30am — B. Curgus (U. of Sarajevo, Yugoslavia)
"Indefinite Sturm-Liouville Problems and
Half-Range Completeness"

Wednesday, June 13

9:00am — A. Krall (Penn State U., University Park)
"Boundary Conditions for Systems"

10:30am — A.B. Mingarelli (U. of Ottawa, Canada)
"Some Questions in Oscillation Theory"

Thursday, June 14

9:00am — W. Allegretto (U. of Alberta, Edmonton, Canada)
"Oscillation and Spectral Properties for
Weakly Coupled Elliptic Systems"

10:30am — E.L. Lusk (ANL)
"The Automated Reasoning Effort at Argonne"

Friday, June 15

9:00am — K. Shaw (VPI&SU, Blacksburg, Va.)
"Dirac Systems with Long- and Short-Range Potentials"

10:30am — D. Hinton (U. of Tennessee, Knoxville)
"On a Dirac System of Titchmarsh"

RICCATI TRANSFORMATIONS AND PRINCIPAL SOLUTIONS OF DISCRETE LINEAR SYSYEMS

Calvin D. Ahlbrandt *
John W. Hooker†

Abstract

Consider the second-order linear matrix difference equation

$$\text{(i)} L[X_n] = -\Delta(C_{n-1}\Delta X_{n-1}) + A_n X_n = 0, \quad n = 1,2,3,\ldots,$$

where C_n and A_n are given sequences of positive-definite, Hermitian $r \times r$ matrices, X_n is an $r \times r$ matrix, and Δ is the forward difference operator $\Delta X_n = X_{n+1} - X_n$. Let X_n be a sequence of nonsingular $r \times r$ complex matrices. Then if $W_n = (C_{n-1}\Delta X_{n-1})X_{n-1}^{-1}$, we have

$$\text{(ii)} X_n^* L[X_n] = X_n^* R[W_n] X_n, \; n = 1,2,3,\ldots,$$

where

$$R[W_n] = -W_{n+1} + W_n[W_n + C_{n-1}]^{-1}C_{n-1} + A_n,$$

a Riccati matrix difference operator.

We give a definition of ***principal*** and ***anti-principal***, or ***recessive*** and ***dominant***, solutions of (i) and use the relationship (ii) to prove the existence of principal and anti-principal solutions of (i) and the essential uniqueness of principal solutions.

1. Introduction

Discrete-time linear systems and related discrete matrix Riccati equations arise in a variety of applied problems, in particular, in discrete linear optimal control and filtering problems (cf. Vaughan [1970]). We discuss here the concept of *principal* (sometimes called *recessive*, or *subdominant*) solutions of a linear vector difference equation

$$l[x_n] = -\Delta(C_{n-1}\Delta x_{n-1}) + A_n x_n = 0, \quad n = 1,2,3,\ldots, \tag{1.1}$$

where A_n and C_n are given sequences of positive-definite, Hermitian $r \times r$ matrices with complex entries. A solution of (1.1) is a sequence x_n of $r \times 1$ complex vectors satisfying (1.1), and Δ is the forward difference operator defined by

$$\Delta U_n = U_{n+1} - U_n$$

for any sequence of matrices or vectors U_n.

Instead of working directly with the vector operator $l[x_n]$ of (1.1), we find it more convenient to make use of the related matrix operator

$$L[X_n] = -\Delta(C_{n-1}\Delta X_{n-1}) + A_n X_n\,, \tag{1.2}$$

where X_n is an $r \times r$ matrix, $n = 0,1,2,\ldots$. We will also make use of a discrete

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Riccati matrix operator which will be defined in Section 2. In Section 3 we define and discuss existence of principal and anti-principal solutions of $L[X_n] = 0$. The essential uniqueness of principal solutions is discussed in Section 4. In Section 5, some further properties of solutions of the Riccati equation $R[W_n] = 0$ are developed. Section 6 connects this work with previous transformation theory for differential operators.

Our results on principal solutions are, for the most part, discrete analogues of results for differential systems discussed by Hartman [1957], Reid [1958, 1963], Ahlbrandt [1972], and others.

An alternative form for the equation $L[X_n] = 0$ is

$$L[X_n] = -C_n X_{n+1} - C_{n-1} X_{n-1} + B_n X_n = 0, \tag{1.3}$$

where $B_n = C_n + C_{n-1} + A_n$, $n=1,2,3,\ldots$. From (1.3) it is clear that, given an integer $M \geq 0$, the solution of $L[X_n] = 0$ for given initial matrices X_M and ΔX_M (or, equivalently, X_M and X_{M+1}) is defined for all $n \geq 0$ and is unique. Indeed, the restriction to non-negative integers is unnecessary, and solutions may be defined for all integers n by use of the recurrence relation (1.3).

Before proceeding, we state for reference some elementary properties of the forward difference operator Δ. If A_n and B_n are sequences of $r \times r$ matrices, then

$$\Delta(A_n B_n) = A_n \Delta B_n + (\Delta A_n) B_{n+1} = A_{n+1} \Delta B_n + (\Delta A_n) B_n, \tag{1.4}$$

$$\sum_{i=M}^{N} \Delta A_i = A_{N+1} - A_M, \tag{1.5}$$

and

$$\sum_{i=M}^{N} A_i \Delta B_i = A_N B_{N+1} - A_{M-1} B_M - \sum_{i=M}^{N} (\Delta A_{i-1}) B_i, \tag{1.6}$$

an analogue of "integration-by-parts" which follows readily from (1.4) and (1.5). By using (1.4), one can also easily show that the operators $l[x_n]$ and $L[X_n]$ satisfy discrete analogues of the Lagrange identity for differential operators; hence, we say that these operators are ***formally self-adjoint*** (cf. Hartman [1964: 385]). We also state without proof the following variation-of-parameters lemma.

LEMMA 1.1. *If X_n is a solution of the homogeneous matrix difference equation*

$$\Delta X_{n-1} + A_n X_n = 0, \tag{1.7}$$

and X_n is nonsingular for all $n \geq M-1$ for some $M>0$, then the solution of

$$\Delta Y_{n-1} + A_n Y_n = B_n, \quad n \geq M,$$

with initial value Y_{M-1}, is

$$Y_n = X_n (X_{M-1}^{-1} Y_{M-1} + \sum_{i=M}^{n} X_{i-1}^{-1} B_i), \quad n \geq M-1,$$

where the summation equals 0 if $n = M-1$.

In the discussion below, $r \times r$ matrices with complex entries will always be denoted by capital letters, I is the $r \times r$ identity matrix, and m,n,M,N,i,j will always denote non-negative integer indices.

2. A Riccati Difference Operator and Prepared Solutions of $L[X_n] = 0$.

Throughout this paper, we assume that the coefficients of the operator $L[X_n] = 0$ satisfy the following condition:

$$A_n \text{ and } C_n \text{ are Hermitian and positive definite for all } n \geq 0. \tag{2.1}$$

The concept of a ***prepared*** (*self-conjoined, isotropic*) solution of a linear self-adjoint system of differential equations has been employed by many writers [Coppel, 1971; Hartman, 1957; Reid, 1958]. An analogous concept for difference operators follows.

For an arbitrary ordered pair of sequences U_n and V_n of $r\times r$ complex matrices, we define a "bracket function"

$$\{U_n, V_n\} = U_n^* C_{n-1}\Delta V_{n-1} - [C_{n-1}\Delta U_{n-1}]^* V_n$$

$$= U_{n-1}^* C_{n-1} V_n - U_n^* C_{n-1} V_{n-1},$$

where U_n^* denotes the transpose conjugate of U_n. It is easily verified that

$$\{U_n, V_n\} = -\{V_n, U_n\}^*. \tag{2.2}$$

It is also readily verified that if U_n and V_n are solutions of $L[X_n] = 0$, then $\Delta\{U_n, V_n\} = 0$ for all n. Hence we have the following lemma.

LEMMA 2.1. *For any solutions U_n and V_n of $L[X_n] = 0$, $\{U_n, V_n\}$ is constant.*

DEFINITION. A sequence of $r\times r$ complex matrices is ***prepared***, or ***isotropic***, if

$$\{X_n, X_n\} = 0 \text{ for all } n \geq M \text{ for some } M \geq 0, \tag{2.3}$$

or, equivalently, if $X_{n-1}^* C_{n-1} X_n$ is Hermitian for $n \geq M$.

In preparation for the development of a principal solution of $L[X_n] = 0$, we first introduce a Riccati difference operator. By analogy with differential systems (cf. Reid [1972]), if X_n is a sequence of nonsingular $r\times r$ matrices, one expects a matrix identity of the form $X_n^* L[X_n] = X_n^* R[W_n]X_n$ for some non-linear operator R. Now

$$L[X_n] = -\Delta(C_{n-1}\Delta X_{n-1}) + A_n X_n$$

$$= -C_n\Delta X_n + C_{n-1}\Delta X_{n-1} + A_n X_n$$

$$= [-(C_n\Delta X_n)X_n^{-1} + (C_{n-1}\Delta X_{n-1})X_n^{-1} + A_n]X_n. \tag{2.4}$$

We let

$$W_n = (C_{n-1}\Delta X_{n-1})X_{n-1}^{-1}, \tag{2.5}$$

which is analogous to the usual Riccati transformation $w = cx'/x$ for the differential equation $-(cx')' + px = 0$. (For other discrete analogues of the Riccati transformation, see Gautschi [1967], Hooker and Patula [1981], Hooker, Kwong, and Patula [1984], and Arscott [1981].) Then (2.4) becomes

$$L[X_n] = (-W_{n+1} + W_n X_{n-1} X_n^{-1} + A_n)X_n. \tag{2.6}$$

Now $W_n = C_{n-1}X_n X_{n-1}^{-1} - C_{n-1}X_{n-1}X_{n-1}^{-1}$. Hence, $X_n X_{n-1}^{-1} = C_{n-1}^{-1}(W_n + C_{n-1})$, so $X_{n-1}X_n^{-1} = (W_n + C_{n-1})^{-1}C_{n-1}$. Substitution into (2.6) gives

$$L[X_n] = (-W_{n+1} + W_n(W_n + C_{n-1})^{-1}C_{n-1} + A_n)X_n.$$

Hence,

$$X_n^* L[X_n] = X_n^* R[W_n]X_n, \tag{2.7}$$

where

$$R[W_n] = -W_{n+1} + W_n(W_n + C_{n-1})^{-1}C_{n-1} + A_n, \tag{2.8}$$

a Riccati matrix difference operator of the type that occurs in problems of discrete linear optimal control and filtering [Vaughan, 1970].

THEOREM 2.1. *Assume condition (2.1). Given an integer $M > 0$, let X_n be the unique solution of the initial value problem*

$$L[X_n] = 0, \quad X_{M-1} = I, \quad C_{M-1}\Delta X_{M-1} = I. \tag{2.9}$$

Then X_n and ΔX_n are nonsingular for all $n \geq M-1$, W_n defined by (2.5) is positive definite for all $n \geq M$, D_n defined by

$$D_n = X_n^* C_n X_{n+1} \tag{2.10}$$

is Hermitian and positive definite for all $n \geq M-1$, and X_n is a prepared solution.

PROOF. From the initial conditions we have $X_{M-1} = I$ and $\Delta X_{M-1} = X_M - X_{M-1} = C_{M-1}^{-1}$. Hence,

$$X_M = I + C_{M-1}^{-1}, \tag{2.11}$$

so X_M is Hermitian. Then

$$D_{n-1} - D_{n-1}^* = X_{n-1}^* C_{n-1} X_n - X_n^* C_{n-1} X_{n-1} = \{X_n, X_n\},$$

a constant matrix by Lemma 2.1. But

$$\{X_M, X_M\} = X_M^* C_{M-1}\Delta X_{M-1} - [C_{M-1}\Delta X_{M-1}]^* X_M$$

$$= X_M^* I - I X_M = 0,$$

since X_M is Hermitian. Therefore, D_n is Hermitian for all $n \geq M-1$. Hence, X_n is a prepared solution.

Thus we have X_{M-1}, X_M nonsingular, and $W_M = I$ is positive definite. Also $D_{M-1} = X_{M-1}^* C_{M-1} X_M$, so from (2.11) we obtain

$$D_{M-1} = C_{M-1}(I + C_{M-1}^{-1}) = C_{M-1} + I,$$

so D_{M-1} is positive definite. We then proceed by induction, assuming that X_{k-1} and X_k are nonsingular and W_k and D_{k-1} are positive definite. Now for W_{k+1} defined by (2.5), we have $R[W_k] = 0$ by (2.7). Thus

$$W_{k+1} = W_k(W_k + C_{k-1})^{-1}C_{k-1} + A_k$$

$$= (C_{k-1}^{-1}(W_k + C_{k-1})W_k^{-1})^{-1} + A_k$$

$$= (C_{k-1}^{-1} + W_k^{-1})^{-1} + A_k,$$

so W_{k+1} is positive definite.

To show that D_k is positive definite, consider

$$X_k^* W_{k+1} X_k = X_k^*(C_k \Delta X_k) X_k^{-1} X_k$$

$$= X_k^* C_k X_{k+1} - X_k^* C_k X_k.$$

Then

$$D_k = X_k^* C_k X_{k+1} = X_k^* W_{k+1} X_k + X_k^* C_k X_k,$$

so D_k is positive definite and, furthermore, X_{k+1} is nonsingular, which completes the induction. Finally, since $W_{n+1} = (C_n \Delta X_n) X_n^{-1}$ is positive definite for all

$n \geq M-1$, ΔX_n is nonsingular for all $n \geq M-1$, which completes the proof.

We will return in Section 5 to further relationships between solutions of $L[X_n] = 0$ and corresponding solutions of $R[W_n] = 0$.

3. Principal Solutions of $L[X_n] = 0$.

We now make use of Theorem 2.1 to establish the existence of *principal* (sometimes called *recessive*, *subdominant*, or *distinguished*) solutions of $L[X_n] = 0$. The definition we use is analogous to that of Hartman [1957] for Sturm-Liouville differential operators. It stems from the property

$$\int^{\infty} (cu^2)^{-1} dt = \infty,$$

which characterizes principal solutions of scalar self-adjoint differential equations (cf. Hartman [1964]). Principal solutions for second-order and nth-order scalar difference equations have been discussed by Patula [1979] and Hartman [1978], respectively. For computation of principal solutions of scalar difference equations, see Gautschi [1967] or Olver and Sookne [1972]. Our discussion of principal solutions is similar to that of Hartman [1957] and Reid [1958].

DEFINITION. A *principal* (or *recessive*) solution of $L[X_n] = 0$ is a solution X_n that is nonsingular for $n \geq M$, for some $M \geq 0$, and that satisfies the conditions

$$X_n^* C_n X_{n+1} \text{ is positive definite for } n \geq M, \tag{3.1}$$

and

$$\sum_{i=M}^{n} k^*(X_i^* C_i X_{i+1})^{-1} k \to \infty \text{ as } n \to \infty \tag{3.2}$$

for every unit vector k. A solution is *anti-principal* if (3.1) is satisfied and

$$\sum_{i=M}^{\infty} k^*(X_i^* C_i X_{i+1})^{-1} k$$

converges for every unit vector k.

LEMMA 3.1. *Assume condition (2.1) and let X_n be a prepared solution of $L[X_n] = 0$ which is nonsingular for all $n \geq M-1$ for some $M > 0$. Then every solution Y_n of $L[Y_n] = 0$ for $n \geq M-1$ is given by*

$$Y_n = X_n\Big(P + \sum_{i=M}^{n} (X_{i-1}^* C_{i-1} X_i)^{-1} Q\Big), \quad n \geq M-1, \tag{3.3}$$

for constant matrices

$$P = X_{M-1}^{-1} Y_{M-1} \text{ and } Q = \{X_{M-1}, Y_{M-1}\} \equiv \{X_n, Y_n\}, \quad n \geq M-1, \tag{3.4}$$

where the summation in (3.3) equals 0 if $n = M-1$.

Conversely, if P and Q are given constant matrices, (3.3) defines a solution of $L[Y_n] = 0$ for $n \geq M-1$.

PROOF. Given X_n and Y_n as stated, we know

$$\{X_n, Y_n\} = X_n^* C_{n-1} \Delta Y_{n-1} - [C_{n-1} \Delta X_{n-1}]^* Y_n = Q,$$

a constant matrix (i.e., Q is independent of n). Since X_n is nonsingular for $n \geq M-1$, we can write

$$C_{n-1} \Delta Y_{n-1} - X_n^{*-1} [C_{n-1} \Delta X_{n-1}]^* Y_n = X_n^{*-1} Q.$$

Hence

$$\Delta Y_{n-1} - C_{n-1}^{-1}[(C_{n-1}\Delta X_{n-1})X_n^{-1}]^* Y_n = C_{n-1}^{-1} X_n^{*-1} Q. \tag{3.5}$$

Since X_n is a prepared solution of $L[X_n] = 0$, X_n is a solution of the homogeneous equation related to (3.5) (i.e., substituting X_n for Y_n makes the left-hand side of (3.5) equal 0). Then by variation of parameters, Lemma 1.1, the solution Y_n of the non-homogeneous equation (3.5) can be written as

$$Y_n = X_n(X_{M-1}^{-1}Y_{M-1} + \sum_{i=M}^{n} X_{i-1}^{-1}C_{i-1}^{-1}X_i^{*-1}Q),$$

which is of the form (3.3) with $P = X_{M-1}^{-1}Y_{M-1}$. Conversely, given P and Q, one may verify that (3.3) is a solution of $L[Y_n] = 0$, which completes the proof.

LEMMA 3.2. *The solution Y_n of $L[Y_n] = 0$ given by (3.3) is a prepared solution if and only if*

$$P^*Q = Q^*P. \tag{3.6}$$

PROOF. We need to show that $Y_{n-1}^*C_{n-1}Y_n$ is Hermitian for $n \geq M$ and, by Lemma 2.1, it suffices to show this for $n = M$. From (3.3) with $n = M$, we have

$$Y_{M-1}^*C_{M-1}Y_M = Y_{M-1}^*C_{M-1}X_MP + Y_{M-1}^*C_{M-1}X_M(X_{M-1}^*C_{M-1}X_M)^{-1}Q. \tag{3.7}$$

Since $Y_{M-1}^* = P^*X_{M-1}^*$ (from (3.3) with $n = M-1$), (3.7) can be written

$$Y_{M-1}^*C_{M-1}Y_M = P^*X_{M-1}^*C_{M-1}X_MP + P^*X_{M-1}^*C_{M-1}X_M(X_{M-1}^*C_{M-1}X_M)^{-1}Q \tag{3.8}$$

Since X_n is a prepared solution, the first term on the right in (3.8) is Hermitian and the second term equals P^*Q. Thus $Y_{M-1}^*C_{M-1}Y_M$ is Hermitian if and only if P^*Q is Hermitian, which completes the proof.

We now proceed to establish the existence of principal and anti-principal solutions of $L[X_n] = 0$, assuming Condition (2.1). If $X = X_n$ is a prepared solution of $L[X_n] = 0$, which is nonsingular for $n \geq M-1$, for brevity of notation we define

$$S_n(X) = \begin{cases} 0, & \text{if } n = M-1 \\ \sum_{i=M}^{n} (X_{i-1}^*C_{i-1}X_i)^{-1}, & \text{if } n \geq M \end{cases} \tag{3.9}$$

Then the solution (3.3) given in Lemma 3.1 can be written

$$Y_n = X_n(P + S_n(X)Q), \quad n \geq M-1. \tag{3.10}$$

In particular, let $P = Q = I$ in (3.10), and let X_n be the solution of the initial value problem (2.9). Then (3.10) becomes

$$Y_n = X_n(I + S_n(X)), \quad n \geq M-1, \tag{3.11}$$

where, by Theorem 2.1, the solution X_n is prepared and $S_n(X)$ is a positive definite, increasing sequence for $n \geq M$ (where matrix inequality is defined in the usual sense of positive definiteness).

By Lemma 3.2, Y_n is a prepared solution. Also, Y_n is nonsingular for all $n \geq M-1$, since $Y_{M-1} = X_{M-1} = I$ and $S_n(X)$ is positive definite for $n \geq M$. Thus we may reverse the roles of X_n and Y_n in Lemma 3.1, which then tells us that

$$X_n = Y_n(\tilde{P} + S_n(Y)\tilde{Q}), \quad n \geq M-1, \tag{3.12}$$

where $\tilde{P} = Y_{M-1}^{-1}X_{M-1} = I$ and $\tilde{Q} = \{Y_n, X_n\}$, a constant matrix. By (2.2),

$$\tilde{Q} = \{Y_n, X_n\} = -\{X_n, Y_n\}^* = -Q^* = -I,$$

so (3.12) becomes

$$X_n = Y_n(I - S_n(Y)).$$

Hence

$$I = X_n^{-1}X_n = X_n^{-1}Y_n(I - S_n(Y)). \tag{3.13}$$

Substituting (3.11) into (3.13), we obtain

$$I = (I + S_n(X))(I - S_n(Y)). \tag{3.14}$$

Thus $I - S_n(Y)$ is the sequence of inverses of the increasing, positive-definite sequence $I + S_n(X)$, and $I + S_n(X) > I$ for $n \geq M$. It follows that $I - S_n(Y)$ is a decreasing, positive-definite sequence satisfying $0 < I - S_n(Y) < I$ for all $n \geq M$. Hence $S_n(Y)$ is an increasing sequence with a positive-definite limit $S_\infty(Y) \leq I$ as $n \to \infty$. Now define the sequence

$$Z_n = Y_n(S_\infty(Y) - S_n(Y)), \quad n \geq M-1. \tag{3.15}$$

This is of the form (3.3) with $P = S_\infty(Y)$ and $Q = \{Y_n, Z_n\} = -I$, so by Lemma 3.1, Z_n is a solution of $L[Z_n] = 0$. We will show that Z_n is prepared and recessive.

Since $S_n(Y)$ is Hermitian for all $n \geq M$, $S_\infty(Y)$ is Hermitian; hence, by Lemma 3.2, Z_n is prepared. Also, $S_\infty(Y) - S_n(Y)$ is positive definite and Y_n is nonsingular for each $n \geq M$, so Z_n is nonsingular. Thus Z_n satisfies the hypotheses of Lemma 3.1, and we may write Y_n in terms of Z_n as

$$Y_n = Z_n(\hat{P} + S_n(Z)\hat{Q}), \quad n \geq M-1, \tag{3.16}$$

where, by (3.4), $\hat{P} = Z_{M-1}^{-1}Y_{M-1} = S_\infty(Y)^{-1}$ and $\hat{Q} = \{Z_n, Y_n\}$. By (2.2), $\hat{Q} = \{Z_n, Y_n\} = -\{Y_n, Z_n\}^* = I$, so (3.16) becomes

$$Y_n = Z_n(S_\infty(Y)^{-1} + S_n(Z)), \quad n \geq M-1.$$

Proceeding as in Steps (3.13) and (3.14), we obtain

$$I = (S_\infty(Y) - S_n(Y))(S_\infty^{-1}(Y) + S_n(Z)). \tag{3.17}$$

But $S_\infty(Y) - S_n(Y)$ is positive definite and decreasing, and tends to 0 as $n \to \infty$. It then follows from (3.17) that $S_n(Z)$ is increasing for $n \geq M-1$, and for every unit $r \times 1$ vector k, $k^* S_n(Z)k$ is positive and increasing as n increases, and $\lim_{n\to\infty} k^* S_n(Z)k = \infty$. Therefore Z_n is a principal solution of $L[Z_n] = 0$, by definition.

Similarly, from the discussion above, we know that $S_n(Y)$ is a positive-definite, increasing sequence for $n \geq M$, with a positive-definite limit $S_\infty(Y) \leq I$, and it follows that Y_n is an anti-principal solution. We have thus established the following theorem.

THEOREM 3.1. *Assume Condition (2.1). Then $L[X_n] = 0$ has a principal (recessive) solution and an anti-principal (dominant) solution.*

4. Reid's Construction of Recessive Solutions

In the matrix differential equation case, Reid [1958] gave a construction of the principal solution at ∞ by means of a limiting case of a solution of a two-point boundary value problem. Reid's construction for the system

$$U' = AU + BV, \quad V' = CU - A^*V$$

started with a solution $U_{st}(x)$, $V_{st}(x)$ defined by

$$U_{st}(s) = I, \quad V_{st}(t) = 0.$$

He showed that $V_{st}(s)$ had a limit $V_{s\infty}(s)$ as t became infinite. Then $(U_{s\infty}(x), V_{s\infty}(x))$ was a principal solution at ∞. Constructions of this type have been used in the construction of recessive solutions of three-term recurrence relations [Gautschi, 1967; Olver and Sookne, 1972]. In particular, such methods have been employed in numerical evaluation of Bessel functions $J_n(x)$ from their recurrence relation $-J_{n+1}(x) - J_{n-1}(x) + 2n/x\, J_n(x) = 0$.

We now use the same idea to construct a recessive solution at ∞. Of course, the previously constructed Z_n is recessive, but for any nonsingular matrix K, $Z_n K$ is also recessive. One application of the following construction is to show *essential uniqueness* of Z_n, i.e., uniqueness up to nonsingular constant multiples.

Let M and N be integers with $M < N$. For Z_n constructed as in Section 3, define $S_n(Z)$ as before. Then $S_n(Z)$ is increasing in n, positive definite for $n \ge M$ and $(S_n(Z))^{-1} \to 0$ as $n \to \infty$. We define $X_n(M,N)$ as the solution defined by the two-point boundary value problem

$$X_{M-1}(M,N) = I,\ X_N(M,N) = 0. \tag{4.1}$$

If such a solution exists, it must have the form

$$X_n(M,N) = Z_n(P + S_n(Z)Q),\ n \ge M-1. \tag{4.2}$$

But the choice

$$I = Z_{M-1}P,\ Q = -(S_N(Z))^{-1}P \tag{4.3}$$

uniquely determines $X_n(M,N)$ as

$$X_n(M,N) = Z_n(I - S_n(Z)[S_N(Z)]^{-1})Z_{M-1}^{-1} \tag{4.4}$$

with the consequence

$$X_n(M,N) \to Z_n Z_{M-1}^{-1} \text{ as } N \to \infty. \tag{4.5}$$

Since $P^*Q = -P^*[S_N(Z)]^{-1}P$ is Hermitian, $X_n(M,N)$ is prepared for each N and since

$$X_n(M,\infty) \equiv Z_n K \text{ for } K = Z_{M-1}^{-1}\ . \tag{4.6}$$

$X_n(M,\infty)$ is prepared and recessive at ∞. We now show that the recessive solution at ∞ is essentially unique. Let Y_n be any other recessive solution at ∞. Without loss of generality, assume M is sufficiently large that Y_n is nonsingular for $n \ge M-1$. Replace Z_n in the previous derivation of $X_n(M,\infty)$ by Y_n for the conclusion

$$X_n(M,\infty) \equiv Y_n Y_{M-1}^{-1} = Z_n Z_{M-1}^{-1} \tag{4.7}$$

i.e.,

$$Y_n = Z_n H \text{ for } H = Z_{M-1}^{-1} Y_{M-1}. \tag{4.8}$$

THEOREM 4.1. *The recessive solution at ∞ is unique up to nonsingular constant multiples and is determined as*

$$\lim_{N \to \infty} X_n(M,N)$$

where $X_n(M,N)$ is the unique solution of the boundary value problem (4.1).

5. Riccati Solutions

We now relate the recessive solutions to a solution of the Riccati equation.

THEOREM 5.1. *Suppose that X_n is any recessive solution of $L[X_n] = 0$ with X_n nonsingular for $n \geq M-1$. Then, for $n \geq M$,*

$$W_n = (C_{n-1}\Delta X_{n-1})X_{n-1}^{-1} < 0, \tag{5.1}$$

$$W_{n+1} = A_n + [C_{n-1}^{-1} + W_n^{-1}]^{-1}, \tag{5.2}$$

and

$$W_n^{-1} = -C_{n-1}^{-1} + (W_{n+1} - A_n)^{-1}. \tag{5.3}$$

Consequently,

$$W_n > -C_{n-1}. \tag{5.4}$$

PROOF. Without loss of generality, assume $X_{M-1} = I$. Let $W_M(M,N) = C_{M-1}\Delta X_{M-1}(M,N)$. Use the summation-by-parts formula (1.6) on

$$0 = \sum_{i=M}^{N} X_i^* L[X_i] = -\sum_{i=M}^{N} X_i^*\Delta(C_{i-1}\Delta X_{i-1}) + \sum_{i=M}^{N} X_i^* A_i X_i$$

to obtain

$$-W_M(M,N) = \sum_{i=M}^{N} (\Delta X_{i-1}(M,N))^* C_{i-1}\Delta X_{i-1}(M,N)$$

$$+ X_i^*(M,N)A_i X_i(M,N).$$

In particular,

$$-W_M(M,N) \geq X_M^*(M,N)A_M X_M(M,N).$$

Let $N \to \infty$ for

$$-W_M(M,\infty) \geq X_M^* A_M X_M > 0.$$

Hence $W_M < 0$ for all M. Use $0 = L[X_n]$ to obtain

$$-C_n(\Delta X_n)X_n^{-1} + C_{n-1}(\Delta X_{n-1})X_n^{-1} + A_n = 0$$

and

$$W_{n+1} - A_n = C_{n-1}(\Delta X_{n-1})X_n^{-1} = W_n X_{n-1} X_n^{-1}.$$

But

$$W_n = C_{n-1}\Delta X_{n-1}X_{n-1}^{-1} = C_{n-1}[X_n X_{n-1}^{-1} - I]$$

and $X_n X_{n-1}^{-1} = C_{n-1}^{-1} W_n + I$, which is nonsingular. Thus

$$\begin{aligned} W_{n+1} - A_n &= W_n(C_{n-1}^{-1}W_n + I)^{-1} \\ &= W_n(W_n + C_{n-1})^{-1}C_{n-1} \\ &= [C_{n-1}^{-1}(W_n + C_{n-1})W_n^{-1}]^{-1} \\ &= [W_n^{-1} + C_{n-1}^{-1}]^{-1}. \end{aligned}$$

But $W_{n+1} - A_n < 0$ since $W_{n+1} < 0$, $A_n > 0$. Consequently,

$$W_n^{-1} + C_{n-1}^{-1} < 0,$$

$$W_n^{-1} + C_{n-1}^{-1} = (W_{n+1} - A_n)^{-1}.$$

and

$$W_n^{-1} = -C_{n-1}^{-1} + (W_{n+1} - A_n)^{-1} < -C_{n-1}^{-1} < 0.$$

6. A Connection with Transformation Theory

A general discussion of transformation theory for linear differential operators was presented by Ahlbrandt, Hinton, and Lewis [1982]. Lemma 3.1 above is related to an analogous transformation theory for difference operators.

Let X_n be a prepared solution of (1.3) with X_n nonsingular for $n \geq M-1$. A variable change of the form

$$Y_n = X_n V_n, \quad n \geq M-1 \tag{6.1}$$

induces the operator identity

$$X_n^* L[Y_n] = -\Delta(D_{n-1}\Delta V_{n-1}) + E_n V_n \equiv \tilde{L}[V_n], \quad n \geq M, \tag{6.2}$$

where $D_n = X_n^* C_n X_{n+1}$ and $E_n = X_n^* L[X_n] = 0$. Hence $L[Y_n] = 0$ if and only if there exists a constant matrix Q such that

$$D_{i-1}\Delta V_{i-1} = Q, \quad i \geq M. \tag{6.3}$$

Thus, by summing (6.3) and using (1.5), one obtains

$$V_n = V_{M-1} + (S_n(X))Q, \tag{6.4}$$

which gives relation (3.3). Choice of $n = M$ implies that Q is the indicated bracket function.

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OSCILLATION AND SPECTRAL PROPERTIES OF WEAKLY COUPLED ELLIPTIC SYSTEMS

W. Allegretto *

Abstract

We establish the connection between the nonoscillation of the weakly coupled elliptic system $l\vec{u} = -\Delta\vec{u} + A\vec{u}$ and the finiteness of the negative spectrum $S_-(L)$ of the associated Friedrich's extension L. We next show how criteria for the finiteness of $S_-(L)$ can be obtained from comparison with scalar equations and from comparison with systems of ordinary differential equations. As an application, we then give an extension to l of Kneser's classical nonoscillation theorem.

Let G be an unbounded domain in R^n, with $n > 2$ for convenience, and smooth boundary ∂G. Consider first in $C_0^\infty(G)$ the expression

$$l\varphi = -\Delta\varphi + q\varphi$$

with q a real function. Here Δ is the Laplace operator and q is assumed to satisfy the following conditions:

(i) $q \in L^\alpha(S)$ with $\alpha = \alpha(S) > n$ for any bounded subdomain S of G, and

(ii) for all $x \in \partial G$ there is a neighborhood R such that $q \in C^\beta(R)$, $\beta = \beta(R) > 0$.

These conditions may be relaxed somewhat, but they are simple and allow a unified presentation. Let B be the form associated with l in the usual way: $B(\varphi,\varphi) = \int_G \sum_i (D_i\varphi)^2 + q\varphi^2$. Following Glazman [1965], we term B (or l) nonoscillatory iff there exists a neighborhood N of ∞ such that if $S \subset N \cap G$, S a bounded domain, then $B(\varphi,\varphi) \geq k(\varphi,\varphi)$ with $k = k(S) > 0$, $\varphi \in C_0^\infty(S)$. This is clearly a localized property. In the middle to late 1970s, it was shown that B was nonoscillatory iff there existed $\{\psi_j\}_{j=1}^r$ in $L^2(G)$ such that $B(\varphi,\varphi) \geq 0$ if $\varphi \perp \psi_j$ in L^2 for $j = 1,\ldots,r$ and $\varphi \in C_0^\infty(G)$. From this it followed that if L was the Friedrichs extension of l and $S_-(L)$ denoted the negative spectrum of L, then $S_-(L)$ was finite iff B was nonoscillatory [Allegretto, 1981; Moss and Piepenbrink, 1978]. Hence the finiteness of $S_-(L)$ was determined near the singularity: ∞. One could change the boundary conditions, coefficients, domain G, in a compact set without affecting this property. Observe, however, that these results do not determine when $S_-(L)$ is finite in any specific case. For this purpose oscillation or nonoscillation criteria are needed. There is a considerable amount of literature on this subject; we refer in particular to the books of Swanson [1968] and Kreith [1973] and to the more recent survey articles [Swanson, 1975 and 1978].

Motivated by these considerations, we recently looked at a related problem: the case of a weakly coupled system. Consider now the expression

$$l\vec{u} = -\Delta\vec{u} + A\vec{u}, \tag{1}$$

where $A = (a_{ij})$ is an $m \times m$ symmetric matrix and $\vec{u} = (u_1, \ldots, u_m)^T$. For convenience, assume that a_{ij} satisfies the same regularity properties as q

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for $i,j = 1,\ldots,m$.

If (1) decomposes into smaller subsystems, then what follows can be applied subsystem by subsystem. We thus consider explicitly only cases where this does not happen. Specifically, it suffices, but is not necessary, to assume that there exists a permutation σ of $\{1,\ldots,m\}$ such that $a_{\sigma(i),\sigma(i+1)}$, $a_{\sigma(m),\sigma(1)}$ are not identically zero near infinity, $i = 1,\ldots,m-1$. Whether A is irreducible at any specific point is irrelevant. Nonoscillation is defined in the obvious way, and B,L are as before. The following result was recently established [Allegretto, 1984], and the proof will appear elsewhere.

THEOREM 1. *Assume* $m = 2$, *and set* $q = (-a_{12},(a_{22}-a_{11})/2)$, $Q = q/|q|$ ($|\;|$ *is the Euclidean norm) with domain* $Q = \{x\,|\,q(x) \neq 0\}$. *Suppose there exists a neighborhood* N *of* ∞ *such that* $\{Q(x)\,|\,x\in N\cap \mathrm{dom}(Q)\}$ *is inside an arc of angle* $\leq \pi$. *Then* $S_-(L)$ *is finite iff* B *is nonoscillatory.*

It is not known if the result is valid without specifying the relative behavior of the a_{ij} near ∞. Note that if a_{12} is of fixed sign near ∞, then the condition on Q is satisfied.

These remarks indicate that, as was the case for $m = 1$, it is of interest to establish oscillation and nonoscillation criteria for (1) as a tool for looking at $S_-(L)$. Observe, however, that we are not concerned here with determinants of prepared matrices nor with h-oscillatory solutions (see Kreith [1973] and Swanson [1968, 1975, and 1978] for clarification of these concepts). As far as the relative behavior of a_{ij} is concerned, note that if $B_1(\vec{\phi},\vec{\phi}) \geq B(\vec{\phi},\vec{\phi})$ for *all* $\vec{\phi}\in C_0^\infty(G)$ and $S_-(L) < \infty$, then $S_-(L_1) < \infty$, where L_1 is the operator associated with B_1, regardless of whether or not the coefficients of L_1 satisfy the conditions at infinity. This leads naturally to a comparison with the scalar equation $-\Delta u + \mu u$, where μ is the least eigenvalue of A. Reversing these arguments leads to a comparison with $-\Delta u + \delta u$ and $S_-(L) = \infty$ criteria, where δ is the largest eigenvalue of A. Such scalar comparisons can obviously be optimal in some cases, but they may be misleading in others. We illustrate this remark with the following theorem, based on considerations of Martynov [1965] for ordinary systems and the related problem of spectrum discreteness.

THEOREM 2. *Let* $m = 2$. *Assume* λ_1,λ_2 *are the eigenvalues for* A *with associated normalized (in* R^m*) eigenvectors* $\vec{\phi}_1,\vec{\phi}_2$. *Let* $p(x)$ *measure the rotation of* $\vec{\phi}_1,\vec{\phi}_2$: $\dfrac{\partial}{\partial x_k}(\vec{\phi}_1) = \vec{\phi}_2\dfrac{\partial p}{\partial x_k}$ *and* $\dfrac{\partial}{\partial x_k}(\vec{\phi}_2) = -\vec{\phi}_1\dfrac{\partial p}{\partial x_k}$ *for* $k = 1,\ldots,n$. *Assume that* λ_1,λ_2,p *are smooth and for some positive constants* ε,δ *with* $\varepsilon^2 < 1/2$ *the inequalities*

$$\lambda_1 + |\nabla p|^2 - \delta^2 \geq -\frac{(n-2)^2}{4|x|^2}[1-2\varepsilon^2]$$

$$\lambda_2 + |\nabla p|^2(1-\frac{2}{\varepsilon^2}) - \frac{1}{\delta^2}(\Delta p)^2 \geq \frac{-(n-2)^2}{4|x|^2}$$

hold near infinity. Then l *is nonoscillatory and* $S_-(L)$ *is finite.*

PROOF. Let $\vec{\psi}\in C_0^\infty(G)$ be given, and set $\vec{\psi} = \alpha\vec{\phi}_1 + \beta\vec{\phi}_2$. A simple calculation shows

$$B(\vec{\psi},\vec{\psi}) = \int\{\sum_i[(\frac{\partial\alpha}{\partial x_i})^2 + (\frac{\partial\beta}{\partial x_i})^2] + \alpha^2[\lambda_1 + |\nabla p|^2]$$

$$+ \beta^2[\lambda_2 + |\nabla p|^2]\} + \int \sum_i [\alpha \frac{\partial \beta}{\partial x_i} - \beta \frac{\partial \alpha}{\partial x_i}] 2 \frac{\partial p}{\partial x_i}$$

$$= I_1 + I_2.$$

Now the divergence theorem yields

$$I_2 = \int -\sum_i 4\beta \frac{\partial \alpha}{\partial x_i} \frac{\partial p}{\partial x_i} - 2\alpha\beta\Delta p.$$

Estimating this in the usual way leads to

$$B(\vec{\psi},\vec{\psi}) \geq \int (1-2\varepsilon^2)|\nabla\alpha|^2 + (\lambda_1 + |\nabla p|^2 - \delta^2)\alpha^2$$

$$+ |\nabla\beta|^2 + \beta^2[\lambda_2 + |\nabla p|^2(1-\frac{2}{\varepsilon^2}) - \frac{1}{\delta^2}(\Delta p)^2].$$

Since the right-hand side represents a diagonal system, we merely apply Kneser's theorem to each component.

Observe that if $\Delta p = 0$, we may choose $\delta = 0$. In any case, if $|\nabla p|$ and $\lambda_2 - \lambda_1$ are sufficiently large (depending on p) near ∞, then λ_1 may be chosen arbitrarily, so that the scalar comparison fails. Analogously, we have the following theorem.

THEOREM 3. *If one of the following conditions holds near infinity, then B is oscillatory and $S_-(L)$ is infinite:*

a. $\lambda_1 + |\nabla p|^2 + \delta^2 \leq \frac{c}{4|x|^2}(1+2\varepsilon^2)$;

b. $\lambda_2 + |\nabla p|^2(1+\frac{2}{\varepsilon^2}) + \frac{1}{\delta^2}(\Delta p)^2 \leq \frac{c}{4|x|^2}$,

where $\varepsilon, \delta > 0$, $c < -(n-2)^2$.

This shows that, once again, one of the eigenvalues may be chosen arbitrarily. Consequently, the scalar comparison fails again.

To indicate how further criteria may be obtained, we introduce the following notation. We write $A \stackrel{\bullet}{\geq} B$ iff $b_{ij} \leq a_{ij}$ for all i,j, while $A \geq B$ signifies that $A - B$ is nonnegative definite. Clearly $A \stackrel{\bullet}{\geq} B$ and $A \geq B$ are different concepts. Assume G is an exterior domain.

THEOREM 4. *Let $a_{ij} \leq 0$, $i \neq j$. Assume $(\tilde{a}_{ij}) = \tilde{A} \stackrel{\bullet}{\leq} A$, $\tilde{a}_{ij} \in C^\infty$ for convenience, and let $\tilde{B}$ be the associated form. If $\tilde{B}$ is nonoscillatory, then $S_-(L)$ is finite.*

PROOF. Observe that $\tilde{B}$ nonoscillatory implies that B is nonoscillatory, since $\tilde{B}(|\vec{\phi}|,|\vec{\phi}|) \leq B(|\vec{\phi}|,|\vec{\phi}|) \leq B(\vec{\phi},\vec{\phi})$ by the sign condition, where $|\vec{\phi}| = \vec{\phi}^+ + \vec{\phi}^-$. We now apply the general version of Theorem 1; see Allegretto [1984].

Let $C_0^{\infty,R}$ denote the radial C_0^∞ functions, and assume $\tilde{a}_{ij}$ are radial functions. The obvious choice is $\tilde{a}_{ij}(r) \leq \inf_{|x|=r} [a_{ij}(x)]$.

THEOREM 5. *Let $a_{ij} \leq 0$, $i \neq j$. If $\tilde{B} \uparrow C_0^{\infty,R}$ is nonoscillatory, then : $\tilde{B} \uparrow C_0^\infty$ is nonoscillatory.*

PROOF. Under these conditions there exists a radial vector $\vec{v} > \vec{0}$ such that

$-\Delta\vec{v} + \tilde{\lambda}\vec{v} = \vec{0}$ near infinity. From this it follows that $\vec{B} \uparrow C_0^\infty$ is nonoscillatory; see again Allegretto [1984].

Theorems 4 and 5 show that criteria for the nonoscillation of ordinary systems may be used to obtain finiteness criteria for the negative spectrum of the partial differential operator L. This is a classic way of dealing with the problem if $m = 1$ (see, again, Kreith [1973] and Swanson [1968]). The sign condition $a_{ij} \leq 0$ may be modified by transformations, but this may mean that *inf* should be replaced by $\max_{|x|=r}$ in some cases. This will happen, e.g., if $m = 2$ and $a_{12} \geq 0$.

We observe that the proof of Theorem 5 also establishes the connection between positive supersolutions of the homogeneous equation and the finiteness of $S_-(L)$. From this observation, finiteness conditions may easily be established by selecting a $\vec{v} > \vec{0}$ *a priori*. As a very special example, we formulate a Kneser theorem.

THEOREM 6. *Let* $m = 2$, $a_{12} \leq 0$, *G an exterior domain. If there exists a constant c,* $-\infty < c < \infty$, *and a neighborhood N of* ∞ *in which*

$$a_{11} + a_{12}|x|^c \geq -\frac{(n-2)^2}{4|x|^2},$$

$$a_{22} + a_{12}|x|^{-c} \geq \left[\frac{4c^2-(n-2)^2}{4|x|^2}\right],$$

then $S_-(L)$ *is finite.*

PROOF. We merely select $\vec{v} = (r^\alpha, r^\beta)^T$ with $\alpha = \frac{2-n}{2}$ and $\beta = \alpha + c$, $r = |x|$. The assumed inequalities then imply $-\Delta\vec{v} + A\vec{v} \geq \vec{0}$, and the result follows.

One obvious choice for c in Theorem 6 is $c = 0$, but clearly other choices of c may be more advantageous in cases where a_{11}, a_{12}, and a_{22} grow at different rates.

It would be most desirable to obtain nonoscillation theorems that take advantage of the rotation of the eigenvectors, as was done in Theorems 2 and 3. It is not clear, however, how this can be done in any generality for large m, without the calculation of the eigenvalues.

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POTENTIALS HAVING EXTREMAL EIGENVALUES SUBJECT TO p-NORM CONSTRAINTS

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Abstract

We consider the Sturm-Liouville operator $H_V = \frac{-d^2}{dt^2} + V(t)$ on certain subsets of the real line with various selfadjoint boundary conditions. We find the optimal upper and lower bounds for the eigenvalues of H_V when the potential V obeys a constraint of the form $\|V\|_p \le M$. We characterize the extremizing potentials in those cases where they exist. Analysis of this one-dimensional problem is facilitated by interpreting it in terms of a classical oscillator.

1. Introduction

In this paper we address the problem of finding optimal bounds for the eigenvalues of the operator

$$H_V = \frac{-d^2}{dt^2} + V(t) \tag{1.1}$$

on certain subsets Ω of the real line (finite interval, half-line, line) with a variety of boundary conditions subject to p-norm constraints on the potential function V. To be more precise, having fixed an interval, a set of boundary conditions, and an index $k \ge 0$, we find optimal upper and lower bounds for $E_k(V)$ where V is allowed to range over the set $S = \{V \in L^p(\Omega) \mid \|V\|_p \le M\}$. Here $E_k(V)$ denotes the $(k+1)$th eigenvalue of H_V as defined by the min-max principle [Reed and Simon, 1972-79]. These bounds depend on S only through the constant M and, as will be made clear shortly, give upper and lower bounds for $E_k(V)$ in terms of $\|V\|_p$.

Our interest in such problems was first stimulated by a problem list of A. G. Ramm [1982] in which the problem of maximizing $E_0(V)$, where H_V acts on a finite interval, has Dirichlet boundary conditions, and V is subjected to a 1-norm constraint, was posed. In particular, in an earlier paper [Harrell, 1984], the maximization problem was analyzed for $E_k(V)$ on a finite interval with various selfadjoint boundary conditions, while laying the foundations for a solution to the problem with general p-norm constraints and also for multidimensional problems, i.e., for $H_V = -\Delta + V(x)$ acting on a set $\Omega \subset \mathbf{R}^d$, $d \ge 2$, with suitable boundary conditions. Much of the groundwork for the present study was laid in that paper, and henceforth we shall refer to it as article I. In a paper currently in preparation, we shall give the results of our investigations into the multidimensional case, as well as further material and some of the proofs dealing with the one-dimensional case. The multidimensional case turns out to be closely related to the problem of best constants in Sobolev's inequality and certain nonlinear elliptic partial differential equations which have been the subject of much current interest [Brézis and Nirenberg, 1983; Lions, 1982].

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Following the publication of Ramm's problem list, several other authors solved the problem posed above and, in some cases, pursued generalizations, restrictions, or related problems of their own. Solutions of which we are aware are those by Essén [1983], Farris [1982], and Talenti [1983]. Talenti, in particular, solved not only the problem posed by Ramm but also the problem of minimizing $E_0(V)$ under the same hypotheses and of minimizing $E_0(V)$ under the conditions $V \geq 0$, $\|V\|_1 = M$, and $\|V\|_\infty = B$. The extremizing potentials that Talenti finds have more than a passing resemblance to those found by Krein [1955] in his investigation of a similar problem for the equation of the vibrating string, $y'' + \lambda\rho(x)y = 0$ on $[0,l]$ subject to $y(0) = y(l) = 0$.

Independently of this, there accumulated over the last 15 years or so a body of literature among workers in ordinary differential equations giving lower bounds for the operator H_V in terms of a given p-norm of V. The relevant papers are those by Everitt [1972], Eastham [1972-72], Evans [1981], and Veling [1982 and 1983]. Each of these authors obtained a lower bound for H_V acting on $L^2(0,\infty)$ of the form $-c\|V\|_p^\alpha$ where c and α are constants depending on p. Each had the correct exponent $\alpha = 2p/(2p-1)$, but Veling was the first to find the optimal value of the constant c. All of these authors dealt with a Dirichlet boundary condition at $t = 0$ and, to varying extents, certain other standard boundary conditions. In particular, Veling [1982] gives the optimal lower bound of the form $-c\|V\|_p^\alpha$ for H_V on $L^2(0,\infty)$ with either a Dirichlet or Neumann boundary condition at $t = 0$. Also, Veling [1983] states the optimal bound for H_V on $L^2(\mathbb{R})$. Not surprisingly, there is a close connection between the three bounds discussed by Veling.

There is yet another line of work that is closely related to our current investigation. This work has been pursued in the mathematical physics community in an effort to get accurate bounds on the number of bound states of a Schrödinger operator and the slightly more restricted problem of obtaining optimal conditions for absence of bound states. The work most closely bearing on our own is that of Glaser, Martin, Grosse, and Thirring [1976], Glaser, Grosse, and Martin [1978], and Lieb and Thirring [1976]. These papers treat problems by methods that are similar in many respects to our own, though since they have somewhat different objectives, our results are largely disjoint from theirs.

Finally, in a forthcoming book by Trubowitz [1984] the problem of extremizing $E_k(V)$ for H_V acting on $L^2(0,1)$ with Dirichlet boundary conditions and with V subjected to a 2-norm constraint is posed and its solution is outlined in hints. One finds in this case that the extremizing potentials have explicit representations in terms of elliptic functions. We shall see shortly that the case $p = 3$ also leads to elliptic functions and, moreover, that qualitatively the solutions in the case of general p are very much the same. This situation is brought out most clearly by discussing the general problem in the context of classical mechanics. It is also worthy of note that elliptic functions arise in the problem of minimizing resonance widths within a suitable class of potentials [Harrell and Svirsky, 1984] and that the potentials for which Hill's equation has precisely one nonvanishing finite instability interval are elliptic functions [Hochstadt, 1976].

2. General Remarks

Since many of our arguments are not special to one dimension, we find it appropriate to include them in our longer paper [Ashbaugh and Harrell, 1984] and only to summarize them here. In addition we present those results of Harrell [1984] on which we base our current analysis.

In any problem involving maximizing or minimizing a functional, one is immediately confronted with the following questions:

1. (Semiboundedness) Does the appropriate supremum or infimum exist?

2. Can we find (or estimate) this value?

3. (Existence) Is there an optimizing function, i.e., a function at which the functional attains its sup (inf)?

4. (Characterization) What are the optimizing functions?

5. (Uniqueness) Is there a unique optimizing function?

General results [Ashbaugh and Harrell, 1984] give affirmative answers to questions 1, 3, and 5 in most cases of interest. Exceptions for questions 3 and 5 do arise and will be discussed at the appropriate point. Our main thrust in this paper will be toward answering question 4 and, to a lesser extent, 2. It will transpire that our answer to question 4 will often answer question 5 as a byproduct. This is because our approach to characterization is to study the equation

$$-u'' \pm (\operatorname{sgn} u)|u|^{(p+1)/(p-1)} = Eu \tag{2.1}$$

which, together with appropriate boundary conditions, was shown in article I (with the + sign only, for maximizers) to be a necessary condition for $\pm|u|^{2/(p-1)}$ to be an optimizing potential for $p > 1$. (For additional comments on the sense in which this equation holds and on the domain on which it holds, see Ashbaugh and Harrell [1984].) Thus, for instance, if we already have existence and can show that equation (2.1) has only one solution of the required type, then uniqueness follows immediately.

One further remark about the formulation of our problem seems appropriate here. While the requirement that the potential function V be locally L^1 is often regarded as the weakest reasonable condition (see, for example, the comments in Eastham and Kalf [1982: p. 4]), we have occasion to consider the operator H_μ, where μ represents a Borel measure. As pointed out to us by Barry Simon, this provides a reasonable operator since one can show that μ is a relatively form-compact perturbation of $H_0 = -d^2/dt^2$ using Fourier transforms. In fact, for H_μ acting on $L^2(\mathbf{R})$ in Fourier transform space, the kernel of $(H_0+1)^{-1/2}\mu(H_0+1)^{-1/2}$,

$$K(k_1,k_2) = (k_1^2+1)^{-1/2}\hat{\mu}(k_1-k_2)(k_2^2+1)^{-1/2}, \tag{2.2}$$

is easily shown to be Hilbert-Schmidt since $\hat{\mu}$ is a bounded continuous function. (Essentially we are defining the operator H_μ by means of quadratic forms in Fourier transform space.) The cases where H_μ has other domains are handled similarly by suitable choice of "Fourier transform." Allowing V to be a measure is crucial to the eigenvalue minimization problem when $p = 1$ since the ball of radius $M > 0$ in L^1 has no extreme points, but it is easy to see that an eigenvalue minimizer must be an extreme point using the Rayleigh-Ritz inequality. Thus when $p = 1$, minimizing potentials do not exist. However, if we allow V to lie in the larger class of all finite Borel measures, then we can obtain an existence result. For example, as exhibited by Talenti [1983], the minimizing potential for a finite interval with Dirichlet boundary conditions is a centered δ-function. With slight modifications the above relative compactness argument also applies to $V \in L^p$, $1 \le p \le 2$. This observation is useful in the one-dimensional case since our general methods and results handle only $p \ge 2$.

Even after restricting attention to the one-dimensional case, there are quite a variety of problems to be considered. First, one can consider the problem either of maximization or of minimization over a set $S = \{V \in L^p(\Omega) \mid \|V\|_p \le M\}$. Since by the min-max principle it is easy to show that a maximizing (minimizing) potential satisfies $V \ge 0$ ($V \le 0$) and $\|V\|_p = M$, it is a small step to consider what we shall call the misère problem of minimizing within the class $V \ge 0$, $\|V\|_p = M$ (maximizing within the class $V \le 0$, $\|V\|_p = M$). We will see, in fact, that the misère problems do not have extremizers and that the optimal bounds are the appropriate $V = 0$ eigenvalues. Second, one has the three choices of domain to consider: finite interval, half-line, and line. Third, one can impose a variety of boundary conditions at the finite endpoints of the domain. Those with which we shall deal are Dirichlet, Neumann, separated (i.e., $\alpha u(t_i) + \beta u'(t_i) = 0$ where t_i is an endpoint), and "compact-support" boundary conditions. Since this last terminology is not standard, we explain: These are the boundary conditions one gets at $\pm l$ if one allows $-\infty < t < \infty$ but requires V to have support in the interval $[-l, l]$. In particular, they take the form

$$u'(-l) = \mp\sqrt{-E}\,u(\pm l).$$

Lastly, one can concentrate on any eigenvalue $E_k(V)$ for $k = 0,1,2,\ldots$. The ground state $E_0(V)$ is perhaps the most interesting, and in fact we can get more detailed results about it (partly because more tools are available for studying it). The ground state is also unique compared to higher states in that for a given problem certain results will hold for the ground state but for no excited states. For example, the finite-interval $p = 1$ maximization problem has a unique maximizer for $E_0(V)$ but not for $E_k(V)$, $k \ge 1$ [Harrell, 1984]. As a second example, on $\mathbf{R}$ with $p > 1$ there exists a ground-state minimizer (unique up to translations), but minimizers for the higher states do not exist. However, the general method and viewpoint presented here lend a degree of unity to the various cases and problems outlined above. In particular, the method applies to a large extent equally to the ground and excited states.

3. The Classical Oscillator Viewpoint

While we chose time as the independent variable with the classical oscillator interpretation in mind, we find it convenient here to set forth other standard notations from the perspective of classical mechanics. For a modern and more comprehensive discussion of classical mechanics, we refer the reader to the recent book by Thirring [1978]. By viewing equation (2.1) as Newton's equation for motion in one dimension (u represents position), we can identify the classical potential energy as

$$W(u;E) = \frac{1}{2}Eu^2 \mp \left(\frac{p-1}{2p}\right) |u|^{2p/(p-1)}. \tag{3.1}$$

Note that the quantum energy E appears as a coefficient in this classical potential. A first integral for this system is given by

$$\frac{1}{2}\left(\frac{du}{dt}\right)^2 + W(u;E) = h, \tag{3.2}$$

where we have let h denote the classical (total) energy of our oscillator. Our convention for the ambiguous sign in all equations —(2.1) and (3.1) thus far —is that we take upper signs when considering maximization problems and lower signs when considering minimization problems.

Though we will refer to the above equations as describing an oscillator, for certain choices of the sign referred to above and the sign of E one will not have oscillations or will have oscillations only for suitable initial values. For the most common boundary conditions (Dirichlet, Neumann) only the truly oscillatory solutions will enter, but with more complicated conditions other solutions can sometimes come into play.

We will refer to the curves given parametrically by $(u(t), u'(t))$, where u solves equation (2.1) as trajectories in phase space. Of course, the oscillatory solutions referred to above are just the closed orbits in phase space. In phase space, separated boundary conditions (Dirichlet and Neumann included) can be viewed geometrically as the condition that a trajectory start on a given line through the origin and end on a second line through the origin (possibly the same) at a specified later time. When the interval is finite, we choose it as $[0,l]$, $l > 0$, or sometimes $[-l,l]$; for the half-line we choose $[0,\infty)$.

4. Minimization on the Line and the Half-Line

We begin our detailed discussion with these cases since from the classical oscillator viewpoint the constant h must be 0, which simplifies the analysis. Also these are the cases that have drawn attention previously. Now since u is an L^2 solution to $H_V u = Eu$, where $V = -|u|^{2/(p-1)} \in L^p$, we can be sure from the theory of Schrödinger operators [Reed and Simon, 1972-79; Richtmyer, 1978] that u and u' go to 0 as t goes to ∞. Thus on infinite intervals our only concern is with classical oscillator solutions having total energy $h = 0$, and we need only solve the equation

$$\frac{1}{2}\left(\frac{du}{dt}\right)^2 + \frac{1}{2}Eu^2 + \left(\frac{p-1}{2p}\right)u^{2p/(p-1)} = 0. \tag{4.1}$$

This equation is readily integrated, with the result that

$$u(t) = \left(\frac{-pE}{p-1}\right)^{(p-1)/2} \operatorname{sech}^{p-1}\left[\frac{\sqrt{-E}\,(t-c)}{p-1}\right], \tag{4.1}$$

and hence

$$V(t) = \frac{pE}{p-1}\operatorname{sech}^2\left[\frac{\sqrt{-E}\,(t-c)}{p-1}\right]. \tag{4.3}$$

Here c is the constant of integration. For the minimization problem on the line, it represents the expected fact that a minimizing potential cannot be unique because of translation invariance. For half-line problems, the constant would have to be chosen so that u satisfies the boundary condition at the origin. We shall see shortly that this has the interesting consequence that no minimizers exist for certain choices of the boundary condition. But first let us finish our discussion of the standard cases.

For the full line minimization problem one can compute

$$\|V\|_p^p = \frac{p^p(-E)^{(2p-1)/2}}{(p-1)^{p-1}}B(p,\tfrac{1}{2}), \tag{4.4}$$

or, solving for E,

$$E = -\left[\frac{(p-1)^{p-1}}{p^p B(p,\frac{1}{2})}\right]^{2/(2p-1)} \|V\|_p^{2p/(2p-1)}. \tag{4.5}$$

Here $B(p,\frac{1}{2})$ represents a beta function in standard notation. This formula is that given by Veling [1983] except for a misprint of $(1-\vartheta)\vartheta^{\vartheta/(1-\vartheta)}$ as $(1-\vartheta)^{\vartheta/(1-\vartheta)}$.

For the half-line problem with Neumann boundary condition one must take $c = 0$ in equation (4.3). The computation can be carried out as before, yielding

$$E = -2^{2/(2p-1)}\left[\frac{(p-1)^{p-1}}{p^p B(p,\frac{1}{2})}\right]^{2/(2p-1)} \|V\|_p^{2p/(2p-1)}, \tag{4.6}$$

again agreeing with a result of Veling [1982].

We now consider the general boundary condition

$$u'(0) = mu(0) \tag{4.7}$$

From equation (4.2) this reduces to

$$m = \sqrt{-E}\tanh(\sqrt{-E}c/(p-1)) \tag{4.8}$$

which has a solution for c if and only if $\sqrt{-E} > |m|$. Holding E fixed, we see that as $m \to \sqrt{-E}$ from below, $c \to \infty$, and that as $m \to -\sqrt{-E}$ from above, $c \to -\infty$. Thus as $m \to -\sqrt{-E}$ our sech^2-potential well translates off to the left, "leaving" the positive half-axis, and as $m \to \sqrt{-E}$ it translates to the right into the positive half-axis. We can better understand what is happening here if we note that the potential $V = 0$ with boundary condition (4.7) has a negative eigenvalue at $E = -m^2$ if $m < 0$. Thus a fixed $E < 0$ will not be minimal for the operator H_V on $L^2(0,\infty)$ with boundary condition (4.7) for $m < 0$ until m increases to $-\sqrt{-E}$. At that value of m, E will be minimal for $\|V\|_p = M = 0$. For $|m| < \sqrt{-E}$, E will be minimal for $\|V\|_p$ fixed as required by equations (4.3) and (4.8). One could write the relation betwen E and $\|V\|_p$ for this range of m in terms of the incomplete beta function, but we refrain from doing so here. When m exceeds $\sqrt{-E}$, one no longer has a minimizing potential, but a minimizing sequence of potentials is easily constructed by taking a sequence of V's given by equation (4.6) with c's going to ∞ and suitably modified on $[0,1]$, say, to meet the boundary condition at $t = 0$. This latter situation also includes the case of Dirichlet boundary conditions. In these cases the value E in equation (4.5) is a strict lower bound for the ground state and hence also for the operator H_V.

We close this section with some cursory remarks about higher eigenvalues. To obtain a minimizing sequence of potentials for a higher eigenvalue, one "pastes on" more sech^2-potential wells out near infinity. The modifications required in the pasting can be shown to have vanishing effect as the spacing between consecutive wells is sent to infinity. We note that the potentials in the minimizing sequence for the k-th eigenvalue approach k-fold degeneracy, i.e., the first k eigenvalues come together in the limit. The appropriate eigenfunction in this case is much like the potential (to the power $(p-1)/2$) except that we flip its sign each time we paste on a new piece; on $[0,\infty)$ we also must rescale the left-most bump so that its L^2 norm is the same as all the others. As an illustration one obtains the bound

$$E_1(V) > -2^{-2/(2p-1)}\left[\frac{(p-1)^{p-1}}{p^p B(p,\frac{1}{2})}\right]^{2/(2p-1)} \|V\|_p^{2p/(2p-1)}$$

in the case of the second eigenvalue of H_V acting on $L^2(\mathbf{R})$.

To those familiar with high-energy physics, there is more than a passing similarity between the above construction of minimizing sequences and the construction of a multiple instanton configuration. We also remark that the sech^2 form of our potential is precisely a soliton solution to the Korteweg de Vries (KdV) equation. There is an extensive literature detailing the intimate connections between the KdV equation and the Schrödinger equation; we content ourselves with noting that the article [Lieb and Thirring, 1976] presents some particularly pertinent observations of P. Lax.

5. Minimization on a Finite Interval

When one seeks to find eigenvalue minimizers on a finite interval, one must consider equation (3.2) with all allowed values of the classical energy h. We adopt the following strategy in this discussion: with fixed $p > 1$ and interval $[0,l]$, we pick a possible optimal eigenvalue E and choose suitable boundary conditions; then we look for those values of h that allow u to meet the boundary conditions at $t = 0$ and $t = l$; and finally we determine the value $M = \|V\|_p$ for which $V = -|u|^{2/(p-1)}$ is a possible minimizer. If at the end of this process we have only one candidate, then, having already proved existence of a minimizer [Ashbaugh and Harrell, 1984], we can conclude that we have found the unique minimizer. Even if we find several candidates, the existence result guarantees that at least one of them will be a minimizer. Existence of minimizers on a finite interval when V is allowed to range over the class of Borel measures μ satisfying $\int d|\mu| \le M$ is shown in our longer paper. This result handles the minimization question when $p = 1$.

We begin our discussion by considering Dirichlet boundary conditions and taking $E < 0$. Then the only h's for which Dirichlet conditions can be met are $h > 0$, and the time required for one excursion (half the period of the orbit) is

$$T(h,E)/2 = \sqrt{2}\int_0^{u_1}[h - W(u;E)]^{-1/2}du, \tag{5.1}$$

where u_1 represents the positive turning point of the motion, i.e., $W(u_1;E) = h$, $u_1 > 0$. To see how $T(h,E)$ varies with h we eliminate h in favor of u_1 while noting that the mapping $h \to u_1$ is an increasing function from $(0,\infty)$ onto $(u_{1,\min},\infty)$ where $u_{1,\min}$ satisfies $0 = W(u_{1,\min};E)$. One has

$$T = 2\sqrt{2}\int_0^{u_1}[W(u_1;E) - W(u;E)]^{-1/2}du \tag{5.2a}$$

$$= 2\sqrt{2}\int_0^{u_1}[E(u_1^2-u^2)/2 + (p-1)\{u_1^{2p/(p-1)} - u^{2p/(p-1)}\}/2p]^{-1/2}du$$

$$= 2\sqrt{2}\int_0^{1}[(p-1)u_1^{2/(p-1)}(1-s^{2p(p-1)})/2p + E(1-s^2)/2]^{-1/2}ds, \tag{5.2b}$$

where we changed variables to $s = u/u_1$ to arrive at the last expression. Thus one sees that T decreases from ∞ to 0 as h increases from 0 to ∞. Since to accommodate the $(k+1)$th eigenvalue E_k we need

$$(k+1)T(h,E_k)/2 = l \tag{5.3}$$

to be satisfied, we see that any $E < 0$ can be a minimal $(k+1)$-th eigenvalue for any $k \ge 0$. A similar analysis leads to the same conclusion when $E = 0$. When $E > 0$, one finds that the period T decreases from $2\pi/\sqrt{E}$ to 0 as h increases from 0 to ∞. Thus if $E > (k+1)^2\pi^2/l^2$, then E cannot be a minimal E_k, whereas if $E \le (k+1)^2\pi^2/l^2$, it will be attainable as a minimal E_k. If one notes that $E_k(0) = (k+1)^2\pi^2/l^2$, the reasonableness of these conditons is apparent. Actually, to complete this discussion, we must look at the equilibrium solutions, i.e., the critical points in the phase plane. These solutions are exceptional in that there is not a fixed period associated with them. For the above, the only critical point solution of relevance is $u = 0$, which is trivial to analyze.

With Neumann boundary conditions the same considerations apply for the orbits and their periods as discussed above. However, there are additional orbits having $h < 0$ to be considered in the case of $E < 0$, including another equilibrium solution corresponding to the minimum of $W(u;E)$. This complicates the indexing of the eigenvalues somewhat, but Sturm's theorem on nodes of eigenfunctions suffices to sort things out. The orbits considered previously lead to candidates for minimal E_k, $k \ge 1$, under the condition

$$kT(h,E_k)/2 = l, \tag{5.4}$$

and the newly considered orbits lead to candidates for E_0 since they give nodeless solutions. Again any $E \le 0$ can be a minimal Neumann E_k, $k \ge 0$, but for

$E > 0$, $E > k^2\pi^2/l^2$ precludes E from being a minimal E_k and $E \le k^2\pi^2/l^2$ allows it. That all allowed E's are actually assumed as minimal E_k's for some choice of $M = \|V\|_p$ follows from continuity considerations which are taken up by Ashbaugh and Harrell [1984].

Other choices of separated boundary conditions at $t = 0$ and $t = l$ will force us to consider more complicated conditions than (5.3) or (5.4) for meeting the boundary conditions. In fact, trajectories that are not closed orbits will even enter: the appropriate point of view is that we need to find those trajectories that take time l to pass from one line through the origin to a second line through the origin in phase space. Periodic or antiperiodic boundary conditions lead back to the same orbits as were discussed in the Neumann case, as do separated boundary conditions of "periodic type": $u'(0) = mu(0)$, $u'(l) = mu(l)$, $m \in \mathbf{R}$

6. Maximization on a Finite Interval

The analysis of the maximization problem differs only in detail from that of the minimization problem. The most significant difference is that the potential $W(u;E)$is now upside down; in particular, $W \to -\infty$ as $u \to \infty$. This has the effect that for all standard boundary conditions only $E \ge 0$ need be considered. By analyzing $T(h,E)$, one finds in this case that $2\pi/\sqrt{E} \le T(h,E) < \infty$ for the permissible values of h. Thus $E < (k+1)^2\pi^2/l^2$ implies that E cannot be an extremal $(k+1)$-th eigenvalue for the Dirichlet problem whereas $E \ge (k+1)^2\pi^2/l^2$ can be. As should be clear, the discussion of this problem parallels almost exactly that of the previous section, so we conclude it here.

7. Misère Problems

We turn now to a brief discussion of the misère problem, that of minimizing (respectively, maximizing) a given eigenvalue when V is constrained to the class $S = \{V \mid V \ge 0, \|V\|_p = M\}$ (resp., $S = \{V \mid V \le 0, \|V\|_p = M\}$). We shall confine the majority of our remarks to the case of the ground state for Dirichlet boundary conditions which we shall denote by $E(V)$.

We begin by considering the minimization problem with $V \ge 0$ where $\Omega \subset \mathbf{R}^d$ is bounded and has smooth boundaries. The case of unbounded domains for this minimization problem is of no interest since $E(V)$ (as defined by the min-max principle) is then always $0 = E(0)$. We shall show that (1) $E(V) > E(0)$ for all $V \in S$ and (2) $\inf_S E(V) = E(0)$. Thus there is no V that is a minimizer for this misère problem. To obtain (1), we simply use the Rayleigh-Ritz inequality for $-\Delta$ with ϕ_V, the normalized ground-state eigenfunction of H_V, as trial function: $E(V) = (\phi_V, (-\Delta + V)\phi_V) = (\phi_V, -\Delta\phi_V) + \int_\Omega V|\phi_V|^2 > E(0)$. To prove (2), note that since the ground state, ϕ_0, of $-\Delta$ on Ω with Dirichlet boundary conditions goes to 0 on $\partial\Omega$ and since $\partial\Omega$ is smooth, we can find a sequence of sets $B_n \subset \Omega$ satisfying (i) $\sup_{B_n} |\varphi_0| \le 1/n$ and (ii) $0 < |B_n| < K$, K a constant independent of n. Then with $V_n = M|B_n|^{-1/p}\chi_{B_n}$ and again using Rayleigh-Ritz, we compute

$$E(V_n) < (\phi_0, (-\Delta + V_n)\phi_0) = E(0) + \int_{B_n} |\phi_0|^2 M|B_n|^{-1/p} \le E(0) + MK^{1-1/p}/n^2,$$

which goes to $E(0)$ with increasing n.

The problem of maximizing over $S = \{V \mid V \le 0, \|V\|_p = M\}$, $p \ge 1$, is more difficult to analyze, but leads to much the same result. That $E(V) \le E(0)$ is again a consequence of Rayleigh-Ritz or, more precisely, the min-max principle. When $E(0)$ is in the discrete spectrum, this inequality is strict; in any event, there is no $V \in S$ for which $-\Delta + V(x)$ has $E(0)$ as an isolated eigenvalue of finite multiplicity. This, together with the fact that $\sup_S E(V) = E(0)$ (to be shown shortly) shows that this misère problem also lacks an optimizer in the sense of $E(0)$ being attained as a point in the discrete spectrum. In some cases, $E(0)$ is attained by $E(V)$ for some $V \in S$; but in cases where $E(V)$ is in the discrete spectrum for all $V \in S$, this cannot occur. All problems where Ω is bounded and the cases $\Omega = \mathbf{R}^d$, $d = 1$ or 2, fall in this latter class. If Ω is unbounded, one can construct a sequence $\{V_n\}$ of potentials in S having $E(V_n) \to 0 = E(0)$ by using $V_n = M|B_n|^{-1/p}\chi_{B_n}$ where the sets $B_n \subset \Omega$ satisfy $|B_n| \to \infty$. Here we have used wide but shallow square wells in our construction. For bounded domains this avenue is not open to us, so we shall use narrow and deep square wells. We pick a sequence of balls $B_n \subset \Omega$ with $|B_n| \neq 0$ for all n and $|B_n| \to 0$. Then for $p > 1$ and $V_n = -M|B_n|^{-1/p}\chi_{B_n}$ we have $\|V_n\|_1 = M|B_n|^{1-1/p} \to 0$ as $n \to \infty$; and using the fact that our lower bound for $E(V)$ in terms of $\|V\|_1$ goes to $E(0)$ as $\|V\|_1 \to 0$ [Ashbaugh and Harrell, 1984], we see that $E(V_n) \to E(0)$. We remark that this sequence works equally well for Ω unbounded but has the drawback that it does not cover the case $p=1$. The essential observations in the above discussion are that for Ω unbounded there is a sequence $\{V_n\}$ in S also lying in L^∞ with $\|V_n\|_\infty \to 0$ and that for $p>1$ and arbitrary Ω there is a sequence $\{V_n\}$ in S also lying in L^1 with $\|V_n\|_1 \to 0$. These observations would also have sufficed in dealing with the misère minimization problem except for the case $p=1$. Indeed, except for this case, the argument given above could have been concluded just by choosing the B_n's so that $|B_n| \to 0$.

To complete the discussion, we need to treat the case of a bounded domain when $p = 1$. Just as in the misère minimization problem, our argument now rests on our choice of Dirichlet boundary conditions. The idea is to take a sequence η_n approximating a δ-function located on $\partial\Omega$ and argue that for $V = -M\eta_n$, we have $(\phi_n, V_n\phi_n) \to 0$ as $n \to \infty$ where ϕ_n represents the normalized ground state for $-\Delta + V_n$. However, here we shall give a proof only in the case of dimension $d = 1$. In this case we may take $\Omega = [0,l]$, $l > 0$, and we define a sequence of potentials $V_n = -Mn\chi_{[0,1/n]}$. By standard methods found in any elementary quantum mechanics textbook, one could give an explicit argument showing that $E(V_n) \to E(0) = \pi^2/l^2$. Instead, we note that $E(V_n)$ is the first eigenvalue of the three-dimensional problem for $-\Delta + V_n(r)$; we remark that this is where we make use of the Dirichlet boundary condition. As a function in three-space, we have $\|V_n\|_1 = Mn\frac{4}{3}\pi(1/n)^3 = 4\pi M/3n^2 \to 0$ as $n \to \infty$ and thus, by the result alluded to above, $E(V_n) \to E(0)$, where the 0 represents the 0 potential on the ball of radius l in $\mathbf{R}^3$. But, passing back to one dimension, we have $E(0) = \pi^2/l^2$, which completes the proof. Finally, we remark that except when $p = 1$, Dirichlet boundary conditions were not needed; in particular, the last argument works for arbitrary boundary conditions imposed at $t = l$.

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INDEFINITE STURM-LIOUVILLE PROBLEMS

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Abstract

We study regular Sturm-Liouville problems

$$Ly = -(py')' + qy = \lambda ry \quad \text{on } [a,b]$$

with separated, selfadjoint boundary conditions, where r is assumed to change sign and L is not assumed to be positive definite. We look primarily at a special case, which seems to give some insight into the general case, obtaining bounds for the so-called Richardson numbers of our problem and presenting the results of numerical calculations of the spectrum of our problem.

1. Introduction

We consider problems of the following form:

$$Ly = -(pu')' + qy = \lambda ry \quad \text{on } (a,b), \tag{1}$$

where

a,b are finite
$p,q,r\colon [a,b] \to \mathbf{R}$
$p(x) > 0$ for almost all $x \in [a,b]$
$p^{-1} = \frac{1}{p}, q, r \in L^1_{loc}[1,b]$
measure$\{x \in [a,b]\colon r(x) > 0\} > 0$
measure$\{x \in [a,b]\colon r(x) < 0\} > 0$
$r(x) \neq 0$ almost everywhere.

A solution must satisfy (1) as well as separated selfadjoint boundary conditions

$$A_1 y(a) + A_2 p(a) y'(a) = 0$$

$$B_1 y(b) + B_2 p(b) y'(b) = 0. \tag{2}$$

Such problems are called "indefinite" because the operator L is not required to be positive definite. As the following proposition shows, however, indefinite problems differ from the definite ones only for finitely many eigenvalues, a fact first proved by Richardson [1918].

PROPOSITION 1. *For problem (1), (2), if $|\lambda|$ is sufficiently large, the number of zeros of the eigenfunctions associated with two successive eigenvalues differs by precisely one.*

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PROOF. Define $\vartheta(x,\lambda)$ by

$$\tan\vartheta = \frac{y}{py'} \tag{3}$$

with $\vartheta(a,\lambda)$ fixed. (This is the Prufer transformation.) We find that

$$\vartheta' = p^{-1}\cos^2\vartheta + (\lambda r - q)\sin^2\vartheta \qquad (\,' = \frac{d}{dx}). \tag{4}$$

If $r > 0$, we can conclude that ϑ is increasing in λ, for fixed $x \in (a,b)$.

Without assuming that $r > 0$, we argue from (3) that

$$\vartheta_\lambda = \{(py')y_\lambda - (py')_\lambda y\} / \{y^2 + (py')^2\} \quad (\lambda = \frac{\partial}{\partial\lambda}). \tag{5}$$

Here y, py' are fixed at $x = a$. We find that the numerator in (5) is

$$\int_a^b ry^2 dx. \tag{6}$$

Hence ϑ is increasing in λ if (6) is positive.

For suitable boundary conditions, in particular for (2), a solution of (1), (2) will satisfy

$$\lambda\int_a^b ry^2 dx = \int_a^b \{py'^2 + qy^2\} dx. \tag{7}$$

Hence, if the quadratic form

$$\int_a^b \{py'^2 + qy^2\} dx \tag{8}$$

is positive for nontrivial differentiable functions satisfying (2), then $\vartheta(b,\lambda)$ is increasing when λ is a positive eigenvalue and decreasing when λ is a negative eigenvalue.

In general, the form (8), when diagonalized, may have a finite number of negative squares, the number of which may be ascertained by replacing r in (1), (2) by a nonnegative weight function, e.g., $|r|$, or 1. If N is the number in question, then for at most N eigenvalues of (1), (2) we can have

$$\vartheta_\lambda(b,\lambda) < 0. \tag{9}$$

The result follows because $y(x)$ vanishes if and only if $\vartheta(x)$ is an integral multiple of π. ///

In light of Prop. 1, we may define

$$\lambda^+ = \inf\{x \in \mathbf{R} \mid \forall\lambda > x,\ \vartheta_\lambda(b,\lambda) > 0\}$$

$$\lambda^- = \sup\{x \in \mathbf{R} \mid \forall\lambda < x,\ \vartheta_\lambda(b,\lambda) < 0\}. \tag{10}$$

We may interpret these quantities as follows: λ^+ is the smallest number such that the real eigenvalues greater than λ^+ behave as in a "typical" Sturm-Liouville problem; that is, an eigenvalue is uniquely associated with its oscillation number (the number of zeros of the associated eigenfunction). λ^- is interpreted similarly.

Note that if $r(x) > 0$, for all $x \in [a,b]$, then as was shown in the proof of Prop. 1, $\lambda^+ = \lambda^- = -\infty$. If r is as in (1) and L is positive definite, $\lambda^+ = \lambda^- = 0$. In the indefinite case, the determination of these numbers is a very significant problem; Mingarelli [1982] has quite appropriately called these quantities "the Richardson numbers" of the problem (1), (2). We can determine what we believe to be a sharp bound on λ^+ and λ^- for the following specific problem:

$$Ly = -y'' + qy = \lambda r y \quad \text{on } [-1,1]$$

$$y(-1) = y(1) = 0 \tag{11}$$

$$r(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$q(x) = q_0,\ q_0 \in \mathbb{R}$$

Note that for $q_0 > \frac{-\pi^2}{4}$, the operator L in (11) is positive definite; we are thus interested in the case where $q_0 < \frac{-\pi^2}{4}$. We now proceed to show that for (11), with $q_0 < -\frac{\pi^2}{4}$, $\lambda^+ \leq |q_0| - \frac{\pi^2}{4}$ (hence, $\lambda^- \geq -|q_0| + \frac{\pi^2}{4}$ by symmetry).

LEMMA 1. *In* $(0,1)$ *let*

$$y'' = -\mu y,\ \mu < \frac{\pi^2}{4}$$

$$y(0) = 0,\ y'(x) > 0 \text{ in } [0,1].$$

Then $\int_0^1 y^2(t)dt < \frac{1}{2}y^2(1)$.

PROOF. Suppose first that $\mu \leq 0$. Then

$$0 < y',\ x \in (0,1],$$

and so

$$0 \leq y(x) \leq xy(1),\ x \in [0,1].$$

Hence $\int_0^1 y^2(t)dt \leq \frac{1}{3}y^2(1)$. Suppose next that

$$0 < \mu < \frac{\pi^2}{4}.$$

Then

$$y(x) = C\sin kx,\ 0 \leq x \leq 1$$

$$C > 0,\ 0 < k < \frac{\pi}{2}.$$

$$\int_0^1 y^2(t)dt = C^2\int_0^1 \sin^2 kt\, dt = \frac{1}{2}C^2\left(1 - \frac{1}{2k}\sin 2k\right).$$

Since $\frac{1}{2k}\sin 2k > \cos^2 k$, we have the result. ///

LEMMA 2. *In* $(0,1)$ *let*

$$y'' = -\mu y,\ \mu > 0$$

$$y(0) = 0,\ y(1)y'(1) < 0.$$

Then $\int_0^1 y^2(t)dt > \frac{1}{2}y^2(1)$.

PROOF. It is sufficient to suppose that

$$y(x) = \sin kx,\ k > 0.$$

Since $y(1)y'(1) > 0$, we have

$$\sin 2k < 0.$$

But

$$\int_0^1 y^2(t)\mathrm{d}t = \frac{1}{2}(1-\frac{1}{2k}\sin 2k),$$

and here $\frac{\sin 2k}{2k} < 0$. Hence

$$\int_0^1 y^2(t)\mathrm{d}t > \frac{1}{2} > \frac{1}{2}y^2(1). \quad ///$$

PROPOSITION 2. *If in (11)*, $q_0 < -\frac{\pi^2}{4}$, *then* $\lambda^+ \le |q_0| - \frac{\pi^2}{4}$.

PROOF. Let y satisfy (11), i.e.,

$$y'' + (\lambda - q_0)y = 0,\ 0 \le x \le 1$$

$$y'' + (-\lambda - q_0)\lambda = 0,\ -1 \le x < 0$$

with $y(-1) = y(1) = 0$.

If $\lambda > -q_0 - \frac{\pi^2}{4}$, then in $(-1,0)$ we have $-\lambda - q_0 < \frac{\pi^2}{4}$, and so applying Lemma 1 to the interval (-1,0), with shift of origin,

$$\int_{-1}^0 y^2(t)\mathrm{d}t < \frac{1}{2}y^2(1).$$

In (0,1), we use Lemma 2 —replacing (0,1) by (1,0) —and have

$$\int_0^1 y^2(t)\mathrm{d}t > \frac{1}{2}y^2(1).$$

Hence, for an eigenvalue λ such that

$$\lambda > -q_0 - \frac{\pi^2}{4},$$

we have

$$\int_{-1}^1 y^2(t)(\mathrm{sgn}(t))\mathrm{d}t > 0.$$

Now if $\vartheta(x,\lambda)$ is defined by

$$\vartheta(-1,\lambda) = 0$$

$$\tan\vartheta(x,\lambda) = \frac{y(x,\lambda)}{y'(x,\lambda)},$$

where $y(x,\lambda)$ is a nontrivial solution with $y(-1,\lambda) = 0$, we have

$$\frac{\partial \vartheta}{\partial \lambda}(x,\lambda) = \frac{1}{y^2+y'^2} = \int_{-1}^{x} y^2(t)(\mathrm{sgn}(t))dt.$$

Hence for eigenvalues λ such that

$$\lambda > -q_0 - \frac{\pi^2}{4}$$

we have $\frac{\partial \vartheta}{\partial \lambda} > 0$. ///

Another approach to this problem (namely, the determination of λ^+ for (11)) is the following one, which is less precise but slightly more general.

PROPOSITION 3. *Let $y(x,\lambda)$ be an eigenfunction of*

$$y'' + (\lambda r + q)y = 0,\ y(a) = y(b) = 0 \tag{12}$$

(r,q as (1),(2)) with zeros

$$a = x_0 < x_1 < \cdots < x_k = b.$$

For each $j = 0,\dots k-1$, let there be a solution u_j of

$$u_j'' + (\mu_j r + q)u_j = 0$$

such that $u_j(x) > 0$ in $[x_j, x_{j+1}]$. Then λ^+ for (12) is less than λ.

PROOF. The proof follows immediately from the Sturm comparison theorem by considering the function $z_j = \frac{y}{u_j}$ on $[x_j, x_{j+1}]$. ///

If we specialize to the case where for some $c \in (a,b)$, $r(x) > 0$ in (c,b), then we have the following proposition.

PROPOSITION 4. *For some μ, let*

$$u'' + (\mu r + q)u = 0$$

have a solution positive in $[a,d]$ for some $d \in (c,b)$. For some $\lambda^* > \mu$, let there be a solution $y \equiv 0$ with $y(a) = 0$ and also a zero in (c,d). Then $\lambda^+ < \lambda^*$ (i.e., the eigenvalues for which $\lambda > \lambda^*$ are uniquely associated with their oscillation number.) ///

PROPOSITION 5. *The above will be fulfilled if*

$$\mu r + q \le 0 \text{ in } (a,c)$$

$$\mu r + q \ge 0 \text{ in } (c,d)$$

$$(d-c)\sup_{(c,d)}\sqrt{\mu r+q} \le \frac{\pi}{2}$$

$$(d-c)\inf_{(c,d)}\sqrt{\lambda^* r + q} > \pi. \quad ///$$

If we specialize further to the case

$$y'' + (-\lambda+q)y = 0 \text{ in } (-1,0)$$

$$y'' + (\lambda+q) = y = 0 \text{ in}(0,1)$$

$$y(-1) = y(1) = 0. \tag{13}$$

For some M, $M > \frac{\pi^2}{12}$, we assume that

$$q(x) \le M \text{ in } (-1,0)$$

$$-M \le q(x) \le M \text{ in } (0, \frac{\pi}{2\sqrt{3M}}) .$$

There is no lower bound on $q(x)$ in $(-1,0)$, and no bounds in $(\frac{\pi}{2\sqrt{3M}}, 1)$.

PROPOSITION 6. *For (13),* $\lambda^+ > 13M$.

PROOF. This results from the choice $\mu = 2M$. Here

$$a = -1,\ b = 1,\ c = 0,\ d = \frac{\pi}{2\sqrt{3M}}.$$

Thus,

$$(d-c)\sup_{(c,d)}\sqrt{\mu r+q} \le \frac{\pi}{2\sqrt{3M}}\sqrt{3M} = \frac{\pi}{2}$$

$$(d-c)\inf_{(c,d)}\sqrt{\lambda r+q} > \frac{\pi}{2\sqrt{3M}}\sqrt{12M} = \pi$$

as required. ///

This gives a less precise bound for λ^+ in problem (11) (for $q_0 < -\frac{\pi^2}{4}$), namely, $13|q_0|$.

Finally, we investigated problem (11) numerically using the VAX 11/780 computer in the Mathematics and Computer Science Division at Argonne National Laboratory. We wanted to understand how the spectrum changes as q_0 changes. We calculated a great many eigenvalues and eigenfunctions (a selection of eigenvalues is given in the appendix) by solving the dispersion relation for (11)

$$\sqrt{-q-\lambda}\cos\sqrt{-q-\lambda}\sin\sqrt{-q+\lambda} + \sqrt{-q+\lambda}\cos\sqrt{-q+\lambda}\sin\sqrt{-q-\lambda} = 0$$

in the complex plane; the eigenfunction is then given by

$$y = \begin{cases} \dfrac{C\sin[\sqrt{-q-\lambda}(1+x)]}{\sin\sqrt{-q-\lambda}} & \text{if } -1 \le x \le 0 \\ \dfrac{C\sin[\sqrt{-q+\lambda}(1-x)]}{\sin\sqrt{-q+\lambda}} & \text{if } 0 \le x \le 1 \end{cases}$$

C an arbitrary constant.

The 20 diagrams at the end of this paper summarize the behavior of the spectrum. An "x" denotes a multiple eigenvalue (necessarily of algebraic multiplicity two). The arrows indicate how the eigenvalues changed with decreasing q (q is always a constant function). The diagrams are informative; they give us a clearer view of the complexity of indefinite problems. In the region between λ^- and λ^+, real eigenvalues tend to coalesce and then "go off" into the complex plane, only to return to the real axis and once again go in opposite directions. The amazing part of the process is that the associated eigenfunctions *gain a zero* when the eigenvalue again becomes real. This phenomenon explains the irregularities one commonly sees in the oscillation numbers of eigenvalues of indefinite problems. The fact that the eigenvalues "come back" to the real axis also is the reason for the difficulty in predicting the number of non-real eigenvalues and the oscillation numbers of the real eigenvalues.

In summary, two fundamental issues that remain before us in the study of indefinite Sturm-Liouville problems are the determination of λ^+ and λ^- ("Richardson numbers") and the determination of the precise number of non-real eigenvalues and the oscillation numbers of the real eigenvalues. The example we have considered reveals many of the complexities involved in these issues.

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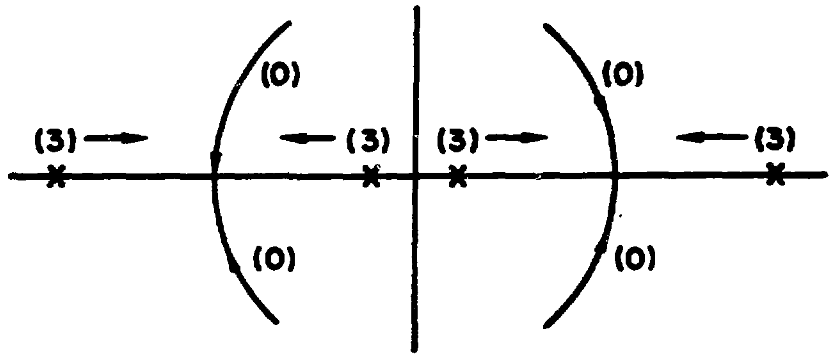
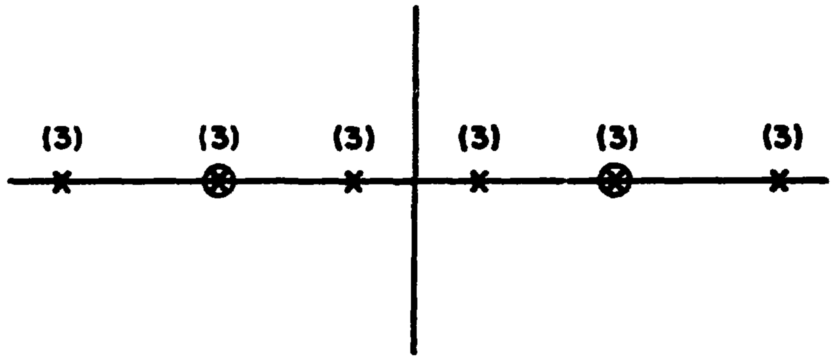
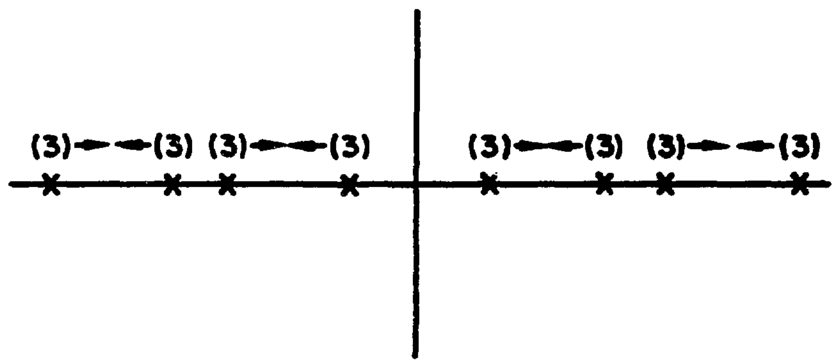
$\sim -41.9093 < q < -4\pi^2$



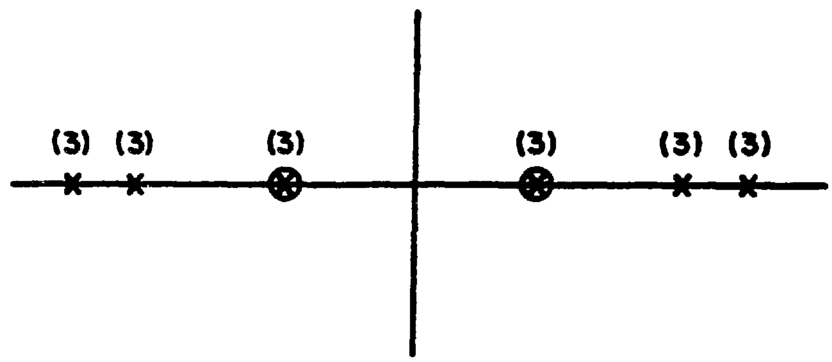
$q = \sim -41.9093$



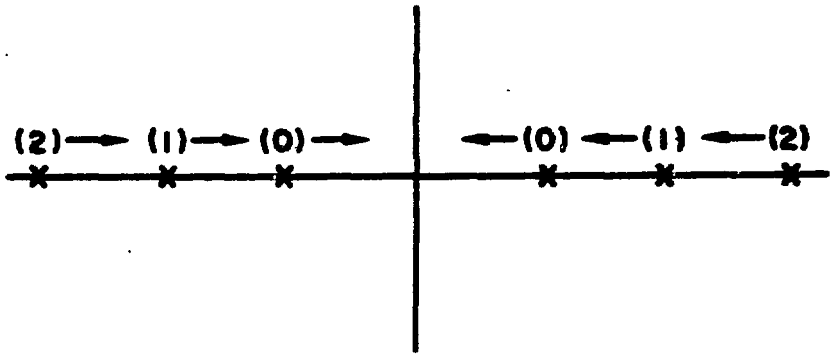
$\frac{-17\pi^2}{4} < q < \sim -41.9093$



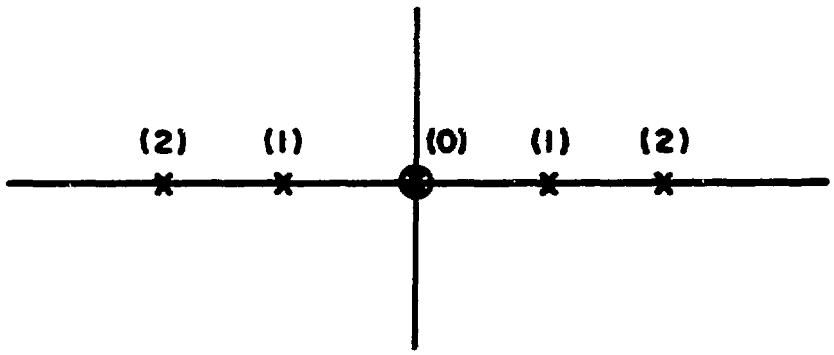
$q = \frac{-17\pi^2}{4}$



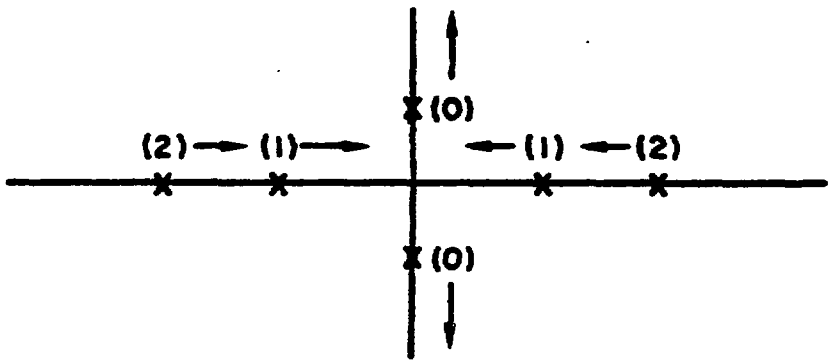
$q > \frac{-\pi^2}{4}$



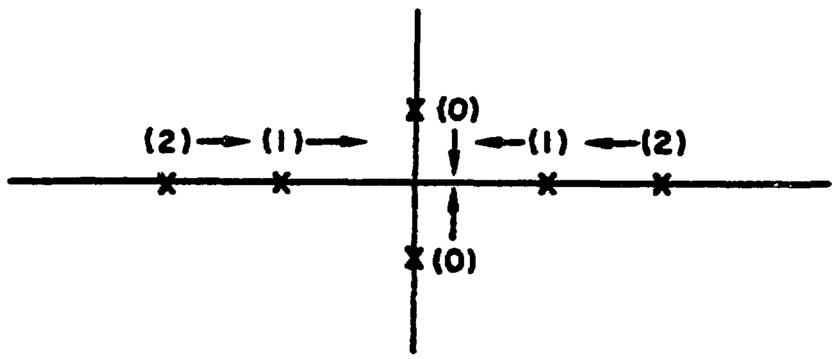
$q = \frac{-\pi^2}{4}$



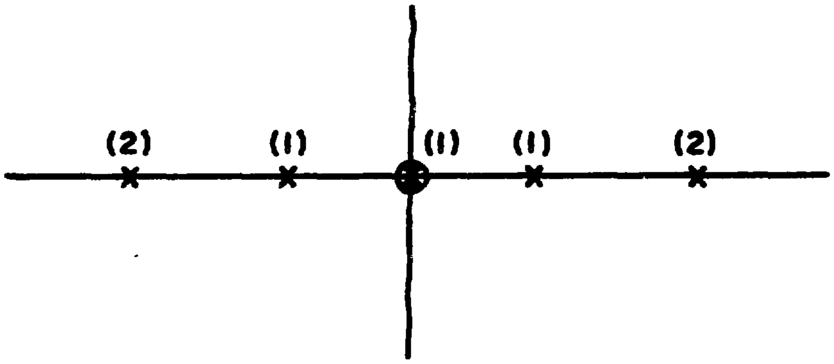
$\sim -6.402 < q < \frac{-\pi^2}{4}$



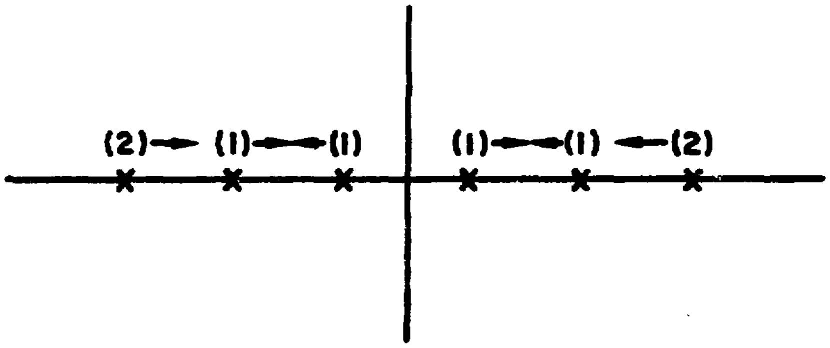
$-\pi^2 < q < \sim -6.402$



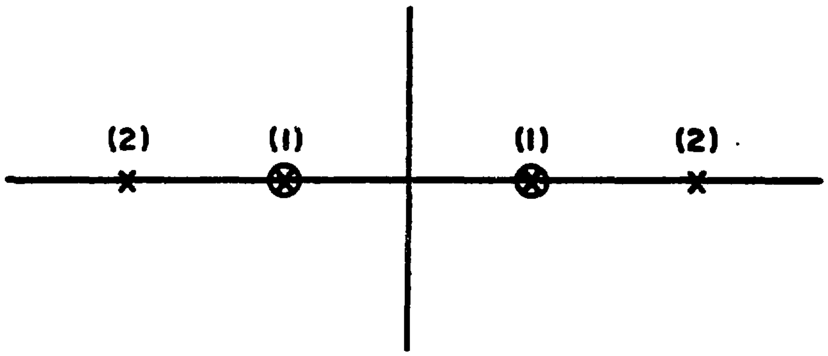
$q = -\pi^2$



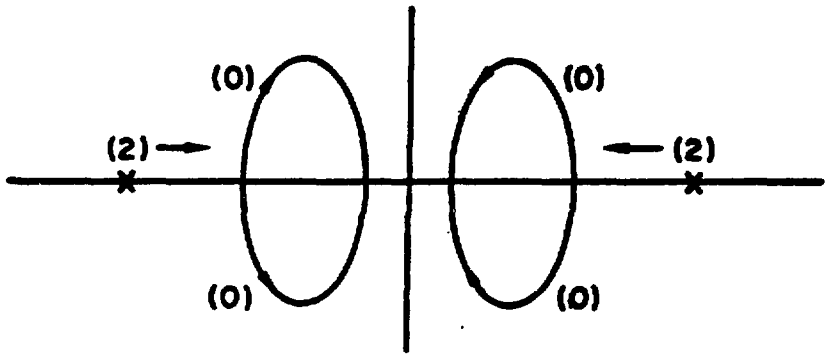
$\frac{-5\pi^2}{4} < q < -\pi^2$



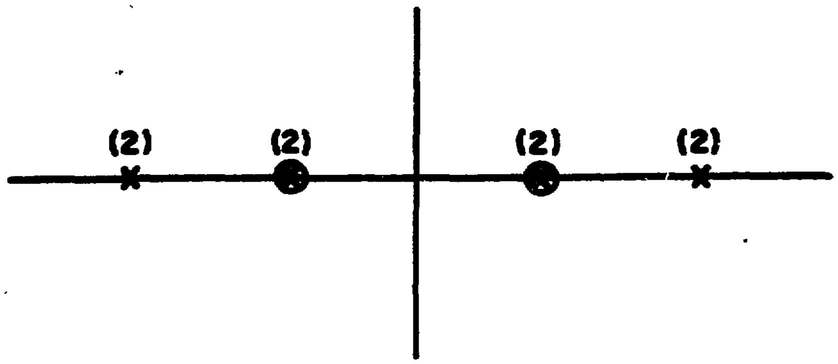
$q = \frac{-5\pi^2}{4}$



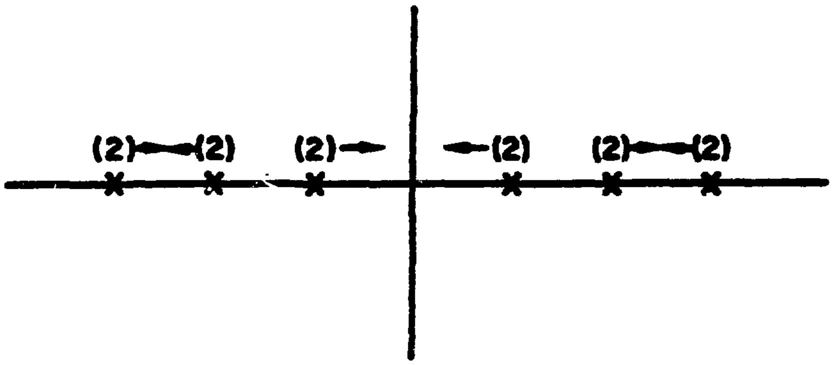
$-21.99604 < q < \frac{-5\pi^2}{4}$



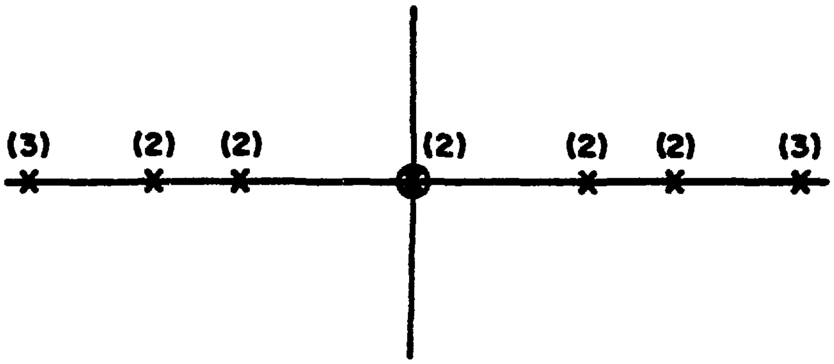
$q = \sim -21.99604$



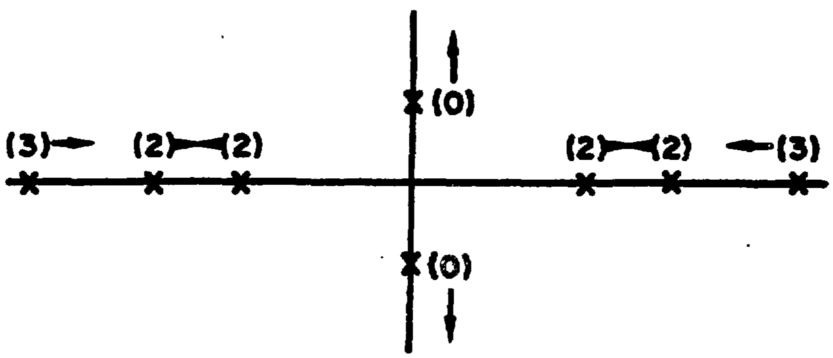
$\frac{-9\pi^2}{4} < q < \sim -21.99604$



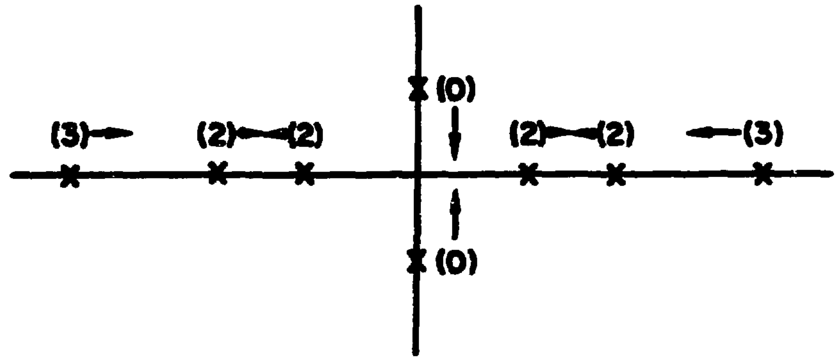
$q = \frac{-9\pi^2}{4}$



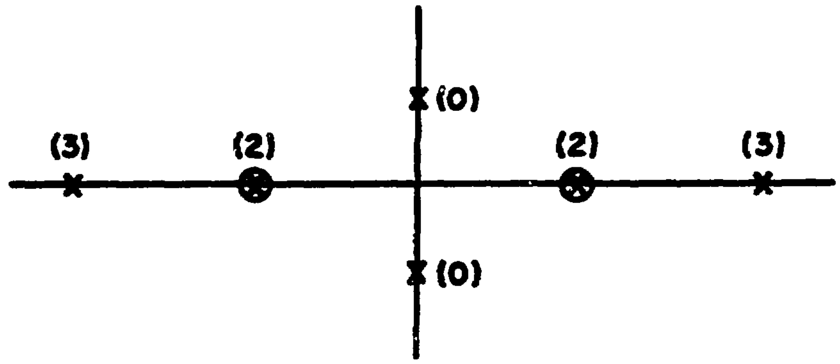
$\sim -31.00 < q < \frac{-9\pi^2}{4}$



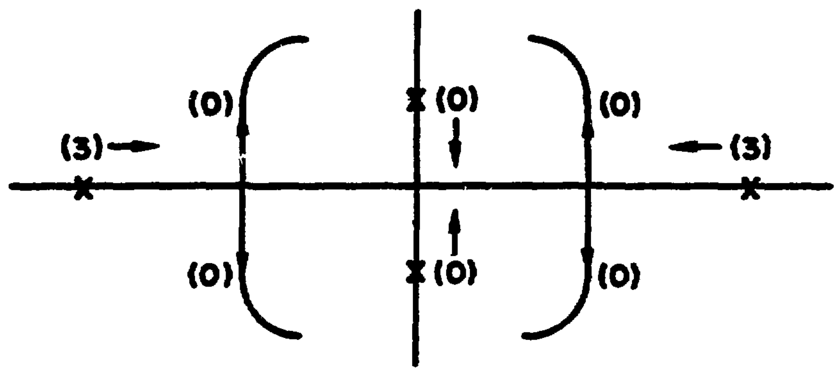
~ −32.076 < q < ~ −31.00



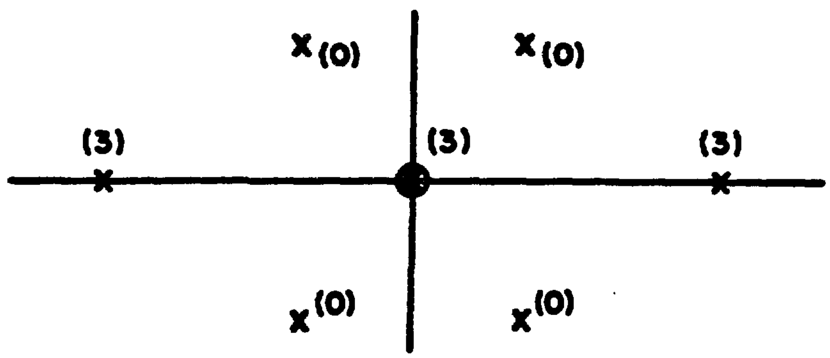
q = ~ −32.076



$-4\pi^2 < q < \sim -32.076$



$q = -4\pi^2$



Appendix

Some eigenvalues for the problem

$$-y'' - qy = \lambda \operatorname{sgn}(x) y \qquad y(-1) = y(1) = 0$$

for varying $q \geq 0$.

q	Re(λ)	Im(λ)
0.0	5.5934	0
0.5	4.8830	0
0.75	4.5085	0
1.0	4.1169	0
1.5	3.2634	0
2.0	2.2057	0
2.25	1.4834	0
2.30	1.298	0
2.35	1.084	0
2.40	0.81884	0
2.45	0.41487	0
2.46	0.27047	0
2.465	0.15395	0
2.467	0.06292	0
2.467401101	0	0
2.467012	0	0.00082
2.47	0	0.35237
2.5	0	0.56613
3.0	0	2.21948
4.0	0	3.51605
5.0	0	4.1666
5.55165	0	4.36280
6.0	0	4.45012
7.0	0	4.41757
8.0	0	4.02714
9.0	0	3.07167
9.5	0	2.11597
9.8	0	0.84955
9.869	0	0.08912
12.0	7.3609	-0.4996×10^{-7}
12.222	8.2926	-0.1049×10^{-6}
12.3	9.0559	-0.1606×10^{-6}
12.32	9.31891	0.1946×10^{-8}
22.33	9.5165	-0.3019×10^{-9}
12.335	9.6808	0.1404×10^{-8}
12.336	9.7359	0.1015×10^{-6}
12.337	9.8597	-0.2×10^{-5}
12.338	9.8696	0.1329
12.34	9.8696	0.2309
12.35	9.8700	0.4801

q	Re(λ)	Im(λ)
13.00	9.87365	3.31141
14.00	9.82986	4.9684
15.00	9.7392	5.9387
16.00	9.6057	6.5421
17.00	9.4264	6.8717
18.00	9.18999	6.9538
19.00	8.87484	6.7718
20.00	8.4337	6.2476
21.00	7.7428	5.12126
21.50	7.18136	4.012
21.00	6.6680	2.7626
21.90	6.43385	2.015
21.95	6.29596	1.42968
21.96	6.26620	1.27163
21.97	6.23559	1.08680
21.98	6.204100	0.857758
21.99	6.171640	0.5294003
21.995	6.155032	0.220283044
21.996	6.151678	0.04295551
21.99603	6.151577	0.0210490
21.99604	6.141657	0.0115332
21.9965	6.003244	≈0
21.9975	5.885123	≈0
22.0	5.7069	≈0
22.20	0.7693241	≈0
32.07	29.26624	≈0
32.076	29.54507	≈0
32.07	29.26624	0
32.0765	29.60877	0.0736369
32.077	29.60895	0.1221131
32.078	29.608539	0.1840865
32.079	29.6083931	0.2299154
32.08	29.608242	0.2680138
32.09	29.60673	0.511246
32.10	29.60522	0.6712813
32.20	29.58995	1.525405
32.30	29.574422	2.042939
32.40	29.55863	2.447615
32.50	29.542570	2.7889968

g	Re(λ)	Im(λ)
32.60	29.5282423	3.0881428
32.70	29.509855	3.356360
32.80	29.4927716	3.600565
32.90	29.475622	3.8253519
32.95	29.466943	3.93151494
33.00	29.4581927	4.0339649
34.00	29.26765	5.5698816
35.00	29.044111	6.5368893
36.00	28.7805101	7.16100644
37.00	28.465735	7.5158925
38.00	28.0814902	7.6201115
39.00	27.5949571	7.4512042
40.00	26.9376319	6.9214995
41.00	25.915600	5.7322556
41.50	25.0331001	4.4803808
41.80	24.050751	2.789358
41.90	23.337294	0.95085883
41.905	23.3205006	0.65836529
41.9075	23.292829	0.43451057
41.908	23.2871669	0.3728994
41.909	23.275707	0.19674029
41.9091	23.274550	0.16913850
41.902	23.2733923	0.1360040
41.9093	23.2722524	0.09151191
41.90935	23.2716517	0.0575207
41.9094	23.2291164	0
41.9095	23.160789	0

STURM-LIOUVILLE PROBLEMS WITH INDEFINITE WEIGHT FUNCTIONS IN BANACH SPACES

*Harold E. Benzinger**

Abstract

Methods of Paley and Wiener and of Levinson for investigating the convergence of non-harmonic Fourier series are applied to the problem of the convergence of half-range expansions for Sturm-Liouville problems with indefinite weight functions in L^p spaces for $p \neq 2$.

1. Introduction

We consider a simple example of a regular Sturm-Liouville problem on $[-\pi,\pi]$, with a weight function that changes sign. The completeness of eigenfunctions and the convergence of eigenfunction expansions in the Banach spaces $L^p(-\pi,\pi)$ and in subspaces are considered, $1 \leq p < \infty$. For the case that $p = 2$, this and other problems have been considered by a number of authors [Beals, 1979 and 1981; Kaper, Kwong, Lekkerkerker, and Zettl, 1983; and Kaper, Lekkerkerker, and Hejtmanek, 1982]. Their results have been obtained using the theory of selfadjoint operators. Lacking this tool when $p \neq 2$, we use here methods from the theory of functions of a complex variable, particularly results from Paley and Wiener [1934] and Levinson [1940] on non-harmonic Fourier series.

Let $\alpha > 0$ be a given constant, and let

$$w(x) = \begin{cases} -\alpha^2, & -\pi \leq x < 0 \\ 1, & 0 < x \leq \pi \end{cases} . \tag{1.1}$$

We consider the boundary value problem consisting of the differential equation

$$-u'' = \lambda w(x) u, \quad -\pi \leq x \leq \pi, \tag{1.2}$$

and the boundary conditions

$$u(-\pi) = 0, \tag{1.3}$$

$$u(\pi) = 0. \tag{1.4}$$

A *solution* of the boundary value problem is a function u in $C^2[-\pi,\pi]$, satisfying the boundary conditions, such that u' is absolutely continuous and u'' is in $L^p(-\pi,\pi)$ for a fixed p, $1 \leq p < \infty$.

2. Eigenvalues, Eigenfunctions, and Green's Function

In (1.2) we make the substitutions

$$\lambda = \rho^2, \quad \sigma = \alpha\rho. \tag{2.1}$$

Let A, B be constants, and let

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$$\varphi(x,\rho) = A\sinh\sigma(\pi+x), \quad \psi(x,\rho) = B\sin\rho(\pi-x). \tag{2.2}$$

Then φ satisfies the differential equation on $[-\pi,0]$ and satisfies the boundary condition (1.3) at $-\pi$, while ψ has the analogous properties for $[0,\pi]$. Thus the function

$$u(x,\rho) = \begin{cases}\varphi(x,\rho), & -\pi\le x<0 \\ \psi(x,\rho), & 0<x\le\pi\end{cases}. \tag{2.3}$$

is an eigenfunction if A,B can be selected to satisfy the interface conditions

$$\varphi(0,\rho) = \psi(0,\rho), \tag{2.4}$$

$$\varphi'(0,\rho) = \psi'(0,\rho). \tag{2.5}$$

Let

$$D(\rho) = \alpha\sin\rho\pi\cosh\sigma\pi + \cos\rho\pi\sinh\sigma\pi. \tag{2.6}$$

A direct computation shows that nontrivial A,B satisfy (2.4), (2.5) if and only if

$$D(\rho) = 0, \tag{2.7}$$

so that $\lambda = \rho^2$ is an eigenvalue if and only if (2.7) holds, and the eigenfunctions are scalar multiples of

$$u(x,\rho) = \begin{cases}\sin\rho\pi\sinh\sigma(\pi+x), & -\pi\le x<0 \\ \sinh\sigma\pi\sin\rho(\pi-x), & 0<x\le\pi\end{cases}. \tag{2.8}$$

To determine the location of the eigenvalues, define γ by

$$\cos\gamma\pi = \alpha/(1+\alpha^2)^{1/2}, \quad \sin\gamma\pi = 1/(1+\alpha^2)^{1/2}. \tag{2.9}$$

Then $0 < \gamma < 1/2$ and

$$D(\rho) = (1+\alpha^2)^{1/2}[\cos\gamma\pi\sin\rho\pi\cosh\sigma\pi+\sin\gamma\pi\cos\rho\pi\sinh\sigma\pi]$$

$$= (1/2)(1+\alpha^2)^{1/2} e^{\sigma\pi}[\sin(\rho+\gamma)\pi+e^{-2\sigma\pi}\sin(\rho-\gamma)\pi]. \tag{2.10}$$

Using identities such as $\cosh ix = \cos x$, we have

$$D(i\rho) = (i/2)(1+\alpha^2)^{1/2} e^{\rho\pi}[\cos(\sigma-\gamma)\pi-e^{-2\rho\pi}\cos(\sigma+\gamma)\pi]. \tag{2.11}$$

THEOREM 2.12. *The eigenvalues $\lambda_k = \rho_k^2$ of the problem (1.1)-(1.4) are real and satisfy the asymptotic relations*

$$\rho_k = k-\gamma+O(e^{-2k\pi\alpha}), \quad k\to\infty, \tag{2.13}$$

$$\alpha\rho_{-k} = -i[k+\gamma+\tfrac{1}{2}+O(e^{-k\pi\alpha})], \quad k\to\infty. \tag{2.14}$$

PROOF. Consider (2.13). The sequence $\{k-\gamma\}$ arises as the zeros of $\sin(\rho+\gamma)\pi$. Applying Rouché's theorem, we see that the error term in (2.13) arises from the term $e^{-2\sigma\pi}\sin(\rho-\gamma)\pi$ in (2.10). The estimate (2.14) follows from similar considerations applied to (2.11).

To see that the λ_k are all real, we note that the spectrum is independent of the particular L^p space being considered, and in the case that $p = 2$, $\{\lambda_k^{-1}\}$ is the sequence of eigenvalues of a selfadjoint operator on a Hilbert space [Kaper, Kwong, Lekkerkerker, and Zettl, 1983: Theorem 2.1]. This completes the proof.

For f in $L^P(-\pi,\pi)$, we consider the nonhomogeneous differential equation

$$-u'' = \lambda w(x)u + f, \tag{2.15}$$

along with the boundary conditions (1.3), (1.4). As is the case for a weight function that does not change sign, the solution is given by

$$u(x,\lambda) = \int_{-\pi}^{\pi} G(x,t,\lambda) f(t)\,dt, \tag{2.16}$$

provided λ is not an eigenvalue. Green's function $G(x,t,\lambda)$ is given by

$$G(x,t,\lambda) = \frac{1}{W(\lambda)} \begin{cases} u_2(x,\lambda)u_1(t,\lambda), & -\pi<t<x<\pi \\ u_1(x,\lambda)u_2(t,\lambda), & -\pi<x<t<\pi \end{cases}. \tag{2.17}$$

where $u_1(x,\lambda)$ is a solution of (1.2) on $[-\pi,\pi]$ satisfying the boundary condition (1.3), $u_2(x,\lambda)$ is a solution on $[-\pi,\pi]$ satisfying (1.4), and $W(\lambda)$ is their Wronskian:

$$W(\lambda) = u_1(\cdot,\lambda)u_2'(\cdot,\lambda) - u_1'(\cdot,\lambda)u_2(\cdot,\lambda). \tag{2.18}$$

One choice for u_1, u_2 is ($\lambda = \rho^2$),

$$u_1(x,\rho) = \begin{cases} \sinh\sigma(\pi+x) & -\pi<x<0 \\ \sinh\sigma\pi\cos\rho x + \alpha\cosh\sigma\pi\sin\rho x & 0<x<\pi \end{cases}. \tag{2.19}$$

$$u_2(x,\rho) = \begin{cases} \sin\rho\pi\cosh\sigma x - (1/\alpha)\cos\rho\pi\sinh\sigma x & -\pi<x<0 \\ \sin\rho(\pi-x) & 0<x<\pi \end{cases} \tag{2.20}$$

Then

$$W(\lambda) = -\rho D(\rho). \tag{2.21}$$

3. Completeness of the Eigenfunctions

DEFINITION 3.1. A sequence $\{\varphi_k\}$ is *complete* in a Banach space X if the set of finite linear combinations of the φ_k is dense in X.

DEFINITION 3.2. The sequence $\{\varphi_k\}$ is a *basis* for X if for each f in X, there exists a unique set of scalars $\{a_k(f)\}$ such that

$$\lim_{N\to\infty} \sum_{|k|\le N} a_k(f)\varphi_a = f.$$

REMARK 3.3. In the literature on problems with indefinite weight functions, the terms "full-range completeness" and "half-range completeness" are customarily used to refer to basis properties of eigenfunction systems [Kaper, Kwong, Lekkerkerker, and Zettl, 1983: pp. 21, 25]. In this paper we use "completeness" in the sense of Definition 3.1.

Let $\{\varphi_k\}$, $k = \pm1, \pm2, \cdots$ denote the eigenfunctions of (1.2)-(1.4), normalized so that $(w\varphi_k, \varphi_k) = 1$.

THEOREM 3.4. *The sequence $\{\varphi_k\}$ is complete in $L^p(-\pi,\pi)$ for $1 \le p < \infty$.*

The proof (given below) is based on Benzinger [1973: Theorem 1.1]. For any f in $L^p(-\pi,\pi)$, the function $u(x,\lambda)$ defined by

$$u(x,\lambda) = \int_{-\pi}^{\pi} G(x,t,\lambda) w(t) f(t)\,dt \tag{3.5}$$

is a meromorphic function of λ, with poles at the eigenvalues λ_k and residue $(wf, \varphi_k)\varphi_k(x)$ at λ_k [Neumark, 1963: p. 33]. If $\{\varphi_k\}$ is not complete in the dual space $L^q(-\pi,\pi)$ $(pq = p+q)$, then there exists $f \neq 0$ in L^p such that $(wf, \varphi_k) = 0$ for all k, and therefore $u(x,\lambda)$ is an entire function of λ for fixed x. Using (2.17), we show below that $u(x,\lambda)$ is then bounded in λ for each x, so u is constant in λ for each x. This is inconsistent with the differential equation unless $f = 0$, yielding a contradictin. Applying this argument to the adjoint problem T^* (which is formally the same as T), we obtain the completeness of $\{\varphi_k\}$ in L^p, $1 \le p < \infty$.

LEMMA 3.1. *Let $\delta > 0$ be given, and let $\lambda = \rho^2$. For $\text{Im}\rho \ge 0$ and $|\rho-\rho_k| \ge \delta$, we have*

$$G(x,t,\lambda) = O(1/\rho), \quad |\rho| \to \infty.$$

PROOF. From (2.17), (2.19), and (2.20), we see that $\rho D(\rho) G(x,t,\lambda)$ is a linear combination of exponentials $e^{\rho z}$, where $z = z(x,t)$ is taken from the rectangle

$$|\text{Re}z| \le a\pi, \quad |\text{Im}z| \le \pi.$$

Also, $D(\rho)$ is a linear combination of these exponentials, where z is taken only from the vertices of the rectangle. Thus for each ρ, x, t, the dominant exponential in ρDG does not exceed the dominant exponential in $D(\rho)$. Factoring the dominant exponential out of $D(\rho)$, and using $|\rho-\rho_k| \ge \delta$, the result follows.

PROOF OF THEOREM 3.4. Let T denote the linear operator defined by

$$Tu = (1/w(x))(-u'')$$

on the domain consisting of functions u in $C^1[-\pi,\pi]$ such that u' is absolutely continuous, u'' is in L^p, and the boundary conditions are satisfied. Then for f in L^p,

$$R(\lambda,T)f = -\int_{-\pi}^{\pi} G(x,f,\lambda) w(t) f(t) dt,$$

and from (3.6),

$$\|R(\lambda,T)\| = O(1/\rho)$$

as $|\lambda| \to \infty$ on an increasing family of circles that are uniformly bounded away from the eigenvalues of T. Thus the conditions of Benzinger [1973: Theorem 1.1] are satisfied.

4. Half-range Expansions

From (2.8) we see that for $0 \le x \le \pi$, the eigenfunctions are scalar multiples of $\sin\rho_k(\pi-x)$. We consider here the completeness and expansion properties of these functions in $L^p(0,\pi)$, using only those ρ_k that are real zeros of $D(\rho)$. Thus from (2.13),

$$\rho_k = k - \gamma + a_k, \quad a_k = O(e^{-2k\pi a}), \; k \to \infty. \tag{4.1}$$

There is no loss of generality in replacing $\pi - x$ by x, so we consider

$$f_k(x) = \sin\rho_k x, \quad k \ge 1. \tag{4.2}$$

Let

$$\mu_k = \begin{cases} \rho_k & k \ge 1 \\ 0 & k = 0 \\ -\rho_{-k} & k \le -1 \end{cases}. \tag{4.3}$$

The expansion properties of $\{f_k\}$, $k \ge 1$ on $[0,\pi]$ are related to those of

$$\varphi_k(x) = e^{i\mu_k x}, \quad k=0,\pm1,\pm2,\cdots \tag{4.4}$$

on $[-\pi,\pi]$. Such problems of non-harmonic Fourier series are considered in Paley and Wiener [1934], Levinson [1940], and Young [1980].

Denote by $\psi_k(x)$, $k=0,\pm1,\cdots$ those functions (if they exist) that are biorthogonal to $\{\varphi_k\}$:

$$(\varphi_k,\psi_j) = \int_{-\pi}^{\pi}\varphi_k(x)\overline{\psi}_j(x)dx = \delta_{kj}. \tag{4.5}$$

Let X denote any of the Banach spaces $L^p(-\pi,\pi)$, $1 \le p < \infty$. Note that as a consequence of $\mu_{-k} = -\mu_k$, we have

$$\varphi_{-k}(x) = \varphi_k(-x). \tag{4.6}$$

LEMMA 4.7. *If $\{\psi_k\}$ is complete in X and if the biorthogonal sequence $\{\psi_k\}$ exists, then*

$$\psi_{-k}(x) = \psi_k(-x).$$

PROOF. Let $w_j(x) = \psi_j(-x)$. Using (4.6), we obtain, for any k,j, that

$$(\varphi_k, w_j-\psi_{-j}) = 0,$$

so from the completeness of $\{\varphi_k\}$, $w_j-\psi_{-j} = 0$.

Let

$$\Phi_k(x) = \varphi_k(x)-\varphi_{-k}(x), \quad 0 \le x \le \pi, \ k \ge 1. \tag{4.8}$$

$$\Psi_k(x) = \psi_k(x)-\psi_{-k}(x) \quad 0 \le x \le \pi, \ k \ge 1. \tag{4.9}$$

Using (4.6), (4.7), we have

$$\int_0^{\pi}\Phi_k(x)\overline{\Psi}_j(x)dx = \delta_{kj}-\delta_{k,-j} = \delta_{kj}, \quad k,j \ge 1.$$

Thus

$$f_k(x) = (1/2i)\Phi_k(x), \tag{4.10}$$

and if $\{\psi_k\}$ exists, then functions biorthogonal to f_k also exist:

$$g_k(x) = 2i\Psi_k(x). \tag{4.11}$$

LEMMA 4.12. *Assume $\{\varphi_k\}$ is complete in X and $\{\psi_k\}$ exists. If f is a function defined on $[0,\pi]$, and if F is its odd extension to $[-\pi,\pi]$, then*

$$\sum_{k=1}^{N}(f,g_k)f_k = \sum_{k=-N}^{N}(F,\psi_k)\varphi_k.$$

PROOF. From (4.6), (4.7), and the oddness of F, we have

$$(F,\psi_k)\varphi_k + (F,\psi_{-k})\varphi_{-k} = (f,g_k)f_k, \quad k \ge 1.$$

THEOREM 4.13. *The set $\{\varphi_k\}$ is complete in $L^p(-\pi,\pi)$, $1 \le p < \infty$.*

PROOF. Let $\nu_k = k-\gamma$ for $k \ge 1$, and $\nu_{-k} = -\nu_k$ for $k \le -1$. Let $n(t)$ denote the number of σ_k satisfying $|\nu_k| \le t$. Then

$$n(t) = 2[t+\gamma]+1,$$

where $[x]$ is the greatest integer in x. Let

$$N(r) = \int_1^r [n(t)/t]dt.$$

Then

$$N(r) = 2r+2\gamma\log r+O(1), \quad r\to\infty,$$

so

$$\lim_{r\to\infty}\sup[N(r)-2r+(1/p)\log r] > -\infty.$$

For $1 < p < \infty$, this condition is sufficient for $e^{i\nu_k x}$ to be complete in $L^p(-\pi,\pi)$ [Levinson, 1940: p. 6; Young, 1980: p. 118]. Clearly, completeness in $L^1(-\pi,\pi)$ follows. The completeness of $\{e^{i\mu_k x}\}$ is transferred from the completeness of $\{e^{i\nu_k x}\}$ since $\sum_{-\infty}^{\infty}|\mu_k-\nu_k| < \infty$ [Young, 1980: p. 132].

COROLLARY 4.14. The system $\{f_k\}$, $k \geq 1$ is complete in $L^p(0,\pi)$, $1 \leq p < \infty$.

PROOF. Let f be in $L^p(0,\pi)$, and let F be its odd extension to $[-\pi,\pi]$. Given $\varepsilon > 0$, there exists a finite sum

$$F_N(x) = \sum_{-N}^{N} c_k\varphi_k$$

such that

$$F(x) = F_N(x)+r(x),$$

where $\|r\|<\varepsilon$. Using (4.6),

$$F_N(-x) = \sum_{-N}^{N} c_k\varphi_{-k}(x).$$

Since F is odd,

$$-F(x) = F_n(-x)+r(-x),$$

so

$$F(x) = (1/2)[F_N(x)-F_N(-x)] + (1/2)[r(x)-r(-x)].$$

Clearly $\|(1/2)[r(x)-r(-x)]\| < \varepsilon$, and the remaining term is a linear combination of f_k's. The result follows upon restricting to $(0,\pi)$.

We next consider basis properties of $\{f_k\}$. Let $1 < p < \infty$, $p^{-1}+q^{-1} = 1$,

$$r = \min(p,q), \tag{4.15}$$

and let

$$\alpha_k(x) = e^{i\nu_k x}. \tag{4.16}$$

THEOREM 4.17. *Let $1 < p < \infty$. The system $\{\alpha_k\}$ is a basis for $L^p(-\pi,\pi)$, equivalent to the ordinary Fourier system $\{e^{ikx}\}$, provided*

$$\gamma\pi < 1/(4+\tan(\pi/2r)). \tag{4.18}$$

PROOF. Let $e_k(x) = e^{ikx}$, and let

$$c_k(f) = (1/2\pi)\int_{-\pi}^{\pi} f(t)\bar{e}_k(t)dt.$$

Consider the operator A defined on $L^p(-\pi,\pi)$ by

$$Af = \sum_{-\infty}^{\infty} c_k(f)\alpha_k(x) \tag{4.19}$$

$$= c_0(f)+e^{-i\gamma x}\sum_{1}^{\infty} c_k(f)e_k(x)+e^{i\gamma x}\sum_{-\infty}^{-1} c_k(f)e_k(x).$$

Let $C:L^p \to L^p$ denote the conjugate function mapping

$$Cf = -i\sum_{-\infty}^{\infty}(\mathrm{sgn}k)a_k(f)e_k(x). \tag{4.20}$$

We have $\|C\|_p = \tan(\pi/2r)$ [Pichorides, 1972]. Let $H:L^p \to L^p$ be defined by

$$Hf = \sum_{0}^{\infty} c_k(f)e_k = (1/2)[c_0(f)+f+iCf]. \tag{4.21}$$

Then

$$(A-I)f = (e^{-i\gamma x}-1)(Hf-c_0(f))+(e^{i\gamma x}-1)(f-Hf),$$

so

$$\|(A-I)f\|_p \le 2\gamma\pi[\|H\|_p+1].$$

If $\|A-I\|_p < 1$, then A is invertible and $Ae_k = \alpha_k$. Now

$$\|H\|_p \le (1/2)[2+\|C\|_p] = 1+(1/2)\tan(\pi/2r).$$

Thus A is invertible if (4.18) holds.

COROLLARY 4.22. *If (4.18) is satisfied, then there exists a sequence $\{\beta_k(x)\}$ in $L^q(-\pi,\pi)$ such that $(\alpha_k,\beta_j) = \delta_{kj}$.*

PROOF. The sequence is

$$\beta_k = A^{*-1}e_k.$$

We note that since $\tan\gamma\pi = 1/\alpha$, condition (4.18) requires that α be selected sufficiently large. A further transplantation of this basis property to the sequence $\{\varphi_k\}$ can be achieved again at the cost of selecting α sufficiently large. From (2.10) and an application of Rouché's theorem, we see that for α sufficiently large, $\sup|\rho_k-\nu_k|$ can be made arbitrarily small. Let $a_k = \rho_k-\nu_k$. Then $a_k = O(e^{-2k\pi\alpha})$, so $\sum_{-\infty}^{\infty}|a_k|$ converges, and this sum can be made arbitrarily small.

THEOREM 4.23. *For α sufficiently large, $\{\varphi_k\}$ is a basis for $L^p(-\pi,\pi)$ equivalent to the Fourier system $\{e_k\}$.*

PROOF. Let

$$Bf = \sum_{-\infty}^{\infty}(f,\beta_k)\varphi_k = \sum_{-\infty}^{\infty}(f,\beta_k)\alpha_k(x)e^{ia_kx}.$$

Since

$$If = \sum_{-\infty}^{\infty}(f,\beta_k)\alpha_k,$$

we have

$$(I-B)f = \sum_{-\infty}^{\infty}(f,\beta_k)\alpha_k(x)(1-e^{ia_k x}),$$

and then

$$\|(I-B)f\|_p \le K_p(\sum_{-\infty}^{\infty}|a_k|)\|f\|_p.$$

Thus, for α sufficiently large, $\|I-B\|_p < 1$, so B is invertible and

$$\varphi_k = BAe_k, \quad \psi_k = (BA)^{*-1}e_k.$$

COROLLARY 4.24. *For α sufficiently large, $\{f_k\}$ is a basis for $L^p(0,\pi)$, $1 < p < \infty$.*

For $-\pi \le x \le 0$ and negative eigenvalues, the eigenfunctions are scalar multiples of

$$\sinh i\sigma_k(\pi+x),$$

where, using (2.14),

$$\sigma_k = k+(1/2)+\gamma+a_k, \quad a_k = O(e^{-2k\pi}), \; k\to\infty. \tag{4.25}$$

Thus we consider

$$g_k(x) = \sin\sigma_k x, \quad 0 \le x \le \pi.$$

The completeness of $\{g_k\}$ in $L^p(0,\pi)$, $1 \le p < \infty$ follows as before, since this is independent of the size of γ. The analysis of basis properties presented here does depend on γ. In this case, to make σ_k sufficiently close to an integer, it is required that γ be close to $1/2$, i.e., α should be sufficiently small. Presumably a more refined technique would remove this dichotomy.

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INTERLACING PROPERTY OF EIGENVALUES OF STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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Abstract

This note contains the proof of an interlacing property of the eigenvalues of boundary value problems described by the Sturm-Liouville differential equation $-y'' + q(t)y = \lambda y$ on $[o,\infty)$.

Recently, Van Duin, Boersma, and Sluÿter [1984] studied the problem of one-dimensional wave propagation in a stratified inhomogeneous medium with a refraction index of the symmetric Epstein type: $\varepsilon_r(z) = 1 + A\ \mathrm{sech}^2(z/2l)$. Possible TM-modes for this problem are given by the eigenfunctions of a Sturm-Liouville problem on the half-line:

$$\tau y \equiv -y'' + q(t)y = \lambda y \quad \text{on } [o,\infty),$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0. \tag{1}$$

Here, $\alpha \in [0,\pi/2]$ is fixed and $q:[o,\infty)\to\mathbf{R}$ is smooth, strictly positive, and rapidly decaying at infinity. The differential expression τ and the boundary condition define a selfadjoint differential operator in $L^2(0,\infty)$, whose spectrum consists of the semi-axis $[0,\infty)$ and a finite (possibly empty) set of eigenvalues $\{\lambda_k(\alpha)\colon k = 0,1,\ldots,n\}$, with

$$-\infty < \lambda_0(\alpha) < \lambda_1(\alpha) < \cdots < \lambda_n(\alpha) < 0. \tag{2}$$

In this note we shall prove an interlacing property of the eigenvalues $\lambda_k(\alpha)$. It is not unlikely that this property can be found somewhere in the vast literature on Sturm-Liouville boundary value problems, but we have not been able to locate it.

THEOREM. *The eigenvalues have the following interlacing property:*

$$\lambda_k(\pi/2) < \lambda_k(0) < \lambda_{k+1}(\pi/2), \quad k = 0,1,2,\ldots. \tag{3}$$

We shall need the following lemma, which is a stronger version of the Sturm comparison theorem.

LEMMA. *Let P and Q be given on $[a,b)$, and let $P(t) \geq Q(t)$ for all $t \in [a,b)$. Suppose u and v satisfy the equations*

$$u'' + P(t)u = 0, \quad v'' + Q(t)v = 0,$$

respectively, on $[a,b)$. If u and v do not vanish on (a,b), and if

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$$\frac{u'(a)}{u(a)} \le \frac{v'(a)}{v(a)},$$

then

$$\frac{u'(t)}{u(t)} \le \frac{v'(t)}{v(t)} \text{ for all } t \in (a,b).$$

REMARK. If $v(a) = 0$, it is understood that $v'(a)/v(a) = \lim_{t \downarrow a} v'(t)/v(t) = +\infty$. Then the assumption $u'(a)/u(a) \le v'(a)/v(a)$ is trivially satisfied.

PROOF. Without loss of generality, it is assumed that u and v are positive on (a,b) and satisfy the condition

$$u'(a)v(a) - u(a)v'(a) \le 0.$$

From the differential equations for u and v it is readily found that

$$u''v - uv'' = -(P-Q)uv.$$

Then by integration from a to t, one has

$$u'(t)v(t) - u(t)v'(t) = u'(a)v(a) - u(a)v'(a) - \int_a^t (P-Q)uv\,dt \le 0$$

for all $t \in (a,b)$. Divide by $u(t)v(t)$. Then

$$\frac{u'(t)}{u(t)} \le \frac{v'(t)}{v(t)} \text{ for all } t \in (a,b). \qquad ///$$

PROOF OF THEOREM. The proof consists of three steps.

Step 1. $\lambda_k(0) \ne \lambda_l(\pi/2)$ for any k, l. That is, the eigenvalues in the cases of Dirichlet and Neumann boundary conditions are different.

Let y_{Dk} be the eigenfunction associated with the eigenvalue $\lambda_k(0)$. Then y_{Dk} has k zeros in $(0,\infty)$ and $y_{Dk}(t) \to 0$ as $t \to \infty$. Let the zeros of $y_{Dk}(t)$ be $t_1, t_2, \ldots, t_k$. Let $\tilde{y}$ be the solution of the boundary value problem

$$\tau y = \lambda_k(0)y \quad \text{on } [0,\infty),$$

$$y(0) = 1, \; y'(0) = 0. \tag{4}$$

It follows from Sturm's separation theorem that $\tilde{y}$ has one zero in $(0,t_1)$, and one zero in (t_i,t_{i+1}) for each $i = 1,2,\ldots,k-1$. Suppose $\tilde{y}$ does not vanish in (t_k,∞). Then

$$\frac{\tilde{y}'(t_k)}{\tilde{y}(t_k)} \le \frac{y_{Dk}'(t_k)}{y_{Dk}(t_k)} = +\infty.$$

Hence, according to the lemma,

$$\frac{\tilde{y}'(t)}{\tilde{y}(t)} \le \frac{y_{Dk}'(t)}{y_{Dk}(t)} \text{ for all } t > t_k.$$

Integration over $[t_k+1, t]$ yields the inequality

$$\frac{\tilde{y}(t)}{\tilde{y}(t_k+1)} \le \frac{y_{Dk}(t)}{y_{Dk}(t_k+1)} \text{ for all } t \ge t_k+1.$$

Because $y_{Dk}(t) \to 0$ as $t \to \infty$, it follows that

$$\tilde{y}(t) \to 0 \text{ as } t \to \infty.$$

Next,

$$\left| \frac{d}{dt} \arctan \frac{\tilde{y}}{y_{Dk}} \right| = \left| \frac{\tilde{y}' y_{Dk} - y'_{Dk} \tilde{y}}{\tilde{y}^2 + y_{Dk}^2} \right| = \left| \frac{\text{const.}(\neq 0)}{\tilde{y}^2 + y_{Dk}^2} \right| \to \infty \text{ as } t \to \infty.$$

Hence,

$$\left| \arctan \frac{\tilde{y}(t)}{y_{Dk}(t)} \right| \to \infty \text{ as } t \to \infty.$$

However, $\tilde{y}(t)/y_{Dk}(t)$ is bounded for all $t \geq t_k + 1$, so certainly

$$\left| \arctan \frac{\tilde{y}(t)}{y_{Dk}(t)} \right| < \frac{\pi}{2}.$$

Here we have arrived at a contradiction, so we must conclude that $\tilde{y}$ vanishes at least once in (t_k, ∞). From Sturm's separation theorem, it follows that $\tilde{y}$ cannot have more than one zero in (t_k, ∞), so $\tilde{y}$ vanishes exactly once in (t_k, ∞). Thus we have shown that $\tilde{y}$ has exactly $k+1$ zeros in $(0, \infty)$; let these zeros be denoted by $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_{k+1}$.

Finally, compare $\tilde{y}$ with the auxiliary function $u(t) = \exp[t\sqrt{-\lambda_k(0)}]$ (recall that $\lambda_k(0) < 0$), which is a solution of the equation

$$u'' + \lambda_k(0)u = 0.$$

Then

$$\frac{u'(\tilde{t}_{k+1})}{u(\tilde{t}_{k+1})} < \frac{\tilde{y}'(\tilde{t}_{k+1})}{\tilde{y}(\tilde{t}_{k+1})} = +\infty.$$

Hence, according to the lemma,

$$\frac{u'(t)}{u(t)} \leq \frac{\tilde{y}'(t)}{\tilde{y}(t)} \text{ for all } t > \tilde{t}_{k+1}.$$

Integration over $[\tilde{t}_{k+1} + 1, t]$ yields the inequality

$$\frac{\tilde{y}(t)}{\tilde{y}(\tilde{t}_{k+1}+1)} \geq \frac{u(t)}{u(\tilde{t}_{k+1}+1)} = \exp[(t - \tilde{t}_{k+1} - 1)\sqrt{-\lambda_k(0)}] \text{ for all } t \geq \tilde{t}_{k+1} + 1.$$

Consequently, $\tilde{y}$ is unbounded as $t \to \infty$.

Since eigenfunctions tend to zero as $t \to \infty$, it is clear from (4) that $\lambda_k(0)$ is not an eigenvalue under the Neumann boundary condition $y'(0) = 0$. Therefore we have the inequality $\lambda_k(0) \neq \lambda_l(\pi/2)$ for any k, l.

Step 2. $\lambda_k(0) < \lambda_{k+1}(\pi/2)$.

Let $\bar{y}$ be the solution of the boundary value problem

$$\tau y = \lambda y \text{ on } [0, \infty),$$

$$y(0) = 1, \; y'(0) = 0, \tag{5}$$

where $\lambda < \lambda_k(0)$. It follows from Sturm's oscillation theorem that $\bar{y}$ cannot have more zeros than $\tilde{y}$ on $(0, \infty)$, so $\bar{y}$ has at most $k+1$ zeros. If $\bar{y}$ has less than $k+1$ zeros, then λ is certainly different from $\lambda_{k+1}(\pi/2)$. Suppose $\bar{y}$ has $k+1$ zeros, denoted by $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{k+1}$. Then $\bar{t}_{k+1} > \tilde{t}_{k+1}$, so neither $\tilde{y}$ nor $\bar{y}$ vanish on $(\bar{t}_{k+1}, \infty)$. Because

$$\frac{\tilde{y}'(\bar{t}_{k+1})}{\tilde{y}(\bar{t}_{k+1})} < \frac{\bar{y}'(\bar{t}_{k+1})}{\bar{y}(\bar{t}_{k+1})} = +\infty,$$

it follows from the lemma that

$$\frac{\tilde{y}'(t)}{\tilde{y}(t)} \le \frac{\bar{y}'(t)}{\bar{y}(t)} \text{ for all } t > \bar{t}_{k+1}.$$

Integration over $[\bar{t}_{k+1} + 1, t]$ yields the inequality

$$\frac{\tilde{y}(t)}{\tilde{y}(\bar{t}_{k+1}+1)} \le \frac{\bar{y}(t)}{\bar{y}(\bar{t}_{k+1}+1)} \text{ for all } t \ge \bar{t}_{k+1} + 1.$$

Because $\tilde{y}(t)$ is unbounded as $t \to \infty$, it follows that

$$|\bar{y}(t)| \to \infty \text{ as } t \to \infty.$$

Hence $\lambda \ne \lambda_{k+1}(\pi/2)$. This result, together with the result of Step 1, implies the inequality

$$\lambda_k(0) < \lambda_{k+1}(\pi/2).$$

Step 3. $\lambda_k(\pi/2) < \lambda_k(0)$.

Let y_{Nk} be the eigenfunction associated with the eigenvalue $\lambda_k(\pi/2)$. Suppose the inequality were not true, i.e.,

$$\lambda_k(\pi/2) \ge \lambda_k(0).$$

Then it follows from Sturm's oscillation theorem that y_{Nk} has at least as many zeros in $(0,\infty)$ as $\tilde{y}$, which, as we have seen, has $k+1$ zeros in $(0,\infty)$. But here we have a contradiction, because y_{Nk} has exactly k zeros in $(0,\infty)$; thus

$$\lambda_k(\pi/2) < \lambda_k(0). \quad ///$$

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A KREIN SPACE APPROACH TO DIRICHLET AND DUAL DIRICHLET INEQUALITIES ASSOCIATED WITH STURM-LIOUVILLE OPERATORS

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Abstract

By embedding the quadratic form associated with a second-order symmetric differential expression defined on $[1,\infty]$ in a Krein space, we derive Dirichlet inequalitites as well as a new class of "Dual Dirichlet inequalities" when the potential may have an unbounded negative part.

1. Introduction

Let I be the real interval $[1,\infty)$, $L^1_{loc}(I)$ the space of complex valued functions, Lebesgue integrable on compact subintervals of I, and $L^2(I)$ the Hilbert space of (equivalence classes of) complex valued functions having square integrable modulus with inner product $[f,g]: = \int_I f\bar{g}$ and norm $\|f\|: = (\int_I |f|^2)^{1/2}$. p_0, p_1 are real measurable functions on I satisfying the "minimal conditions": $p_0 > 0$ a.e., $p_0^{-1}, p_1 \in L^1_{loc}(I)$. $M[y]$ denotes the symmetric differential expression $-(p_0y')'+p_1y$, and L, L_0 the maximal and minimal operators corresponding to M in $L^2(I)$ (for precise definitions of L, L_0 as well as further details concerning the elementary Hilbert space theory of symmetric differential operators, see Kauffman, Read, and Zettl [1977] or Naimark [1968: Chap. V].

We are concerned in this paper with inequalities similar in form to

$$\int_I p_0|y'|^2+p_1|y|^2 \geq \mu_1 \int_I |y|^2, \tag{1.1}$$

$$\int_I |(-p_0^{1/2} z_1)'+p_1^{1/2} z_2|^2 \geq \mu_1 \sup_y\{|\int_I z_1 p_0^{1/2} \bar{y}'+z_2 p_1^{1/2} \bar{y}|^2:$$

$$\int_I p_0|y'|^2+p_1|y|^2 = 1;\ p_0^{1/2} z_1(1^+) = 0\}, \tag{1.2}$$

where in (1.2) (z_1,z_2) belongs to a class of suitable function pairs in $L^2(I)\times L^2(I)$.

Inequality (1.1) is the classical *Dirichlet inequality*. It has had a long history going back at least to R. G. D. Richardson [1910-1912] and Lichtenstein [1919], and in the past decade has attracted renewed interest on the part of several authors (for an illuminating survey and bibliography, see the recent paper of Everitt and Wray [1983]). Inequality (1.2), on the other hand, seems relatively new in spite of a natural relationship between it and (1.1).

When $p_1 \geq \varepsilon > 0$, a satisfactory theory of (1.1) and (1.2) can be constructed in several ways. For example, it is not hard to show that the quadratic "Dirichlet form"

$$\mathrm{t}[y]: = \int_I p_0|y'|^2+p_1|y|^2$$

defined on

$$D: = \{y \in L^2(I): y \text{ locally absolutely continuous ("AC") on } I;\ \mathrm{t}[y] < \infty\}$$

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is *closed* in the sense of Kato [1980: p. 313]. Next the First Representation Theorem [Kato, 1980: Theorem 2.1, p. 322] gives the existence of a selfadjoint extension $T \supset L_0$† defined on a core of t with lower bound μ_1. T has domain

$$D(T) := \{y \in D(L) : p_0 y'(1^+) = 0;\ \mathfrak{t}[y] < \infty\}.$$

Since $D(T)$ is a core of **D**, (1.1) holds on **D** as well as $D(T)$. A somewhat different but equivalent point of view was introduced by Brown [1984]. First define the "maximal" operator $\mathbf{L}: \mathbf{D} \subset L^2(I) \to L^2(I)$ by

$$\mathbf{L}y = \begin{bmatrix} p_0^{1/2} y' \\ p_1^{1/2} y \end{bmatrix}. \qquad (1.3)$$

Since t is closed and $\mathfrak{t}[y] = \|\mathbf{L}y\|^2$, **L** is closed and has closed range. Hence by the closed graph theorem $\mathbf{L}^{-1}$ is bounded. Further $\|\mathbf{L}^{-1}\| = \mu_1^{-1/2}$. Additional analysis shows that T is the selfadjoint operator $\mathbf{L}^*\mathbf{L}$ which exists by a theorem of von Neuman [Kato, 1980: Theorem 3.24, p. 275].

One advantage of the latter approach is a convenient interpretation of the "dual" inequality (1.2). Since $\mathbf{L}^*$ is the adjoint of **L**, it is defined on a certain subspace of pairs $(z_1, z_2) \in L^2(I) \times L^2(I)$; for (z_1, z_2) in this domain $\mathbf{L}^*(z_1, z_2) = -(p_0^{1/2} z_1)' + p_1^{1/2} z_2$. Equation (1.2) then states that the norm of the Hilbert space pseudoinverse†† of $\mathbf{L}^* = \|\mathbf{L}^{-1}\| = \mu_1^{-1/2}$.

For a discussion of a number of other approaches to Dirichlet inequalities, the reader may consult Everitt and Wray [1983].

We pass now to the fundamental problem addressed in this paper: What happens to (1.1) and (1.2) when the potential p_1 is no longer positive and can take on possibly unbounded negative values in a neighborhood of ∞?

If t is closed and bounded below, reasonably straightforward arguments from the theory of sesquilinear forms apply to (1.1) (e.g., see Brown [1984]). The situation, however, is more complicated with respect to dual inequalities. Since $p_1^{1/2}$ need not be real, (1.2) requires restatement and new methods of proof. In particular, the pseudoinverse analogy would appear to break down.

The primary goal of this paper will be to present a unified treatment of both the Dirichlet and dual Dirichlet inequalities in the new setting where p_1 is not necessarily positive or bounded below. We will do this, moreover, in a way that is consistent with our earlier treatment. To this end, we shall introduce the powerful apparatus of the theory of Krein spaces. Specifically, t or $R(\mathbf{L})$ will be embedded in a natural Krein space. Using this device, we can extend the positive coefficient treatment sketched above.

We briefly survey the contents of the paper. In Section 2, the machinery we require is presented. The fundamental Krein space **K** and the new maximal and minimal operator analogues $\mathbf{L}_1, \mathbf{L}_0$ to (1.3) are defined. Several technical lemmas explore the properties of t as a subspace of **K**, especially with reference to the existence/uniqueness of orthogonal projections upon it. Our results for the most part depend on a fundamental assumption "H1" which relates the positive and negative parts of t and which forces t to be closed. Towards the end of Section 2, we determine the adjoints $\mathbf{L}^*$ and $\mathbf{L}_0^*$. Precise statements of the new versions of inequalities (1.1)-(1.2) are given and proved in Section 3 (Theorems 1 and 2). We also state an additional theorem (Theorem 3) and some corollaries to

†T may also be viewed as the *Friedrichs extension* of the symmetric operator $S \supset L_0$ defined on functions of compact support such that $p_0 z'(1^+) = 0$. See Kato [1980: p. 325].

††If $T: H \to H'$ is a closed operator with closed range between Hilbert spaces H and H', the *pseudoinverse* of T is the operator $H' \to h$ giving the least norm solution of the equation $Tx = g$. For a good discussion of pseudoinverses, see Luenberger [1969: Section 6.11].

show how other differential inequalities can be derived from dual inequalities. (To our knowledge, the inequalities presented here are new.) The last section of the paper investigates the situation when H1 is abandoned, and consequently the Dirichlet form t need not be closed. Our results here are less complete than in the previous sections. Nevertheless, we are able to derive some new properties of the ordinary Dirichlet inequality (Theorem 4) and also to prove a new dual inequality (Theorem 5).

Finally, we remark that for the sake of technical simplicity only inequalities corresponding to a second-order Sturm-Liouville operator defined on I are treated in this paper. The reader should have little difficulty in supplying the technical changes required for the higher order case on a different interval, e.g., (0,1].

2. Preliminary Theory Relating to Krein Spaces

Let p_0, p_1 be as above. Define the sesquilinear form $\mathbf{t}[u,v]$ in $L^2(I)\times L^2(I)$ by

$$\mathbf{t}[u,v] := \int_I \mu'\bar{v}' + p_1\mu\bar{v}$$

on D. Set

$$p_1^+ := \max(p_1, 0)$$

$$p_1^- := \max(0, -p_1)$$

and define

$$M^+[y] := -(p_0\mu')' + p_1^+ y$$

$$\mathbf{t}^+[\mu,v] := \int_I p_0\mu\bar{v}' + p_1^+\mu v$$

$$\mathbf{t}^-[\mu,v] := \int_I p_1^-\mu\bar{v},$$

so that $\mathbf{t} = \mathbf{t}^+ - t^-$.

Let C_0^∞ signify the space of infinitely differentiable functions with compact support in I. Then $C_0^\infty \subset \mathbf{D}$ but $D(L)$ need contain no nontrivial element of C_0^∞.*

We now assume the following hypotheses.

H1. *For some* $0 < \vartheta < 1$ *and all* $\varphi \in C_0^\infty$, *there exists a constant* $K > 0$ *(not depending on* φ*) such that*

$$\mathbf{t}^-[\varphi] \le K\|\varphi\|^2 + \vartheta \int_I p_0|\varphi'|^2. \tag{2.1}$$

Since $\vartheta < 1$ and $\mathbf{t}^+[\varphi] > \int_I p_0|\varphi'|^2$, H1$\to \mathbf{t}[\varphi] \ge -K\|\varphi\|^2$. Let μ_1 signify the greatest lower bound of $\mathbf{t}$. For a suitable translation of $\mathbf{t}$, $\mu_1 > 0$. We make this translation if needed, and continue to refer to the translated form by "t."

Summarizing, H1 implies the weaker principle H2.

* $C_0^\infty \subset D(L)$ if p_0, p_1' and p_1 are L^2 integrable on compact subintervals of I.

H2. *For all* φ *in* $C^{\infty}_{[0,\infty)}$

$$\mathfrak{t}[\varphi] \geq \mu_1 \|\varphi\|^2.$$

Examples of H1:

1. p_1^- has compact support.
2. p_1^- is integrable.
3. p_1^- is essentially bounded.
4. $p_0 = t^\beta \quad p_1^- = t^\alpha$ and $\alpha \leq \beta$.
5. For every $\varepsilon > 0$ there exists a positive continuous function f such that

$$\sup_{t\in I} \int_t^{t+\varepsilon f} \frac{p_1^-}{\varepsilon} f < \infty, \quad \sup_{t\in I} \int_t^{t+\varepsilon f} \frac{p_0^{-1}}{\varepsilon f} < \infty.$$

Further sufficient conditions for H1 and its higher derivative analogues may be found in Brown and Hinton [1984].

Let $\tilde{z} := (z_1, z_2, z_3)$, $\tilde{z}' := (z_1', z_2', z_3')$ be elements of $\mathbf{H} := \overset{3}{\underset{i=1}{X}} L^2(I)$. Define the sesquilinear form

$$[\tilde{z}, \tilde{z}']_{\mathbf{K}} = [z_1, z_1'] + [z_2, z_2'] - [z_3, z_3']. \tag{2.2}$$

H endowed with the indefinite inner product (2.2) is a Krein space that we label "K" Clearly $\mathbf{K} = \mathbf{K}^+(+)\mathbf{K}^-$ where (+) is orthogonal direct sum decomposition in terms of (2.2) and

$$\mathbf{K}^+ \subset \mathbf{K} = \{(z_1, z_2, 0)\}$$

$$\mathbf{K}^- \subset \mathbf{K} = \{(0, 0, z_3)\}.$$

K^+, K^- are Hilbert spaces with respect to the inner products $[\cdot,\cdot]_{\mathbf{K}}$ and $-[\cdot,\cdot]_{\mathbf{K}}$. In the terminology of Bognár [1974: p. 71], they are "intrinsically complete" subspaces of K. Given $\tilde{z} \in \mathbf{K}$ we write $\tilde{z} = \tilde{z}^+ + \tilde{z}^-$ where $\tilde{z}^+ \in \mathbf{K}^+$ and $\tilde{z}^- \in \mathbf{K}^-$. The operators $P^+ : \mathbf{K} \to \mathbf{K}^+$ and $P_- : \mathbf{K} \to \mathbf{K}^-$ defined by $P^+\tilde{z} = z^+$ and $P^-\tilde{z} = z^-$ are called the fundamental projectors. Let $J := P^+ - P^-$. Then J is the "fundamental symmetry"; and

$$[\tilde{z}, \tilde{z}']_{\mathbf{H}} := [J\tilde{z}, \tilde{z}']_{\mathbf{K}}$$

is the standard inner product for a Hilbert space structure in H.

The next step is to embed $\mathfrak{t}$ into K via the map

$$\mathbf{L}y := \begin{pmatrix} p_0^{\frac{1}{2}} y' \\ p_1^{+\frac{1}{2}} y \\ p_1^{-\frac{1}{2}} y \end{pmatrix},$$

for $y \in \mathbf{D}$, and to explore the Krein space attributes of $\mathfrak{t}$ considered as "$R(\mathbf{L})$".

If for $\tilde{z} \in \mathbf{K}$ there exists $y \in R(\mathbf{L})$ and $v \in R(\mathbf{L})^{\perp}$ such that $\tilde{z} = y + v$, we call y an ***orthogonal projection*** of $\tilde{z}$ onto $R(\mathbf{L})$ and write $y \in P\tilde{z}$. In Krein spaces the existence and uniqueness of orthogonal projections are difficult questions. For example, $\tilde{z}$ may have nonunique projectors $P_1\tilde{z}$, $P_2\tilde{z}$ which differ by an "isotropic" element $\varphi \in R(\mathbf{L}) \cap R(\mathbf{L})^{\perp}$; such a φ is also neutral, i.e., $[\varphi, \varphi]_{\mathbf{K}} = 0$.

The proof of the following lemma (which is a collection of results concerning the existence of projections) may be found in Bognár [1984].

LEMMA 1. *Statements (i)-(iv) are equivalent:*

(i) A projection $P:\mathbf{K}\to R(\mathbf{L})$ exists.

(ii) $R(\mathbf{L})$ is "orthocomplemented" (i.e., span $<R(\mathbf{L}), R(\mathbf{L})> = \mathbf{K}$

(iii) The functional $\varphi_{\tilde z}(y) := [\tilde z - y, \tilde z - y]_{\mathbf{K}}$ considered for $y\in R(\mathbf{L})$ attains its minimum for every $\tilde z \in \mathbf{K}$

(iv) If $R(\mathbf{L})$ is a positive definite subspace of $\mathbf{K}$ and is "intrinsically complete" (i.e., a Hilbert space under the norm $[\mathbf{L}y, \mathbf{L}y]_{\mathbf{K}}$), for every $\tilde z \in \mathbf{K}$ the functional $\Phi_{\tilde z}(y) := [z, \mathbf{L}y]_{\mathbf{K}}$ is "intrinsically continuous" (or, in Bognar's nomenclature, $R(\mathbf{L})$ is regular; see Bognar [1974: p. 71 and Theorem 9.2, p. 73]).

(v) For every $\tilde z \in \mathbf{K}$ there exists $y_0 \in \mathbf{D}$ such that

$$[\mathbf{L}y_0, \mathbf{L}y_0]_{\mathbf{K}} = [\mathbf{L}y_0, \tilde z]_{\mathbf{K}} = \sup_y \{ |\varphi_{\tilde z}(y)|^2 : [\mathbf{L}y, \mathbf{L}y]_{\mathbf{K}} = 1 \},$$

and $\mathbf{L}y_0 = P\tilde z$.

LEMMA 2. *For all $y \in \mathbf{D}$*

$$t[y] \geq \mu_1 \|y\|^2. \tag{2.3}$$

PROOF. By H2, (2.3) holds on $C_{[0}^{\infty}$. Rewrite (2.3) restricted to $C_{[0}^{\infty}$ in the equivalent form:

$$\int_I p_0 |y'|^2 + p_1^+ |y|^2 \geq \int_I (\mu_1 + p_1^-) |y|^2. \tag{2.4}$$

Consider the differential expression

$$|M|[v] := -(p_0 v')' + (p_1^+ + p_1^-) v,$$

and the maximal and minimal operators $|L|, |L_0|$ which $|M|$ generates in $L^2(I)$. It is known [Kauffman, Read, and Zettl, 1977: Theorem 2.1, p. 23] that $|M|$ is limit point at ∞. This in turn (see Brown [1984] or Kauffman [1979]) implies that the "Dirichlet index" of $|M|+I$ — or the dimension of the space of solutions v such that $t[v] < \infty$ — is 1. However, the minimal Dirichlet index property is equivalent (again see Brown [1984] or Kauffman [1977] for a proof) to $C_{[0}^{\infty}$ being a core of $|t| := |t^+| + |t^-|$. Since $D|t| = \mathbf{D}$, we conclude that for every $y \in \mathbf{D}$ and $\delta > 0$, there exists $\varphi \in C_{[0}^{\infty}$ such that

$$\int_I |y-\varphi|^2 < \delta, \quad \int_I p_1^+ |y-\varphi|^2 < \delta$$

$$\int_I p_1^- |y-\varphi|^2 < \delta, \quad \int_I p_0 |y'-\varphi'|^2 < \delta. \tag{2.5}$$

These inequalities and H2 establish (2.4) and hence (2.3) on $\mathbf{D}$.

REMARK 1. Lemma 2 is a proof of the classical Dirichlet inequality on $\mathbf{D}$ which consists of functions satisfying the minimal smoothness and integrability properties for the inequality to make sense (assuming $\int |p_1| |y|^2 < \infty$). Since the proof depends only on H2 and the fact that $|M|[y]$ has a Dirichlet index 1, it extends immediately to the case where $M[y]$ is a quasi-derivative of order $2n$

based on coefficients $p_0, p_1, \dots p_n$ satisfying minimal conditions if H2 is satisfied and $|L|+I$ has a minimal Dirichlet index n. (Here $|L|$ is the maximal operator corresponding to the quasiderivative $|M|[y]:=y^{[2n]}$ based on the coefficients $\{p_0, p_1^+ + p_1^-, \dots, p_n^+ + p_n^-\}$). The significance of the Dirichlet index concept in the proof of Dirichlet inequalities seems first to have been pointed out in the paper of Bradley, Hinton, and Kauffman [1981].

LEMMA 3. *H1 is true for all y in* D.

PROOF. This follows at once from H1 and the approximation inequalities (2.5) established in Lemma 2.

LEMMA 4. *The Dirichlet form* t *is closed.*

PROOF. Since $\mathbf{t}^+$ majorizes $\mathbf{t}$, the inequality

$$\mathbf{t}^+[y] \geq \mu_1 \|y\| \tag{2.6}$$

holds on D. We show first that $\mathbf{t}^+$ is closed. If $\mathbf{t}^+[y_n - y_m] \to 0$ for sufficiently large m and n (2.6) $\Rightarrow \langle y_n \rangle$ is a Cauchy sequence in $L^2(I)$. Let y be the limit of $\langle y_n \rangle$. Since the maximal operator K determined by $p_0 y'$ is closed (see Brown [1983: Lemma 1]), $y \in D(K)$ and $\int_I p_0 |y' - y_n'|^2 \Rightarrow 0$. Set $\eta := p_1^+ + 1$ and consider the Hilbert space $L_\eta^2(I)$. Since $\mathbf{t}^+[y_n - y_m] \geq \int_I p_1^+ (y_n - y_m)^2$ and by (2.6), $\langle y_n \rangle$ is a Cauchy sequence in $L_\eta^2(I)$ with limit ζ. Since the norm in $L_\eta^2(I)$ dominates that of $L^2(I)$, $y = \zeta$. Thus $y \in \mathbf{D}$ and $\mathbf{t}^+[y - y_n] \to 0$, so that $\mathbf{t}^+$ is closed. Finally, $\mathbf{t}$ is closed by Lemma 3 (see Kato [1980: Theorem 1.33, p. 320]).

LEMMA 5. *All the statements of Lemma 1 are true.*

PROOF. It is enough to show that any single statement of Lemma 1 is true. We consider (iv). Statement (iv) asserts that

$$\varphi_{\tilde{z}}(y) := \int_I p_0^{1/2}\, y' z_1 + p_1^{+1/2}\, y z_2 - p_1^{-1/2}\, y z_3$$

is continuous in the Hilbert space norm $[\mathbf{L}y, \mathbf{L}y]^{1/2}$. (That $R(\mathbf{L})$ is a Hilbert space with this definition of a norm follows from Lemmas 2 and 4.) We must show therefore that $\mathbf{L}(y_n) \to 0 \Rightarrow \varphi_{\tilde{z}}(y_n) \to 0$. Now

$$\|\mathbf{L}y\|^2 = \mathbf{t}[y] = \mathbf{t}^+[y] - \mathbf{t}^-[y].$$

Also from Lemma 3,

$$t^-[y] \leq K\|y\|^2 + \vartheta \mathbf{t}^+[y]. \tag{2.7}$$

From (2.7) and Kato [1980: Problem 1.2 and (1.2), p. 190] we obtain the estimates

$$\mathbf{t}^-[y]^{1/2} \leq (1-\vartheta)^{-1}(K\|y\| + \vartheta\|\mathbf{L}y\|),$$

$$(1-\vartheta)\mathbf{t}^+[y]^{1/2} \leq K\|y\| + \|\mathbf{L}y\|. \tag{2.8}$$

Also by Cauchy's inequality

$$|\varphi_{\tilde{z}}(y_n)| \leq \mathbf{t}^+(y_n)^{1/2}(\|z_1\|^2 + \|z_2\|^2)^{1/2} + \mathbf{t}^-(y_n)^{1/2}\|z_3\|. \tag{2.9}$$

If then $\mathbf{L}(y_n) \to 0$, $\varphi_{\tilde{z}}(y_n) \to 0$ by (2.3) and (2.8), (2.9), and the lemma is proved.

REMARK. Inequality (2.3) of Lemma 2 assures us that projections upon $R(\mathbf{L})$ are unique, since $R(\mathbf{L})$ has no neutral elements. Lemma 5 furthermore assures us of the existence of a projection and that $\mathbf{K} = R(\mathbf{L})(+)R(\mathbf{L})^+$.

DEFINITION 1. *If* $T: L^2(I) \to \mathbf{K}$ *is a densely defined operator,* T^* *is the set of pairs* $(\alpha, T^*\alpha)$ *such that* $[Ty, \alpha]_{\mathbf{K}} = [y, T^*\alpha]$ *for all* y *in* $D(T)$.

DEFINITION 2. *We introduce the following additional operators:* $\mathbf{L}_0' \subset \mathbf{L}$ *on*

$\mathbf{D}_0' := \{y \subset \mathbf{D}: y$ *has compact support in* $L^2(I)\}$.

$\mathbf{L}^+: \mathbf{K} \to L^2(I)$ *defined by*

$$\mathbf{L}^+(\tilde{z}) := -(p_0^{\frac{1}{2}} z_1)' + p_1^{+\frac{1}{2}} z_2 - p_1^{-\frac{1}{2}} z_3$$

on

$\mathbf{D}^+ := \{\tilde{z} \in \mathbf{K}: p_0^{\frac{1}{2}} z_1$ *is* AC; $\mathbf{L}^+(\tilde{z}) \in L^2(I)\}$.

$\mathbf{L}_0^+ \subset \mathbf{L}^+$ *on*

$$\mathbf{D}_0^+ := \{\tilde{z} \in \mathbf{D}^+: p_0^{\frac{1}{2}} z_1(1^+) = 0\}.$$

The proof of the next lemma parallels similar arguments in Brown [1983, 1984]. Therefore we shall omit it.

LEMMA 6.

$$\mathbf{L}^* = \mathbf{L}_0^+;\ L_0^* = \mathbf{L}_0'^* = \mathbf{L}^+;$$

$$L_0^{+*} = \mathbf{L};\ L^{+*} = \mathbf{L}_0 = \bar{\mathbf{L}}_0'.$$

COROLLARY 1. $\mathbf{K} = R(\mathbf{L})(+)N(\mathbf{L}_0^+) = R(\mathbf{l}_0)(+)N(\mathbf{L}^+)$.

LEMMA 7. $\mathbf{L}^+\mathbf{L}_0$ *and* $\mathbf{L}_0^+\mathbf{L}$ *are selfadjoint extensions of* L_0. *Their domains are cores of* $\mathbf{L}_0$ *and* $\mathbf{L}$. *They have the same lower bounds as* $\mathbf{L}_0$ *and* $\mathbf{L}$. *Further* $D\sqrt{(\mathbf{L}^+\mathbf{L}_0)} = \mathbf{D}_0$ *and* $D\sqrt{(\mathbf{L}_0^+\mathbf{L})} = \mathbf{D}$.

PROOF. By inspection $\mathbf{L}^+\mathbf{L}_0$ and $\mathbf{L}_0^+\mathbf{L}$ are exactly the operators i^+i_0 and i_0^+i investigated by Brown [1984: Lemma 5].

3. Dirichlet and Dual Dirichlet Inequalities

With the machinery developed in the previous section, our treatment of inequalities is similar to the positive coefficient case studied in Brown [1983].

THEOREM 1 (The Dirichlet Inequality). *For all* $y \in \mathbf{D}(\mathbf{D}_0)$ *the inequality*

$$\int_I p_0|y'|^2 + p_1|y|^2 \geq \mu_1(\mu_2)\int_I |y|^2 \tag{3.1}$$

holds. Here μ_1 *and* μ_2 *are, respectively, the least elements in the spectrum of* $\mathbf{L}_0^+\mathbf{L}$ *or* $\mathbf{L}^+\mathbf{L}_0$. *Equality holds in (3.1) if and only if* y *is an eigenfunction corresponding to* μ_1 *or* μ_2. *Otherwise there is a sequence of functions* $\langle\varphi_n\rangle$ *such that*

$$\int_I p_0|\varphi_n'|^2 + p_1|\varphi_n|^2 - \mu_1(\mu_2)\int_I |\varphi_n|^2 \to 0. \tag{3.2}$$

PROOF. The inequality (3.1) follows from the First Representation Theorem and Lemma 7. The equation (3.2) expresses the fact that the spectrum of a selfadjoint closed operator is its approximate point spectrum.

REMARK 3. Recall that according to H2, $\mu_1(\mu_2) > 0$. However, Theorem 1 is true for any finite μ_1 or μ_2. To see this, simply subtract what necessary from both sides of (3.1) to get the original μ_1 or μ_2. By the spectral mapping principle these numbers are the infima of the spectra of the corresponding translates of $\mathbf{L}_0^+\mathbf{L}$ or $\mathbf{L}^+\mathbf{L}_0$.

COROLLARY 2. *For all y in $D(\mathbf{L}_0^+\mathbf{L})$ or $D(\mathbf{L}^+\mathbf{L}_0)$*

$$\int_I |-(p_0 y')' + p_1 y|^2 \geq \mu_1(\mu_2)(\int_I p_0|y'|^2 + p_1|y|^2).$$

THEOREM 2 (The Dual Inequality). *For all $\tilde{z} \in \mathbf{D}_0^+(\mathbf{D}^+)$*

$$\int_I |-(p_0^{1/2} z_1)' + p_1^{+1/2} z_2 - p_1^{-1/2} z_3|^2 \geq \mu_1(\mu_2)\sup_y\{|[\tilde{z}, \mathbf{L}(\mathbf{L}_0)y]_{\mathbf{K}}|^2 :$$

$$\|\mathbf{L}(\mathbf{L}_0)y\| = 1\}. \tag{3.3}$$

Equality holds in (3.3) if and only if $\tilde{z} = \mathbf{L}g \bmod N(\mathbf{L}_0^+(\mathbf{L}^+))$ where $\mathbf{L}g$ is an eigenfunction corresponding to $\mu_1(\mu_2)$ of $\mathbf{L}\mathbf{L}_0^+$ $(\mathbf{L}_0\mathbf{L}^+)$. Otherwise there is a sequence $<\tilde{z}_n>$ of elements in $D(\mathbf{L}\mathbf{L}_0^+)$ or $D(\mathbf{L}_0\mathbf{L}^+)$ such that

$$\lim_{n\to\infty}|\int_I |-(p_0^{1/2} z_{1n})' + p_1^{+1/2} z_{2n} - p_1^{-1/2} z_{3n}|^2 - \mu_1(\mu_2)\sup_y\{|[\tilde{z}_n, \mathbf{L}(\mathbf{L}_0)y]_{\mathbf{K}}|^2 :$$

$$\|\mathbf{L}(\mathbf{L}_0)y\| = 1\} = 0.$$

PROOF. Let P be a projection on $R(\mathbf{L})$ which, in view of Lemma 5, exists and is unique. Then for any $\tilde{z} \in \mathbf{K}$, $\tilde{z} = P\tilde{z} + (I-P)\tilde{z}$ where $(I-D)\tilde{z} \in N(\mathbf{L}_0^+)$ (Corollary 1). By Corollary 2

$$\|\mathbf{L}_0^+\tilde{z}\|^2 = \|\mathbf{L}_0^+P\tilde{z}\|^2 \geq \mu_1\|P\tilde{z}\|^2.$$

The equation (3.3) now can be obtained by applying (v) of Lemma 1. To prove the second statement, observe that

$$\|\mathbf{L}_0^+\tilde{z}\|^2 = \mu_1\|P\tilde{z}\|^2 \Leftrightarrow \|\mathbf{L}_0^+P\tilde{z}\|^2 = \mu_1\|P\tilde{z}\|^2$$

$$\Leftrightarrow \|\mathbf{L}_0^+\mathbf{L}g\|^2 = \mu_1\|\mathbf{L}g\|^2 \Leftrightarrow [\mathbf{L}\mathbf{L}_0^+\mathbf{L}g - \mu_1\mathbf{L}g, \mathbf{L}y]_{\mathbf{K}} = 0$$

$$\Leftrightarrow \mathbf{L}\mathbf{L}_0^+(\mathbf{L}g) = \mu_1\mathbf{L}g.$$

Hence $\mathbf{L}g$ is an eigenfunction of $\mathbf{K}$ corresponding to μ_1 and $\tilde{z} = \mathbf{L}y \bmod N(\mathbf{L}_0^+)$. The last statement depends on the same principle of the indentity of the approximate point spectrum and the spectrum of a selfadjoint operator that we used in Theorem 1 to justify a similar statement.

Since the argument is the same for the inequality involving $\tilde{z}$ in $\mathbf{D}^+$ and μ_2, we omit it.

THEOREM 3. *For all $\tilde{z} \in D(\mathbf{L}\mathbf{L}_0^+)$ or $D(\mathbf{L}_0\mathbf{L}^+)$*

$$\int_I p_0|(\mathbf{L}^+\tilde{z})'|^2 + p_1^+|\mathbf{L}^+\tilde{z}|^2 - p_1^-|\mathbf{L}^+\tilde{z}|^2$$

$$\geq \mu_1^2(\mu_2)^2\sup_y\{|[\tilde{z}, \mathbf{L}(\mathbf{L}_0)y]_{\mathbf{K}}|^2 : \|\mathbf{L}(\mathbf{L}_0)y\| = 1\}.$$

PROOF. By Theorem 1 $\|L(L_0^+\tilde{z})\|^2 \geq \mu_0\|L_0^+z\|^2$. Now apply Theorem 2.

COROLLARY 3. *For all v in $D(L_0^+L)$ or $D(L^+L_0)$*

$$\left(\int_I p_0|v'|^2+p_1|v|^2\right)\int_I|(p_0v')'|^2 \geq \mu_1(\mu_2)\left(\int_I p_0|v'|^2\right)^2.$$

PROOF. In Theorem 2 take $z_2 = z_3 = 0$, $z_1 = v$, and $y = v\|Lv\|^{-1/2}$.

COROLLARY 4. *For all y in $D(L_0^+L)$ or $D(L^+L_0)$*

$$\int_I p_0|M[y]')^2 + p_1|M[y]|^2 \geq \mu_1^2(\mu_2^2)\left(\int_I p_0|y'|^2+p_1|y|^2\right).$$

4. Addendum: Inequalities without H1

In spite of the Krein space methods employed in this paper, our results have so far been a not-unexpected generalization of the positive coefficient case. In particular, the Friedrichs extensions L_0^+L and L^+L_0 we have constructed have been of a familiar type. Further, as should be clear from Section 2, the fundamental assumption behind our results is H1. H1 states that the form t^- is relatively bounded with bound <1 with respect to t^+ on an appropriate subspace. Together with H2 this implies that t is a positive definite intrinsic closed subspace of K. Furthermore, a unique orthogonal projection exists upon t considered as $R(L)$. Once this point has been reached, the remaining arguments are standard ones.

Suppose now that we assume only H2. Then L_0 is bounded below on the intersection of its domain with C_0^∞, but t need not be closed. As we have seen, it is still possible to prove the Dirichlet inequality in D (or D_0) (Lemma 2). However, the number μ_2 has no spectral significance. The difficulty lies in the fact that although t is closable, t itself need not be closed; furthermore, $\tilde{t}$ need not have the same structure as t. Thus the inequality not only may be defined on a larger set than D but it may be a "different" inequality. To illustrate these ideas, we appeal to an example of Kalf [1978]. Let $I = (0,\infty)$, $p_0 = 1$, $p_1 = -1/4x^2-\mu$, $\mu \geq 0$. Then

$$\tilde{t}[y] = \int_I|y'-y/2x|^2 \geq \mu_1\int_I|y|^2, \tag{4.1}$$

where μ_1 is the least element of the spectrum of the Friedrichs extension having domain $\{y\in D(L): \int_I|y'-y/2x|^2 < \infty\}$. Now (4.1) => inequality (2.3) on D, but D is strictly contained in $D(\tilde{t})$.*

It would therefore be an interesting problem to determine necessary and sufficient conditions that t be closed and when it is not, to determine the closure, especially when $M[y] = y^{[2n]}$, $n>1$. We can also ask what the dual inequality corresponding to (4.1) is.

In the remainder of this section, we show how something can be salvaged using H2 rather than H1, together with a technique that incorporates the negative part of the potential into the weight of a certain weighted symmetric expression. Although our results do not coincide with Kalf's, they do yield new inequalities and some new properties of known inequalities, and thus may be of independent interest.

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* This example has singular point 0 and so does not fit into our setting. However, it is easy to construct similar examples with singularity at ∞.

THEOREM 4. *Assume H2; then for all $y \in \mathrm{D}(\mathrm{D}_0)$*

$$\mathrm{t}[y] \geq \mu_1(\mu_2)\|y\|^2. \tag{4.2}$$

Equality holds in (4.2) if and only if y is an eigenfunction of $M[y]$ in D with the additional properties

(i) $p_0 y'(1) = 0 \ (p_0 y(1) = 0)$.

(ii) $p_0 y'$ *is AC.*

(iii) $\int_I (p_1^- + \mu_1(\mu_2))^{-1} |-(p_0 y')' + p_1^+ y|^2 < \infty$.

Otherwise there exists a sequence of functions $<\varphi_n>$ in $\mathrm{D}(\mathrm{D}_0)$ satisfying (i)-(iii) such that

$$\lim_{n \to \infty} \int_I p_0 |\varphi_n'|^2 + p_1 |\varphi_n|^2 - \mu_1(\mu_2) |\varphi_n|^2 = 0.$$

PROOF. (4.2) is (2.3) of Lemma 2. Rewrite the inequality in the form (2.4). Let $w := p_1^- + \mu_1$. Then $w > 0$ a.e. and $w \in L^1_{loc}(I)$. Define $M_w[y] := w^{-1}[-(p_0 y')' + p_1^+ y]$. Let L_w, L_{0w} be the associated maximal and minimal operators in $L^2_w(I)$. Since t^+ is closed on $L^2(I)$, it is closed on $L^2_w(I)$.

We are now in the positive coefficient case with respect to the weighted expression M_w. We define $L'_0, L_0, \mathrm{L}: L^2_w(I) \to L^2(I) \times L^2(I) \mathrm{L}^+, \mathrm{L}_0^+$ as in Brown [1983] and repeat all the arguments of that paper. By Theorem 2 of Brown, we end up with (2.4) — and therefore (4.2) — in D (the Friedrichs extension $\mathrm{L}_0^+\mathrm{L}: L^2_w(I) \to L^2_w(I)$ has lower bound 1) and discover that properties (i)-(iii)) express the fact that $y \in D(\mathrm{L}_0^+\mathrm{L})$.

The same reasoning may be used for the inequality on D_0.

Corresponding to Theorem 4 is a dual inequality that is a special case of Theorem 3 in Brown [1983].

THEOREM 5. *For all $(z_1, z_2) \in \mathrm{D}_0^+(\mathrm{D}^+)$ the inequality*

$$\int_I (p_1^- + \mu_1(\mu_2))^{-1} |-(p_0^{1/2} z_1)' + p_1^{+1/2} z_2|^2 \geq \sup_y \{ | \int_I p_0^{1/2} z_1 \bar{y}' + p_1^{+1/2} z_2 \bar{y} |^2 : \|\mathrm{L}(\mathrm{L}_0) y\| = 1 \} \tag{4.3}$$

holds with conditions for inequality as in Theorem 2 above or in Brown [1983: Theorem 3].

REMARK 4. Since Theorems 4 and 5 are true *a fortiori* if H1 is assumed, (4.3) is a new dual inequality that is valid in all circumstances. Moreover, an extremal y for the ordinary Dirichlet inequality (Theorem 1) will always have the additional weighted integrability property (iii) of Theorem 4.

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SPECTRAL PROPERTIES OF SELFADJOINT ORDINARY DIFFERENTIAL OPERATORS WITH AN INDEFINITE WEIGHT FUNCTION

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Abstract

Spectral properties of the equation $l(f) - \lambda rf = 0$ with an indefinite weight function r are studied in $L^2_{|r|}$. The main tool is the theory of definitizable operators in Krein spaces. Under special assumptions on the weight function, for the operator associated with the problem, the regularity of the critical point infinity is proved. Some relations to full- and half-range expansions are indicated.

1. Basic Properties

1.1. We consider the formal differential expression $l(f)$ of order $2n$ on the interval (a,b), $-\infty \le a < b \le +\infty$:

$$l(f) := (-1)^n (p_0 f^{(n)})^{(n)} + (-1)^{n-1}(p_1 f^{(n-1)})^{(n-1)} + \ldots + p_n f,$$

where the functions $p_0,\ldots,p_n$ are real, $p_0 > 0$ a.e. on (a,b) and $1/p_0$, $p_1,\ldots,p_n \in L^1_{loc}(a,b)$. The exact meaning of $l(f)$ under this general assumption is that of the quasi-derivative of order $2n$ (see Krein [1947] and Naimark [1968]): $l(f) := f^{[2n]}$. We study the spectral properties of the equation

$$l(f) - \lambda rf = 0, \tag{1.1}$$

where the real weight function $r \in L^1_{loc}(a,b)$ is *indefinite*, that is, the sets $\Delta_+ := \{x : r(x) > 0\}$, $\Delta_- := \{x : r(x) < 0\}$ are both of positive Lebesgue measure. For the sake of simplicity we assume that $r \neq 0$ a.e. on (a,b). The problem (1.1) is called *regular* if $-\infty < a < b < \infty$ and $\frac{1}{p_0}$, $p_1,\ldots,p_n$, $r \in L^1(a,b)$; the boundary point a (b) is called *singular* if $a = -\infty$ $(b = \infty)$ or at least one of the functions $\frac{1}{p_0}$, $p_1,\ldots,p_n$, r is not summable at a (b, respectively).

By L^2_r we denote the Krein space [Bognár, 1974; Langer, 1982] of all measurable functions f on (a,b) such that $\int_a^b |f|^2 |r| \,dx < \infty$, equipped with the indefinite and definite inner products

$$[f,g] := \int_a^b f\bar{g} r \,dx \quad \text{and} \quad (f,g) := \int_a^b f\bar{g}\,|r|\,dx, \text{ resp.} \tag{1.2}$$

Evidently, the operator J

$$(Jf)(x) := (\operatorname{sgn} r(x)) f(x) \qquad (x \in (a,b))$$

is the fundamental symmetry connecting the scalar products in (1.2).

By D^0 we denote the set of all $f \in L^2_r$ which vanish identically in neighborhoods of a and b and have absolutely continuous quasi-derivatives up to order

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$2n-1$ such that

$$f^{[2n]} = l(f) = |r|g$$

with some $g \in L_r^2$. On $\mathbf{D}^0$ we define the operators $B_{\min}^0$ and $A_{\min}^0$: $\mathbf{D}(A_{\min}^0) := \mathbf{D}(B_{\min}^0) := \mathbf{D}^0$

$$B_{\min}^0 f := g \text{ if } l(f) = |r|g, \ g \in L_r^2$$

and $A_{\min}^0 := JB_{\min}^0$. Evidently, $A_{\min}^0 f = g$ if and only if for $f \in \mathbf{D}^0$, $g \in L_r^2$ we have $l(f) = rg$. It is easy to see that the definition of these operators is correct. The closure of $A_{\min}^0$ in L_r^2 exists; it is denoted by $A_{\min}$ and called the *minimal operator* associated with the problem (1.1). It is easy to see that $A_{\min}^0$ and $A_{\min}$ are Hermitian with respect to the inner product $[\cdot,\cdot]$, that is, they are Hermitian in the Krein space L_r^2 (for the definition of Hermitian and selfadjoint operators in Krein spaces, we refer to Bognár [1974] and Langer [1982]).

1.2. Recall that an inner product on a linear space $\mathbf{L}$ is said to have a finite number κ of negative squares, if it is negative definite on a κ-dimensional subspace of $\mathbf{L}$ and there exists no $(\kappa+1)$-dimensional subspace with this property. In this paper we study the problem (1.1) under the following assumptions (A1) and (A2).

(A1). *The inner product* $\{\cdot,\cdot\}$, *defined on* $\mathbf{D}^0$ *by*

$$\{f,g\} := [A_{\min}^0 f, g] \; (= \sum_{j=0}^{n} \int_a^b p_{n-j} f^{(j)} \overline{g^{(j)}} dx)$$

has a finite number of negative squares.

PROPOSITION 1. *The condition (A1) is satisfied in each of the following cases:*

(a) *The problem (1.1) is regular.*

(b) *For each singular boundary point a or b of the problem (1.1), there exists $a' \in (a,b)$ or $b' \in (a,b)$ such that the inner product $\{\cdot,\cdot\}$ is nonnegative definite on the set of all functions $f \in \mathbf{D}^0$ which vanish outside of (a,a') or (b',b), respectively.*

To prove the first statement, we observe that $[A_{\min}^0 f, g] = (B_{\min}^0 f, g)$ $(f, g \in \mathbf{D}^0)$ and use M. G. Krein's results that $B_{\min}^0$ is bounded from below and that an arbitrary selfadjoint extension of $B_{\min}$ in $L_{|r|}^2$ has discrete spectrum (see Krein [1947] and Naimark [1968]). The second statement follows if we use the decomposition method of I. M. Glazman [1967], restricting $A_{\min}^0$ to all functions $f \in \mathbf{D}^0$ with the property $f(b') = f^{[1]}(b') = \cdots = f^{[2n-1]}(b') = 0$ (if, for example, b is singular), and use statement (a).

1.3. The Hermitian operator $A_{\min}$ in the Krein space L_r^2 has selfadjoint extensions in L_r^2. In fact, A is a selfadjoint extension of $A_{\min}$ if and only if the operator $B := JA$ is a selfadjoint extension of the Hermitian operator $B_{\min}$ in the Hilbert space $L_{|r|}^2$. Therefore the selfadjoint extensions A of $A_{\min}$ are completely described by boundary conditions at a and b which are the same for A and $B = JA$ and which can be found, for example, in Naimark [1968]. Now we formulate the second assumption.

(A2). *For some (and hence for all) selfadjoint extensions A of $A_{\min}^0$ in L_r^2, the resolvent set is nonempty.*

We mention (cf. Daho and Langer [1977]) that this condition is equivalent to each of the following:

(A2') *For some (and hence for all)* $\lambda \in \mathbf{C}$, *the range* $\mathbf{R}(A_{\min} - \lambda I)$ *is closed.*

(A2'') *For some selfadjoint extension* A *of* $A_{\min}$ *and for some* $\lambda \in \mathbf{C}$, *the range* $\mathbf{R}(A - \lambda I)$ *is closed.*

PROPOSITION 2. *The condition (A2) is satisfied in each of the following cases:*

(a) *The problem (1.1) is regular.*

(b) *For each singular boundary point* a *or* b *of the problem (1.1) there exists* $a' \in (a,b)$ *or* $b' \in (a,b)$ *such that the weight function* r *is of constant sign a.e. on* (a,a') *or* (b',b), *respectively.*

Here the statement (a) is a classical result, and (b) follows again from an application of Glazman's decomposition method [Glazman, 1967].

2. Definitizability of the Selfadjoint Extensions

2.1. Recall that a selfadjoint operator A in a Krein space $\mathbf{K}$ is said to be *definitizable* [Langer, 1982] if $\rho(A) \neq \phi$ and there exists a polynomial p such that $[p(A)f,f] \geq 0$ for all $f \in \mathbf{D}(A^k)$, where k is the degree of p.

THEOREM 1. *Suppose that the operator* $A^0_{\min}$ *in Section 1.1 satisfies the conditions (A1) and (A2). Then every selfadjoint extension* A *of* $A^0_{\min}$ *in* L^2_r *is definitizable.*

Indeed, it is easy to see that for such a selfadjoint extension A the inner product $[Af,g]$ $(f,g \in \mathbf{D}(A))$ has a finite number of negative squares. Hence we can apply Langer [1982: 1.3(c)], and the statement follows.

Suppose, for example, that the problem (1.1) is regular. Then we have for $f,g \in \mathbf{D}(A)$

$$[Af,g] = \sum_{j=0}^{n} \int_a^b p_{n-j} f^{(j)} \overline{g^{(j)}} dx + \mathbf{b}(f,g), \tag{2.1}$$

where $\mathbf{b}(f,g)$ ("the boundary form") is an inner product, depending for a regular (singular) boundary point only on the values of f,g and their first $2n-1$ quasi-derivatives at this point (in the neighborhood of this point, respectively). The number of negative squares of the inner product, given by the first term on the righthand side of (2.1), coincides with the number of negative squares of the inner product $[A^0_{\min}f,g]$ on $\mathbf{D}^0$. Thus, the number of negative squares of $[Af,g]$ $(f,g \in \mathbf{D}(A))$ is not greater than the sum of the negative squares of $[A^0_{\min}f,g]$ $(f,g \in \mathbf{D}^0)$ and of the boundary form $\mathbf{b}(f,g)$ $(f,g \in \mathbf{D}(A))$.

2.2. Here we suppose that the conditions of Theorem 1 are satisfied and A is an arbitrary selfadjoint extension of $A^0_{\min}$ in the Krein space L^2_r. By κ_A we denote the number of negative squares of the inner product $[Af,g]$ $(f,g \in \mathbf{D}(A))$. The following spectral properties of A are immediate consequences of the definitizability of A (see Langer [1982]).

(1) *The operator* A *has at least* κ_A *eigenvalues* λ *(counted according to their algebraic multiplicities) in the closed upper half-plane with the following property: If* $\lambda > 0$ $(\lambda < 0)$ *there exists an eigenelement* f *of* A *corresponding to* λ *such that* $[f,f] \leq 0$ $([f,f] \geq 0)$.

(2) *The nonreal spectrum of A consists of pairs of isolated eigenvalues $\lambda,\bar{\lambda}$; the linear span of the root spaces corresponding to these eigenvalues λ in the upper half-plane is neutral and hence of dimension $\leq \kappa_A$.*

We mention that for any selfadjoint operator A in a Krein space the root spaces, corresponding to two eigenvalues λ,μ are orthogonal with respect to the indefinite inner product if $\lambda \neq \bar{\mu}$, and skewly linked if $\mu = \bar{\lambda}$ and λ,μ are isolated points of $\sigma(A)$.

(3) *The operator A has positive and negative spectrum, both of infinite multiplicity. If, in particular, $\sigma(A)$ is discrete, it contains infinitely many positive eigenvalues s_j^+ and infinitely many negative eigenvalues s_j^-, $j = 1,2,\ldots$, and the root spaces, corresponding to the real eigenvalues of A are nondegenerated with respect to the indefinite inner product.*

We denote the signature of the root space corresponding to the real eigenvalue λ of A by $(\kappa_-(\lambda), \kappa_+(\lambda))$; for an arbitrary eigenvalue λ of A, its algebraic multiplicity is denoted by $\nu(\lambda)$.

(4) *If $\sigma(A)$ is discrete, we have*

$$\sum_j \kappa_+(s_j^-) + \sum_j \kappa_-(s_j^+) + \sum_{\substack{\lambda\in\sigma(A)\\ \mathrm{Im}(\lambda)>0}} \nu(\lambda) \leq \kappa_A,$$

where the sign = holds if 0 is not an eigenvalue of A.

We mention that these statements imply some results of Mingarelli [1983a and 1983b].

The spectral theory of definitizable operators in Krein spaces yields the existence of a spectral function with critical points (see Langer [1982]) for A. It can also be shown that there exists a scalar or matrix spectral measure that has, possibly, certain singularities; in a special situation this spectral measure was considered by Daho and Langer [1977]. Moreover, expansions of arbitrary elements of L_r^2 with respect to eigenelements or generalized eigenelements of A hold. However, they become more complicated than in the case of a positive weight function as the integrals need a regularization at the singular critical points of A (see Daho and Langer [1977] for the case of second-order operators). In Section 3 we shall show that these expansions are "nice" if the spectrum of A is discrete, r has only finitely many turning points, and at these turning points some condition —going back to Beals [1984] —is satisfied. Recall that the points of $\bar{\Delta}_+\cap\bar{\Delta}_-$ are called the *turning points* of r.

2.3. The following result can also be proved by means of Glazman's decomposition method, using Theorem 1 of Jonas and Langer [1979].

PROPOSITION 3. *Suppose that the condition (A1) is satisfied and that r has only a finite number of turning points. If the set Δ_- has a positive distance from all the singular boundary points a or b, then $\sigma(A)\cap(-\infty,0)$ is discrete with the only accumulation point $-\infty$.*

For a special differential operator, this structure of $\sigma(A)$ was established in Mikulina [1971].

Finally, we mention that in the special case $\kappa_A = 0$ (that is, $[Af,f] \geq 0$ for $f \in D(A)$) and $0 \notin \sigma(A)$ the eigenvalues $s_j^\pm$, $j = 1,2,\ldots$, can be characterized by means of minimax principles (see Phillips [1970] and Textorius [1974]).

3. Regularity of the Critical Point Infinity

3.1. The turning point x_0 of the weight function r is said to be *n-simple* if there exists an interval I_0 around x_0 such that for $x \in I_0 \setminus \{x_0\}$ representation

$$r(x) = \operatorname{sgn}(x-x_0) \cdot |x-x_0|^{\alpha}\rho(x) \tag{3.1}$$

holds with some $\alpha > -\frac{1}{2}$ and a function ρ:

$$\rho(x) := \rho_+(x), \quad x > x_0,$$

$$\rho(x) := \rho_-(x), \quad x < x_0,$$

where $\rho_+ (\rho_-)$ is defined and of class C^n on $I_0 \cap [x_0, \infty)$ ($I_0 \cap (-\infty, x_0]$, resp.), $0 \neq \operatorname{sgn}\rho_+(x_0) = \operatorname{sgn}\rho_-(x_0)$ and for the one-sided derivatives at x_0 we have

$$\rho_\pm'(x_0) = \rho_\pm''(x_0) = \ldots = \rho_\pm^{(n-1)}(x_0) = 0 \quad \text{if } n > 1.$$

THEOREM 2. *Suppose that the following conditions are satisfied:*

1. *The problem (1.1) is regular.*

2. *The weight function r has only a finite number of turning points that are all n-simple.*

3. *There exists a $\delta > 0$ such that for each turning point x_0 of r we have*

$$0 < \inf_{\substack{|x-x_0| \leq \delta \\ x \in (a,b)}} p_0(x) \leq \sup_{\substack{|x-x_0| \leq \delta \\ x \in (a,b)}} p_0(x) < \infty.$$

Then infinity is not a singular critical point for every selfadjoint extension A of $A^0_{\min}$ in L^2_r.

We shall only sketch the proof. Propositions 1 and 2 and Theorem 1 imply that A is definitizable. We show that for A there exists an operator W with the properties given by Curgus [1984: Remark 3.6], and an application of a proposition given in that paper [Curgus, 1984: Proposition 3.5] yields the desired result.

To simplify the construction of W, we suppose $a < 0 < b$ and that $x_0 = 0$ is the only turning point of r. Let $\delta > 0$ be such that $(-\delta, \delta) \subseteq (a, b)$ and I_0 in (3.1) can be chosen to be $(-\delta, \delta)$.

We choose $2n$ mutually distinct points $t_1, \ldots, t_{2n} \in (1,2)$ and define the functions

$$h_j(x) := \frac{1}{t_j^{\alpha}} \frac{\rho(x)}{\rho(-t_j x)} \quad (x \in [-\frac{\delta}{2}, \frac{\delta}{2}], \ x \neq 0).$$

By $\mathbf{D}$ we denote the set of all functions $f \in L^2_r$ which have an absolutely continuous $(n-1)$-st derivative and for which

$$\int_a^b p_0 |f^{(n)}|^2 |r| \, dx < \infty.$$

Further, we choose $\varphi \in C^n(a,b)$, which is constant in a neighborhood of zero, $\varphi(0) = 1$, and $\operatorname{supp}\varphi \subseteq [-\frac{\delta}{2}, \frac{\delta}{2}]$.

Now we define linear operators $X_\pm$, $Y_\pm$ in L_r^2 as follows:

$$(X_+u)(x) := u(x), \qquad x \in (0,b]$$

$$(X_+u)(x) := \varphi(x)\sum_{j=1}^{2n} \alpha_j t_j u(-t_j x), \quad x \in [a,0),$$

$$(Y_+u)(x) := 0, \qquad x \in [a,0),$$

$$u(x) + \sum_{j=1}^{2n} \alpha_j (\varphi h_j u)(-\frac{x}{t_j}), \quad x \in (0,b],$$

where $\alpha_1, \ldots, \alpha_{2n}$ are reals to be chosen below. It is not hard to see that X_+, Y_+ are bounded in L_r^2. Moreover, the numbers $\alpha_1, \ldots, \alpha_{2n}$ can be chosen such that X_+, Y_+ map **D** into itself. In order to see this with $u \in \mathbf{D}$, we form the first n derivatives of X_+u on $[a,0)$ and on $(0,b]$. Then X_+u will have $n-1$ absolutely continuous derivatives on $[a,b]$ if and only if for the first $n-1$ derivatives the limits from the left and from the right at zero coincide, which is equivalent to the equations

$$\sum_{j=1}^{2n} \alpha_j t_j^{k+1} = (-1)^k \quad (k = 0,1,\ldots,n-1). \tag{3.2}$$

A similar reasoning for Y_+ yields the equations

$$\sum_{j=1}^{2n} \alpha_j t_j^{-k-a} = (-1)^{k+1} \frac{\rho_+(0)}{\rho_-(0)} \quad (k = 0,1,\ldots,n-1). \tag{3.3}$$

The system (3.2), (3.3) determines the numbers α_j, $j = 1,2,\ldots,2n$ uniquely. It is easy to check that the operators X_+, Y_+ satisfy the relation $X_+ = Y_+^*J$ where * denotes the adjoint in $L_{|r|}^2$.

In the same way, exchanging the roles of $[a,0)$ and $(0,b]$, operators X_-, Y_- with similar properties are defined. Finally, put

$$W := Y_+X_+ + Y_-X_-.$$

As in Curgus [1984: Remark 3.6], it follows that W is positive, bounded, and boundedly invertible in the Krein space L_r^2. Moreover,

$$(Wu)(x) = u(x) \quad \text{if } x \in [a,b] \setminus (-\frac{\delta}{2}, \frac{\delta}{2}).$$

We mention that $X_\pm$, $Y_\pm$ here do not necessarily have the property (a) given in Curgus [1984: Remark 3.6].

The set $\mathbf{D}[JA]$ (see Krein [1947] and Curgus [1984]) consists of those functions of **D** that satisfy the essential boundary conditions. As the function Wu coincides near a and b with u, it satisfies the same boundary conditions as u; hence $W\mathbf{D}[JA] \subset \mathbf{D}[JA]$. Thus W has all the desired properties.

The construction of the operators $X_\pm$, $Y_\pm$ follows [Beals, 1984: Lemma 1].

3.2. Under the conditions of Theorem 2 we denote by $P_{j,\pm}$ the orthogonal projection in the Krein space L_r^2 onto the root space of A corresponding to $s_j^\pm$, $j = 1,2,\ldots$, and by P_0 the orthogonal projection onto the (finite dimensional) span of the root spaces corresponding to the (possible) eigenvalue zero and to the nonreal eigenvalues of A.

COROLLARY. *Under the conditions of Theorem 2 we have for arbitrary* $f \in L_r^2$

$$f = P_0 f + \sum_{j=1}^{\infty} P_{j,+} f + \sum_{j=1}^{\infty} P_{j,-} f ,$$

where both sums converge in the norm of $L^2_{|r|}$.

We mention that for all the points $s_j^{\pm}$ with the property $\kappa_+(s_j^+) = \kappa_+(s_j^-) = 0$ there can be chosen an orthogonal basis of eigenvectors $e^{\pm}_{j,k}$ in $P_{j,\pm} L^2_r$, $k = 1,\ldots,\nu_j^{\pm}$, $j = 1,2,\ldots$, such that

$$P_{j,\pm} = \sum_{k=1}^{\nu_j^{\pm}} \frac{[\,\cdot\,, e^{\pm}_{j,k}]}{[e^{\pm}_{j,k}, e^{\pm}_{j,k}]} .$$

The corollary contains, for example, the full-range expansions of the "regular" examples in Kaper, Kwong, Lekkerkerker, and Zettl [1984]. We mention that the above construction of W and hence the statement of Theorem 2 can also be extended to some singular operators. This extension will be considered elsewhere.

3.3. Suppose now for a moment that (under the conditions of Theorem 2) we have $\kappa_A = 0$ and $0 \notin \sigma_p(A)$. Then $\sigma(A)$ consists of the two sequences (s_j^+), (s_j^-), and we have $\kappa_-(s_j^+) = \kappa_+(s_j^-) = 0$, $j = 1,2,\ldots$ Moreover, the subspace

$$c.l.s.\,\{P_{j,+} L^2_r : j = 1,2,\ldots\}$$

is a maximal nonnegative subspace of the Krein space L^2_r (see Bognár [1974] and Langer [1982]). If we denote by $\mathbf{K}_+$ the subspace

$$\mathbf{K}_+ := \{f \in L^2_r : f(x) = 0 \quad \text{if } x \in \Delta_-\}$$

and by P_+ the orthogonal projection onto $\mathbf{K}_+$ in L^2_r, it follows that for arbitrary $f_+ \in \mathbf{K}_+$ we have

$$f_+ = \sum_{j=1}^{\infty} P_+ P_{j,+} f_+, \tag{3.4}$$

where the series converges again in the norm of $L^2_{|r|}$. This is an abstract form of the half-range expansion considered, for example, in Beals [1984] and Kaper, Kwong, Lekkerkerker, and Zettl [1984].

If $\kappa_A \geq 0$ we consider for arbitrary $f_+ \in \mathbf{K}_+$ the sum

$$\sum_{j:\kappa_-(s_j^+)=0} P_+ P_{j,+} f_+ .$$

It converges in the norm of $L^2_{|r|}$; however, it equals f_+ only for $f_+ \in \mathbf{K}_+'$, where $\mathbf{K}_+'$ is a subspace of K_+ such that $\dim \mathbf{K}_+ / \mathbf{K}_+' < \infty$. To expand arbitrary elements of $\mathbf{K}_+$, we have to add to the elements of $P_+ P_{j,+} L^2_r$, $\kappa_-(s_j^+) = 0$, finitely many elements h_k^+ that are the projections onto K_+ of root vectors h_k of A corresponding to the possible eigenvalue zero, the eigenvalues $s_j^{\pm}$ with $\kappa_-(s_j^+) > 0$, $\kappa_+(s_j^-) > 0$ and to nonreal eigenvalues. A minimal set of such elements h_k^+ which have to be added can easily be found from the condition that the linear span of these root vectors and of c.l.s. $\{P_{j,+} L^2_r : \kappa_-(s_j^+) = 0,\ j = 1,2,\ldots\}$ is a maximal nonnegative subspace of the Krein space L^2_r.

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LINEAR RELATIONS IN INDEFINITE INNER PRODUCT SPACES

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Abstract

The spectral theory of selfadjoint operators in Pontryagin spaces is well developed; see Bognar [1974], Iohvidov, Krein, and Langer [1982], and Krein and Langer [1971/72]. Even in Krein spaces there is a spectral theory for a class of selfadjoint operators, namely, the definitizable operators; cf. Langer [1982]. However, in a number of problems it is not sufficient to consider operators; instead one has to consider relations (multi-valued operators). The spectral theory of such relations has been studied by Coddington, Dijksma, and de Snoo (among others), in the case of Hilbert spaces. In the more general case of Pontryagin spaces or Krein spaces, recent work has been done by Ricner [1982] and Sorjonen [1978/79 and 1980]. Here we report on some results for relations in such spaces. Details will appear in Dijksma and de Snoo [1984]. Also, we report on some results of our joint work with Langer [Dijksma, Langer, and de Snoo, 1984]. This last paper contains applications to boundary value problems. Other applications can be found in the area of "left-definite" eigenvalue problems; see Coddington and de Snoo [1981] for the Hilbert space case, and also Langer [1972].

1. Some General Results

Let K be a Banach space and provide K^2 with the usual topology. Let $A \subset K^2$ be a linear relation or linear manifold. We define the set of ***points of regular type*** by

$$\gamma(A) = \{\lambda \in \mathbf{C} \mid (A-\lambda)^{-1} \text{ is a bounded operator}\},$$

and the ***resolvent set*** $\rho(A)$ by

$$\rho(A) = \{\lambda \in \gamma(A) \mid \mathbf{R}(A-\lambda) \text{ is dense in } K\}.$$

It can be shown that $\gamma(A)$ and $\rho(A)$ are open sets. In addition, we assume that A is a subspace, i.e., a closed linear manifold, such that $\rho(A) \neq \phi$. We define the resolvent operator $R_A : \rho(A) \to [K]$ by

$$R_A(\lambda) = (A-\lambda)^{-1}, \quad \lambda \in \rho(A),$$

where $[K]$ is the set of all bounded linear operators from all of K into K. We have the ***resolvent equation***

$$R_A(\lambda) - R_A(\mu) = (\lambda-\mu) R_A(\lambda) R_A(\mu), \quad \lambda, \mu \in \rho(A),$$

which implies that $R_A : \rho(A) \to [K]$ is analytic. Note that

$$\nu(R_A(\lambda)) = A(o), \quad \lambda \in \rho(A),$$

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where $\nu(R_A(\lambda))$ indicates the null-space of $R_A(\lambda)$ and

$$A(o) = \{g \in K \mid \{o,g\} \in A\}.$$

REMARK. If $\phi \neq \Omega \subset \mathbf{C}$ and $R:\Omega \to [K]$ is a ***pseudo-resolvent***, i.e.,

$$R(\lambda) - R(\mu) = (\lambda-\mu)R(\lambda)R(\mu), \quad \lambda,\mu \in \Omega,$$

then there exists a unique subspace $A \subset K^2$ such that

$$\Omega \subset \rho(A), \quad R = R_A|_\Omega.$$

Let $p:\mathbf{C} \to \mathbf{C}$ be a complex-valued polynomial $p(\lambda) = \sum_{k=0}^{n} c_k(\lambda-\alpha)^k$, $\alpha \in \mathbf{C}$, $c_n \neq 0$. Then we define for the linear manifold $A \subset K^2$

$$p(A) = \sum_{k=0}^{n} c_k (A-\alpha)^k.$$

If $A \subset K^2$ is a subspace with $\rho(A) \neq \phi$, we have the following results which we state for later reference:

(i) $(p(A)$ is a subspace (closed linear manifold),

(ii) $R_A(\lambda)^n p(A) R_A(\mu)^n \in [K]$, $\lambda,\mu \in \rho(A)$.

We define the ***semi-Fredholm sets*** $\Phi_\pm(A)$ by

$$\Phi_+(A) = \{\lambda \in \mathbf{C} \mid \mathbf{R}(A-\lambda) \text{ is closed, } \dim \nu(A-\lambda) < \infty\},$$

and

$$\Phi_-(A) = \{\lambda \in \mathbf{C} \mid \mathbf{R}(A-\lambda) \text{ is closed, } \dim K/\mathbf{R}(A-\lambda) < \infty\}.$$

THEOREM 1. *Let $A \subset K^2$ be a subspace. Then*

(i) $\Phi_\pm(A)$ is open,

(ii) $\dim \nu(A-\lambda) - \dim K/\mathbf{R}(A-\lambda)$ is constant on components of $\Phi_\pm(A)$, and

(iii) $\dim \nu(A-\lambda)$ and $\dim K/\mathbf{R}(A-\lambda)$ are constant on components of $\Phi_\pm(A)$ except at isolated points, where they are strictly larger than the said constants.

REMARK. Let U be a component of $\Phi_+(A)$. Then for all $\lambda \in U$

$$\dim(\nu(A-\lambda) \cap \bigcap_{n \in \mathbf{N}} \mathbf{R}(A-\lambda)^n)$$

is equal to the constant value that $\dim \nu(A-\lambda)$ assumes on U, except at isolated points. Hence it is clear that $\mu \in U$ is not an exceptional point if and only if

$$\nu(A-\mu) \subset \mathbf{R}(A-\mu)^n, \quad n \in \mathbf{N}.$$

A similar remark applies to $\dim K/\mathbf{R}(A-\lambda)$.

COROLLARY (Constancy of Deficiency Index). *Let $A \subset K^2$ be a subspace. Then*

$$\dim K/\mathbf{R}(A-\lambda),$$

is constant on components of $\gamma(A)$, the set of points of regular type.

As to the proof of the theorem, it is possible to reduce the given result to a corresponding result for a pair of bounded linear operators, as given by Gohberg and Krein [1960] and by Kato [1966].

For a subspace $A \subset K^2$, we define $F(A)$ to be the set of all complex-valued functions which are analytic in some neighborhood of $\tilde{\sigma}(A)$, the extended spectrum of A. Note $\tilde{\sigma}(A) = \sigma(A) \cup \{\infty\}$, when $A \notin [K]$; here $\sigma(A)$ denotes the spectrum of A: $\sigma(A) = \mathbf{C} \setminus \rho(A)$. If $A \subset K^2$ is a subspace with $\rho(A) \neq \phi$ and $A \notin [K]$, then we define for $f \in F(A)$

$$f(A) = f(\infty)I - \frac{1}{2\pi i} \int_\Gamma f(\lambda) R_A(\lambda) d\lambda,$$

where Γ is a suitable contour, surrounding the extended spectrum $\tilde{\sigma}(A)$. Then $f(A) \in [K]$ and has the usual properties. In particular, if $\sigma \subset \sigma(A)$ is a bounded spectral set of A, and

$$E_A(\sigma) = -\frac{1}{2\pi i} \int_\Gamma R_A(\lambda) d\lambda,$$

where Γ is a closed contour in $\rho(A)$, such that σ is inside Γ and $\sigma(A) \setminus \sigma$ is outside Γ, then $E_A(\sigma)$ is a projection, the so-called *Riesz-projection*.

If $A \subset K^2$ is a linear manifold and $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a 2×2 complex-valued matrix, we define the linear manifold MA by

$$MA = \{\{\alpha f + \beta g, \gamma f + \delta g\} \mid \{f, g\} \in A\}.$$

The Cayley and inverse Cayley transforms are defined by the matrices

$$\begin{pmatrix} -\mu & 1 \\ -\bar{\mu} & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1 \\ -\bar{\mu} & \mu \end{pmatrix},$$

respectively.

If $A \subset K^2$ is a subspace with $\rho(A) \neq \phi$ and if $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is such that $\beta \neq 0$ and $-\frac{\alpha}{\beta} \in \rho(A)$, then

$$MA = \frac{\delta}{\beta} I - \frac{\det M}{\beta^2} R_A\left(-\frac{\alpha}{\beta}\right).$$

In addition, if $A \notin [K]$ and

$$f(\lambda) = \frac{\delta\lambda + \gamma}{\beta\lambda + \alpha}, \quad \lambda \neq -\frac{\alpha}{\beta},$$

then $f \in F(A)$ and $f(A) = MA$.

2. Krein Spaces and Pontryagin Spaces

Let K be a linear space over $\mathbf{C}$, and let $[\ ,\ ]: K \times K \to \mathbf{C}$ be a sesquilinear form:

$$\begin{cases} \forall g \in K, \ f \to [f, g] \text{ is linear,} \\ \forall f, g \in K, \ [f, g] = \overline{[g, f]}, \end{cases}$$

which is non-degenerate:

$$[f, g] = 0 \quad \forall g \in K \Rightarrow f = 0.$$

Then $(K, [\ ,\ ])$ is called a *Krein space* if $K = K_+ + K_-$, where $K_\pm \subset K$ are linear manifolds, such that

$$(K_\pm, \pm[\,.\,]\,|_{K_\pm \times K_\pm})$$ are Hilbert spaces,

and

$$[K_+, K_-] = 0.$$

In a Krein space the decomposition $K = K_+ + K_-$ is a direct sum. Let $P_\pm : K \to K_\pm$ be the corresponding projections and put $J = P_+ - P_-$. Then $(\,.\,) : K \times K \to \mathbf{C}$ defined by

$$(f, g) = [Jf, g]$$

is an inner product on K and $(K, (\,.\,))$ is a Hilbert space. In general, there are many decompositions. If $K = K'_+ + K'_-$ with $P'_\pm$ and J' is another such decomposition, then the norms corresponding to $(\,.\,)$ and $(\,.\,)'$, defined by $(f, g)' = [J'f, g]$ are equivalent and $\dim K_\pm = \dim K'_\pm$. We shall use the topological notions with respect to one of these norms, and hence with respect to all of them.

$(K, [\,.\,])$ is called a Pontryagin space if $\dim K_+ < \infty$ or $\dim K_- < \infty$. In the sequel we shall always assume for Pontryagin spaces that $\kappa = \dim K_- < \infty$ and denote such spaces by $\mathbf{H}_\kappa$.

Orthogonality with respect to $[\,.\,]$ is denoted by $[\perp]$; for $L \subset K$ we define

$$L^{[\perp]} = \{f \in K \mid f [\perp] L\}.$$

A linear manifold $L \subset K$ is called ***non-negative*** if $[f, f] \geq 0$, $f \in L$, ***non-positive*** if $[f, f] \leq 0$, $f \in L$, and ***neutral*** if $[f, f] = 0$, $f \in L$. Maximality with respect to each of these notions is defined in the usual way. A linear manifold L is maximal neutral if and only if L is non-negative and non-positive and maximal for one of these notions. A linear manifold L is hypermaximal neutral if L is maximal non-negative and maximal non-positive.

Example. In $\mathbf{C}^p$ we consider a symmetric, non-degenerate form $[\,.\,]$, given by

$$[f, g] = g^* Q f, \quad f, g \in \mathbf{C}^p,$$

where Q is a symmetric, invertible operator. Let (σ_+, σ_-) be the signature of $[\,.\,]$; then $\sigma_+ + \sigma_- = p$. For a linear manifold $L \subset \mathbf{C}^p$ we have

L maximal non-negative	$\Leftrightarrow$ L non-negative,	$\dim L = \sigma_+$,
L maximal non-positive	$\Leftrightarrow$ L non-positive,	$\dim L = \sigma_-$,
L maximal neutral	$\Leftrightarrow$ L neutral,	$\dim L = \min(\sigma_+, \sigma_-)$,
L hypermaximal neutral	$\Leftrightarrow$ L neutral,	$\dim L = \sigma_+ = \sigma_-$.

Note that

L neutral	$\Leftrightarrow$ $L \subset L^{[\perp]}$,
L hypermaximal neutral	$\Leftrightarrow$ $L = L^{[\perp]}$.

We have the following result. Let $L \subset \mathbf{C}^p$ be a linear manifold. If

(i) L is maximal non-negative,
(ii) L is maximal non-positive,
(iii) L is maximal neutral,
(iv) L is hypermaximal neutral,

then there exists a $\sigma \times p$ matrix $\mathbf{M}$ such that

(+) rank $\mathbf{M} = \sigma$,

with the following properties, respectively,

(i') $\sigma = \sigma_-$, $\quad \mathbf{M}Q^{-1}\mathbf{M}^* \leq O_\sigma^\sigma$,
(ii') $\sigma = \sigma_+$, $\quad \mathbf{M}Q^{-1}\mathbf{M}^* \geq O_\sigma^\sigma$,
(iii') $\sigma = \max(\sigma_+, \sigma_-)$, $\quad \mathbf{M}Q^{-1}\mathbf{M}^* = O_\sigma^\sigma$,
(iv') $\sigma = \sigma_+ = \sigma_-$, $\quad \mathbf{M}Q^{-1}\mathbf{M}^* = O_\sigma^\sigma$,

such that

(++) $\quad L = \{f \in \mathbf{C}^p \mid \mathbf{M}f = O^{1\sigma}\}$.

The matrix $\mathbf{M}$ is unique, up to multiplication on the left by an invertible $\sigma\times\sigma$ matrix. Conversely, if we are given a $\sigma\times p$ matrix $\mathbf{M}$ such that (+) holds, and which satisfies (i'), (ii'), (iii'), or (iv'), and the linear manifold L is defined by (++), then L has the property (i), (ii), (iii), or (iv), respectively. These results can be applied to boundary value problems; cf. Coddington and de Snoo [1981] and Langer and Textorius [1982].

3. Linear Relations in Krein Spaces

Let K be a Krein space with inner product $[\ ,\ ]$. Let $A \subset K^2$ be a linear manifold. Then its adjoint A^+ is defined by

$$A^+ = \{\{f,g\} \in K^2 \mid [g,h] = [f,k] \text{ for all } \{h,k\} \in A\}.$$

Note that

$$A^+ = JA^*J,$$

where A^* is the adjoint in the Hilbert space $(K, [J\ \cdot\ ,\ \cdot\ ])$.

If $A \subset K^2$ is a subspace, then we have

$$\mathbf{R}(A-\lambda) \text{ closed} \Leftrightarrow \mathbf{R}(A^+ - \bar{\lambda}) \text{ closed};$$

cf. Coddington and Dijksma [1978].

A linear manifold $A \subset K^2$ is called

dissipative if $\operatorname{Im}[g,f] \geq 0$, $\{f,g\} \in A$,

symmetric if $\operatorname{Im}[g,f] = 0$, $\{f,g\} \in A$, or equivalently if $A \subset A^+$,

selfadjoint if $A = A^+$.

A linear manifold $A \subset K^2$ is called

contractive if $[g,g] \leq [f,f]$, $\{f,g\} \in A$,

isometric if $[g,g] = [f,f]$, $\{f,g\} \in A$, or equivalently if $A^{-1} \subset A^+$,

unitary if $A^{-1} = A^+$.

We have introduced the Cayley transform $C_\mu(A)$ of a linear manifold $A \subset K^2$,

$$C_\mu(A) = \{\{g - \mu f, g - \bar{\mu} f\} \mid \{f,g\} \in A\}, \quad \mu \in \mathbf{C}.$$

Note that for $\mu \in \mathbf{C} \setminus \mathbf{R}$, we have

$-A$ is dissipative $\Leftrightarrow$ $C_\mu(A)$ is contractive,
A is symmetric $\Leftrightarrow$ $C_\mu(A)$ is isometric,
A is selfadjoint $\Leftrightarrow$ $C_\mu(A)$ is unitary.

REMARK. We may provide K^2 with a new inner product, given by $-i <\,.\,,\,.\,>$, where $<\,.\,,\,.\,>$ is given by

$$<\{f,g\},\{h,k\}> = [g,h] - [f,k], \quad \{f,g\},\{h,k\} \in K^2.$$

We denote orthogonality with respect to this inner product by $<\perp>$. For a linear manifold $A \subset K^2$, we then have

$$A^+ = A^{<\perp>}.$$

We observe that the notions dissipative, symmetric, and selfadjoint are equivalent to the notions non-negative, neutral, and hypermaximal neutral, respectively.

A similar remark may be applied to the notions contractive, isometric, and unitary if we provide K^2 with the inner product

$$[f,h] - [g,k], \quad \{f,g\},\{h,k\} \in K^2.$$

Example. The above remark in connection with the example in Section 2 may be used to characterize maximal dissipative, selfadjoint, maximal contractive, and unitary linear manifolds in finite dimensional spaces. Consider the inner product space $(\mathbf{C}^p,[\,.\,,\,.\,])$, where $[f,g] = g^*Qf$, $f,g \in \mathbf{C}^p$ as in the example of Section 2. Let $A \subset (\mathbf{C}^p)^2$ be a linear manifold. If

(i) A is maximal dissipative (selfadjoint),
(ii) A is maximal contractive (unitary),

then there exist $p \times p$ matrices M,N, such that

(+) rank $(M:N) = p$,

with the following properties, respectively,

(i') $\operatorname{Im} MQ^{-1}N^* \leq O_p^p$ $(\operatorname{Im} MQ^{-1}N^* = O_p^p)$,
(ii') $MQ^{-1}M^* - NQ^{-1}N^* \leq O_p^p$ $(MQ^{-1}M^* - NQ^{-1}N^* = O_p^p)$,

such that

(++) $A = \{\{f,g\} \in (\mathbf{C}^p)^2 \mid Mf + Ng = O_p\}$.

The matrices M,N are unique, up to simultaneous multiplication on the left by an invertible $p \times p$ matrix. Conversely, if we are given $p \times p$ matrices M,N such that (+) holds and which satisfy (i') or (ii'), and the linear manifold A is defined by (++), then A has the property (i) or (ii), respectively.

In the rest of the paper we shall concentrate on symmetric and selfadjoint linear manifolds in Krein spaces (and in Pontryagin spaces). If A is a symmetric subspace in K^2 with $\rho(A) \neq \phi$, then we may choose $\mu \in \rho(A) \setminus \mathbf{R}$, and $C_\mu(A)$ is an isometric operator belonging to $[K]$. Conversely, if $U \in [K]$ is isometric and $\mu \in \mathbf{C} \setminus \mathbf{R}$, then the inverse Cayley transform $F_\mu(U)$ is a symmetric subspace, such that $\mu \in \rho(F_\mu(U))$. These and similar observations can be used to translate properties from one situation to another. We shall not state further results in this direction, but we refer to Dijksma and de Snoo [1984].

A selfadjoint subspace $A \subset K^2$ is called *definitizable* if $\rho(A) \neq \phi$ and if there exists a real polynomial p such that $p(A) \geq 0$, meaning that

$$[g,f] \geq 0 \quad \text{for all } \{f,g\} \in p(A).$$

THEOREM 2. *Let $A\subset K^2$ be selfadjoint with $\rho(A)\neq\phi$, and $\alpha\in\rho(A)$. Let p be a real polynomial of degree n. Then*

$$p(A)\geq 0 \iff R_A(\alpha)^n p(A) R_A(\bar{\alpha})^n \geq 0.$$

As a consequence we can prove, by means of the operational calculus, the following corollary.

COROLLARY. *If the selfadjoint subspace $A\subset K^2$ is definitizable, then A^{-1} is also definitizable.*

It turns out that for definitizable subspaces a spectral theory can be developed; for the case of definitizable operators we refer, for instance, to Langer [1982]. We shall not present the details here, but conclude with the following observation.

PROPOSITION. *Let K be a Pontryagin space and let $A\subset K^2$ be selfadjoint with $\rho(A)\neq 0$. Then A is definitizable.*

4. Symmetric Relations in Pontryagin Spaces

We now consider relations in Pontryagin spaces. We shall use the notation $\mathbf{H}_k$ for a Pontryagin space with k negative squares ($k<\infty$). First we note that if $A\subset\mathbf{H}_k^2$ is a linear manifold with $\mathbf{R}(A-\lambda)$ closed, then we have

$$\dim \mathbf{H}_k/\mathbf{R}(A-\lambda) = \dim \nu(A^+-\bar{\lambda});$$

see Iohvidov, Krein, and Langer [1982: Theorem 3.4] or Bognar [1974]. Hence we observe

$$\Phi_+(A) = \Phi_-(A^+),$$

and the index

$$\dim \nu(A-\lambda) - \dim \nu(A^+-\bar{\lambda})$$

is constant on components of $\Phi_\pm(A)$. A special case of this situation for non-densely defined multi-valued differential relations may be found in Coddington and de Snoo [1981: Theorem 7.3].

Let $A\subset\mathbf{H}_k^2$ be a symmetric subspace. Then we may prove

$$\mathbf{C}_0\backslash\sigma_p(A)\subset\gamma(A),$$

and

$$\mathbf{R}(A-\lambda) \quad \text{is closed for } \lambda\in\mathbf{C}_0,$$

where $\mathbf{C}_0$ denotes $\mathbf{C}^+\cup\mathbf{C}^-$, and $\sigma_p(A)$ denotes the point spectrum of A: the set of all $\lambda\in\mathbf{C}$ for which there exists a non-trivial f such that $\{f,\lambda f\}\in A$.

If $A\subset\mathbf{H}_k^2$ is symmetric, then it is not difficult to see that $\nu(A-\lambda)$, for $\lambda\in\mathbf{C}_0$, is a neutral subspace of $\mathbf{H}_k$, and hence

$$\dim \nu(A-\lambda)\leq k, \quad \lambda\in\mathbf{C}_0;$$

see Iohvidov, Krein, and Langer [1982: Lemma 1.2]. This implies

$$\mathbf{C}^+\subset\Phi_+(A), \quad \mathbf{C}^-\subset\Phi_+(A).$$

THEOREM 3. *Let $A\subset\mathbf{H}_k^2$ be a symmetric linear manifold. Then $\dim\nu(A-\lambda)$ is constant on $\mathbf{C}^+$ (or $\mathbf{C}^-$), say $\dim\nu(A-\lambda)=n$, with the exception of at most k-n points in $\mathbf{C}^+$ (or $\mathbf{C}^-$). Hence also $\dim\nu(A^+-\bar{\lambda})$ is constant on $\mathbf{C}^+$(or $\mathbf{C}^-$) outside the same exceptional set.*

A proof of this theorem may be given using the remark following Theorem 1 and the fact that

$$\nu(A-\lambda)\subset \bigcap_{n\in\mathbb{N}} \mathbf{R}(A-\mu)^n, \quad \lambda,\mu\in\mathbf{C}, \quad \lambda \neq \mu.$$

In particular, if in Theorem 3 we have $n = 0$, then the defect index $\dim(A^+-\bar{\lambda})$ is constant in $\mathbf{C}^+$ (or $\mathbf{C}^-$), say $\dim\nu(A^+-\bar{\lambda}) = m$, with the exception of at most k points in $\mathbf{C}^+$ (or $F\mathbf{C}^-$), where

$$\dim\nu(A^+-\bar{\lambda}) = m + \dim\nu(A-\lambda);$$

cf. Iohvidou, Krein, and Langer [1982: Theorem 6.1].

If in Theorem 3 $n > 0$, then we can show $\sigma_p(A) = \mathbf{C}$, and hence we have the following alternative. Let $A\subset\mathbf{H}_k^2$ be a symmetric linear manifold. Then either $|\sigma_p(A)\cap\mathbf{C}^+| \leq k$ and $|\sigma_p(A)\cap\mathbf{C}^-| \leq k$, or $\sigma_p(A) = \mathbf{C}$.

An eigenvalue $\lambda\in\sigma_p(A)$ is called *critical* if there exists an element $\varphi\in\mathbf{H}_k$, $\varphi \neq 0$, such that $\{\varphi,\lambda\varphi\}\in A$ and $[\varphi,\varphi] \leq 0$. Similarly ∞ is called a *critical* eigenvalue if there exists an element $\varphi\in\mathbf{H}_k$, $\varphi \neq 0$, such that $\{0,\varphi\}\in A$ and $[\varphi,\varphi] \leq 0$.

THEOREM 4. *Let $A\subset\mathbf{H}_k^2$ be a symmetric linear manifold with $\sigma_p(A) \neq \mathbf{C}$. Then A has at most k critical eigenvalues in $\mathbf{C}^+\cup\mathbf{R}\cup\{\infty\}$, and at most k critical eigenvalues in $\mathbf{C}^-\cup\mathbf{R}\cup\{\infty\}$.*

We shall not give further properties concerning eigenvalues and root manifolds here, but state a result concerning maximal symmetric linear manifolds.

THEOREM 5. *Let $A\subset H_k^2$ be a symmetric subspace. Then $\rho(A) \neq \phi$ if and only if A is maximal and $\sigma_p(A) \neq \mathbf{C}$. In particular, if A is a selfadjoint subspace, then $\rho(A) \neq \phi$ if and only if $\sigma_p(A) \neq \mathbf{C}$.*

We close this section with the remark that in the case of symmetric relations the situation with regard to the degeneracy of the spaces $\nu(A^+-\lambda)$ is more complicated than in the case of densely symmetric operators; cf. Krein and Langer [1971/72].

5. Compressed Resolvents

This section contains some results of our work with Langer [Dijksma, Langer, and de Snoo [1984]. Let $\mathbf{H}$ be a Hilbert space, and let $\mathbf{H}_k$ be a Pontryagin space with k negative squares. We use the notation $\mathbf{H}\subset_s\mathbf{H}_k$ to indicate that $\mathbf{H}$ is a subspace of $\mathbf{H}_k$ and that the indefinite inner product of $\mathbf{H}_k$ restricted to $\mathbf{H}$ coincides with the Hilbert space scalar product of $\mathbf{H}$. Let A be a subspace in $\mathbf{H}_k^2$ with $\rho(A) \neq \phi$. On $\rho(A)$ we study the locally holomorphic $[\mathbf{H}]$-valued function R defined by

$$R(\lambda) = PR_A(\lambda) \mid_{\mathbf{H}}, \quad \lambda\in\rho(A),$$

where P denotes the orthogonal projection from $\mathbf{H}_k$ onto $\mathbf{H}$. This function R is called the *compressed resolvent* of A in $\mathbf{H}$. It is possible to characterize the compressed resolvents of various subspaces in $\mathbf{H}_k^2$. We shall give the result for the situation where $A\subset A^+$, $\rho(A)\neq\phi$. First we introduce the following notions.

If $A\subset\mathbf{H}_k^2$ is a symmetric subspace with $\rho(A)\cap\mathbf{C}^+ \neq \phi$, and $\mathbf{H}\subset_s\mathbf{H}_k$, where $\mathbf{H}$ is a Hilbert space, then A and $\mathbf{H}$ are said to be *closely upper-connected*, if the closed linear span of $\mathbf{H}$ and all ranges $(A-\lambda)^{-1}\mathbf{H}$, $\lambda\in\rho(A)\cap\mathbf{C}^+$ is the space $\mathbf{H}_k$.

Let $\mathbf{D}\subset\mathbf{C}^+$, $\mathbf{D} \neq \phi$ and let $K:\mathbf{D}\times\mathbf{D}\to[\mathbf{H}]$ be a mapping. We say K has k *negative squares* if

(i) it is Hermitian, i.e., $K(l,\lambda) = K(\lambda,l)^*$, $l,\lambda \in \mathbf{D}$,
(ii) for all choices of $n \in \mathbf{N}$, $l_i \in \mathbf{D}$, $f_i \in \mathrm{H}$, $i = 1,\dots,n$, the matrix

$$([K(l_i,l_j)f_i, Ff_j])_{i,j=1,\dots,n},$$

has at most k and for at least one such choice it has exactly k negative squares. Let R be a function defined on some set $\mathbf{D} \subset \mathbf{C}^+$, $\mathbf{D} \neq \phi$, and with values in $[\mathrm{H}]$. By $\mathbf{R}_R$ we denote the kernel

$$\mathbf{R}_R(l,\lambda) = \frac{R(l)-R(\lambda)^*}{l-\bar{\lambda}} - R(\lambda)^* R(l), \quad l,\lambda \in \mathbf{D}.$$

The set of all functions R, which are defined and meromorphic in $\mathbf{C}^+$ with values in $[\mathrm{H}]$, such that the kernel $\mathbf{R}_R$ has k negative squares on the domain of holomorphy of R, will be denoted by $\mathbf{R}_k^+(H)$.

THEOREM 6.

(i) *Let H_k be a Pontryagin space, and let $A \subset \mathbf{H}_k^2$ be a symmetric subspace with $\rho(A) \cap \mathbf{C}^+ \neq \phi$. If $\mathrm{H} \subset_s \mathrm{H}_k$ is a Hilbert space and $R: \rho(A) \to [W]$ is the compressed resolvent in* H *associated with A, then $R \in \mathbf{R}_{k'}^+(H)$ for some $k', 0 \leq k' \leq k$. If A and* H *are closely upper-connected, then $k' = k$.*

(ii) *If $R \in \mathbf{R}_k^+(\mathrm{H})$ is given, then there exists a Pontryagin space H_k, $\mathrm{H} \subset_s \mathrm{H}_k$, and a symmetric subspace $A \subset \mathbf{H}_k^2$ with $\rho(A) \cap \mathbf{C}^+ \neq \phi$, such that R is the compressed resolvent in* H *associated with A. Here H_k and A can be chosen such that A and H_k are closely upper-connected, in which case A and H_k are uniquely determined up to isomorphisms, which restricted to* H *are equal to the identity operator on* H.

Similar results are valid for the case where A is selfadjoint in $\mathbf{H}_k^2$ with $\rho(A) \neq \phi$. Also these results can be translated into results for the associated *family of Štraus subspaces*, defined by

$$T(l) = R(l)^{-1} + l, \quad l \in \rho(A).$$

Assume we are given a symmetric subspace $S \subset \mathbf{H}^2$, where H is a Hilbert space. Let A be a selfadjoint subspace in $\mathbf{H}_k^2$ with $\rho(A) \neq \phi$, such that $\mathrm{H} \subset_s \mathrm{H}_k$ and A extends S, i.e., $S \subset A$. Then we have

$$S \subset T(l) \subset S^*, \quad l \in \rho(A).$$

Now $T(l)/S$ can be completely determined in the boundary space S^*/S, analogous to the above theorem. In this way it is possible to associate subspaces in Pontryagin spaces with concrete boundary value problems. For details, we refer the reader to Dijksma, Langer, and de Snoo [1984].

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SPECTRUM OF SELFADJOINT AND POSITIVE OPERATORS WITH COMPACT INVERSE

J. Fleckinger

Abstract

This paper is a rough English translation of a seminar held in Orsay several years ago. It surveys different methods for studying the spectrum of elliptic operators. In particular, two methods for estimating $N(\lambda)$ are given: 1) the transform of the spectral function, and 2) the "max-min" formula. An appendix is devoted to the variational formulation of elliptic problems.

Notations and Main Results on Sobolov Spaces

We introduce some notations used throughout the paper:

Let m and n be integers greater than or equal to 1.

Let Ω be an open set in $\mathbf{R}^n$ with boundary $\partial\Omega$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we denote by D^α the derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$: $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ with $D_j = \partial / \partial x_j$.

$H^m(\Omega)$ denotes the usual Sobolev space of order m; i.e., $H^m(\Omega) = \{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}$. $H^m(\Omega)$ equipped with the norm $\|u\|_{H^m(\Omega)} = \{\int_\Omega (\sum_{|\alpha| \le m} |D^\alpha u(x)|^2) dx\}^{1/2}$ is a Hilbert space. Recall that if $\partial\Omega$ is "smooth enough," $H^m(\Omega)$ is the restriction to Ω of elements of $H^m(\mathbf{R}^n)$.

$H_0^m(\Omega)$ is the completion of $\mathbf{C}_0^\infty(\Omega)$ with respect to the norm of $H^m(\Omega)$; here $\mathbf{C}_0^\infty(\Omega)$ — or $\mathbf{D}(\Omega)$ — is the space of infinitely differentiable functions with compact support in Ω.

REMARK. When $\partial\Omega$ is "smooth enough,"

$$H_0^m(\Omega) = \{u \in H^m(\Omega) \mid \operatorname{supp} u \subset \Omega\}.$$

The following results are well known:

- When Ω is bounded, the imbedding of $H_0^m(\Omega)$ into $L^2(\Omega)$ is compact.
- When Ω is bounded, there exists $c(\Omega) > 0$ such that

$$\|u\|^2_{L^2(\Omega)} \le c(\Omega) \sum_{|\alpha| = m} \|D^\alpha u\|^2_{L^2(\Omega)} \qquad \text{for all } u \in H^m(\Omega).$$

1. Introduction

Let H be a separable Hilbert space, and let A be a selfadjoint and unbounded linear operator in H with domain $D(A)$. We suppose that the imbedding of $D(A)$, equipped with the graph norm, into H is compact.

Moreover, we suppose that A is positive; this is always possible by considering $A + tI$ or A^2 in place of A.

When these assumptions are satisfied, the operator $B = A^{-1}$ from H into H is compact, and it is possible to apply the classical results of the spectral theory of selfadjoint operators with compact inverse; see, e.g., Akhiezer and Glazman [1967], Yosida [1968], and Goulaouic [1973].

PROPERTIES.

1.1. The spectrum of A, denoted by $\sigma(A)$, consists only of eigenvalues.

1.2. The eigenvalues of A are real and positive.

1.3. The set of eigenvalues is a countable sequence

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$$

and λ_k tends to $+\infty$ as k tends to $+\infty$.

1.4. The eigenvectors associated respectively with two distinct eigenvalues are orthogonal.

1.5. The eigenspaces L_k are of finite dimension.

1.6. The Hilbert space H is the (Hilbertian) direct sum of the eigenspaces L_k.

2. Spectral Decomposition of an Operator

For further information on spectral decomposition, the reader is referred to Akhiezer and Glazman [1967] and Yosida [1968].

DEFINITION 2.1. *Let L be a subspace in H, and let M be such that $L \oplus M = H$. One says that L "reduces" A if*

(i) *L and M are invariant under A, and*

(ii) *For all $x \in D(A)$, $P_L(x)$ is in $D(A)$; here P_L denotes the orthogonal projection on L.*

REMARK. It follows that $P_M(x)$ is also in $D(A)$.

We now have the following results.

PROPERTIES.

2.1. The subspace L reduces A if and only if P_L and A commute.

2.2. Let $E_1, E_2, \ldots, E_k, \ldots$ be a sequence of orthogonal subspaces such that each E_k reduces A; then $E = \oplus E_k$ reduces A too.

2.3. If λ is an eigenvalue of A, then L_λ, the associated eigenspace, reduces A.

Let us denote by $0 < \lambda_I < \lambda_{II} < \cdots < \lambda_K < \cdots$ the (strictly) increasing sequence of eigenvalues of A, by L_K the associated eigenspaces, and by P_K the projections on L_K. We then have the following property.

2.4. Each $x \in D(A)$ can be written as

$$x = \sum_{K=1}^{\infty} x_K \qquad \text{with } x_K \in L_K$$

and

$$Ax = K = \sum_{1}^{\infty} \lambda_K x_K.$$

DEFINITION 2.2. *A "resolution of the identity" is a family of projections E_λ which depend on the real parameter λ, such that*

(i) *If $\mu \leq \lambda$, then $E_\lambda E_\mu = E_\mu E_\lambda = E_\mu$.*

(ii) *E_λ is right continuous in λ, and*

(iii) *$E_\lambda \to I$ as $\lambda \to +\infty$ and $E_\lambda \to 0$ as $\lambda \to -\infty$ in the strong topology.*

Let us now introduce, like a Stieljes integral, the quantity

$$(Sx,y) = \int_{-\infty}^{+\infty} \lambda \mathrm{d}(E_\lambda x, y), \qquad x \in H,\, y \in H,$$

which defines the operator S.

Coming back to the operator A, we consider

$$E_\lambda = \sum_{\lambda_K \leq \lambda} P_K;$$

E_λ is a resolution of the identity which is called the "spectral family" of A.

THEOREM 2.1. *It is possible to associate with each selfadjoint operator T in a Hilbert space a unique spectral family E_λ such that*

(i) *E_λ is a resolution of the identity, and*

(ii) $D(T) = \{x \in H \mid \int_{-\infty}^{+\infty} \lambda^2 \mathrm{d}(E_\lambda x, x) < \infty\}$.

Then $Tx = \int_{-\infty}^{+\infty} \lambda \mathrm{d}E_\lambda x$ (this is a semiconvergent integral). Conversely, every operator defined by such an integral is selfadjoint.

REMARKS. When T is positive, all eigenvalues are positive and

$$Tx = \int_0^{\infty} \lambda \mathrm{d}E_\lambda x \quad \text{with } D(T) = \{x \in H \mid \int_0^{\infty} \lambda^2 \mathrm{d}(E_\lambda x, x) < \infty\}.$$

If we consider the positive operator $T^{1/2} = \sqrt{T}$ (such that $\sqrt{T}^2 = T$), we have in the same way

$$\sqrt{T}x = \int_0^{\infty} \sqrt{\lambda} \mathrm{d}E_\lambda x \text{ with } D(\sqrt{T}) = \{x \in H \mid \int_0^{\infty} (\sqrt{\lambda})^2 \mathrm{d}(E_\lambda x, x) < \infty.$$

More generally, when f is a continuous function on $\mathbf{R}^+$, it is possible to define

$$fT(x) = \int_0^{\infty} f(\lambda) \mathrm{d}E_\lambda x \text{ with } D(fT) = \{x \in H \mid \int_0^{\infty} [f(\lambda)]^2 \mathrm{d}(E_\lambda x, x) < \infty\}.$$

If λ is a regular value for T, it is possible to define the resolvent $R_\lambda = (T-\lambda I)^{-1}$ by

$$R_\lambda x = \int_0^\infty \frac{dE_\mu x}{\mu-\lambda}.$$

THEOREM 2.2. *The regular values λ of T are such that there exists an interval $(\lambda-\varepsilon, \lambda+\varepsilon)$ in $\mathbf{R}$ on which E_λ is constant.*

THEOREM 2.3. *The eigenvalues λ of T are the real numbers where E_λ has a jump; $E_{\lambda+0} - E_{\lambda-0}$ is the projection on the associated eigenspace.*

3. Transforms of the Spectrum

For further information on spectral transforms, the reader is directed to Boutet de Monvel [1970] and Hörmander [1968].

Let Ω be a bounded open set in $\mathbf{R}^n$; let $\mathbf{A}$ be a differential operator in $\mathbf{D}'(\Omega)$. We suppose that $\mathbf{A}$ is elliptic of order $2m$, formally selfadjoint, and positive on Ω. We consider a boundary value problem in Ω associated with $\mathbf{A}$, and we denote by A its "realization in $L^2(\Omega)$."* We suppose that A is a positive selfadjoint and unbounded operator in $L^2(\Omega)$.

E_λ is defined by a kernel $e(x,y,\lambda)$ which is called the "spectral function" of A:

$$E_\lambda f(x) = \int_\Omega e(x,y,\lambda) f(y) dy.$$

If we denote by $N(\lambda)$ the number of eigenvalues of A less than or equal to λ (each eigenvalue counted according to its multiplicity), $N(\lambda)$ is the trace of E_λ. Therefore, it is possible to use transforms of the spectral function in the study of $N(\lambda)$. Estimates of $N(\lambda)$ can be deduced from estimates of the transform by means of (primarily) Tauberian theorems.

i) **The Stieljes transform**: It leads to the resolvent; for $z \notin \mathbf{R}^+$, we consider $G_1(z) = \int_0^\infty (\lambda-z)^{-1} dE_\lambda = (A-zI)^{-1}$; an asymptotic estimate for $G_1(z)$ can be obtained outside an angle $|\arg z| \le \varepsilon$ (see, e.g., Agmon-Kannaï [1967]).

ii) **The Laplace transform**: It leads to the fundamental solution of the diffusion equation associated with A: $G_2(t) = \int_0^\infty e^{-t\lambda} dE_\lambda$ satisfies $(\frac{\partial}{\partial t} + A) G_2(t) = 0$ with $G_2(0) = I$, $t \ge 0$ (see, e.g., Hörmander [1968] and Minakshisunderam [1953]).

iii) **The Riemann transform**: $G_3(s) = \int_0^\infty \lambda^{-s} dE_\lambda$ when $\mathrm{Re}\, s > n/2m$; $G_3(s)$ is an operator with a kernel $g_3(x,y,s)$. Let us define $\zeta(s) = \mathrm{Tr}\, G_3(s)$; then

$$\zeta(s) = \int_\Omega g_3(x,x,s) dx = \sum_{k=1}^\infty \lambda_k^{-s}.$$

where λ_k denotes the kth eigenvalue (each eigenvalue counted according to its multiplicity); see, e.g., Minakshisundaram [1943] and Minakshisundaram and Pleijel [1949].

*Throughout this paper we write "realization" instead of "realization in $L^2(\Omega)$", i.e., $D(A) = \{u \in L^2(\Omega) \mid \mathbf{A}u = Au \in L^2(\Omega)\}$.

iv) **The Fourier transform:** $G_5(t) = \int_0^\infty e^{it\lambda^{1/2m}} dE_\lambda$ has been introduced by Hörmander [1968]. G_5 satisfies $(i \frac{\partial}{\partial t} - A^{1/2m})G_5(t) = 0$ with $G_5(0) = I$. The spectral function can be deduced from G_5 by the inverse Fourier transform, which is easier to use than a Tauberian theorem as in the previous cases. But this method requires the use of Fourier integral operators: $A^{1/2m}$, which is defined by its spectral function, is no longer a differential operator. In his paper Hörmander shows that there exists a Fourier integral operator $L_{\psi q}(u)$ which describes all the singularities of G_5:

$$L_{\psi q}(u) - G_5(u) \in C^\infty.$$

Then he proves the following theorem.

THEOREM 3.1. $e(x,x,\lambda) = \lambda^{n/2m} c(x) + o(\lambda^{(n-1)/2m})$ with $c(x) = (2\pi)^{-n} \int_{A'(x,\xi)<1} d\xi$; $A'(x,\xi)$ is the symbol of the leading part of **A**.

4. Example: Elliptic Degenerate Operators

For further information, the reader is directed to Baouendi-Goulaouic [1969].

Let Ω be a smooth bounded open set in $\mathbf{R}^n$. We suppose, as above, that **A** is a differential operator in $\mathbf{D}'(\Omega)$, which is properly elliptic of order $2m > n$, with coefficients in $\mathbf{C}^\infty(\Omega)$. Let us denote by A the realization of a boundary value problem which satisfies the hypothesis of Section 3 and such that $H_0^{2m}(\Omega) C \; D(A) C \; H^m(\Omega)$. A is an isomorphism from $D(A)$ into $L^2(\Omega)$.

4.1. Local Estimate

Let us associate with $(A+tI)$ the Green operator $(A+tI)^{-1} = G_t$, which is an operator with a kernel $G_t(x,y) = G_{tx}(y)$, such that

$$f(x) = \int_\Omega G_t(x,y)(A+tI)f(y)dy \tag{4.1}$$

for all $x \in \Omega$ and for all $f \in D(A)$. It follows that

$$\|G_{tx}\|_{L^2(\Omega)} = \sup_{f \in D(A)} \frac{|f(x)|}{\|(A+tI)f\|_{L^2(\Omega)}}. \tag{4.2}$$

Therefore, there exists a positive constant C such that $\|G_{tx}\|_{L^2(\Omega)} \leq C$. A is an isomorphism from $D(A)$ continuously imbedded in $H^m(\Omega)$ into $L^2(\Omega)$; hence, the eigenvalues of A form a nondecreasing sequence $(\lambda_i)_{i\in\mathbf{N}}$, tending to $+\infty$ at infinity:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$$

(each eigenvalue repeated according to its multiplicity). Let $(\varphi_i)_{i\in\mathbf{N}}$ be an orthonormal and Hilbertian basis in $L^2(\Omega)$ of eigenfunctions associated respectively with the eigenvalues λ_i. For x and y in Ω and for $t \geq 0$ we have

$$G_{tx} = \sum_{i=0}^\infty (\lambda_i + t)^{-1} \varphi_i(x) \overline{\varphi}_i \quad \text{in } L^2(\Omega). \tag{4.3}$$

Let us now consider the operator $(A+tI)^{1/2}$, which is selfadjoint also. If we denote by $g_{tx}(y)$ its Green's kernel, we have for all $x \in \Omega$ and $f \in D((A+tI)^{1/2})$:

$$f(x) = \int_\Omega g_{tx}(z)(A+tI)^{\frac{1}{2}} f(z)\,dz, \tag{4.4}$$

$$\|g_{tx}\|_{L^2(\Omega)} = \sup_{f \in D((A+tI)^{\frac{1}{2}})} \frac{|f(x)|}{\|(A+tI)^{\frac{1}{2}} f\|_{L^2(\Omega)}}. \tag{4.5}$$

(Obviously, these equalities are analogous to 4.1 and 4.2.)

It follows from 4.1 and 4.4 that

$$G_{tx}(x) = \int_\Omega g_{tx}(z) g_{tx}(x)\,dz. \tag{4.6}$$

Moreover, $g_{tx}(x) = \overline{g_{tx}(z)}$. Hence

$$G_{tx}(x) = \|g_{tx}\|^2_{L^2(\Omega)},$$

and, by 4.5,

$$G_{tx}(x) = \sup_{f \in D((A+tI)^{\frac{1}{2}})} \frac{|f(x)|^2}{((A+tI)f, f)_{L^2(\Omega)}}. \tag{4.7}$$

Let us now write 4.4 for the eigenfunction φ_i:

$$(\lambda_i + t)^{-\frac{1}{2}} \varphi_i(x) = \int_\Omega g_{tx}(z)\varphi_i(z)\,dz.$$

By Parseval's equality we obtain

$$G_{tx}(x) = \sum_{i=0}^{\infty} \frac{\varphi_i(x)\,\overline{\varphi_i(y)}}{\lambda_i + t}. \tag{4.8}$$

A local estimate of the Green's kernel is obtained in Baouendi-Gaoulouic by proving the following results:

a. **Behavior of the Green's kernel outside the diagonal.** For each compact $K \subset \Omega \times \Omega$ that does not intersect the diagonal, the function $t \to \sup_{(x,y)\in K} |G_t(x,y)|$, which is defined for t large enough, decreases rapidly when t tends to $+\infty$.

b. **Comparison of Green's kernels of two operators having the same symbol.** Let Ω_1 be a smooth open set such that $x \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega$. Let us denote by G^1_{tx} the Green's kernel in x of the realization in $L^2(\Omega_1)$ of the Dirichlet boundary value problem in Ω_1 defined by **A**.

In each open set Ω_2 such that $\overline{\Omega_2} \subset \Omega_1$, the function $t \to \sup_{y\in\Omega_2} |G_{tx}(y) - G^1_{tx}(y)|$ decreases rapidly when t tends to $+\infty$.

c. **Asymptotic estimate of the Green's kernel on the diagonal.**

THEOREM 4.1. *For all* $x \in \Omega$, $\lim_{t\to+\infty} t^{1-n/2m} G_{tx}(x) = l(x)$ *with*

$$l(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + \mathbf{A}'(x,\xi))^{-1} d\xi,$$

where $\mathbf{A}'(x,\xi)$ *is the symbol of the leading part of* **A**.

This theorem is known for operators with constant coefficients. The general result is established by comparison of A with $A(x_0, D)$, which has constant coefficients: the coefficients of A that are "frozen" in x_0.

From 4.8 and Theorem 4.1, we deduce the following.

THEOREM 4.2. *For all $x \in \Omega$,*

$$\lim_{t\to\infty} t^{1-(n/2m)} \sum_{i=0}^{\infty} (\lambda_i+t)^{-1} |\varphi_i(x)|^2 = l(x).$$

Moreover, for $x \neq y$, $x \in \Omega$, $y \in \Omega$, the function $t \to \sum_{i=0}^{\infty} (\lambda_i+t)^{-1} \varphi_i(x)\overline{\varphi_i(y)}$ decreases rapidly as t tends to $+\infty$.

We then apply to Theorem 4.2 the following Tauberian theorem.

THEOREM 4.3. *Let σ be an increasing function from $\mathbf{R}^+$ into $\mathbf{R}$, α a real number in $(0,1)$, and c a positive number. If*

$$\int_0^1 (\lambda+t)^{-1} d\sigma(\lambda) = ct^{\alpha-1} + o(t^{\alpha-1}), \text{ as } t \to +\infty,$$

then

$$\sigma(\lambda) = c\,\frac{\sin \pi\alpha}{\pi\alpha}\lambda^{\alpha} + o(\lambda^{\alpha}) \text{ as } \lambda \to +\infty.$$

If we choose $\sigma(\lambda) = \sum_{\lambda_i \le \lambda} |\varphi_i(x)|^2$, we obtain the following theorem.

THEOREM 4.4. *For all $x \in \Omega$, $y \in \Omega$, $x \neq y$,*

$$\lim_{\lambda\to+\infty} \lambda^{-(n/2m)} \sum_{\lambda_j \le \lambda} \varphi_j(x)\overline{\varphi_j(y)} = 0.$$

4.2. Global Estimates

We now suppose that $D(A)$ is continuously imbedded in $H^{2m}(\Omega)$. It is possible to obtain the asymptotic behavior of $\int_\Omega G_t(x,x)dx$.

THEOREM 4.5. *There exists $c \ge 0$ such that for all $x \in \Omega$, $t \ge 0$, $f \in D(A^{1/2})$,*

$$|f(x)|^2 \le ct^{(n/2m)-1}\,[\|A^{1/2}f\|^2_{L^2(\Omega)} + t\|f\|^2_{L^2(\Omega)}].$$

It follows that

$$\sup_{\substack{t\ge 0\\ x\in\Omega}} |G_t(x,x)| \le ct^{(n/2m)-1}. \tag{4.9}$$

Integrating and using Theorems 4.1 and 4.2, we have

$$\lim_{t\to+\infty} t^{1-(n/2m)} \int_\Omega G_t(x,x)dx = \lim_{t\to+\infty} t^{1-(n/2m)} \sum_{i=0}^{\infty} (\lambda_i+t)^{-1} = \int_\Omega l(x)dx.$$

Applying again the Tauberian theorem, we find

$$\lim_{\lambda\to+\infty} \lambda^{-n/2m} N(\lambda) = h(m,n)\int_\Omega l(x)dx \quad \text{as } \lambda \to +\infty,$$

with

$$h(m,n) = \frac{2m}{\pi n}\sin\frac{\pi n}{2m}.$$

This is equivalent to

$$\lambda_j \sim c j^{2m/n} \text{ with } c = [h(m,n)\int_\Omega l(x)\mathrm{d}x]^{-2m/n} \text{ as } j \to +\infty.$$

5. The Max-Min Principle

5.1. The Classical Formulae

We suppose that the hypotheses of Section 1 are satisfied. Let $\mu_1, \mu_2, \ldots, \mu_p, \ldots$ denote the nonincreasing sequence of the eigenvalues of $B = A^{-1}$ (which is selfadjoint, positive, and compact). We suppose that each eigenvalue is repeated according to its multiplicity. Let $\varphi_1, \varphi_2, \ldots, \varphi_p, \ldots$ be an orthonormal system of associated eigenfunctions. Let us denote by W_p the subspace spanned by the first p eigenfunctions $\{\varphi_1, \ldots, \varphi_p\}$, and by $\mathbf{F}_p$ the set of all p-dimensional subspaces in H.

THEOREM 5.1. *The eigenvalues of B are characterized by*

$$\mu_p = \sup_{\substack{u \in W_{p-1}^\perp \\ \|u\|=1}} (Bu,u) = \inf_{E_{p-1} \in \mathbf{F}_{p-1}} \sup_{\substack{u \in E_{p-1}^\perp \\ \|u\|=1}} (Bu,u) = \sup_{E_p \in \mathbf{F}_p} \inf_{\substack{u \in E_p \\ \|u\|=1}} (Bu,u).$$

This theorem implies an analogous theorem for A which is an unbounded operator. The eigenvalues of A are $\lambda_p = \mu_p^{-1}$.

THEOREM 5.2. *The eigenvalues of A are characterized by*

$$\lambda_p = \inf_{\substack{u \in W_{p-1}^\perp \\ \|u\|=1}} (Au,u) = \sup_{E_{p-1} \in \mathbf{F}_{p-1}} \inf_{\substack{u \in E_{p-1}^\perp \\ \|u\|=1}} (Au,u) = \sup_{\substack{u \in W_p \\ \|u\|=1}} (Au,u) = \inf_{E_p \in \mathbf{F}_p} \sup_{\substack{u \in E_p \\ \|u\|=1}} (Au,u).$$

PROOF. Let $v \in D(A)$. Then $(Av,v) = \sum_{i=1}^{\infty} \lambda_i (v,\varphi_i)^2$. When $v \in W_{p-1}^\perp$, $(Av,v) = \sum_{i=p}^{\infty} \lambda_i (v,\varphi_i)^2 \geq \lambda_p \sum_{i=p}^{\infty} (v,\varphi_i)^2$ (because the $(\lambda_p)_{p \in \mathbb{N}}$ are nondecreasing). When $\|v\| = 1$, i.e., $\sum_{i=1}^{\infty} (v,\varphi_i)^2 = 1$, we obtain

$$\inf_{\substack{v \in W_{p-1}^\perp \\ \|v\|=1}} (Av,v) \geq \lambda_p.$$

By choosing $v = \varphi_p$, the infimum λ_p is reached and the first equality is proved. In the same way, $v \in W_p$ implies $(Av,v) = \sum_{i=1}^{p} \lambda_i (v,\varphi_i)^2 \leq \lambda_p \sum_{i=1}^{p} (v,\varphi_i)^2$ and $\sup_{\substack{v \in W_p \\ \|v\|=1}} (Av,v) \leq \lambda_p$. The third equality is then obtained by taking $v = \varphi_p$.

To prove the second equality, we notice that for all $E_{p-1} \in \mathbf{F}_{p-1}$, there exists $u_0 \in W_p$ such that $u_0 \in E_{p-1}^\perp$ and $\|u_0\| = 1$. Let $c_p = (u_0, \varphi_p)$. We have $u_0 = \sum_{i=1}^{p} c_i \varphi_i$ and $\sum_{i=1}^{p} c_i^2 = 1$. Therefore

$$\inf_{\substack{\|u\|=1 \\ u \in E_{p-1}^\perp}} (Au_0,u) \leq (Au,u_0) = \sum_{i=1}^{p} \lambda_i c_i^2 \leq \lambda_p,$$

and

$$\sup_{E_{p-1}\in \mathbf{F}_{p-1}} \inf_{\substack{\|u\|=1\\ u\in E_{p-1}^{\perp}}} (Au,u) \le \lambda_p .$$

If we choose $E_{p-1} = W_{p-1}$, we obtain the equality

The fourth equality is proved analogously. For all $E_p \in \mathbf{F}_p$, there exists $u_0 \in W_{p-1}^{\perp}$, $\|u_0\|=1$; hence $u_0 = \sum_{i=p}^{\infty} c_i \varphi_i$ and

$$\sup_{\substack{u\in E_p\\ \|u\|=1}} (Au,u) \ge (Au_0,u_0) = \sum_{i=p}^{\infty} \lambda_i c_i^2 \ge \lambda_p .$$

Therefore

$$\inf_{E_p\in \mathbf{F}_p} \sup_{\substack{u\in E_p\\ \|u\|=1}} (Au,u) \ge \lambda_p .$$

The equality is obtained by choosing $E_p = W_p$

5.2. Other Formulations of the Same Theorem

DEFINITION. Let K be a compact set in H. The "p-width of K in H," which is denoted by $d_p(K,H)$, is defined by

$$d_p(K,H) = \inf_{E_p\in \mathbf{F}_p} \sup_{u\in K} \inf_{f\in E_p} \|u-f\|.$$

This definition has been introduced by Kolmogorov (see, e.g., Boutet de Monvel and Grisvard [1971] and Singer [1970]).

If K is an ellipsoid with diameters δ_p (we suppose that $\delta_1 \ge \delta_2 \ge \cdots$) and if $(\varphi_p)_{p\in \mathbf{N}}$ is an orthonormal basis in H, with

$$K = \{u = \sum \alpha_k \varphi_k / \sum \alpha_k^2 \delta_k^{-2} \le 1\},$$

then the following theorem holds.

THEOREM 5.3. $d_p(K,H) = \delta_{p+1}$.

PROOF. Observe that $\inf_{f\in E_p} \|u-f\| = u - P_{E_p} u$.

For all $E_p \in \mathbf{F}_p$, there exists $u_1 \in E_p^{\perp} \cap W_{p+1}$ such that $u_1 = \sum_{i=1}^{p+1} \alpha_i \varphi_i$ and $\sum_{i=1}^{p+1} \alpha_i^2 \delta_i^{-2} = 1$. Hence

$$\sup_{u\in K} \inf_{f\in E_p} \|u-f\| \ge \inf_{f\in E_p} \|u_1 - f\| = \|u_1\| \ge \delta_{p+1}.$$

As above, the theorem follows from

$$\delta_{p+1} = \sup_{u\in K} \inf_{f\in W_p} \|u-f\| \ge \inf_{E_p\in \mathbf{F}_p} \sup_{u\in K} \inf_{f\in E_p} \|u-f\| \ge \delta_{p+1},$$

by taking $E_p = W_p$ and $u = \delta_{p+1}\varphi_{p+1}$.

It is possible to characterize the eigenvalues of the operator $B = A^{-1}$ by means of this definition. Let

$$K_B = \{Bf \mid \|f\| \le 1\} = \{u \mid \|Au\| \le 1\}.$$

K_B is an ellipsoid:

$$K_B = \{u = \sum_{p=1}^{\infty} \alpha_p \varphi_p / \sum_{p=1}^{\infty} \alpha_p^2 \mu_p^{-1} \le 1\}.$$

The diameters of K_B are the eigenvalues μ_p of B. The following theorem is the usual max-min formula written in a different way.

THEOREM 5.4. $\mu_p = \lambda_p^{-1} = d_{p-1}(K_B, H)$ with $K_B = \{Bf \mid \|f\| \le 1\}$.

PROOF. If we write Theorem 5.1 for B^2, we have

$$\mu_{p+1}^2 = \inf_{E_p \in \mathscr{V}_p} \sup_{\substack{\|u\| \le 1 \\ u \in E_p^\perp}} (B^2 u, u) = \sup_{\substack{\|u\| \le 1 \\ u \in \mathscr{V}_p^\perp}} (B^2 u, u) = \sup_{\substack{\|u\| \le 1 \\ u \in \mathscr{V}_p^\perp}} \|Bu\|^2.$$

Moreover, u is in $\mathscr{W}_p^\perp$ if and only if $Bu \in \mathscr{W}_p^\perp$. Hence

$$\mu_{p+1} = \sup_{\substack{\|u\| \le 1 \\ Bu \in \mathscr{V}_p^\perp}} \|Bu\| = \sup_{\substack{\|u\| \le 1 \\ Bu \in \mathscr{V}_p^\perp}} \inf_{f \in \mathscr{V}_p} \|Bu - f\|$$

$$\le \sup_{\|u\| \le 1} \inf_{f \in \mathscr{V}_p} \|Bu - f\| = \sup_{v \in K_p} \inf_{f \in \mathscr{V}_p} \|v - f\|$$

$$= \inf_{E_p \in \mathscr{V}_p} \sup_{v \in K_p} \inf_{f \in E_p} \|v - f\| = d_p(K_B, H).$$

The equality can be proved as above. Of course the eigenvalues of A can be characterized in the same way: $\lambda_p^{-1} = d_{p-1}(L_A, H)$ with $L_A = K_{A^{-1}} = \{u \mid \|Au\| \le 1\}$.

5.3. Consequences

COROLLARY 5.1. *Suppose that the eigenvalues ν_p of a positive and selfadjoint operator T in H are explicitly known and that $(Tu, u) \le (Au, u)$ for all $u \in D(A)$. Then $\nu_p \le \lambda_p$.*

COROLLARY 5.2. *Let $E \subset D(B)$ and $\tilde{B} = B|_E$. Let $\tilde{\mu}_p$ be the eigenvalues of $\tilde{B}$. Then $\tilde{\mu}_p \le \mu_p$.*

6. Application to Variational Problems

Let (V, H, a) be a variational triple with V compactly imbedded in H. The operator A associated with this triple is positive, selfadjoint, and invertible, with compact inverse; $V = D(A^{1/2})$. Therefore we have the following theorem.

THEOREM 6.1. $(\lambda_{p+1})^{1/2} = d_p(L_{A^{1/2}}, H)$ *with*

$$L_{A^{1/2}} = \{u \in V \mid \|A^{1/2} u\| \le 1\} = \{u \in V \mid a(u, u) \le 1\}.$$

6.1. Boundary Value Problems

Let us now consider a variational boundary value problem: $H = L^2(\Omega)$; $H_0^m(\Omega) \subset V \subset H^m(\Omega)$. Therefore

$$\delta^{-1/2} \underline{V} \subset L_{A^{1/2}} \subset \gamma^{-1/2} \underline{V},$$

where γ and δ are positive constants of continuity and coerciveness for a and where $\underline{V}$ denotes the unit ball of V. Hence

$$\delta^{-\frac{1}{2}}\underline{H_0^m(\Omega)}\subset\delta^{-\frac{1}{2}}V\subset L_{A^{\frac{1}{2}}}\subset\gamma^{-\frac{1}{2}}V\subset\gamma^{-\frac{1}{2}}\underline{H^m(\Omega)}.$$

An estimate of the widths of $\underline{H_0^m(\Omega)}$ and $\underline{H^m(\Omega)}$ gives a lower bound and an upper bound for the eigenvalues of A. These widths have been computed by El Kolli [1971].

THEOREM 6.2. *Let Ω be a bounded open set in $\mathbf{R}^n$, and let s be a positive real number. Then*

$$0<\liminf_{p\to+\infty} p^{s/n} d_p(\underline{H_0^s(\Omega)},\ L^2(\Omega)) \le \limsup_{p\to+\infty} d_p(H^s(\Omega),\ L^2(\Omega)).$$

This theorem can be proved by means of the isomorphism between $L^2(T^n)$ and l^2, where T^n denotes the torus in $\mathbf{R}^n$, which, in turn, implies an isomorphism between $H^m(T^n)$ and $l^2_{j^{-(2m/n)}}$, where $\underline{l^2_{j^{-(2m/n)}}} = \{(a_j)\in\mathbf{N}^n / \sum j^{2m/n} a_j^2 \le 1\}$. We have an ellipsoid whose widths in l^2 are known. Using the isomorphism, we deduce the result for the torus. Moreover, there exists a continuous linear extension P from $H^m(\Omega)$ into $H^m(T^n)$ such that $RP = I|_{H^m(\Omega)}$, where R is the restriction to Ω. Therefore, $\lambda_p^{-\frac{1}{2}} \sim cp^{-m/n}$.

THEOREM 6.3.

$$0<\liminf_{p\to+\infty} p^{-(2m/n)}\lambda_p \le \limsup_{p\to+\infty} p^{-(2m/n)}\lambda_p < \infty.$$

6.2. Consequences

Let us consider a variational triple (V,H,a), and let us denote by $N(\lambda,V,a)$ the number of eigenvalues λ_p (defined by Theorem 6.1) less than or equal to λ.

Under the same hypotheses as above, we have

$$N(\lambda,V,a+t) = N(\lambda-t,V,a) \quad \text{for all } t\in\mathbf{R}$$

$$N(\lambda,V,ta) = N(t^{-1}\lambda,V,a) \quad \text{for all } t\in\mathbf{R}_+.$$

COROLLARY 6.1. *If (V,H,a_1) and (V,H,a_2) are two variational triples such that $a_1(u,u)\le a_2(u,u)$ for all $u\in V$, then*

$$N(\lambda,V,a_2)\le N(\lambda,V,a_1).$$

COROLLARY 6.2. *Let $(V_i,H_i,a_i)_{i\in I\subset\mathbf{N}}$ be a sequence of variational triples for which there exists $c>0$ such that $a_i(u_i,u_i)\ge c\,\|u_i\|^2_{H_i}$ for all $i\in I$, $u_i\in V_i$. Let*

$$\mathbf{H} = \{u=(u_i)_{i\in I}\,|\,u_i\in H_i;\ \ \|u\|^2_{\mathbf{H}} = \sum_i \|u_i\|^2_{H_i} < \infty\}$$

and

$$\mathbf{V} = \{u=(u_i)_{i\in I}\,|\,u_i\in V_i;\ \ a(u,u) = \sum_i a_i(u_i,u_i) < \infty\}.$$

The triple $(\mathbf{V},\mathbf{H},a)$ is a "variational triple" and

$$N(\lambda,v,a) = \sum_i N(\lambda,v_i,a_i) \quad \text{for all } \lambda>0.$$

Let Ω be an open set in $\mathbf{R}^n$ (Ω is not necessarily bounded), and consider a "variational triple" $(V(\Omega),\ L^2(\Omega),a)$ where $V(\Omega)$ is a "weighted Sobolev space." Let us denote by $V^0(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm of $V(\Omega)$. When $\omega\subset\Omega$, $V^1(\omega)$ is the set of the restrictions to ω of elements of $V^0(\Omega)$.

COROLLARY 6.3. *Let Ω_1 and Ω_2 be two disjoint open sets in Ω such that $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$. If $V^0(\Omega_1) \oplus V^0(\Omega_2) \subset V^0(\Omega)$ and $V^1(\Omega_1) + V^1(\Omega_2) \supset V^1(\Omega)$, then*

$$N(\lambda, V^0(\Omega_1), a) + N(\lambda, V^0(\Omega_2), a) \leq N(\lambda, V^0(\Omega), a)$$

$$\leq N(\lambda, V^1(\Omega), a) \leq N(\lambda, V^1(\Omega_1), a) + N(\lambda, V^1(\Omega_2), a).$$

7. Example: Eigenvalues of Operators of Schrödinger Type

For further information on Schrödinger-type operators, see Reed and Simon [1978], Rozenbljum [1975], and Fleckinger [1981].

Let $\Omega = \mathbf{R}^n$ ($n \geq 1$), and let a be an integrodifferential form on $H^1(\mathbf{R}^n)$ which is Hermitian and continuous:

$$a(u,v) = \int_\Omega b(x)(\nabla u \nabla \bar{v} + u\bar{v})dx$$

with b uniformly continuous and positive and bounded away from zero. Let q be a positive and continuous function on $\mathbf{R}^n$ tending to $+\infty$ at infinity and such that for all $\varepsilon > 0$ there exists $\sigma > 0$ such that $|q(x) - q(y)| \leq \varepsilon q(x)$ whenever $|x-y| < \sigma$. When $\Omega \subset \mathbf{R}^n$, let $V_q^0(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{q,\Omega} = \{\int_\Omega [\sum_{i=1}^n |\frac{\partial u(x)}{\partial x_i}|^2 + q(x)|u(x)|^2]dx\}^{1/2}.$$

The space $V_q^0(\mathbf{R}^n)$ equipped with this norm is a Hilbert space, and the imbedding of $V_q^0(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$ is compact.

Moreover, when $\omega \subset \mathbf{R}^n$, we can define $V_q^1(\omega)$ as in Section 6, and Corollary 6.3 can be applied.

Let A_q be the realization of the variational problem $(V_q^0(\mathbf{R}^n), L^2(\mathbf{R}^n), a+q)$.

REMARK. When $b(x) \equiv 1$, $Aq = -\Delta + I + q = -\Delta + V$ is the usual Schrödinger operator.

We deduce from the compactness of the imbedding of $V_q^0(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$ that the spectrum of A_q is discrete.

THEOREM 7.1. *If $\int_{\mathbf{R}^n} b(x)^{n/2}dx = +\infty$, then*

$$N(\lambda, A_q, \mathbf{R}^n) \sim (2\pi)^{-n}\omega_n \int_{\{x\in\mathbf{R}^n | q(x)<\lambda\}} b(x)^{n/2}(\lambda - q(x))^{n/2}dx \quad \text{as } \lambda \to +\infty,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$.

PROOF. We consider a partition of $\mathbf{R}^n$ into nonoverlapping cubes $(Q_\zeta)_{\zeta\in\mathbf{Z}^n}$ with centers x_ζ and sides η. By Corollary 6.3 we have

$$\sum_{\zeta\in\mathbf{Z}^n} N(\lambda, V_q^0(Q_\zeta), a+q) \leq N(\lambda, A_q, \mathbf{R}^n) \leq \sum_{\zeta\in\mathbf{Z}^n} N(\lambda, V_q^1(Q_\zeta), a+q). \tag{7.1}$$

Let $\Omega_\lambda = \{x \in \mathbf{R}^n | q(x) < \lambda\}$, $J = \{\zeta \in \mathbf{Z}^n | \bar{Q}_\zeta \cap \bar{\Omega}_\lambda \neq \phi\}$ and $I = \{\zeta\in\mathbf{Z}^n | \bar{Q}_\zeta \subset \Omega_\lambda\}$. We suppose that q is such that Ω_λ is a "Jordan contented set" with meas $(\partial\Omega_\lambda) = 0$ and $\int_{\Omega_\lambda} b(x)^{n/2}dx \leq \gamma_1 \int_{\Omega_{\lambda/2}} b(x)^{n/2}dx$ for some $\gamma_1 > 0$.

It follows immediately from the max-min principle that $N(\lambda, V_q^1(Q_\zeta), a+q) = 0$ when $\zeta \notin J$. Therefore 7.1 can be written as follows:

$$\sum_{\zeta\in I} N(\lambda, V_q^0(Q_\zeta), a+q) \le N(\lambda, A_q, \mathbb{R}^n) \le \sum_{\zeta\in J} N(\lambda, V_q^1(Q_\zeta), a+q). \quad (7.2)$$

On each cube we compare $a+q$ with an integral form with constant coefficients:

$$((a_\zeta+q_\zeta)u,v) = \int_{Q_\zeta} (b(x_\zeta)\nabla u \nabla \bar{v} + (q(x_\zeta)+1)u\bar{v})dx.$$

We deduce from the hypothesis on b and q that, for all $\varepsilon > 0$, there exists $\eta > 0$ such that $|((a+q)u,u) - ((a_\zeta+q_\zeta)u,u)| \le \varepsilon((a+q)u,u)$. Hence, by Corollaries 6.1 and 6.2, we have

$$\sum_{\zeta\in I} N((1-\varepsilon)\lambda, V_q^0(Q_\zeta), a_\zeta+q_\zeta) \le N(\lambda, A_q, \mathbb{R}^n) \le \sum_{\zeta\in J} N((1+\varepsilon)\lambda, V_q^1(Q_\zeta), a_\zeta+q_\zeta). \quad (7.3)$$

$N(\lambda, V_q^i(Q_\zeta), a_\zeta+q_\zeta)$ is the number of eigenvalues less than λ of the operator $L_\zeta = b(x_\zeta)(-\Delta+I) + q(x_\zeta)$ defined on the cube Q_ζ with Dirichlet or Neumann boundary conditions, and it is easy to prove that

$$|N(\lambda, V_q^i(Q_\zeta), a_\zeta+q_\zeta) - (2\pi)^{-r}\omega_n b(x_\zeta)^{n/2}\eta^n(\lambda-q(x_\zeta))^{n/2}| \le \gamma_2(1+\lambda^{(n-1)/2}) \quad (7.4)$$

By choosing $\eta = \lambda^{-1/4n}$ and by letting ε tend to 0, we obtain Theorem 7.1.

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Appendix: Variational Boundary Value Problems

A.1 Definition

A "variational triple" (V, H, a) is defined by

i) Two Hilbert spaces V and H such that

- V is continuously imbedded in H.
- V is dense in H.

ii) A sesquilinear form $a(\,,\,)$ which is continuous and coercive on V; i.e., there exist two positive contants γ and δ such that

$a(u,v) \leq \delta \|u\|_V \|v\|_V$ for all $(u,v) \in V\times V$

$|a(u,u)| \geq \gamma \|u\|_V^2$ for all $u \in V$.

We have $V \subset H \subset V'$. By the Lax-Milgram Theorem, it is possible to define an isomorphism A from V onto V' by

$$<Au, \bar{v}>_{V'\times V} = a(u,v) \quad \text{for all} \quad (u,v) \in V\times V.$$

Thus, $A \in \mathbf{L}(V, V')$. A can be considered as an unbounded operator in H with $D(A) = \{u \in V \mid Au \in H\}$.

A.2 Examples: Boundary Value Problems

Let Ω be an open set in $\mathbf{R}^n$ $(n \geq 1)$ and set $H = L^2(\Omega)$. The following variational problems will be studied.

1. $V_1 = H_0^1(\Omega)$; $a_1(u,v) = \int_\Omega (u\bar{v} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i}) dx$. By Green's formula (or integration by parts), it is easy to see that

$$a_1(u,v) = \int_\Omega (u\bar{v} - (\Delta u)\bar{v}) dx = ((I-\Delta)u, v) = (A_1 u, v)$$

for all $v \in H_0^1(\Omega) = V_1$ and for all $u \in H^2(\Omega) \cap H_0^1(\Omega) = D(A_1)$. Therefore we have associated the differential operator $A_1 = I - \Delta$ with Dirchlet boundary conditions and $D(A_1) = H^2(\Omega) \cap H_0^1(\Omega)$ with the "variational triple" $(H_0^1(\Omega), L^2(\Omega), a_1)$.

2. Ω is bounded, and φ is a function that is equivalent to the distance at the boundary. (For example, when Ω is the unit ball in $\mathbf{R}^n$, $\varphi(x) = 1 - |x|^2$.)

$V_2 = \{u \in H \mid \sqrt{\varphi} D_i u \in H,\ i = 1, \ldots, n\}$.

$a_2(u,v) = \int_\Omega (\varphi \nabla v \nabla \bar{v} + u\bar{v}) dx$.

The associated differential operator is $A_2 = I - \mathrm{div}(\varphi \,\mathrm{grad})$, which is an extension of the Legendre operator.

3. Ω is unbounded, and q is a positive function tending to $+\infty$ at infinity.

$V_3(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{V_3} = \left(\int_\Omega \left(\sum_{i=1}^{n} |D_i u|^2 + q(x)|u(x)|^2\right)dx\right)^{1/2};$$

$a_3(u,u) = \|u\|^2_{V_3}$.

$A_3 = -\Delta + q$ is the Dirichlet Schrödinger operator.

A.3 Properties

It is obvious that if the imbedding of V into H is compact, then the imbedding of $D(A)$ into H is compact too.

PROPOSITION 1. *The operator A is selfadjoint (resp., positive and selfadjoint) if and only if a is Hermitian (resp., positive and Hermitian).*

The proposition follows from the fact that $D(A^*) = \{u \in V | v \to (Av, u)$ is continuous on $D(A)$ for the norm of $H\}$.

REMARK. When A is a linear operator in the Hilbert space H with $D(A)$ continuously imbedded in H and dense range, we can define the variational triple (V_4, H, a_4) where $V_4 = D(A)$, and $a_4(u,v) = (Au, Av)$. The operator associated with this "triple" is A^2.

ASYMPTOTICS OF EIGENVALUES OF VARIATIONAL ELLIPTIC PROBLEMS WITH INDEFINITE WEIGHT FUNCTION

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Abstract

This paper is concerned with spectral theory of "right non-definite" elliptic boundary value problems, i.e., $Au = \lambda gu$ on $\Omega \subset \mathbb{R}^n$, with Dirichlet homogeneous boundary conditions, when g changes sign. Asymptotics are obtained for the eigenvalues in two cases: 1) when Ω is bounded, and 2) when Ω is unbounded. In case 1, the usual Weyl-Courant estimate is generalized; in case 2, the de Wett-Mandl formula is extended for operators of "Schrödinger type."

1. Introduction

Since the beginning of the century, a vast amount of research has been carried out on elliptic problems with indefinite weight function in the one-dimensional case. Until recently, however, the multidimensional case has attracted less attention. The first paper we know of on the subject is by Holmgrem [1904]; it is concerned with the Dirichlet problem on Ω, a bounded domain in $\mathbb{R}^2$: $\Delta u + \lambda g(x,y)u = 0$. In this case there are an infinite number of positive and negative eigenvalues. Holmgren proved that these eigenvalues can be defined as the solutions of a problem in the calculus of variations, exactly like the well-known Courant-Fisher result when g is positive.

In 1942, Pleijel [1942] gave an estimate for these eigenvalues; he improved the Weyl-Courant estimate on the asymptotics of the number of eigenvalues less than λ [Courant and Hilbert, 1953; Weyl, 1911]. Two years later Pleijel studied the nature of the spectrum for a Schrödinger problem: $+\Delta u - qu + \lambda gu = 0$ when q is positive and when Ω is not necessarily bounded [Pleijel, 1944].

More recently, Weinberger [1974] established the variational characterization of eigenvalues for abstract problems whose right member is not necessarily definite. Also, Manes and Micheletti [1973] applied Weinberger's results for the homogeneous Dirichlet boundary value problem

$$(1.0) \quad Au = \lambda gu \text{ on } \Omega; \quad u/_{\partial\Omega} = 0.$$

When g changes sign, Ω is bounded, and **A** is an elliptic operator of order 2, Manes and Micheletti prove the "max min" ("min max") principle for the positive (negative) eigenvalues. Similar results were given in the de Figueiredo [1982] survey paper. Recently, Lapidus [1984] improved Pleijel's results on the asymptotics of the number of eigenvalues of the Laplacian when the weight is not necessarily smooth.

In the last decade, the eigenvalue problem (1.0) with an indefinite weight function has been intensively studied, primarily as a result of its connection with the theory of semilinear elliptic boundary value problems (see de Figueiredo [1982] and the references there). Nevertheless, most of this work has focused

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on establishing the existence of a positive principal eigenvalue.

Our goal here is very different. This paper is devoted to the asymptotics of the number of eigenvalues of the variational problem associated with (1.0) in two cases:

1. When Ω is bounded and **A** is of order $2m$ (we generalize the earlier papers of Pleijel and Lapidus).

2. When Ω is unbounded and when $\mathbf{A} = L + q$ is an operator of "Schrödinger type."

In the one-dimensional case, when A is degenerate, asymptotics are given in Kaper, Lekkerkerker, Kwong, and Zettl [1984].

Notation. The following notation is used throughout the paper:

- $H^m(\Omega)$ denotes the usual Sobolev space of order m.
- When $\alpha = (\alpha_1,\ldots,\alpha_n)\in\mathbf{N}^n$, D^α is the derivative of order $|\alpha| = \alpha_1+\ldots+\alpha_n$:
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}.$$
- The norm in $H^m(\Omega)$ will be
$$\|u\|_{H^m(\Omega)} = \Big(\int_\Omega \sum_{|\alpha|\le m} |D^\alpha u(x)|^2\, dx\Big)^{\frac{1}{2}}.$$
- $(\ ,\ )$ is the usual inner product in $L^2(\Omega)$.

We will now give more precise results.

(1.1). Let Ω be an open set in $\mathbf{R}^n$ and **A** a formally selfadjoint elliptic operator of order $2m$, defined on Ω: $\mathbf{A} = \sum_{\substack{|\alpha|\le m\\|\beta|\le m}} D^\alpha(a_{\alpha\beta}D^\beta)$, where $a_{\alpha\beta} = \bar a_{\beta\alpha}\in L^\infty(\Omega)$ when $|\alpha+\beta|<2m$ and $a_{\alpha\beta}\in C^0(\bar\Omega)$ when $|\alpha+\beta| = 2m$.

Let us denote by $D(A) = \{u\in L^2(\Omega)/\mathbf{A}u\in L^2(\Omega); u/_{\partial\Omega} = 0\}$ and by A, the positive selfadjoint and unbounded realization in $L^2(\Omega)$ of the homogeneous Dirichlet boundary value problem associated with **A**.

(1.2). We suppose that the embedding of $D(A)$ (equipped with the norm graph) into $L^2(\Omega)$ is compact.

(1.3). Let g be a continuous function on Ω, which changes sign. Let us denote by $\Omega_+ = \{x\in\Omega/g(x) > 0\}$ and $\Omega_- = \{x\in\Omega/g(x) < 0\}$. We suppose that $|\Omega_+| > 0$ and $|\Omega_-| > 0$ where $|\cdot|$ denotes the Lebesgue measure.

We study the spectrum of the boundary value problem

$$(P)\quad \begin{cases}\mathbf{A}u = \lambda g u & \text{on } \Omega,\ u\in D(A)\\ u/_{\partial\Omega} = 0\end{cases}$$

We first prove that this spectrum is discrete. It consists of two countable sequences (one positive and one negative) of eigenvalues, tending to infinity:

$$\ldots\le\lambda_{j+1}^-\le\lambda_j^-\le\cdots\le\lambda_2^-\le\lambda_1^- < 0\le\lambda_1^+\le\lambda_2^+\le\cdots\le\lambda_j^+\le\lambda_{j+1}^+\le\ldots.$$

We write the "max min" and "min max" formulae and then estimate the asymptotics of $N^{\pm}(\lambda,A,g,\Omega)$ where $N^{+}(\lambda,A,g,\Omega)$ $[N^{-}(\lambda,A,g,\Omega)]$ denotes the number of positive (negative) eigenvalues less (greater) than λ.

Under suitable assumptions on the regularity, we prove the following estimates:

1. When Ω is bounded and **A** is an elliptic operator of order $2m$ such that $H_o^{2m}(\Omega)\quad D(A)\quad H^{2m}(\Omega)$, then

$$(1.4)\qquad N^{+}(\lambda,A,g,\Omega) \sim \int_{\Omega_+} (\lambda g(x))^{\frac{n}{2m}}\ \mu'_A(x)\mathrm{d}x, \quad \lambda\to+\infty,$$

where $\mu'_A(x)$ is the usual "Browder-Gårding" density:

$$\mu'_A(x) = (2\pi)^{-n} \text{ meas } \{\xi\in\mathbf{R}^n / \mathbf{A}'(x,\xi) < 1\};$$

$\mathbf{A}'(x,\xi)$ denotes the symbol of the leading part of **A**.

$$(1.5)\qquad N^{-}(\lambda,A,g,\Omega) \sim \int_{\Omega_-} (\lambda g(x))^{\frac{n}{2m}}\mu'_A(x)\mathrm{d}x, \quad \lambda\to-\infty.$$

REMARKS

1. We notice that in (1.5), λ and $g(x)$ are negative; hence $\lambda g(x)$ is positive.

2. It is very easy to deduce (1.5) from (1.4) because $Au = \lambda gu$ can be written as $Au = (-\lambda)(-g)u$. This remark will be used throughout the paper. All results proved for positive eigenvalues imply analogous results for negative eigenvalues.

3. Both (1.4) and (1.5) can also be written

$$N^{\pm}(\lambda,A,g,\Omega) \sim \int_{\Omega} (\lambda g_{\pm}(x))^{\frac{n}{2m}}\ \mu'_A(x)\mathrm{d}x, \quad \lambda\to\pm\infty$$

where $g_+(x) = \max(g(x),0)$, and $g_-(x) = \min(g(x),0)$.

Of course, when g is positive (i.e., $|\Omega_-| = 0$), we have no negative eigenvalues, and we find the usual "Browder-Gårding" estimate where $N^{+}(\lambda,A,g,\Omega)$ is usually denoted by $N(\lambda,A,g,\Omega)$. When $g(x)\equiv1$, we write $N(\lambda,A,\Omega)$ instead of $N(\lambda,A,1,\Omega)$ or $N^{+}(\lambda,A,1,\Omega)$. Hence, (1.4) and (1.5) extend earlier estimates [Browder, 1953; Courant and Hilbert, 1953; Fleckinger and Métivier, 1973; Gårding, 1953; Lapidus, 1984; Manes and Micheletti, 1973; Métivier, 1977; Pleijel, 1942; Reed and Simon, 1978, and Weyl, 1911].

2. When Ω is unbounded, we assume that A is an operator of "Schrödinger type," i.e., $A = L + q$, where L satisfies (1.1) and (1.2).

(1.6). q is a positive and continuous function defined on Ω, tending to $+\infty$ at infinity.

Then, under suitable hypothesis, a) when $\int_\Omega g^{\frac{n}{2m}} < \infty$, (1.4) holds; and when $\int_\Omega g^{n/2m}$ is infinite:

$$(1.7)\qquad N^{+}(\lambda,L+q,g,\Omega) \sim \int_{\{x\in\Omega_+/q(x)<\lambda g(x)\}} \mu'_L(x)(\lambda g(x) - q(x))^{\frac{n}{2m}}\mathrm{d}x, \quad \lambda\to+\infty$$

(1.8) $N^{-}(\lambda, L+q, g, \Omega) \sim \int_{\{x\in\Omega_{-}/q(x)<\lambda g(x)\}} \mu'_L(x)(\lambda g(x)-q(x))^{\frac{n}{2m}}dx, \quad \lambda\to-\infty.$

Again, we notice that (1.7) and (1.8) can be written as

$$N^{\pm}(\lambda, L+q, g, \Omega) \sim \int_{\{x\in\Omega/q(x)<\lambda g_{\pm}(x)\}} \mu'_L(x)(\lambda g_{\pm}(x)-q(x))^{\frac{n}{2m}}dx, \quad \lambda\to\pm\infty.$$

Both these formulae generalize de Wett-Mandl asymptotics of $N(\lambda, -\Delta+q, \mathbb{R}^n)$ [Courant and Hilbert, 1953; Reed and Simon, 1978; and Titchmarsh, 1958] and its later extensions for $N(\lambda, -\Delta+q, \Omega)$ [Rozenbljum, 1975] or $N(\lambda, A=L+q, \Omega)$ [Fleckinger, 1981]. (Of course, we write as above $N(\lambda, A, \Omega)$ instead of $N(\lambda, A, 1, \Omega)$.

2. An Abstract " Max Min Principle "

To prove the above estimates, we first write a variational characterization of eigenvalues for an abstract problem. Our proofs in this part are almost the same as in Pleijel [1942], Weinberger [1974], Manes and Micheletti [1973], and de Figueiredo [1982].

(2.1). Let H be a separable Hilbert space with inner product (,) and norm $\|\ \|_H$.

(2.2). Let us now consider a "variational triple" (V, H, a) where V is dense in H with compact embedding and a is a Hermitian continuous and coercive form defined on V:

$$\exists\alpha>0 \quad |a(u,v)| \le \alpha\|u\|_V\|v\|_V \quad \forall(u,v)\in V\times V$$

$$\exists\beta>0 \quad a(u,u) \ge \beta\|u\|_V^2 \quad \forall u\in V.$$

We associate to (V, H, a) the operator A which is linear, positive, selfadjoint, and unbounded in H, with compact inverse: $V = D(A^{\frac{1}{2}})$ and

$$\forall u\in D(A) \quad \forall v\in V \quad a(u,v) = (Au, v).$$

(2.3). C is a linear and selfadjoint operator in H which is "A bounded," i.e.,

$$\exists\gamma>0, \quad |(Cu,u)| \le \gamma a(u,u) \quad \forall u\in V.$$

In this part we are concerned with the abstract variational eigenvalue problem to find $(\lambda, u)\in\mathbb{C}\times H$ *st*.

(2.4) $\quad Au = \lambda Cu \quad u\in V.$

It follows immediately from the assumptions (2.3) that 0 cannot be an eigenvalue of (2.4) and that a defines an inner product on V; hence, using the Frechet-Riesz representation, we prove the following proposition.

PROPOSITION 1. *The operator T defined on V by*

(2.5) $\quad (Cu, v) = a(Tu, v) \quad \forall(u,v)\in V\times V$

is linear and continuous in V.

This is a simple consequence of (2.2) and (2.3):

$$\beta\|Tu\|_V^2 \le a(Tu, Tu) = (CTu, u) \le \gamma a(u, Tu) \le \alpha\gamma\|Tu\|_V\|u\|_V$$

and then

$$\|Tu\|_V \le \frac{\alpha\gamma}{\beta}\|u\|_V \quad \forall u \in V.$$

(2.6). We suppose that T is compact.

Indeed, we note that formally $T = A^{-1}C$ and hypothesis (2.6) will be satisfied, for example, when C is bounded. Moreover, T is "a-selfadjoint," i.e., selfadjoint on V equipped with the inner product $a(\ ,\ )$:

$$a(Tu,v) = a(u,Tv) \quad \forall (u,v) \in V \times V.$$

Now we introduce the eigenvalue problem.

(2.7) $Tu = \mu u \quad u \in V$. We can apply to T the usual spectral theory of selfadjoint compact operators in Hilbert spaces.

PROPOSITION 2.

i) T has no continuous spectrum except, possibly, at 0.

ii) The eigenvalues of T are real.

iii) If μ_i and μ_j are two distinct eigenvalues of (2.7) with φ_i and φ_j the associated eigenfunctions, then $a(\varphi_i,\varphi_j) = 0$ (φ_i and φ_j are "a-orthogonal").

iv) The spectrum of T (except, possibly, 0) consists of two countable sequences of eigenvalues (one positive and one negative) tending to 0:

$$\mu_1^- \le \mu_2^- \dots \le \mu_\gamma^- \le \cdots \le 0 \le \cdots \le \mu_{\gamma+1}^+ \le \mu_\gamma^+ \le \dots \le \mu_2^+ \le \mu_1^+.$$

The a-selfadjointedness of T implies ii) and iii): If $Tu = \mu u$, $a(Tu,u) = a(u,Tu)$ implies that μ is real and if $Tv = \nu v$, we obtain $(\mu-\nu)a(u,v) = 0$.

To prove i) and iv), we use the following lemma, which is a consequence of the T compactness

LEMMA 1 *There exists a non-zero function $\varphi_1^+ \in V$ for which*

$$\mu_1^+ = \sup_{\substack{u \in V \\ a(u,u)=1}} (Cu,u).$$

Then $T\varphi_1^+ = \mu_1^+\varphi_1^+$

Let us take the restriction of T to the orthogonal complement (for the inner product defined by a) of the linear subspace generated by φ_1^+. We obtain a selfadjoint and compact operator and hence

$$\mu_2^+ = \sup_{\substack{u \in V \\ a(u,\varphi_1^+)=0 \\ a(u,u)=1}} (Cu,u)\dots$$

(2.8) $$\mu_j^+ = \sup_{\substack{u \in V \\ a(u,\varphi_i^+)=0,\ i\in\{1,\dots,j-1\} \\ a(u,u)=1}} (Cu,u).$$

In the same way we can prove that

(2.9) $$\mu_1^- = \inf_{\substack{u \in V \\ a(u,u)=+1}} (Cu,u).$$

Let us denote by φ_j^+ the associated eigenfunctions such that $a(\varphi_j^+,\varphi_j^+) = 1$.

PROPOSITION 3. *The eigenvalues of (2.7) are characterized by the variational principle*

(2.10) $$\mu_{j+1}^+ = \inf_{E_j \in \mathbf{A}_j} \sup_{\substack{u \in V \\ u \perp E_j}} \frac{(Cu,u)}{a(u,u)}$$

where $\mathbf{A}_j$ denotes the set of all j dimensional subspaces in V:

$$\mu_{j+1}^- = \sup_{E_j \in \mathbf{A}_j} \inf_{\substack{u \in V \\ u \perp E_j}} \frac{(Cu,u)}{a(u,u)}$$

Let $E_j \in \mathbf{A}_j$, and choose $u = \sum_{k=1}^{j+1} C_j \varphi_j^+$ such that u is orthogonal to E_j for the inner product associated to a. We suppose that $a(u,u) = \sum_{k=1}^{j+1} C_j^2 = 1$.

$$(Cu,u) = \sum_{k=1}^{j+1} \mu_k^+ C_j^2 \le \mu_{j+1}^+ \sum_{k=1}^{j+1} C_j^2 = \mu_{j+1}^+ .$$

and

$$\sup_{u \perp E_j} \frac{(Cu,u)}{a(u,u)} \le \mu_{j+1}^+ .$$

We obtain the equality by taking E_j the linear subspace generated by $\varphi_1^+, \ldots, \varphi_j^+$ and $u = \varphi_{j+1}^+$.

We notice that the eigenfunctions of (2.4) are eigenfunctions of (2.7) with $\lambda = \frac{1}{\mu}$, the associated eigenvalues.

Let us denote by $\mu_j^+(V,a)$, in place of μ_j^+, the "min max" of $(Cu,u)/a(u,u)$ introduced in 2.10. We deduce from Proposition 2.3 the following corollaries which we will use throughout this paper.

COROLLARY 1. *If (V_1,H,a) and (V_2,H,a) are two "variational triples" with $V_1 \subset V_2$, then*

$$\mu_j^+(V_1,a) \le \mu_j^+(V_2,a).$$

COROLLARY 2. *If (V,H,a_1) and (V,H,a_2) are two "variational triples" such that $a_1(u,u) \le a_2(u,u)$ $\forall u \in V$, then*

$$\mu_j^+(V,a_1) \ge \mu_j^+(V,a_2).$$

3. Application to Boundary Value Problems Defined on Bounded Domains

(3.1). Let Ω be a bounded domain in $\mathbf{R}^n$. We consider on Ω the variational eigenvalue problem:

(P) $Au = \lambda g u$ on Ω; $u/_{\partial\Omega} = 0$,

where A and g are satisfying assumptions (1.1) to (1.3).

We will apply the previous results with $H = L^2(\Omega)$, C defined by $Cu(x) = g(x)u(x)$ and $V = H_0^m(\Omega)$, where $H_0^m(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ for the norm of H^m. We will denote by $a(\ ,\ )$ the integrodifferential form defined by $a(u,v) = (Au,v)$ $\forall u \in D(A)$ $\forall v \in V = H_0^m(\Omega)$ and problem (P) is studied in its variational formulation:

(3.2) $a(u,v) = \lambda \int_\Omega g(x)u(x)\overline{v(x)}dx \quad \forall v \in H_0^m(\Omega).$

THEOREM 1. *We suppose that hypotheses (1.1) to (1.3) and (3.1) are satisfied. Under these hypotheses, (P) has a purely point spectrum. It consists of two sequences of eigenvalues, one positive and one negative, tending to infinity:*

$$\cdots \le \lambda_{j+1}^- \le \lambda_j^- \le \cdots \le \lambda_2^- \le \lambda_1^- < 0$$

$$< \lambda_1^+ \le \lambda_2^+ \le \cdots \le \lambda_j^+ \le \lambda_{j+1}^+ \le \ldots; |\lambda_j^\pm| \to +\infty_{|j\to+\infty}.$$

Moreover,

$$\lambda_{j+1}^+ = \inf_{E_j \in \mathbf{A}_j} \sup_{\substack{u \in H_0^m(\Omega) \\ u \perp E_j}} \frac{(gu,u)}{a(u,u)},$$

where (,) denotes the usual $L^2(\Omega)$ inner product and $\mathbf{A}_j$ the set of j dimensional subspaces of $H_0^m(\Omega)$

$$\lambda_{j+1}^- = \sup_{E_j \in \mathbf{A}_j} \inf_{\substack{u \in H_0^m(\Omega) \\ u \perp E_j}} \frac{(gu,u)}{a(u,u)},$$

and estimates (1.4) and (1.5) hold.

PROOF. The first part of Theorem 1 follows immediately from the former abstract theory. Hence we have only to prove the estimate on $N^\pm(\lambda,A,g,\Omega)$.

When g is positive, this estimate is well known [Weyl, 1911]. We deduce (exactly as in the positive case) from Corollaries 1 and 2

(3.3) $\lambda_j^+(g,\Omega) \le \lambda_j^+(g,\Omega')$ if $\Omega' \subset \Omega$

(3.4) $\lambda_j^+(g,\Omega) \le \lambda_j^+(g_1,\Omega)$ if $g_1 \le g$

where $\lambda_j^+(g,\Omega)$ denotes the jth positive eigenvalue of (3.2). These inequalities are equivalent to the following:

(3.5) $N(\lambda,A,g,\Omega') \le N(\lambda,A,g,\Omega)$ *when* $\Omega' \subseteq \Omega$.

(3.6) $N(\lambda,A,g_1,\Omega) \le N(\lambda,A,g,\Omega)$ *when* $g_1 \le g$.

Let us denote by $g_+ = \max(g,0)$; g_+ is non-negative on Ω, and it follows from (3.5) and (3.6) that

$$\forall \varepsilon > 0 \quad N(\lambda,A,g,\Omega_+) \le N^+(\lambda,A,g,\Omega) \le N(\lambda,A,g_+ + \varepsilon,\Omega).$$

Let us denote by

$$\varphi(g,\Omega,\lambda) = \varphi(\lambda) = \lambda^{\frac{n}{2m}} \int_{\Omega_+} (g(x))^{\frac{n}{2m}} \mu'_A(x)dx.$$

The function g being positive on Ω_+, we have $N^+(\lambda,A,g,\Omega_+) = N(\lambda,A,g,\Omega_+)$, and

(3.7) $\lim_{\lambda\to+\infty} \varphi^{-1}(\lambda)N(\lambda,A,g,\Omega_+) = 1.$

The same limit holds for $\varphi^{-1}(g_+ + \varepsilon,\Omega,\lambda)N(\lambda,A,g_+ + \varepsilon,\Omega)$. Hence,

$$\forall \varepsilon > 0, \quad \varphi^{-1}(\lambda)N(\lambda,A,g,\Omega_+) \le \varphi^{-1}(\lambda)N^+(\lambda,A,g,\Omega)$$

$$\le (\varphi^{-1}(\lambda)\varphi(g_+ + \varepsilon,\Omega,\lambda))\varphi^{-1}(g_+ + \varepsilon,\Omega,\lambda)N(\lambda,A,g_+ + \varepsilon,\Omega).$$

By letting ε tend to 0 and using Lebesgue's theorem, we prove that $\varphi^{-1}(\lambda)\varphi(g_+ + \varepsilon,\Omega,\lambda)$ tends to 1. We notice that $\varphi(g_+,\Omega,\lambda) = \varphi(g,\Omega_+,\lambda)$. By letting λ tend to $+\infty$, we have (3.7).

4. Application to Operators of Schrodinger Type

We now suppose that Ω is unbounded in $\mathbb{R}^n$.

(4.1). Let q be a positive and continuous function defined on $\mathbb{R}^n$, tending to $+\infty$ at infinity.

(4.2). Let L be a differential operator of order $2m$ defined on Ω satisfying hypothesis (1.1).

(4.3). Let g be a continuous function defined on Ω, such that $g(x)q^{-1}(x)$ tends to 0 at infinity.

We consider the variational eigenvalue problem

$$(Q) \quad \begin{cases} Au = (L+q)u = \lambda g u & \text{on } \Omega \\ u/_{\partial\Omega} = 0 \end{cases}.$$

To obtain estimates (1.7) and (1.8), we will apply again the results of the second part and work exactly as above. First, however, we write the estimate $N(\lambda, L+q, g_+, \Omega)$. This study is divided in three parts. To begin, we discuss the right definite case and the right non-definite case when $\int g^{n/2m} = \infty$. Then we give the estimate when $\int g^{n/2m} < \infty$. We now suppose that the following assumptions are satisfied:

(4.4). There exist two positive numbers ε_0 and λ' such that for all $\delta > 0$, q, $g_+ + \delta$ and $a_{\alpha\beta}$ (with $|\alpha + \beta| = 2m$) can be extended to $\Omega_+^{\sim} = \{x \in \mathbb{R}^n / \operatorname{dist}(x,\Omega_+) < \varepsilon_0\}$ and

$$\forall \varepsilon \in]0\varepsilon_0[, \quad \forall \lambda \ge \lambda', \exists \eta_\lambda \, \forall \eta \le \eta_\lambda,$$

$|x-y| < \sqrt{n}\,\eta$. Then

$$|q(x) - q(y)| \le \varepsilon q(x)$$

$$|g_+(x) - g_+(y)| \le \varepsilon |(g(x)+\delta)|$$

$$|a_{\alpha\beta}(x) - a_{\alpha\beta}(y)| \le \varepsilon |a_{\alpha\beta}(x)|.$$

(4.5). For any positive number λ, $\Omega_\lambda = \{x \in \Omega / q(x) < \lambda g(x)\}$ is a Lebesgue measurable set and $\exists \lambda'' \ge \lambda'$ $\exists \gamma_1 > 0$ such that $[\Omega_\lambda] \le \gamma_1 [\Omega_{\lambda/2}]$ for all $\lambda \ge \lambda''$ where

$$[\Omega_\lambda] = \int_{\Omega_\lambda} g^{\frac{n}{2m}}(x)\,dx.$$

(4.6). We consider a partition of $\mathbb{R}^n$ into non-overlapping cubes $(Q_\xi)_{\xi \in \mathbb{Z}^n}$ with side η and centers x_ξ. We suppose that

$$\frac{\sum_{\xi\in J\,I} g_\xi^{n/2m}}{\sum_{\xi\in J} g_\xi^{n/2m}} \to 0_{\eta\to 0} \quad \forall\lambda \geq \lambda'' ,$$

where $g_\xi = g(x_\xi)$, $I = \{\xi\in\mathbb{Z}^n / \bar{Q}_\xi\subset\bar{\Omega}_\lambda\}$ and $J = \{\xi\in\mathbb{Z}^n / \bar{Q}_\xi\cap\bar{\Omega}_\lambda \neq \Phi\}$.

(4.7). We suppose that there exists $\gamma_2 > 0$ such that, for all

$$\omega\subset\Omega, \; (\mathbf{L}'u,u) \geq \gamma_2\|u\|^2_{H^m(\omega)} \quad \forall u\in H^m(\omega),$$

where $\mathbf{L}'$ is the leading part of $\mathbf{L}$. We note that (because of 4.1) when $\lambda > 0$, $\Omega_\lambda\subset\Omega_{+\lambda} = \{x\in\Omega_{+\lambda} = \{x\in\Omega_+ / q(x) < \lambda g(x)\}$. We first give two results when $\int g^{n/2m}$ is infinite.

A. The Right Definite Case

(4.8). We suppose that $g(x) \geq \delta > 0$. Let us denote by $W(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W(\Omega)} = [\|u\|^2_{H^m(\Omega)} + (qu,u)]^{1/2}.$$

$W(\Omega)$ is a Hilbert space, and we deduce from (4.1) the following proposition.

PROPOSITION 5. *The embedding $W(\Omega)$ into $L^2(\Omega)$ is compact.*

This proposition is a consequence of the usual criterion of compactness for unbounded domains. We have

$$\|u\|^2_{L^2(\Omega'_R)} \leq \sup_{\Omega'_R} \frac{1}{q(x)} \int_{\Omega'_R} qu^2 \,(x)\mathrm{d}x \leq \varepsilon(R)\|u\|^2_{W(\Omega'_R)} .$$

where $\Omega'_R = \{x\in\Omega / |x|>R\}$ and $\varepsilon(R)$ tends to 0 when R tends to $+\infty$.

We now prove the following theorem.

THEOREM 2. *When the hypotheses (4.1) to (4.8) are satisfied, and if $\int_\Omega g^{n/2m}$ is infinite, then*

$$N(\lambda,L+q,g,\Omega) \sim \int_{\Omega_\lambda=\{x\in\Omega/ g(x)<\lambda g(x)\}} \mu'_L(x)\,(\lambda g(x) - q(x))^{\frac{n}{2m}}\,\mathrm{d}x \quad \lambda\to+\infty.$$

This estimate hold, for example, for

$$(-\Delta + (1+|x|^2)^{2k})u = \lambda(1+|x|^2)^{-k}\, u \quad \text{on } \mathbb{R}^n.$$

Theorem 2 will be proved in Section C.

B. The Right Non-definite Case

We do not further assume that g is positive. We will obtain estimates (1.7) and (1.8) as in the third part. We apply the results of the second part with $H=L^2(\Omega)$; $V = W(\Omega)$. The Hermitian form a is defined on $W(\Omega)$ by $a(u,u) = (Lu,u) + (qu,u)$. The operator C is defined as previously by $Cu(x) = g(x)u(x)$. We note that (2.3) follows from (4.3).

THEOREM 3. *We suppose that hypotheses (4.1) to (4.7) are satisfied. When $\int_\Omega g^{n/2m}$ is infinite, estimates (1.7) and (1.8) hold:*

$$N^{\pm}(\lambda, L+q, g, \Omega) \sim \int_{\Omega_\pm} \mu_L'(\lambda g - q)_+^{n/2m}, \quad \lambda \to +\infty.$$

PROOF OF THEOREM 3. The estimate (1.7) holds when $g(x) \ge \delta > 0$. We deduce from Corollaries 1 and 2

$$N(\lambda, L+q, g, \Omega) \le N(\lambda, L+q, g, \Omega) \le N(\lambda, L+q, g_+ + \delta, \Omega)$$

and the estimate follows as in Section 3 by letting $\delta \to 0$.

C. Proof of Theorem 2

The proof of Theorem 2 will be almost the same as that given by Fleckinger [1981]. We will use the two following results:

PROPOSITION 5. *There exists two positive numbers C' and C'' such that*

$$C'\lambda^{n/2m}[\Omega_\lambda] < \varphi(\lambda) < C''\lambda^{n/2m}[\Omega_\lambda]$$

where

$$\varphi(\lambda) = \int_\Omega \mu'_L(\lambda g - q)^{n/2m}.$$

PROOF OF PROPOSITION 5. The upper bound is obvious by use of (4.7). It follows from (1.1) that $\mu'_L(x) \ge C > 0$ and

$$\varphi(\lambda) \ge C \int_{\{x \in \Omega_+ / \frac{\lambda}{2} g(x) - q(x) > 0\}} (\lambda g - q)^{n/2m}.$$

By use of (4.5) we have the result.

When $\omega \subset \tilde{\Omega}$ ($\tilde{\Omega}$ has been introduced in (4.4)), let us denote by $W^1(\omega)$ the restriction to ω of elements of $W(\tilde{\Omega})$, and by $N^+(\lambda, A, g, \omega)$ [$N_1^+(\lambda, A, g, \omega)$] the number of positive eigenvalues less than λ of the variational problem:

$$Au = (L+q)u = \lambda g u \quad \text{in } \omega; \quad u \in W(\omega) \quad [u \in W^1(\omega)].$$

PROPOSITION 6. *If ω_1 and ω_2 are two disjoint open sets in ω such that $\bar{\omega}_1 \cup \bar{\omega}_2 = \bar{\omega}$, then*

$$N^+(\lambda, A, g, \omega_1) + N^+(\lambda, A, g, \omega_2) \le N^+(\lambda, A, g, \omega)$$

$$\le N_1^+(\lambda, A, g, \omega) \le N_1^+(\lambda, A, g, \omega_1) + N_1^+(\lambda, A, g, \omega_2).$$

This is a simple consequence of

$$W(\omega_1) \oplus W(\omega_2) \subset W(\omega) \text{ and } W^1(\omega) \subset W^1(\omega_1) \oplus W^1(\omega_2).$$

When $\zeta \notin J$, $N_1(\lambda, A, g, Q_\zeta) = 0$, because $\lambda r - q < 0$ on Q_ζ. It follows from this remark and from Proposition 6 that

$$\text{(4.8)} \quad \sum_{\zeta \in I} N^+(\lambda, A, g, Q_\zeta) \le N^+(\lambda, A, g, \Omega) \le \sum_{\zeta \in J} N_1^+(\lambda, A, g, Q_\zeta).$$

On each cube we compare $\mathbf{A}$ with $\mathbf{A}_\zeta = \mathbf{L}_\zeta + q_\zeta$ and g with g_ζ, where $r_\zeta = r(x_\zeta)$ and $\mathbf{L}_\zeta = \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x_\zeta) D^{\alpha+\beta}$. It follows from interpolation inequalities and from (4.4) that for all ε small enough, there exist $\eta > 0$ and $C > 0$ such that, for all $\varepsilon \in J$

$$\text{(4.9)} \quad |(Au, u) - (\mathbf{A}_\zeta u, u)| \le \varepsilon(\mathbf{A}_\zeta u, u) + \frac{C\varepsilon^{1-2m}}{g_\zeta} \int_{\varphi_\zeta} g_\zeta u^2.$$

We know that there exist a positive constant C_1 such that for all ζ

(4.10) $|N(\lambda, A_\zeta, g_\zeta, Q_\zeta) - \varphi(\lambda, \zeta)| \le C(\lambda g_\zeta)^{\frac{n-1}{2m}}$

with

(4.11) $\varphi(\lambda,\zeta) = \mu_\zeta \mid Q_\zeta \mid (\lambda g_\zeta - q_\zeta)^{\frac{n}{2m}}$,

where $\mu_\zeta = \mu'_{L_\zeta}$.

The same holds with $N_1(\cdot)$. Choosing $\varepsilon = \lambda^{-\frac{1}{2(2m-1)}}$, prove that

(4.12) $\sup \varphi^{-1}(\lambda) \sum_{\zeta \in I} \varphi((1-\varepsilon)\lambda - \frac{C\varepsilon^{1-2m}}{g_\zeta^2}, \zeta) \to 1; \ \lambda \to +\infty.$

We have an analogous result for the upper bound.

D. The Case $\int_\Omega g_+^{n/2m} < \infty$.

THEOREM 4. *We suppose that hypotheses (4.1) to (4.7) are satisfied and that $\int g_+^{n/2m} < \infty$; then*

$$N(\lambda, L+q, g, \Omega) \sim \int_\Omega (\lambda g_+)^{n/2m}, \quad \lambda \to +\infty.$$

The proof is almost the same as that of Theorem 3. The lower bound is obvious.

For example,

$$(-\Delta + (1+x^2)^s u = \lambda x (1+x^2)^t u \quad \text{on } \Omega = \{(x,y) \in \mathbf{R}^2 / \ |y| < 1\}.$$

If $t = 0$, we apply Theorem 3. If $t < 0$, we apply Theorem 4.

Note: We have recently been told that some estimates can be found in Birman and Solomjak [1973].

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A NONOSCILLATION THEOREM FOR SECOND-ORDER LINEAR EQUATIONS

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Abstract

Let $q:[0,\infty)\to\mathbf{R}$, q not a.e. zero be a locally Lebesgue integrable function. We are interested in the existence of real constants $c \neq 0$ such that the equation $x'' + cq(t)x = 0$ is nonoscillatory on $[0,\infty)$. Applications to Bohr and Stepanoff almost-periodic functions q are included.

1. Introduction

Let $q: [0,\infty)\to\mathbf{R}$ be a locally Lebesgue integrable function and c a real constant. We consider the oscillatory behavior of the second-order linear differential equation

$$x''(t) + cq(t)x(t) = 0, \qquad t \geq 0. \tag{1.1}$$

Equation (1.1) is said to be *oscillatory* if any nontrivial solution vanishes an infinite number of times. Otherwise (1.1) is said to be nonoscillatory.

For some potentials $q(t)$ it may turn out that (1.1) is oscillatory for *every* $c \neq 0$. For example, $q(t) = \sin t$ is such a potential. In fact (see Markus and Moore [1956]) such potentials actually include the class of all nontrivial (Bohr) almost-periodic functions having mean-value equal to zero (for definitions see, for example, Besicovitch [1954]). An interesting problem is to discover classes of functions that either have (cf. Halvorsen and Mingarelli [1984]) or fail to have this property. Below, we give an example of the latter.

2. A Nonoscillation Theorem

THEOREM. *Let there exist positive constants M, T, ε such that*

$$\left|\int_{nT}^{t} q(s)ds\right| \leq M \tag{2.1}$$

for all $t\in[nT,(n+1)T]$ *and* $n = 1,2,3,\ldots$,

$$\int_{nT}^{(n+1)T} \left(\int_{nT}^{t} q(s)ds\right)^2 dt \leq M^2 \tag{2.2}$$

for each $n \geq 1$, *and*

$$\int_{nT}^{(n+1)T} q(s)ds \leq -\varepsilon \tag{2.3}$$

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for all sufficiently large integers n.

Then there exists a constant $c > 0$ such that (1.1) is nonoscillatory on $[0,\infty)$.

REMARK. It is well known that if (1.1) is nonoscillatory for $c = c_1 > 0$, then (1.1) remains nonoscillatory for all $c \in [0,c_1]$ (see, for example, Markus and Moore [1958]).

PROOF. It is well known that the nonoscillation of (1.1) is equivalent to the existence, for some $a \geq 0$, on $[a,\infty)$ of a continuous solution of the Riccati equation

$$r'(t) = -cq(t) - r^2(t), \qquad t \in [a,\infty) \tag{2.4}$$

which results from the change of variable $r(t) = x'(t)/x(t)$ in (1.1). We will exhibit such an $r(t)$ by choosing a suitable initial condition $r(a)$ and then show that the solution can be continued to one that is defined in $[a,\infty)$. The continuability is established by showing that $r(t)$ is always positive.

Without loss of generality we may assume that $T = 1$. We choose

$$c = \min\{(0.01)M^{-1}, (0.01)\varepsilon M^{-2}\}. \tag{2.5}$$

Let $[\alpha,\beta]$ denote the interval $[n,n+1]$. We first show that if

$$r(\alpha) = 3cM, \tag{2.6}$$

then $r(t)$ can be continued up to β and, furthermore, $r(t)$ remains positive and $r(\beta) \geq 3cM$. As usual, from (2.4), one easily derives the equation

$$r(t) = r(\alpha) - \int_\alpha^t cq(s)ds - \int_\alpha^t r^2(s)ds. \tag{2.7}$$

Use of (2.1) and (2.6) readily yields the inequalities

$$2cM - \int_\alpha^t r^2(s)ds \leq r(t) \leq 3cM - \int_\alpha^t cq(s)ds. \tag{2.8}$$

Now suppose that $r(t) = 0$ somewhere in $[\alpha,\beta]$. Let t_0 be the first zero. Then $r(t) > 0$ for $t \in [\alpha,t_0)$ and $r(t_0) = 0$. Now the second inequality in (2.8) implies that

$$\int_\alpha^{t_0} r^2(s)ds \leq \int_\alpha^{t_0} [3cM - \int_\alpha^t cq(s)ds]^2 dt$$

$$\leq 2\int_\alpha^{t_0} [9c^2M^2 + c^2(\int_\alpha^t q(s)ds)^2]dt.$$

It follows from (2.2) and the fact that $t_0 - \alpha \leq 1$ that

$$\int_\alpha^{t_0} r^2(s)ds \leq 20c^2M^2. \tag{2.9}$$

The first inequality in (2.8) and (2.5) now imply that

$$r(t_0) \geq 2cM - 20c^2M^2 > 0,$$

contradicting the definition of t_0. Thus $r(t) > 0$ in $[\alpha,\beta]$. The same arguments as above easily lead to the following refinement of (2.9):

$$\int_{\alpha}^{\beta} r^2(s)ds \le 20c^2M^2. \qquad \textbf{(2.10)}$$

Using (2.7), (2.5), and (2.3), we have

$$r(\beta) \ge 3cM + c\varepsilon - 20c^2M^2$$
$$\ge 3cM$$

as claimed earlier.

If, instead of (2.6), we have

$$r(\alpha) \ge 3cM, \qquad (2.11)$$

using Sturm's comparison theorem or results in the theory of differential inequalities (use of differential inequalities in the theory of oscillation can be found, for example, in Man Kam Kwong and A. Zettl [1982]), we see that the solution of the Riccati equation must lie above that which satisfies (2.6). In particular, if (2.11) is satisfied, then $r(t) > 0$ in $[\alpha,\beta]$ and $r(\beta) \ge 3cM$. The possibility that $r(t)$ may blow up to infinity in $[\alpha,\beta]$ is excluded by the inequality

$$r(t) \le r(\alpha) - \int_{\alpha}^{t} cq(s)ds \le r(\alpha) + cM.$$

We can now complete the proof of the theorem by choosing $r(0) = 3cM$ so that $r(t) > 0$ in $[0,1]$ and $r(t) \ge 3cM$. Induction shows that $r(t)$ can be continued indefinitely on $[n,n+1]$ and $r(t)$ remains positive.

COROLLARY 1. *If q is uniformly bounded on* $\mathbb{R}$ *and (2.3) holds, then (1.1) is nonoscillatory for some* $c > 0$.

COROLLARY 2. *If q is the restriction on $[0,\infty)$ of a (real) Bohr almost-periodic function on $(-\infty,\infty)$ with non-zero mean-value, then (1.1) is nonoscillatory for some* $c \ne 0$.

PROOF. Suppose that the mean-value of $q(t)$ is $\mu < 0$ (If $\mu > 0$, we choose a negative c.) One of Bohr's fundamental theorems states that [Besicovitch, 1954]

$$\lim_{T\to\infty} \frac{1}{T}\int_{\alpha}^{\alpha+T} q(s)ds = \mu < 0 \qquad (2.12)$$

exists uniformly in α. Thus there exists a T large enough so that

$$\frac{1}{T}\int_{\alpha}^{\alpha+T} q(s)ds \le \frac{1}{2}\mu < 0$$

for each α. This yields (2.3) for $\varepsilon = (|\mu| T)/2$. Since Bohr almost-periodic functions are uniformly bounded on $(-\infty,\infty)$ [Besicovitch, 1954], the conditions (2.1) and (2.2) are certainly satisfied for suitable M.

REMARK. In fact, the nonoscillation of (1.1) with a Bohr almost-periodic potential q implies that (1.1) is disconjugate on $[0,\infty)$ (see Markus and Moore [1956]).

There is an extension of the class of Bohr almost-periodic functions to a class that includes functions that are not necessarily continuous. The generalization we refer to is due to Stepanoff (see Besicovitch [1954] for definitions, etc.). It is known that functions in the Stepanoff class S_l^1 are uniformly bounded in the Stepanoff metric, i.e., for some $L > 0$,

$$\|q\|_S = \sup_{s \in \mathbb{R}} \frac{1}{L} \int_s^{s+L} |q(s)|\, ds < \infty. \tag{2.13}$$

As in the case of Bohr almost-periodic functions, these almost-periodic functions of Stepanoff admit a mean-value, as in (2.12), that is uniform with respect to α. Since this is the case, it is clear from above considerations that (2.3) will be satisfied for such functions. In fact the following corollary holds.

COROLLARY 3. *If $q \in S_L^1$ is the restriction on $[0,\infty)$ of a (real) Stepanoff almost-periodic function on $(-\infty,\infty)$ with a non-zero mean-value, then (1.1) is non-oscillatory (in fact, disconjugate — see Halvorsen and Mingarelli [1984] — on $[0,\infty)$ for some $c \neq 0$.*

PROOF. Let $T = L > 0$. It now follows immediately from (2.13) that there holds (2.1) and (2.2) with $M = \max\{\|q\|_S T,\ \|q\|_S T^{3/2}\}$. The result follows.

Note: The techniques used abovve yield a different proof of Corollary 2 and Corollary 3, both of which are to be found in Halvorsen and Mingarelli [1984]. Indeed, it is shown therein that the above corollaries remain valid even for certain classes of Weyl and Besicovitch almost-periodic functions.

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SOME PROBLEMS OF TRANSPORT THEORY

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Abstract

In this paper we discuss the linear transport equation and a rather abstract operator V that occurs in the study of an associated eigenvalue problem. The eigenvalue problem is or is comparable to an indefinite Sturm-Liouville problem.

1. Introduction

Many of the discussions during the Argonne workshop on Sturm-Liouville operators considered the Sturm-Liouville eigenvalue problem

$$Av = \lambda Tv, \tag{1}$$

where A is a selfadjoint Sturm-Liouville operator and T a real-valued function of varying sign. In linear transport theory a similar eigenvalue problem is considered, and the questions studied are essentially the same. The operator A is, however, of considerably simpler structure in the latter case, and the physical problems that lead to the study of (1) are rather transparent. In discussing indefinite Sturm-Liouville problems, some acquaintance with the underlying physical problems seems as desirable as in the case of definite Sturm-Liouville problems, where one assumes some knowledge of the relation with standard physical problems like heat transfer. For these reasons we think it appropriate to spend some time on the transport equation and its physical background.

The following are two examples of transport equations:

I. Neutron transport in a homogeneous slab:

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) = -\psi(x,\mu) + \frac{c}{2}\int_{-1}^{+1} f(\mu,\mu')\psi(x,\mu')\,d\mu', \tag{2}$$

where $\mu \in [-1,+1]$, $x \in J$ (an open real interval), $c \in (0,1)$. The function f is nonnegative, symmetric in the pair (μ,μ'), and satisfies

$$\frac{1}{2}\int_{-1}^{+1} f(\mu,\mu')\,d\mu' = 1.$$

II. Electron transport in a homogeneous metal plate:

$$\mu \frac{\partial \psi}{\partial x}(x,\mu) = \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial \psi}{\partial \mu}(x,\mu), \tag{3}$$

where μ and x range over the same intervals as in I. The function ψ is required to remain bounded for $\mu \to \pm 1$.

Both equations describe time-independent problems. For this reason we prefer the use of the variable x instead of t in the derivatives occurring in the left-hand members of (2) and (3). Actually, the variable x stands for a position coordinate (see Section 2). Equations (2) and (3) are of the type $T\psi' = A\psi$.

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An application of the two-sided Laplace transform or separation of variables (or just plain experience) could lead one to study the associated eigenvalue problems (1).

There do exist many more examples than the two given here. In particular, Beals and Protopopescu [1984] and Kaper [1979] discuss situations where μ ranges over the whole real line instead of over $[-1,+1]$.

2. Physical Meaning of the Transport Equation

Let us give a brief description of the physical meaning of equation (2). We consider a plane-parallel homogeneous slab of width a through which inert particles (neutrons) move along straight lines. These particles may collide with nuclei of the medium within the slab. In principle, we need six independent variables to represent the position and velocity of a particle. We use three Cartesian coordinates for the position and three spherical coordinates for the velocity. A line perpendicular to the slab is taken as the x-axis. The positive x-direction is also taken as the direction of the polar axis for the spherical coordinates of the velocity. We assume a situation in which the particle density ψ actually depends only on the position coordinate x and on the polar velocity angle φ, i.e., the angle between velocity direction and the positive x-direction. In other words, we assume translational invariance lateral to the slab, rotational invariance about the x-direction, and all particles to have the same speed. In this way the phase space is reduced to a two-dimensional space in which points are represented by coordinate pairs (x,μ), where $x\in[0,a]$ and $\mu = \cos\varphi\in[-1,+1]$.

The particle density ψ is then defined as a nonnegative function of the pair (x,μ) such that $\psi(x,\mu)\mathrm{d}x\,\mathrm{d}\mu$ equals the number of particles in a region of volume $\mathrm{d}x\,\mathrm{d}\mu$ at the point (x,μ).

The derivation of equation (2) is similar to the derivation of the heat equation with convection term. One sets up a balance equation for a region in phase space by counting the number of particles that enter and leave the region through its boundaries, taking into account the losses and gains resulting from collisions with nuclei of the medium. In this way we can interpret the terms occurring in (2) as follows:

• First term on right-hand side: convection term due to the fact that the particles move.

• First term on left-hand side: loss of particles due to collisions with nuclei of the medium. The coefficient equals -1 because of an appropriate choice of the unit in which x is measured.

• Second term on right-hand side: gain of particles because not all collisions with nuclei lead to a complete loss. The particle may be scattered into another direction, or the collision may lead to the emission of one or more new particles. The constant c is the mean number of particles released per collision. The function $f(\mu,\mu')$ is a probability density giving the probability that a secondary particle has velocity direction μ when it is the result of the collision of a primary particle with velocity direction μ'.

The physical meaning of equation (3) is similar, with the different right-hand side resulting from a different behavior of electrons as compared to neutrons.

Typical boundary conditions for each of the two equations are the following:

Finite slab (i.e., $J = (0,a)$):

$\psi(0,\mu) = g_+(\mu)$, $0 < \mu \le 1$: incoming density at $x = 0$;
$\psi(a,\mu) = g_-(\mu)$, $-1 \le \mu < 0$: incoming density at $x = a$.

Half space (i.e., $J = (0,\infty)$):

$\psi(0,\mu) = g_+(\mu)$, $0 < \mu \le 1$: incoming density at $x = 0$;
$\lim_{x\to\infty}\psi(x,\mu) = 0$, $-1 \le \mu \le 1$.

Anticipating the Hilbert space formalism which we introduce in the next section, we write these boundary conditions as follows. Let $H = H_+ \oplus H_-$, where $H_+ = L_2[(0,1); d\mu]$ and $H_- = L_2[(-1,0); d\mu]$, and let P_+, P_- denote the projections associated with this direct sum. The above boundary conditions are then interpreted as follows:

Finite slab: $P_+\psi(0) = g_+ \in H_+$, $P_-\psi(a) = g_- \in H_-$;

Half space: $P_+\psi(0) = g_+ \in H_+$, $\lim_{x\to\infty}\psi(x) = 0$ in H.

3. Hilbert Space Formalism

Let $H = L_2[(-1,+1); d\mu]$. Then we may interpret equations (2) and (3) as ordinary differential equations in H of the form

$$T \frac{d\psi}{dx}(x) = -A\psi(x), \quad x \in J. \tag{4}$$

In both cases T and A are selfadjoint relative to the usual inner product in H. Of importance here are the spectral properties of A:

- For equation (2): $\sum(A) \subset [1-c, 1+c]$. There are three cases:

  $c < 1$ (absorbing medium): $\sum(A)$ is positive and bounded away from 0.

  $c = 1$ (scattering medium): A has 0 as an eigenvalue. $\sum(A)\setminus\{0\}$ is bounded away from 0.

  $c > 1$ (multiplying medium): $\sum(A)$ has a negative part.

- For equation (3): $\sum(A) \subset \mathbf{R}_+$. A has 0 as an eigenvalue. $\sum(A)\setminus\{0\}$ is bounded away from 0.

One should note the correspondence between the cases of equation (2), $c = 1$, and equation (3). The case of equation (2), $c < 1$, is by now well understood, up to the point that some people call it the trivial case. The case of equation (2), $c = 1$, was (within the bounds of the Hilbert space formalism) first attacked by C. G. Lekkerkerker [1976] for isotropic scattering (i.e., $f \equiv 1$ in (2)). This same case was resolved by C. van der Mee [1981] for very general functions f. The case of equation (3) (and also equation (2), $c = 1$, but in a more general setting than van der Mee's contribution) was solved by R. Beals [1984]. The case of equation (2), $c > 1$, remains quite hazardous and poorly understood, notwithstanding some well-intended attempts.

In the remainder of this paper, we restrict ourselves to equation (2), $c < 1$. We do not include equation (2), $c = 1$, and equation (3) because these cases are marred by (apparently unavoidable) technical complications.

In the case of equation (2), $c < 1$, the operator A has a bounded inverse. We rewrite (4) as

$$\frac{d\psi}{dx}(x) = -S^{-1}\psi(x),\ x \in J, \tag{5}$$

where $S = A^{-1}T$ and the existence of S^{-1} is implied by the existence of T^{-1}. The associated eigenvalue problem (1) takes the form

$$S^{-1}v = \lambda v. \tag{6}$$

One may attempt to solve (5) subject to specific boundary conditions by expanding $\psi(x)$ in a series of eigenvectors of S^{-1} (in this case, eigenfunctionals since S^{-1} has not a discrete spectrum). Following this route, one has to prove, as a first step, the completeness of the system of eigenfunctionals. Though he did not use the Hilbert space formalism expounded here, it is exactly this route that K. M. Case [1960] followed for isotropic scattering. Somewhat later [Hangelbroek, 1976] we avoided the hardships of having to prove this completeness property (also for $f = 1$) by introducing a new inner product,

$$(v,w)_A = (Av,w),\ v,w \in H.$$

This inner product is equivalent to the original $(\ ,\ )$. Since S is selfadjoint relative to $(\ ,\ )_A$, we were able to invoke the Spectral Theorem. We do not have the time here to discuss the details of Case's contribution, but we must mention his half-range completeness proof. Our endeavors to avoid this latter proof ultimately led to the introduction of the operator V [Hangelbroek and Lekkerkerker, 1977].

The spectrum of S^{-1} contains the intervals $(-\infty,-1]$ and $[1,\infty)$. Hence, S^{-1} does not generate a semigroup of operators by which (5) can be solved. Using the Spectral Theorem, however, we can decompose H in a direct sum

$$H = H_p \oplus H_m \tag{7}$$

of closed subspaces such that H_p (H_m) is the maximal invariant subspace in which S is positive (negative). Solving (5) separately in H_p and H_m, we obtain as the general solution in the case of a finite slab,

$$\psi(x) = (e^{-xS^{-1}}P_p + e^{(a-x)S^{-1}}P_m)h,\ 0<x<a, \tag{8}$$

where P_p and P_m denote the projections associated with (7) and h is some element of H to be determined from the boundary conditions.

In the case of a half space we obtain

$$\psi(x) = e^{-xS^{-1}}h_1,\ x>0, \tag{9}$$

where h_1 must be in H_p in order that $\lim_{x\to\infty}\psi(x) = 0$. Substitution of (9) in the boundary condition stated at the end of Section 2 yields that h_1 has to be a solution of the equation $(P_+P_p + P_-P_m)h_1 = g_+$, i.e.,

$$Vh_1 = g_+, \tag{10}$$

where we defined $V = P_+P_p + P_-P_m$. In the same way we find by substitution of (8) in the boundary conditions for the finite slab that h must be a solution of the equation

$$V_a h = g_+ + g_-, \tag{11}$$

where $V_a = V + (1-V)e^{-a|S^{-1}|}$, $|S^{-1}| = (P_p - P_m)S^{-1}$.

We conclude that in order to be able to solve the half space and finite slab problems, we need to prove the existence of V^{-1} and V_a^{-1}.

4. Properties of the Operator V

In the following observations, we assume the existence of V^{-1} for the time being.

Since $V = P_+P_p + P_-P_m$, we have $V: H_p \to H_+$ and $H_m \to H_-$. Then $V^{-1}: H_+ \to H_p$ and $H_- \to H_m$. Actually, $V^{-1}P_+$ is a projection onto H_p along H_- and $V^{-1}P_-$ is a projection onto H_m along H_+. The latter statements follow from the properties

$$P_+V^{-1}P_+ = P_+VV^{-1}P_+ = P_+ \tag{12}$$

and similarly,

$$P_-V^{-1}P_- = P_-.$$

One may call V^{-1} an extension operator. It extends each element $g_+ \in H_+$ to an element $V^{-1}g_+ \in H_p$ by adding an element in H_-,

$$V^{-1}g_+ = g_+ + P_-V^{-1}g_+.$$

Let $I = \int dE_\lambda$ be the resolution of the identity for the operator S. If $g_+ \in H_+$, then

$$V^{-1}g_+ = \int_{\mathbb{R}_+} dE_\lambda V^{-1}g_+,$$

since $V^{-1}g_+ \in H_p$. Hence, because of (12),

$$g_+ = P_+\int_{\mathbb{R}_+} dE_\lambda V^{-1}g_+ = \int_{\mathbb{R}_+} d(P_+E_\lambda V^{-1})g_+,$$

which proves the so-called half-range completeness of the system of eigenfunctionals associated with the operator S.

We have left to discuss the existence of V^{-1} and V_a^{-1}. There are two different approaches:

(i) We use the fact that the integral term in (2) defines a compact operator, C say, in H. Thus, $A = 1-C$ is a compact perturbation of the identity. Assuming that C is a trace class operator [Hangelbroek, 1980] or, more generally, that $C = |T|^\varepsilon D$ for some $0 < \varepsilon < 1$ with D compact [van der Mee, 1981], one can prove that $1-V$ is compact. It is not difficult to show that V is injective, either by a direct argument or by inference from the classical result that the half space problem has a unique solution. Using Fredholm's Alternative, we then conclude that V^{-1} exists in H. The same argument yields that V_a^{-1} exists since

$$V_a = V(1+(V^{-1}-1)e^{-a|S^{-1}|}) \tag{13}$$

with $V^{-1}-1$ compact.

(ii) We follow the approach initiated by R. Beals [1984], in which the compactness of $C = 1-A$ is not used. We introduce two new inner products,

$$(v,w)_T = (|T|v,w) \text{ and } (v,w)_s = (|S|v,w)_A,$$

where $|T| = (P_+-P_-)T$ and $|S| = (P_p-P_m)S$. The equivalence of the inner products $(\,,\,)$ and $(\,,\,)_A$ implies the equivalence of $(\,,\,)_T$ and $(\,,\,)_S$ [Beals, 1984; Hangelbroek, 1984]. The completion of H relative to $(\,,\,)_T$ (or $(\,,\,)_S$) is $\tilde{H} = L_2[(-1,+1);|\mu|)d\mu]$. We present a proof that V^{-1} exists in $\tilde{H}$ using an argument that is simpler than the one used by Beals.

The relation between $(\,,\,)_T$ and $(\,,\,)_S$ is

$$(v,w)_S = ((2V-1)v,w)_T.$$

Hence, in terms of the partial ordering of selfadjoint operators in a Hilbert space,

$$\varepsilon < 2V-1 < \varepsilon^{-1} \text{ relative to } (\,,\,)_T$$

for some $\varepsilon > 0$. Then

$$\frac{2\varepsilon}{1+\varepsilon} < V^{-1} < \frac{2}{1+\varepsilon} \text{relative to } (\,,\,)_T.$$

We conclude that V^{-1} exists with $\|V^{-1}\|_T < 2$. Since

$$-1 < V^{-1}-1 < 1 \text{ relative to } (\,,\,)_T \text{ and } (\,,\,)_S,$$

we obtain from (13) that V_s^{-1} also exists as a bounded operator in $\hat{H}$ since $\|(V^{-1}-1)e^{-s|S^{-1}|}\|_S < 1$.

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Note: In the above list, we attempted to minimize the number of references, restricting ourselves to publications that, in our opinion, are the most important or informative within the context of this paper. Such a selection may hurt many feelings. Especially, we feel that (unavoidable) injustice is done to the many serious and very able scientists and mathematicians who are not mentioned but whose studies have had considerable influence on our investigations.

Below, we give four additional references that are important to the understanding of equations (2) and (3).

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ASYMPTOTIC BEHAVIOR OF SEMIGROUPS

J. Hejtmanek *

Abstract

A motivation is given for the study of the spectral mapping theorem for the exponential function. Also, some results of this study are summarized.

1. Semigroups

Let X be a Banach space. Usually we have in mind an L^2-Hilbert space or an L^1-Banach space. Both types of spaces have an ordering structure through the cone of positive functions X_+. Both types of spaces are Banach lattices. An operator $T: X \to X$ is called positive iff $TX_+ \subseteq X_+$.

Let $W = [W(t): t \in \mathbf{R}_+]$ be a strongly continuous one-parameter semigroup with generator A; $W(t) = \exp(At)$. It follows from the Feller-Mijadera-Phillips Theorem that the spectrum $\sigma(A)$ of the generator is contained in some left half-plane. We define the spectral bound of the generator A

$$s := \sup\{\text{Re } \lambda : \lambda \in \sigma(A)\},$$

and the type of the semigroup W

$$\omega_0 := \inf\{\omega \in \mathbf{R}: \text{ there exists an } M \geq 1 \text{ such that } \|W(t)\| \leq M\exp(\omega t) \text{ for all } t \geq 0\}.$$

It follows from the Laplace transform that $s \leq \omega_0$, but in general we do not have equality.

THEOREM. $[W(t)x_0 : t \in \mathbf{R}_+]$ *is the solution of the Cauchy problem*

$$\dot{x}(t) = Ax(t), \; x(0) = x_0 \in \text{dom } A.$$

A semigroup is called a semigroup of positive operators iff $W(t)X_+ \subseteq X_+$ for all $t \geq 0$.

We assume that the reader is familiar with the standard theory of linear semigroups; references are the texts by Hille and Phillips [1957], Davies [1980], Goldstein [1983], and Pazy [1983].

2. The Time-Dependent Neutron Transport Equation

Let the physical domain of a neutron transport system be a convex subset Ω of $\mathbf{R}^3$, and let the velocity domain be a sphere or spherical shell

$$S = \{\xi \in \mathbf{R}^3 : 0 \leq v_0 \leq |\xi| \leq v_1 \leq \infty\}.$$

Then $f(x,\xi,t)\,dx\,d\xi$ is the expected number of neutrons in the volume element $dx\,d\xi$ at $(x,\xi) \in \Omega \times S$ at time t. We assume that the particle density function $f(\cdot,\cdot,t)$ is an element of $L^1(\Omega \times S)$ or $L^2(\Omega \times S)$. The linear transport equation is a balance equation which has the following form:

$$\frac{\partial t}{\partial t} = -\frac{\partial}{\partial x} \cdot \xi f(x,\xi,t) - h(x,\xi) f(x,\xi,t) + \int_S k(x,\xi \leftarrow \xi') f(x,\xi' t)\,d\xi'.$$

The operator defined by the right-hand side is a sum of a partial differential

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operator of order one, a multiplication operator, and a partial identity.

In addition, boundary conditions are prescribed: (i) in the reactor problem Ω is a bounded subset of $\mathbb{R}^3$, and the incoming flux on the boundary $\partial\Omega$ vanishes; (ii) in the multiple scattering problem, we have $\Omega = \mathbb{R}^3$, and a bounded and convex target D is embedded in Ω. In both cases, we can write the linear transport equation in functional form,

$$\dot{f}(t) = (-T-A_1+A_2)f(t), \quad f(0) = f_0 \in \text{dom } T.$$

We assume that the linear operators A_1 and A_2 are bounded.

The solution of the differential equation $\dot{f}(t) = (-T-A_1)f(t)$ has the explicit form

$$W_1(t)f_0(x,\xi) = \exp(-\int_0^t h(x-\xi s,\xi)ds)f_0(x-t\xi,\xi), \quad \text{if } x-t\xi\in\Omega,$$

and zero otherwise.

The solution of the differential equation $\dot{f}(t) = (-T-A_1+A_2)f(t)$ can be written as a Dyson-Phillips expansion of the corresponding perturbation problem:

$$W(t)f_0 = W_1(t)f_0 + \int_0^t W_1(t-s)A_2W_1(s)f_0 ds + \ldots .$$

We refer to the monograph by Kaper, Lekkerkerker, and Hejtmanek [1982] for details. There sufficient conditions on the functions h and k can be found such that the linear transport operator $-T-A_1+A_2$ has a strictly dominant eigenvalue. The existence of such an eigenvalue, which had been tacitly assumed since Fermi's famous criticality experiment with the CP-1 reactor at the University of Chicago on December 12, 1942, implies that the solution of the linear transport equation, for large t, is determined by a decay constant λ_0 and a fundamental mode $\varphi(x,\xi)$:

$$W(t)f_0(x,\xi) \sim e^{\lambda_0 t}\varphi(x,\xi).$$

The problem of predicting such asymptotic behavior from the knowledge of the spectrum of $-T-A_1+A_2$ is, however, still unsolved. It requires the decomposition of the semigroup W into an asymptotic part $e^{\lambda_0 t}P_0$ and a transient part $Z_0(t)$

$$W(t) = e^{\lambda_0 t}P_0 + Z_0(t)(I-P_0),$$

where $\omega_0(Z_0) < \omega_0(W) = \lambda_0$. We remark that the equality $\omega_0(W) = \lambda_0$ is not true for general semigroups, but it is true for semigroups of positive operators in an L^1-Banach lattice (Theorem of Derndinger).

3. Spectral Mapping Theorem

The validity of the spectral mapping theorem for the exponential function would enable us to predict the desired asymptotic behavior for linear transport processes.

THEOREM. *If* $[W(t): t>t_0]$ *is continuous in the uniform operator topology for some* $t_0 > 0$, *then* $\exp(\sigma(A)t) = \sigma(\exp(At))\setminus\{0\}$ *for all* $t \geq 0$. *In particular, if the semigroup is holomorphic, then the spectral mapping theorem is true.*

THEOREM. *If* W *is a semigroup of positive operators in an* L^1- *or* L^2-*Banach lattice, then* $s = \omega_0$. See Nagel [1984].

Several examples have been given of semigroups for which $s < \omega_0$. See Hille-Phillips [1957], Fojas [1973], Greiner, Voigt, and Wolff [1981], Wolff [1981], and Nagel [1982].

We remark that the semigroup generated by the linear transport operator is not holomorphic, but that it can happen that $[W(t): t > t_0]$ is continuous in the uniform operator topology for some $t_0 > 0$.

Furthermore, we remark that there are semigroups for which $\exp(\sigma(A)t) \neq \sigma(\exp(At))\backslash\{0\}$ for all $t > 0$ (e.g., if $s < \omega_0$), and that there are semigroups for which equality holds for rational t, but inequality for irrational t.

4. Additional Examples

4.1. Schrödinger Semigroups

In general, it is not true that the rotation of the generator of a semigroup, especially by 90°, results in the generator of another semigroup. If the generator of the Schrödinger equation $-iH$ is rotated by 90°, then $-H$ is generator of a holomorphic semigroup $[\exp(-Ht): t \geq 0]$, which is called the Schrödinger semigroup; see B. Simon [1982]. In this case the spectral mapping theorem is true, and *a fortiori* $s = \omega_0$.

4.2. Stability Theory

We refer to the articles of Pritchard and Zabczyk [1981], Wolff [1981], and Nagel [1984]. Let us assume that $s \leq \omega_0 \leq 0$.

THEOREM. *If $s = \omega_0$, and if $s < 0$, then the semigroup is exponentially asymptotically stable in the uniform operator topology, i.e., there exist constants $u > 0$ and $M \geq 1$ such that $\|W(t)\| \leq M \exp(-ut)$ for all $t \geq 0$.*

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SOME EXTENSIONS OF RESULTS OF TITCHMARSH ON DIRAC SYSTEMS

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Abstract

Spectral properties are considered for a one-dimensional Dirac system with singularities at zero and infinity. Of primary concern is the asymptotic behavior of solutions near zero. From the form of these solutions, criteria are given for the singular point zero to be in the limit point or limit circle case. The Titchmarsh-Weyl m-coefficient at zero is shown to be meromorphic. This result, when combined with known behavior at infinity, is used to establish the location and class $C^{(1)}$ nature of the essential spectrum.

1. Introduction

The system considered is

$$Ly(x) := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \{y'(x) - \begin{bmatrix} k/x & c+V_1(x) \\ c-V_2(x) & -k/x \end{bmatrix} y(x)\}$$

$$= \lambda y(x),\ 0 < x < \infty,\ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{1.1}$$

where V_i is real and locally Lebesgue integrable on $(0,\infty)$, and k, λ, and c are constants with $c > 0$ and k,λ possibly complex. Denote by $H(I)$ the Hilbert space of all equivalence classes of complex vector-valued functions $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ on an interval I such that $\int_I(|f_1|^2 + |f_2|^2)dx < \infty$. The operator L determines (for k real) in $H(0,\infty)$ certain selfadjoint operators (cf. Weidmann [1971]) by suitably restricting it to functions in $H(0,\infty)$ satisfying certain boundary conditions. The number of such boundary conditions at each singular endpoint may be determined by the method of Weyl (cf. Levitan and Sargsjan [1975]); thus, following the terminology of this method, we say L is limit-point (LP) at 0 (limit-circle (LC) at 0) if the number of linearly independent solutions of $Ly = \lambda y$ in $H(0,1)$ for $\text{Im}\lambda \neq 0$ is exactly 1 (2). Similar definitions apply at $x = \infty$, although in the setting here L is always LP at ∞ [Levitan and Sargsjan, 1975: p. 492]. When L is LP at a singular point, no boundary conditions are required there; and when L is LC at a singular point, one boundary condition is required.

Another problem of interest in connection with L is the location of the essential spectrum of the selfadjoint operators determined by L. Under fairly general conditions for V_1, V_2 "small" at ∞, this essential spectrum is $(-\infty,-c]\cup[c,\infty)$; when V_1, V_2 are "large" at ∞, this essential spectrum is $(-\infty,\infty)$. Results of this nature may be found in Behncke [1980], Hinton and Shaw [1984a], Titchmarsh [1961 and 1962], and Weidmann [1971 and 1982].

For $V_1 = V_2$, the system (2.2) was investigated by Titchmarsh [1961 and 1962]. For the singular endpoint $x = 0$, we extend Titchmarsh's LP-LC criteria

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to allow for highly oscillatory potentials as well as having $V_1 \neq V_2$. In Section 3 we show how these criteria, derived by asymptotic methods, also allow extensions of criteria for essential spectra to be $(-\infty,-c]\cup[c,\infty)$ or $(-\infty,\infty)$. Actually, we show the essential spectrum is of class $C^{(1)}$ in its interior. Finally, in Section 4 we prove that the essential spectrum is simple in its interior. Other extensions of Titchmarsh's *LP-LC* criteria, which are complementary to those given here, have been given by Kalf [1972] and Behncke [1980].

The operator (1.1) arises from the 3-dimensional Dirac operator by a separation of variables, and its spectral properties can be related to those of the 3-dimensional Dirac operator (cf. Behnke [1980] and Rejto [1971a and 1971b]). Considerable literature exists on spectral properties of the 3-dimensional case; we refer to Barut and Kraus [1976]; Burnap, Brysk, and Zweifel [1981]; Kalf, Schmincke, Walter, and Wüst [1975]; Klaus [1980]; Klaus and Wüst [1979]; Schmincke [1972 and 1973]; and Wüst [1977] for a partial listing of such properties.

2. Asymptotics at 0

We use here a theorem from Hinton [1984] for the system

$$\eta'(t) = [\Omega'(t)\Omega^{-1}(t) + B(t) + C(t)]\eta(t)),\ a \le t < \infty, \tag{2.1}$$

where Ω is a diagonal matrix each with Ω_i nonvanishing and essentially increasing or essentially decreasing on $[a,\infty)$ (cf. Hinton [1984: p. 294]).

THEOREM 2.1 [Hinton, 1984]. *Suppose in (2.1) that* Ω, B, *and* C *are locally Lebesgue integrable and that*

i. $B_0(t) := \int_t^\infty B(s)ds$ *exists.*

ii. $\int_a^\infty \|C(t)\|dt < \infty$.

iii. $\int_a^\infty \|G(t)\|dt < \infty$, *where* $G = -\Omega'\Omega^{-1}B_0 + B_0\Omega'\Omega^{-1} + B_0B$.

Then there is a fundamental matrix Γ *of (2.1) such that*

$$\lim_{t\to\infty}\Gamma(t)\Omega^{-1}(t) = I.$$

To apply Theorem 2.1 to (1.1) the singular point is transformed to ∞ by

$$\vec{y}(x) = \begin{bmatrix} -x^m & 0 \\ 0 & x^{-m} \end{bmatrix} \vec{z}(t),\ t = 1/x,\ 0 < x < 1. \tag{2.2}$$

Then $\vec{z}$ satisfies $(\cdot = d/dt)$

$$\vec{z}\cdot(t) = \begin{bmatrix} (m-k)/t & (\lambda+\tilde{V}_1(t)+c)t^{2m-2} \\ (-\lambda-\tilde{V}_2(t)+c)t^{-2m-2} & -(m-k)/t \end{bmatrix} \vec{z}(t), \tag{2.3}$$

where $\tilde{V}_i(t) = V_i(1/t)$. We state three theorems that distinguish the cases: i) k/x dominates V_1, V_2; ii) k/x and V_1, V_2 are of comparable size; and iii) V_1, V_2 dominate k/x.

THEOREM 2.2. *Suppose in (1.1) that for* $i = 1,2$, *i.* $\mathbf{V}_i(x) := \int_0^x V_i(\xi)d\xi$ *exists,*

ii. $\int_0^1 x^{-1}|\mathbf{V}_i(x)|dx < \infty,$

iii. $\int_0^1 \{|\mathbf{V}_1(x)V_2(x)| + |\mathbf{V}_2(x)V_1(x)|\}dx < \infty.$

Then there is a fundamental matrix $Y(x,\lambda)$ of (1.1) such that

$$\lim_{x\to 0^+} Y(x,\lambda)\begin{bmatrix} -x^{-k} & 0 \\ 0 & x^k \end{bmatrix} = I. \tag{2.4}$$

Moreover, for k real, L is LC at 0 iff $|k| < 1/2$.

PROOF. We write (2.3) for $m = 0$ as

$$z\cdot(t) = \{\frac{1}{t}\begin{bmatrix} -k & 0 \\ 0 & k \end{bmatrix} + \frac{\tilde{V}_1(t)}{t^2}M_1 + \frac{\tilde{V}_2(t)}{t^2}M_2 + C(t)\}\, z(t), \tag{2.5}$$

where

$$M_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \quad C(t) = t^{-2}\begin{bmatrix} 0 & \lambda+c \\ -\lambda+c & 0 \end{bmatrix}.$$

To apply Theorem 2.1 to (2.4), we set

$$\Omega(t) = \begin{bmatrix} t^{-k} & 0 \\ 0 & t^k \end{bmatrix}, B(t) = \frac{\tilde{V}_1(t)}{t^2}M_1 + \frac{\tilde{V}_2(t)}{t^2}M_2.$$

By defining $\tilde{\mathbf{V}}_i(t) = \mathbf{V}_i(1/t) = \int_t^\infty s^{-2}\tilde{V}_i(s)ds$, the matrix G in (iii) of Theorem 2.1 is computed to be

$$G(t) = 2kt^{-1}\begin{bmatrix} 0 & \tilde{\mathbf{V}}_1(t) \\ \tilde{\mathbf{V}}_2(t) & 0 \end{bmatrix} - t^{-2}\begin{bmatrix} \tilde{\mathbf{V}}_1(t)\tilde{V}_2(t) & 0 \\ 0 & \tilde{\mathbf{V}}_2(t)\tilde{V}_1(t) \end{bmatrix}.$$

It is readily computed that Theorem 2.1 applies and there is a fundamental matrix $Z(t,\lambda)$ of (2.5) such that

$$\lim_{t\to\infty} Z(t,\lambda)\begin{bmatrix} t^k & 0 \\ 0 & t^{-k} \end{bmatrix} = I. \tag{2.6}$$

The transformations (2.2) and (2.6) yield (2.4). The asymptotics (2.4) yield immediately the "moreover" part.

A prototype example for Theorem 2.1 is

$$V_1(x) = V_2(x) = a/x^n + bx^\alpha \sin x^\beta,\ 0 < x < 1,$$

where a, b, n, α, β are real constants with $n < 1$, $\beta < 0$, and $\alpha > (\beta/2) - 1$.

THEOREM 2.3. *Suppose in (1.1) that for $i = 1,2$,*

i. $V_i(x) = a_i/x + q_i(x)$, *a_i constant.*

ii. $\mu_0 := \sqrt{k^2 - a_1a_2} \neq 0.$

iii. $Q_i(x) := \int_0^x q_i(\xi)d\xi$ *exists.*

iv. $\int_0^1 x^{-1}|Q_i(x)|\,dx < \infty$.

v. $\int_0^1 \{|Q_1(x)q_2(x)| + |Q_2(x)q_1(x)|\}\,dx < \infty$.

Then there is a fundamental matrix $Y(x,\lambda)$ *of (1.1) such that*

$$\lim_{x\to 0^+} S^{-1}Y(x,\lambda)\begin{bmatrix} x^{-\mu_0} & 0 \\ 0 & x^{\mu_0}\end{bmatrix} = I, \quad S = \begin{bmatrix} -a_1 & -(k+\mu_0) \\ k+\mu_0 & a_2 \end{bmatrix}.$$

Moreover, if a_1, a_2, *and* k *are real, then* L *is LC at 0 iff* $|\mathrm{Re}\mu_0| < 1/2$.

The proof of this theorem is similar to that of Theorem 2.2. It is first necessary to diagonalize the leading matrix in (2.5) by the transformation $\hat{z}(t) = S^{-1}z(t)$.

Results in case (iii), i.e., when V_1, V_2 dominate k/x may be found in Hinton and Shaw [1984a]. For example, the following theorem is a corollary of the asymptotics in Hinton and Shaw [1984a].

THEOREM 2.4. *Suppose in (1.1) for* $i = 1,2$ *and some* $a > 0$,

(i) $V_1(x), V_2(x) \to \infty$ *as* $x \to 0$, $\int_0^a [x^2 V_i(x)]^{-1}\,dx < \infty$,

(ii) $\int_0^a [(V_2(x)-c)/(V_1(x)+c)]^{1/2}\,dx < \infty$,

$\int_0^a [(V_1(x)+c)/(V_2(x)-c)]^{1/2}\,dx < \infty$,

(iii) $\eta(x) := [(V_1(x)+c)/(V_2(x)-c)]^{1/4}x^{-k}$,

$\Delta(x) := \eta'(x)/\eta(x)[(V_1(x)+c)(V_2(x)-c)]^{1/2}$,

satisfy $\Delta(x)\to 0$ *as* $x\to 0$ *and* $\int_0^a |\Delta'(x)|\,dx < \infty$. *Then there is a fundamental matrix* $Y(x,\lambda)$ *of (1.1) such that as* $x\to 0^+$,

$$T\begin{bmatrix} -x^{-k}/\eta(x) & 0 \\ 0 & \eta(x)x^k \end{bmatrix} Y(x,\lambda) \begin{bmatrix} E(x)^{-1} & 0 \\ 0 & E(x) \end{bmatrix} \to I,$$

where for some $b > 0$,

$$E(x) := \exp\left\{\int_x^b i\mu_0(x)x^{-2}\,dx\right\}$$

$$\mu_0(x) := \sqrt{1-\Delta(x)^2}$$

$$T := \frac{1}{2i}\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}.$$

Moreover, (1.1) is LC at 0.

Note that for $V_1(x) = V_2(x) = k/x^n$, these hypotheses are satisfied for $k > 0$ and $n > 1$.

3. Essential Spectrum Criteria

For the computation of essential spectra, it is sufficient to consider each endpoint separately since the essential spectrum for a two-singular endpoint problem is the union of two separate one-singular endpoint problems (cf. Weidmann [1971]). However, it is also convenient to consider the two-singular endpoint problem from the viewpoint of the Titchmarsh-Weyl m-coefficient which may be defined as follows.

Define solutions $\vec{\vartheta}, \vec{\varphi}$ of (1.1) by the initial values

$$(\vec{\vartheta}(1,\lambda), \vec{\varphi}(1,\lambda)) = \begin{bmatrix} \vartheta_1(1,\lambda) & \varphi_1(1,\lambda) \\ \vartheta_2(1,\lambda) & \varphi_2(1,\lambda) \end{bmatrix} = I.$$

Then the one-singular endpoint m-coefficients are given for $\mathrm{Im}\lambda \neq 0$ by

$$m^+(\lambda) = \lim_{x\to\infty} \frac{\vartheta_1(x,\lambda)}{\varphi_1(x,\lambda)}, \quad m^-(\lambda) = \lim_{x\to 0} \frac{\vartheta_1(x,\lambda)}{\varphi_1(x,\lambda)}. \tag{3.1}$$

Where $x = 0$ is in the limit circle case, the limit for m^- in (3.1) is a sequential limit as m^- is not unique; different sequential limits correspond to different boundary conditions at 0. The m-coefficient (cf. Hinton and Shaw [1984b]) for a selfadjoint operator associated with (1.1) is given by

$$M(\lambda) = [m^-(\lambda) - m^+(\lambda)]^{-1} \begin{bmatrix} 1 & (m^+(\lambda)+m^-(\lambda))/2 \\ (m^+(\lambda)+m^-(\lambda))/2 & m^+(\lambda)m^-(\lambda) \end{bmatrix}. \tag{3.2}$$

For the Sturm-Liouville operator, fundamental relations between the singular structure of the m-coefficient and the spectrum have been derived by Chaudhuri and Everitt [1968]. These results were extended to systems by Hinton and Shaw [1982 and 1984b]. We describe now the relations that we utilize here. let T be a selfadjoint operator associated with (1.1) whose m-coefficient is M. We say M is analytic (has a simple pole) at a real λ_0 if M has an extension that is analytic (has a simple pole) at λ_0. Let $\rho(T)$ be the resolvent set of T, $\sigma(T)$ be the spectrum of T, and $P(T)$ be the isolated points of $\sigma(T)$. Define $E(T) = \sigma(T) - P(T)$ as the essential spectrum of T. The set $PC(T) \subset E(T)$ consisting of eigenvalues in $E(T)$ is called the point-continuous spectrum and $C(T) = E(T) - PC(T)$ is called the continuous spectrum. The following relations are established in Hinton and Shaw [1984b]:

(i) $\lambda_0 \in \rho(T) \Leftrightarrow M(\lambda)$ is analytic at λ_0. (3.3)

(ii) $\lambda_0 \in P(T) \Leftrightarrow M(\lambda)$ has a simple pole at λ_0.

(iii) $\lambda_0 \in C(T) \Leftrightarrow M(\lambda)$ is not analytic at λ_0 and $\lim_{\nu\to 0^+} \nu M(\lambda_0 + i\nu) = 0$.

(iv) $\lambda_0 \in PC(T) \Leftrightarrow \lim_{\nu\to 0^+} \nu M(\lambda_0 + i\nu) = S \neq 0$, and $M(\lambda) - iS(\lambda-\lambda_0)^{-1}$ is not analytic at λ_0.

An examination of the proof of Theorem 2.1 shows that the matrix $Y(x,\lambda)$ in Theorems 2.2 and 2.3 is entire in λ for each fixed x. Since $(\vec{\vartheta}(x,\lambda), \vec{\varphi}(x,\lambda)) = Y(x,\lambda)Y(1,\lambda)^{-1}$ we may then conclude that under the hypotheses of Theorem 2.2 or 2.3, $m^-(\lambda)$ is meromorphic on **C**. Under the hypotheses of Theorem 2.4, $m^-(\lambda)$ is meromorphic on **C** since (1.1) is LC at 0. *We assume for the remainder of this section that $m^-(\lambda)$ is meromorphic on* **C**.

Under rather general conditions for V_1, V_2 "small" at ∞, $m^+(\lambda)$ satisfies the following:

(i) $m^+(\lambda)$ is meromorphic on $\{\lambda: -c < \mathrm{Re}\lambda < c\}$. (3.4)

(ii) $m^+(\lambda)$ has a continuous extension to $\{\lambda: \mathrm{Re}\lambda \notin [-c,c],\ \mathrm{Im}\lambda \geq 0\}$ such that $\mathrm{Im}m^+(\lambda) > 0$ for λ real and $\lambda \notin [-c,c]$.

For example, if $V_i = V_{i1} + V_{i2}$ with $V_{i1}(x) \to 0$ as $x \to \infty$ and $V_{i1}', V_{i2} \in L(1,\infty)$, m^+ satisfies (3.4) [Hinton and Shaw, 1984a]. We now note some consequences of m^+ satisfying (3.4).

If we write $m^\pm(\lambda) = \sigma^\pm(\lambda) + i\gamma^\pm(\lambda)$ on $\{\lambda: \mathrm{Im}\lambda > 0\}$ with $\sigma^\pm$, $\gamma^\pm$ real, then we compute that (with λ suppressed),

$$\mathrm{Im}M = \frac{1}{(\gamma^+-\gamma^-)^2 + (\sigma^+-\sigma^-)^2}\begin{bmatrix}\gamma^+-\gamma^- & \gamma^+\sigma^- - \gamma^-\sigma^+ \\ \gamma^+\sigma^- - \gamma^-\sigma^+ & \gamma^+(\sigma^-+\gamma^-)^2 + \gamma^-(\sigma^++\gamma^+)^2\end{bmatrix}. \quad (3.5)$$

From (3.2), (3.4), and (3.5) we have the following:

(α) M is meromorphic on $\{\lambda: -c < \mathrm{Re}\lambda < c\}$; hence the spectrum of T purely discrete on $(-c,c)$.

(β) For λ_0 in $(-\infty,-c)\cup(c,\infty)$ a regular point of m^-, there is an interval $(\lambda_0-\delta,\lambda_0+\delta)$ on which $\gamma^- \equiv 0$ and

$$\mathrm{Im}M = \frac{1}{(\gamma^+)^2 + (\sigma^+-\sigma^-)^2}\begin{bmatrix}\gamma^+ & \gamma^+\sigma^- \\ \gamma^+\sigma^- & \gamma^+(\sigma^-)^2\end{bmatrix}.$$

(γ) For λ_0 in $(-\infty,-c)\cup(c,\infty)$ a pole of m^-,

$$\lim_{\varepsilon\to 0} M(\lambda_0 + i\varepsilon) = \begin{bmatrix}0 & 1/2 \\ 1/2 & m^+(\lambda_0)\end{bmatrix}.$$

(δ) If ρ is the spectral matrix of T, then by the Titchmarsh-Kodaira formula, i.e.,

$$\rho(\lambda) - \rho(\mu) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_\mu^\lambda \mathrm{Im}M(s+i\varepsilon)ds,$$

we have on $(-\infty,-c)\cup(c,\infty)$ that ρ is of class $C^{(1)}$ and $\rho'(\lambda)$ has a rank 1.

For V_1, V_2 "large" at ∞ and of like sign, we have under general conditions, e.g., Hinton and Shaw [1984a: Theorem 2], that $m^+(\lambda)$ has a continuous extension to $\{\lambda: \mathrm{Im}\lambda \geq 0\}$ with $\mathrm{Im}m(\lambda) > 0$ on $(-\infty,\infty)$. Under these circumstances, the spectral matrix of ρ of T is of class $C^{(1)}$ on $(-\infty,\infty)$ with rank $\rho'(\lambda) = 1$ on $(-\infty,\infty)$.

Theorems of the above type for the one-dimensional and three-dimensional Dirac operator have suggested the following problems. However, results seem to be scarce; one may be found in Glazman [1965: p. 207].

(i) Find conditions on the coefficients of (1.1) that ensure that $\sigma(T)\cap(-c,c)$ is finite.

(ii) Find conditins on the coefficients of (1.1) that ensure that $c(-c)$ is a limit point of $P(T)$.

For the three-dimensional Dirac operator, sufficient conditions for there to exist $[-d,d]$ so that $\sigma(T)\cap[-d,d] = \phi$ are given in Klaus and Wüst [1979], Schmincke [1973], and Wüst [1977]. Bounds on the eigenvalues in the one-dimensional case are given in Evans and Harris [1981].

4. Simple Spectrum

The expansion formula for (1.1) may be written for $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in H(0,\infty)$ as (cf. Levitan and Sargsjan [1979])

$$f(x) = \int_{-\infty}^{\infty} F_\vartheta(\lambda)\{\vec{\vartheta}(x,\lambda)d\rho_{11}(\lambda) + \vec{\varphi}(x,\lambda)d\rho_{12}(\lambda)\}$$

$$+\int_{-\infty}^{\infty} F_\varphi(\lambda)\{\vec{\vartheta}(x,\lambda)d\rho_{21}(\lambda) + \vec{\varphi}(x,\lambda)d\rho_{22}(\lambda)\}, \tag{4.1}$$

where

$$F_\vartheta(\lambda) = \int_0^\infty f^T(x)\vec{\vartheta}(x,\lambda)dx,\ F_\varphi(\lambda) = \int_0^\infty f^T(x)\vec{\varphi}(x,\lambda)dx.$$

For simplicity we will say an interval $I \subset C(T)$ is *simple* (see Naimark [1968] for the definition of simple spectrum) provided the contribution to (4.1) over I can be expressed in the form

$$\int_I F(\lambda)\vec{\psi}(x,\lambda)d\mu(\lambda),\ F(\lambda) = \int_0^\infty f^T(x)\vec{\psi}(x,\lambda)dx, \tag{4.2}$$

where $\vec{\psi}$ is a solution of (1.1) and μ is a monotone nondecreasing scalar function on I.

To see that these conditions hold for $I \subset (-\infty,-c)\cup(c,\infty)$ for V_1, V_2 "small" or $I \subset (-\infty,\infty)$ for V_1, V_2 "large," first suppose I contains no pole of $m^-(\lambda)$. Then from (β) above, $(\rho_{21}'(\lambda), \rho_{22}'(\lambda) = \sigma^-(\lambda),(\rho_{11}'(\lambda),\rho_{12}'(\lambda))$ and since ρ is symmetric, $\rho_{12}' = \rho_{21}' = \sigma^-\rho_{11}'$. Thus the contribution to (4.1) on I can be written as

$$\int_I \{F_\vartheta(\lambda) + \sigma^-(\lambda)F_\varphi(\lambda)\}\{\vec{\vartheta}(x,\lambda) + \sigma^-(\lambda)\vec{\varphi}(x,\lambda)\}\rho_{11}'(\lambda)d\lambda, \tag{4.3}$$

which is of the form of (4.2). At a pole λ_0 of $m^-(\lambda)$, we have in a neighborhood of λ_0, $\rho_{12}' = \rho_{21}' = \rho_{22}'/\sigma^-$ and $\rho_{11}' = \rho_{22}'/(\sigma^-)^2$, and a formula similar to (4.3) can be derived.

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SEMIGROUPS GENERATED BY ORDINARY DIFFERENTIAL OPERATORS

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Abstract

Certain regularity properties of semigroups generated by ordinary differential operators are considered. It is shown that semigroups generated by a general class of such operators (which allows for singular coefficients) are continuous, both pointwise and in L^p. The semigroups are infinitely smoothing in the scale of L^p spaces, but only partially so in the scale of Sobolev spaces.

1. Introduction

This paper will address some questions in the study of expansions in eigenfunctions of ordinary differential operators. Some of the results in Section 2 are essentially generalizations to higher order operators of results proved by Kon and Raphael [1983] for Sturm-Liouville expansions on the half-line. Our initial motivation arises from the heat equation $-\infty < x < \infty$:

$$-\frac{\partial^2 u}{\partial x^2} = -k\ \frac{\partial u}{\partial t}; \qquad u(x,0) = u_0(x); \qquad (0 \le t < \infty). \tag{1}$$

This of course describes heat flow on an infinite rod. The equation on a finite or semi-infinite interval with homogeneous boundary conditions can be treated using similar techniques to those presented here. Our development will be directed by the following questions.

QUESTION 1: In what sense does

$$u(x,t) \underset{t \to 0}{\to} u_0(x)? \tag{2}$$

The answer will be the best possible in a much wider context; namely, (2) will be shown to hold for a large class of differential operators replacing $(\partial^2)/(\partial x^2)$, in all L^p-spaces as well as almost everywhere. Related results for the heat equation and eigenfunction expansions in general have been studied by Benzinger [1970, 1979]. This problem can, of course, be viewed as one of convergence to boundary values in partial differential equations, and our techniques can in fact be adapted to study certain elliptic boundary value problems. There has been recent interest in this type of result, for example, in the study of the Laplace and heat equations on domains with C^1 boundary; see Fabes and Rivière [1979] and Jodeit [1979].

It is well known that forward time translation in (1) is infinitely smoothing, i.e., that if $u_0(x) \in L^p$, then $u(x,t) \in \mathbf{L}^p_m$ in x for $t > 0$ and any $m > 0$. Here, $\mathbf{L}^p_m$ is the L^p-Sobolev space of order m, i.e., functions with m derivatives in L^p. However, if (1) is replaced by, say,

$$(-\frac{\partial^2}{\partial x^2} + q(x))u = -\frac{\partial u}{\partial t}, \tag{3}$$

or a more general even-order operator with singular coefficients is used, then such smoothing fails to occur. An equation such as (3), incidentally, describes heat flow, with a position-dependent rate q of heat loss; a physically interesting

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"drift term" $a(x)\frac{\partial u}{\partial x}$, relevant to heat diffusion in a fluid, can be added as well.

Thus, we will also address a general version of

QUESTION 2: How does the smoothness (in x) of solutions in (2) depend on that of q?

The question of smoothing has been studied in Schrödinger semigroup theory (see Simon [1982] and Carmona [1979]), as well as for hyperbolic equations, where extremely precise results have been obtained (see Marshall, Strauss, and Wainger [1980] and Peral [1980]). It divides neatly into a simple and a difficult problem, namely, that of smoothing in the scale of L^P spaces, and that in the scale of Sobolev spaces.

The solution of (3) is, of course,

$$u(x,t) = e^{-tA}u_0(x). \tag{4}$$

Alternatively, if A is selfadjoint (for example),

$$u(x,t) = \int e^{-t\lambda}v(x,\lambda)\tilde{u}_0(\lambda)d\rho(\lambda), \tag{5}$$

where $v(x,\lambda)$ is an eigenfunction of A, $\tilde{u}_0$ is a generalized Fourier transform of u, and $\rho(\lambda)$ is the spectral function. Equation (5) shows that Question 1 is actually one in summability theory. The integration is over the spectrum of A, which in the case of the full line contains a twofold copy of $\mathbf{R}^+$; it may equivalently be considered a sum of two integrals over the real line (see Levitan and Sargsjan [1975]).

For brevity we will concentrate here on results, and sketch some proofs.

2. Convergence to Initial Values

Consider the semigroup generated by

$$A = (-1)^{m/2}\frac{d^m}{dx^m} + b_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + \dots + b_0(x), \tag{6a}$$

where $m > 0$ is even and

$$b_l(x) \in L^\infty + L^{r_l} \qquad (d \equiv \sup_{l \le m-1}\frac{1}{r_l} + l < m). \tag{6b}$$

The sum in (6b) means that b_l must be expressible as a sum of two functions in the indicated L^P-classes.

Let $u(x,t) = e^{-tA}u_0(x)$, and define the kernel $K_t(x,y)$ by

$$e^{-tA}u_0 = \int_{-\infty}^{\infty} K_t(x,y)u_0(y)dy.$$

Our approach to studying K_t (as well as other analytic functions of A) first leads to analysis of the resolvent $R_\zeta = (\zeta - A)^{-1}$, for $\zeta \in \mathbf{C}$.

THEOREM 1 (cf. Gurarie and Kon [1984]). *The L^P $(1 \le p \le \min r_l)$ spectrum $\sigma(A)$ of A is contained in a complex domain*

$$\Omega = \{\zeta = \rho e^{i\vartheta}: C\rho^{\frac{d}{m}-1} < |\sin\frac{\vartheta}{2}|^2\},$$

where d is given by (6b). Outside Ω, the resolvent $R_\zeta = (\zeta - A)^{-1}$ has a kernel bounded by

$$|R_\zeta(x,y)| \le \frac{C}{|\sin\frac{\vartheta}{2}|^2} h(x-y), \tag{7}$$

where

$$h(x) = \begin{cases} 1; & |x| \le 1 \\ |x|^{-t}; & |x| \ge 1 \end{cases}, \tag{8}$$

with t any number greater than 1.

We sketch the proof. Let

$$A_0 = (-1)^{\frac{m}{2}} \frac{d^m}{dx^m}, \qquad B = \sum_{i=0}^{m-1} b_i(x) \left(\frac{d}{dx}\right)^i,$$

and $R_\zeta^0 = (\zeta - A_0)^{-1}$. If appropriate convergence occurs, we have the standard representation

$$R_\zeta = R_\zeta^0 \sum_{k=0}^{\infty} (BR_\zeta^0)^k. \tag{9}$$

Under Fourier transformation R_ζ^0 is multiplication by $(\zeta+\xi^m)^{-1}$, where ξ is the dual variable of x. If we insert the definition of B into $R_\zeta^0(BR_\zeta^0)^k$, and multiply out, then a typical term in the product will be

$$R_\zeta^0 b_{i_1} D^{i_1} R_\zeta^0 b_{i_2} D^{i_2} R_\zeta^0 \cdots b_{i_k} D^{i_k} R_\zeta^0, \tag{10}$$

where b_i represents multiplication, $D^i = (\frac{d}{dx})^i$, and $i_1,\ldots,i_k$ are chosen from $(0,1,\ldots,m-1)$. If $K_i(x)$ is the convolution kernel of $D^i R_\zeta^0$ $(i < m)$, then (4) has kernel

$$L_{i_1}\ldots_{i_k}(x,y) = R_\zeta^0(x) b_{i_1}(x) K_{i_1}(x) * b_{i_2}(x) K_{i_2}(x) * \cdots * b_{i_k}(x) K_{i_k}(x-y). \tag{11}$$

The factors b_{i_j} can be removed from the integral implied in (11) through iteration of Hölder's inequality. The remaining convolution of functions, each bounded by multiples of (8), can be shown again to be bounded by a multiple of (8). Summation of the resulting bound on (11) over all collections $i_1, \cdots, i_k$ and finally over all k leads to a geometric series whose sum bounds the kernel of (9) and is bounded by the right side of (7). This constitutes the proof of the theorem, since the spectrum of A clearly lies in the complement of the domain of convergence of the above sum; this domain of convergence is precisely $\sim\Omega$. Some consequences of Theorem 1 are discussed below.

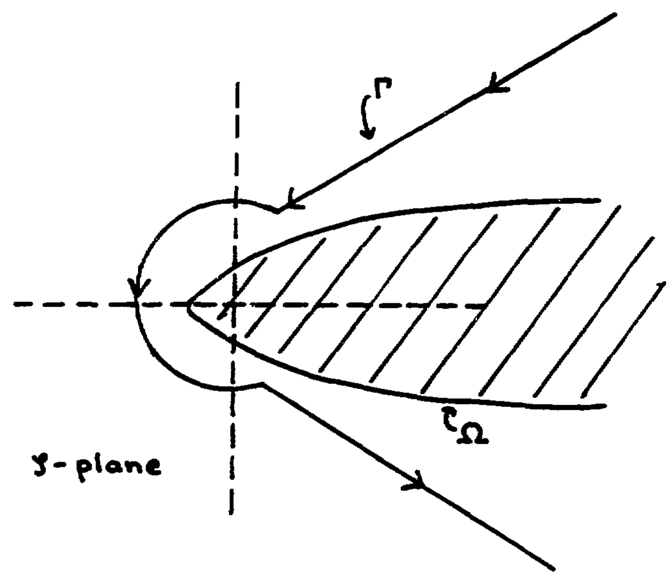
THEOREM 2. *If A is formally selfadjoint, and* $\min r_i \ge 2$, *then it is essentially selfadjoint on any essential domain of* $\frac{d^m}{dx^m}$.

Theorem 2 is proved by first using the bounds of Theorem 1 to show that the lower order terms in A are a relatively bounded perturbation of the leading term, and then applying the Kato-Rellich theorem (see Kato [1980]).

We can also construct and find appropriate bounds on the semigroup generated by A.

THEOREM 3 (cf. Gurarie and Kon [1984]). *The operator A generates an* L^p-*continuous semigroup* e^{-tA} $(1 \le p \le \min r_\alpha)$ *analytic in the right half t-plane, and strongly continuous in* L^p *at* $t = 0$.

To prove pointwise as well as strong continuity, we construct e^{-tA} explicitly in terms of the resolvent; this will bound the semigroup kernel. Let D_r be the disk of radius r centered at 0, and $\Omega_{\vartheta_1} = \{\zeta: \arg\zeta \le \vartheta_1\}$. Choose $r > 0$ and $0 < \vartheta_1 < \frac{\pi}{2}$, so that $\Omega \subset D_r \cup \Omega_{\vartheta_1}$. Thus the positively oriented contour $\Gamma = \partial(D_r \cup \Omega_{\vartheta_1})$ contains the spectrum of A, as shown below:



We use the Cauchy representation

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma \frac{e^{-t\zeta}}{\zeta - A}\,d\zeta. \tag{12}$$

By Theorem 1, the integral converges in the uniform operator topology on L^p $(1 \le p \le \min_l r_l)$ and the kernel K_t of e^{-tA} is bounded by

$$|K_t(x,y)| = \frac{1}{2\pi}\left|\int_\Gamma e^{-t\zeta} R_\zeta(x,y)\,d\zeta\right|,$$

$$\le c\int_\Gamma |e^{-t\zeta}|\ |\zeta|^{\frac{1}{m}-1} h(|\zeta|^{\frac{1}{m}}(x-y))\ |d\zeta|, \tag{13}$$

where h is given by (8). Essentially a change of variables shows that

$$|K_t(x,y)| \le c\,|t|^{-\frac{1}{m}} h(|t|^{-\frac{1}{m}}(x-y)),$$

uniformly in t for $|\arg t| \le \vartheta_1 < \pi$. This indicates that K_t behaves like an approximate identity as $t \to 0$, and a modification of some standard results in harmonic analysis (see Stein and Weiss [1971: Theorem 1.25]) gives Theorem 4.

THEOREM 4. *If A is given by (6), and $u(x,t)$ solves*

$$-Au(x,t) = \frac{\partial u}{\partial t}, \qquad u(x,0) = u_0(x) \in L^P, \tag{14}$$

then

$$u(x,t) = e^{-tA} u_0(x) \underset{t\to 0}{\to} u_0(x) \tag{15}$$

in L^P $(1 \le p < \min r_i)$ and almost everywhere in x.

If φ is an analytic function defined on Γ and its interior, with $\varphi(0) = 1$, then

$$\varphi(tA) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t\zeta)}{(\zeta - A)} d\zeta,$$

if the integration converges absolutely in the uniform operator topology. The question of the convergence $\varphi(tA)f \underset{t\to 0}{\to} f$ is one of summability of eigenfunction expansions, since, in the notation of (5),

$$\varphi(tA)f \sim \int \varphi(t\lambda) \tilde{f}(\lambda) v(x,\lambda) d\rho(\lambda).$$

A statement exactly parallel to (15) holds here, with some minor algebraic decay conditions on φ.

3. Smoothing Properties

We first briefly consider smoothing of the semigroup in the scale of L^P spaces on $\mathbb{R}$. To this end, note that by a simple application of the Dunford operator calculus,

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma \frac{e^{-t\zeta}(C+\zeta)^k}{(\zeta - A)(C+A)^k} d\zeta \qquad (k = 0,1,2,\ldots). \tag{16}$$

That is, the semigroup is expressible in terms of high power of the resolvent. Some investigation of the L^P-smoothing properties of the convolution kernel in (7), and iteration of these through (16), shows that e^{-tA} is infinitely smoothing, i.e., that $e^{-tA} L^q \subset \bigcap_{1 \le p \le \infty} L^P$ $(1 \le q \le \infty)$.

The answer to the corresponding question for the scale of Sobolev spaces depends strongly on the smoothness of A. This is indicated by (16), since a high power of the resolvent is itself the inverse of a differential operator only if the coefficients are sufficiently smooth. We have only a sufficient condition for smoothing. Let L_m^P be the Sobolev space of order m, i.e., the functions with m derivatives in L^P.

THEOREM 5. *If A satisfies (6) and has sufficiently smooth coefficients, i.e., if $b_i(x) \in \mathrm{L}_{km}^{r_i}$, then*

$$e^{-tA}: L^P(\mathbb{R}) \to \mathrm{L}_{(k+1)m}^P(\mathbb{R}) \tag{17}$$

is bounded for all $t > 0$.

We note that one cannot hope that a statement much stronger than (17) will hold, since, for example, the operator

$$A = -\frac{d^2}{dx^2} - |x|^{-\varepsilon} \qquad (0 < \varepsilon < 1)$$

has eigenfunctions $v(x) \sim c_1 |x|^{2-\varepsilon} + c_2 x + c_3$ $(x \to 0)$, so that $e^{-tA}u$, in general, is not in L^2_2, if $u \in L^2$, for $\varepsilon > \frac{1}{2}$. A precise statement of necessary and sufficient conditions for Theorem 5 would be very interesting; it would presumably be intimately connected with smoothness of the eigenfunctions of A.

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PROBLEMS CONCERNING ORTHOGONAL POLYNOMIALS AND SINGULAR STURM-LIOUVILLE SYSTEMS

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Abstract

As singular Sturm-Liouville theory progresses, its application to the boundary value problems describing various classical orthogonal polynomial sets is also beginning to be known. This article outlines what has been done and what remains.

1. Introduction

Anyone who has taught a course in Fourier series or intermediate differential equations has encountered such classic problems as the calculation of the Legendre polynomials, eigenvalues, and eigenfunctions. Most such people, however, fail to recognize that many of these problems (such as the Legendre polynomial problem) are singular and hence do not fit into the context of the regular Sturm-Liouville problem, which is usually discussed at the intermediate level. The reason for the failure is easy to understand. Quite simply, until fairly recently, the singular problem was not fully recognized, and indeed, even today, there are many aspects of the singular problem that remain to be solved.

The purpose of this article is to describe briefly what has been done and to list some still unsolved problems. We shall begin with the classical orthogonal polynomials of mathematical physics, which satisfy second-order ordinary differential equations and for which the theory is well developed. We also list some polynomial problems that satisfy higher order differential equations for which the theory is less well understood. We conclude with some problems that are wide open.

2. Second Order Problems

The Legendre polynomials satisfy the differential equation

$$((1-t^2)u')' + \lambda_n u = 0.$$

Traditionally defined on the interval $[-1,1]$, they form a complete orthogonal set for $L^2[-1,1]$. By so doing, they become part of the blocks forming the solutions of a number of problems in mathematical physics. On $L^2[-1,1]$, the minimal operator associated with the expression $Lu = ((1-t^2)u')'$ is symmetric, with equal deficiency indices $(2,2)$, and so possesses a self-adjoint extension. This extension has a domain that is in part characterized by boundary conditions at ± 1. It is these that have caused considerable confusion over the years. We shall show how they are derived.

Likewise, the Laguerre polynomials satisfy the differential equation

$$-e^t(e^{-t}u')' + \lambda_n u = 0.$$

Traditionally defined on the interval $[0,\infty]$, they form a complete orthogonal set for $L^2([0,\infty); e^{-t})$. The minimal operator associated with $Lu = (-e^{-t}u')'$ has equal deficiency indices $(1,1)$, and so again it possesses a self-adjoint extension,

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whose domain satisfies a boundary condition at 0. It also is not well known. We shall exhibit it as well. There is also a boundary condition at ∞, but since it is automatically satisfied, we shall not need it.

Finally, the Hermite polynomials satisfy the differential equation

$$-e^{t^2}(e^{-t^2}u')' + \lambda_n u = 0.$$

Here the interval is $(-\infty,\infty)$. On $L^2((-\infty,\infty); e^{-t^2})$ the minimal operator associated with $Lu = (e^{-t^2}u')'$ has deficiency indices (0,0), and is therefore already self-adjoint. Elements in the domain of the operator satisfy automatically a boundary condition at both $\pm\infty$, much as in the Laguerre case at ∞. Since these conditions are automatic, we need not concern ourselves with their exact nature.

How are these facts derived? Let us outline the method. Any symmetric second-order differential equation

$$(p_1 u')' + p_0 u + \lambda r u = 0$$

can be put in a symmetric system format:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = [\lambda \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} p_0 & 0 \\ 0 & 1/P_1 \end{pmatrix}] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} .$$

Writing J, A, B for the matrix coefficients, and y for the vector, we see that J is skew Hermitian $J^* = -J = J^{-1}$, and A and B are real and Hermitian. Over an interval $[a,b]$, therefore, we may consider the Hilbert space $L_A^2[a,b]$ generated by the inner product

$$<y,z> = \int_a^b z^* A y \, dt,$$

and on $L_A^2[a,b]$, we may consider the differential expression defined by $Ly = Jy' - By$. The minimal operator is symmetric and has equal deficiency indices. To extend the domain, we need to define appropriate boundary conditions in the sense of Dunford and Schwartz [1964].

Let c be an arbitrary but fixed point in (a,b), and let $Y = \begin{pmatrix} \vartheta & \varphi \\ \vartheta_1 & \varphi_1 \end{pmatrix}$ be a fundamental matrix for the differential equation above. Assume without loss of generality that $Y(c) = I$, the identity. The Weyl limit-point limit-circle theory has established that for complex λ, there exists a solution

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} \vartheta & \varphi \\ \vartheta_1 & \varphi_1 \end{pmatrix} \begin{pmatrix} 1 \\ M \end{pmatrix}$$

in $L_A^2[c,b]$, and a solution

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \vartheta & \varphi \\ \vartheta_1 & \varphi_1 \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix}$$

in $L^2[a,c]$. The coefficients M and m are the much-studied Weyl coefficients. We can use φ and ψ to derive new fundamental matrices that are more suitable for our calculations. Let

$$Z_b = \begin{pmatrix} \psi & \varphi \\ \psi_1 & \varphi_1 \end{pmatrix} = \begin{pmatrix} \vartheta & \varphi \\ \vartheta_1 & \varphi_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix}$$

and

$$Z_a = \begin{pmatrix} \eta & \varphi \\ \eta_1 & \varphi_1 \end{pmatrix} = \begin{pmatrix} \vartheta & \varphi \\ \vartheta_1 & \varphi_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}.$$

Then

$$Y = \begin{pmatrix} \psi & \varphi \\ \psi_1 & \varphi_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -M & 1 \end{pmatrix} = \begin{pmatrix} \eta & \varphi \\ \eta_1 & \varphi_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix}.$$

If we compute Green's formula, we find after setting $Jy' - By = Af$, $Jz' - Bz = Ag$, that

$$<Ly,z> - <y,Lz> = \int_a^b z^*(Jy' - By)dt - \int_a^b (Jz' - Bz)^* y\,dt = (z^*Jy)_a^b.$$

In order for the operator generated by the expression L to be self-adjoint, the term $(z^*Jy)_a^b$ must be made to vanish in such a way that constraints on y and z are the same and at the same time minimal.

LEMMA 2.1. *Let Y, Z_a, Z_b have conjugate transpose matrices, with λ replaced by $\bar{\lambda}$, denoted by $\tilde{Y}$, $\tilde{Z_a}$, $\tilde{Z_b}$, respectively. Then*

$$\tilde{Y}JY = J,\ \tilde{Z_a}JZ_a = J,\ \tilde{Z_b}JZ_b = J.$$

The proof is quite simple. The derivatives of these expressions are 0; hence they are constant. At $t = c$, they all equal J.

LEMMA 2.2.

$$-JYJ\tilde{Y}J = J,\ -JZ_aJ\tilde{Z_a}J = J,\ -JZ_bJ\tilde{Z_b}J = J.$$

PROOF. Note $-(J\tilde{Y})(JY) = I$. So $(JY)^{-1}$ is $-J\tilde{Y}$. Hence these may be permuted. Multiplication by J completes the proof. The proofs for Z_a, Z_b are the same.

Note that

$$\tilde{Z_b} = \begin{pmatrix} \psi & \psi_1 \\ \varphi & \varphi_1 \end{pmatrix}.$$

where $\psi = \psi(t,\lambda)$, $\varphi = \varphi(t,\lambda)$, the same λ as in Z_b, and **not** its conjugate. Let us now consider the term $(z^*Jy)(b)$. Replacing J by $(-JZ_b)J(\tilde{Z_b}J)$, we find it becomes

$$z^*(-JZ_b)J(\tilde{Z_b}J)y = (Z_b^*Jz)^*J(\tilde{Z_b}Jy).$$

Let us examine the expressions in parentheses. Since $JZ_b' = (\lambda A+B)Z_b$, $\tilde{Z_b}$ satisfies

$$-\tilde{Z_b}'J = \tilde{Z_b}(\lambda A+B).$$

Recalling $Jy' = By + Af$, we find

$$\begin{aligned}(\tilde{Z_b}Jy)' &= (\tilde{Z_b}'J)y + \tilde{Z_b}(Jy')) \\ &= -\tilde{Z_b}(\lambda A + B)y + \tilde{Z_b}(By + Af) \\ &= \tilde{Z_b}A(f - \lambda y).\end{aligned}$$

In component form this becomes

$$\begin{bmatrix} y_1\psi'-y_2\psi \\ y_1\varphi'-y_2\varphi \end{bmatrix}' = \begin{bmatrix} \psi r(f_1-\lambda y_1) \\ \varphi r(f_2-\lambda y_1) \end{bmatrix}.$$

Certainly the top component on the right is integrable. If φ is likewise in $L^2([a,b];r)$, then the second component on the right is integrable as well.

THEOREM 2.3. *Let ψ be a solution of $(p_1u')' + p_0u + \lambda ru = 0$, $\mathrm{Im}\lambda \neq 0$, which is in $L^2([a,b];r)$; let y and $[(p_1y')' + p_0y]/r = f$ be in $L^2([a,b];r)$. Then $\lim_{t\to b} W[y,\psi]$ exists.*

Conversely, if y and $[(p_1y'))' + p_0y]/r$ are in $L^2([a,b];r)$ and $\lim_{t\to b} W[y,\varphi]$ exists for all such y's, the φ is in $L^2([a,b];r)$.

We have essentially proved the first part. The proof of the second may be found in Krall [1984].

The conclusion is that the expressions in

$$(z^*Jy)(b) = (Z_b^*J_z)^*J(\tilde{Z_b}Jy)$$

are Wronskians and have limits at b if and only if the solutions ψ,φ generating them are square integrable. These Wronskians are the generalized boundary conditions. In any case, regardless of what happens to the components, the expression $(z^*Jy)(b)$ exists.

Likewise at $t = a$,

$$(z^*Jy)(a) = (Z_a^*Jz)^*J(\tilde{Z_a}Jy).$$

Again the various components have individual limits if and only if the solutions η,φ are square integrable.

THEOREM 2.4 (Green's Formula).

$$\langle Ly,z\rangle - \langle y,Lz\rangle = (Z_b^*Jz)^*J(\tilde{Z_b}Jy)(b)$$
$$- (Z_a^*Jz)^*J(\tilde{Z_b}Jy)(a).$$

The situations at a and b depend on how many solutions η,φ and ψ,φ, respectively, are square integrable. We cite Littlejohn [1984] and Krall [1984]. At $t = b$, if both ψ,φ are square integrable, then the simplest requirement is to set $W[y,\psi](b) = 0$. Since the Green's formula term at b is

$$W[z,\varphi]W[y,\psi] - W[z,\psi]W[y,\varphi],$$

if Wronskians of both y and z with ψ vanish, then the entire term disappears. Actually, this can be generalized considerably [Krall 1984]. In particular, we note that the equation $W[y,\psi] = 0$ is independent of the eigenvalue parameter λ.

At $t = b$, if only ψ is square integrable, then $W[z,\psi](b)$ and $W[y,\psi](b)$ vanish so rapidly that the entire term vanishes, even though $W[z,\varphi]$ and $W[y,\varphi]$ may become infinite. Again see Krall [1984].

The same occurs at $t = a$. So, for simplicity, we require that $W[y,\psi])b) = 0$, $W[y,\eta](a) = 0$ in either case. The actual proof then that the extension of the operators L with domain so constrained is self-adjoint is still quite complicated. The statements of the previous paragraph as well as self-adjointedness require the use of a Green's function. Again we cite Krall [1984]. Nonetheless, at least the nature of the boundary conditions has been exhibited.

The boundary conditions for the orthogonal polynomials are as follows. For the Legendre polynomials, require

$$\lim_{t\to 1}(1-t^2)u'(t) = 0,\ \lim_{t\to -1}(1-t^2)u'(t) = 0.$$

For the Laguerre polynomials, require

$$\lim_{t\to 0} t\,(2u'(t)-u(t)) = 0.$$

At ∞, the constraint is automatic. For the Hermite polynomials, the constraints at $\pm\infty$ are both automatic.

We invite the reader to find the constraints for the Bessel operator at $t = 0$.

3. Fourth Order Problems

Each of the operators associated with the polynomials of the previous section can be squared to generate a fourth order problem. Only the Legendre squared operator has actually been considered [Krall and Fulton, 1982], and it needs to be revised. In addition to these, three other sets of orthogonal polynomials satisfy fourth order differential equations. Found by Krall [1940], they have been named the Legendre type, the Laguerre type, and the Jacobi type polynomials. They are of special interest since their boundary constraints are λ-dependent.

The proper way to discuss these is also in a system format. Since every symmetric fourth-order differential equation

$$(p_2y'')'' + (p_1y')' + (p_0y) + \lambda ry = 0$$

is equivalent to

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}' = [\lambda \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} p_0 & 0 & 0 & 0 \\ 0 & -p_1 & 0 & 1 \\ 0 & 0 & -1/p_2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

the theory outlined in the previous section holds fairly well, with only a few complications. However, none of these complications have been explicitly worked out. As a consequence, our description of the boundary value problems associated with these polynomials is less than satisfactory.

The Legendre type polynomials satisfy the differential equation

$$((t^2-1)^2u'')'' + 4((\alpha(t^2-1)-2)y')' ++ \lambda_n y = 0.$$

The boundary conditions can be expressed as

$$8\alpha u'(1) = \lambda u(1)$$

$$-8\alpha u'(-1) = \lambda u(-1).$$

The Hilbert space setting is $L^2(-1,1)\otimes R\otimes R$, required because of the λ-dependent boundary conditions. These conditions really should be rewritten as indicated earlier.

The Laguerre type polynomials satisfy the differential equation

$$(t^2e^{-t}u'')'' - (([2R+2]t+2)e^{-t}y')' + e^{-t}\lambda_n y = 0.$$

The boundary condition required is

$$-2Ru'(0) = \lambda u(0).$$

This should be recharacterized as well. Since λ again appears in the boundary condition, the setting required is $L^2[[0,\infty);e^{-t}]\otimes R$.

Finally, the Jacobi type polynomials satisfy the differential equation

$$([(1-t)^{\alpha+4}-2(1-t)^{\alpha+3}+(1-t)^{\alpha+2}]u'')''$$
$$+([(2\alpha+2+2M)(1-t)^{\alpha+2}-(2\alpha+4+2M)(1-t)^{\alpha+1}]y')'$$
$$+(1-t)^{\alpha}\lambda_n y = 0.$$

The boundary constraints are more complicated.

1. $-2Mu'(0) = \lambda u(0)$.

2. If $[\cdot,\cdot]$ is the Laguerre bilinear concomitant associated with the differential equation, then

   (a) $\lim_{t\to 1}[u,1] = 0$, $\lim_{t\to 1}[u,t] = 0$, $-1<\alpha<1$.
   (b) $\lim_{t\to 1}[u,1] = 0$, $1 \le \alpha < 3$.
   (c) no requirement, $3 \le \alpha < \infty$.

For those who are familiar with the language, we say that 0 is in the limit-3 case. The point 1 is in the limit-4 case in (a), in the limit-3 case in (b), in the limit-2 case in (c).

All of these need to be recharacterized in a singular format. We cite Krall [1981] for further information concerning these polynomials.

4. Sixth Order Problems

Littlejohn [1982, 1984] and Littlejohn and Krall [1982] have come up with two sets of orthogonal polynomials that satisfy sixth-order differential equations. The first of these, the H. L. Krall polynomials, satisfies the differential equation

$$((t^2-1)^3u''')''' + 3(AC+BC)((t^2-1)u'')''$$
$$+6((t^2-1)(t^2-3)u)'' + 12ABC^2((t^2-1)u')'$$
$$+6AC((t^2-1)(t^2-3)u)' + 6BC((t^2-1)(t^2+3)u')'$$
$$+24u'' + \lambda_n u = 0.$$

The boundary constraints are

$$24BCu''(1) + (24ABC^2+24BC)u'(1) = \lambda u(1),$$
$$24ACu''(-1) + (-24ABC^2-24AC)u'(-1) = \lambda u(-1).$$

The setting for these is $L^2[[-1,1];c]\otimes R\otimes R$.

The second set satisfies the differential equation

$$(t^4e^{-t}u''')'''-6((t^2+t)e^{-t}u'')''$$
$$+((6t^2+12t+12)e^{-t}u')' + 6R(t^2e^{-t}u')' + \lambda_n u = 0.$$

One boundary condition is required:

$$12Ru'(0) = \lambda u(0).$$

The setting is $L^2[[0,\infty);e^{-t}]\otimes R$. The boundary conditions here need further work in a system context. The sixth-order differential equations can be written in the $Jy' = (\lambda A+B)y$ system format. We cite Walker [1974].

5. Remarks

There are many problems suggested by the outline presented here.

1. What are the singular boundary conditions in the system format for the problems described in Sections 3 and 4?
2. Can one develop a simple theory of singular Sturm-Liouville boundary value problems? Niessen [1971, 1972] has paved the way, but his papers are formidable.
3. How far can one go in classifying such polynomial boundary value problems? In particular, are there other sixth order problems unrelated to the second and fourth order problems?
4. Kaper, Kwong, and Zettl [1984] have come up with a way of regularizing singular points. How does this apply to these problems?

These questions shold keep us busy for some time to come!

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SPECTRAL THEORY OF ELLIPTIC PROBLEMS WITH INDEFINITE WEIGHTS

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Abstract

Consider the homogeneous Dirichlet boundary value problem $-\Delta u = \lambda\rho(x)u$ on a bounded open set Ω in $\mathbf{R}^k$ $(k \geq 1)$, where the weight function ρ changes sign in Ω. *Let* λ_n (λ_{-n}) *be the* n*th positive (negative) eigenvalue, and let* $N_+(\lambda)$ $(N_-(\lambda))$ *be the* number of positive (negative) eigenvalues larger than λ (smaller than $-\lambda$).

In this note we give the leading terms of the asymptotic expansions of $\lambda_{\pm n}$ as $n \to \infty$ and $N_\pm(\lambda)$ as $\lambda \to \infty$, as well as estimates for their remainders. We also give lower bounds for $|\lambda_{\pm n}|$ and $N_\pm(\lambda)$ which hold for all n.

1. Introduction

Let ρ be a real-valued function defined on a bounded open set Ω of $\mathbf{R}^k$ $(k \geq 1)$. We consider the linear eigenvalue problem

(P) $\qquad -\Delta u = \lambda\rho(x)u, \quad x \in \Omega,$

with (homogeneous) Dirichlet boundary conditions: $u = 0$ on $\partial\Omega$, in the variational sense, where $\partial\Omega$ denotes the boundary of Ω and $\Delta = \sum_{j=1}^{k} \partial^2 / \partial x_j^2$. We shall assume that ρ changes sign in Ω, in a sense to be made precise below; for this reason, ρ is often called an *indefinite weight function* in the literature. Such an eigenvalue problem is typically obtained by linearization of a semilinear elliptic problem; for instance, once linearized about the origin, the nonlinear eigenvalue problem

$$\Delta u = \lambda f(x,u), \quad x \in \Omega,$$

with $f(x,0) = 0$ and Dirichlet boundary conditions, yields the above linear problem with $\rho(x) = \partial f / \partial x\ (x,0)$; and, clearly, $\rho(x)$ need not keep a constant sign in this case.

These linear and nonlinear problems have recently been the object of much attention. (See, for example, the review article of de Figueiredo [1982] and the references therein, or the paper by Hess and Kato [1980]; a survey of early results on related linear Sturm-Liouville problems $(k = 1)$ can be found in Bocher [1913]. They are of current interest in applied mathematics, physics, and engineering (see, for example, Dee, Grube, and Harper [1972], Kaper, Kwong, Lekkerkerker, and Zettl [1984], Ludford and Robertson [1974], and Ludford and Wilson [1974].)** From a mathematical point of view, they lead to some rather interesting questions in the theory of partial differential equations and operator theory.

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We shall now focus our attention on the linear problem (P). Partly because of the potential applications to nonlinear problems, we want to consider weights that may be discontinuous. Precisely, unless otherwise specified, we assume *that* $\rho \in L^p(\Omega)$ *with* $p \geq k/2$ *if* $k \geq 3$ and $p = 1$ if $k = 1$. Moreover, we suppose that $\Omega_+ = \{x \in \Omega: \rho(x) > 0\}$ and $\Omega_- = \{x \in \Omega: \rho(x) < 0\}$ have positive Lebesgue measure in $\mathbb{R}^k$. The scalar λ is said to be an eigenvalue of the Dirichlet problem (P) *if* there exists a nonzero u *in* $H_0^1(\Omega)$ *satisfying the equation* $-\Delta u = \lambda \rho u$, in the distributional sense;* accordingly, u is then called an eigenfunction of (P) belonging to λ. Under the above assumptions, it is known [Manes and Micheletti, 1973; de Figueiredo, 1982: Chap. I] that (P) has a countable set of positive and negative eigenvalues, written in increasing order according to multiplicity:

$$\cdots \leq \lambda_{-(n+1)} \leq \lambda_{-n} \leq \cdots \leq \lambda_{-2} \leq \lambda_{-1} < (0) < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots.$$

In addition λ_n^{-1} is given by the "max-min principle":

$$\frac{1}{\lambda_n} = \max_{F_n} \min_{u \in F_n} \{\textstyle\int_\Omega \rho u^2 : \|\text{grad } u\|_{L^2(\Omega)} = 1\}, \tag{1.1}$$

where F_n runs through the family of all n-dimensional subspaces of $H_0^1(\Omega)$. Note that, by the obvious symmetry of the problem,

$$\lambda_{-n}(\rho) = -\lambda_n(-\rho). \tag{1.2}$$

Many authors have been studying almost exclusively the positive eigenfunctions of (P); this is entirely justified in the case when an eigenfunction must represent a population density, for example. (See, e.g., Brown [1983].) Recall that a positive eigenfunction of the Dirichlet problem must necessarily belong to the first eigenvalues λ_1 or λ_{-1}, as can be seen by use of the Krein-Rutman theorem (see Manes and Micheletti [1973] and Hess and Kato [1980]). For certain applications, however, the eigenfunctions of interest do change sign — since they represent, for instance, some component of a velocity field; hence, it seems worthwhile to pursue the investigation of higher eigenvalues and eigenfunctions of (P).

In this paper, we determine the asymptotic behavior of the eigenvalues $\lambda_{\pm n}$ of (P) and, under more stringent hypotheses, obtain corresponding remainder estimates. We also give lower bounds for $|\lambda_{\pm n}|$ valid for all n. In addition, we try to indicate some of the many problems that remain to be solved. For expository purposes, we present here the case of the Laplacian with Dirichlet boundary conditions; under appropriate assumptions, however, most results hold for second-order elliptic operators with mixed homogeneous Dirichlet-Neumann boundary conditions. Because of space limitations, the preliminary findings about the eigenfunctions of (P) which were reported at the Argonne workshop will be presented elsewhere. Mathematically, our long-term objective is to extend to the case of indefinite weights some of the main results of the classical spectral theory of linear second-order elliptic operators with weight $\rho \equiv 1$ in bounded as well as unbounded domains of $\mathbb{R}^k$. For the "classical theory," the interested reader might consult Agmon [1965], Gilbarg and Trudinger [1977], Courant and Hilbert [1953: Chaps. IV-VI], Reed and Simon [1978], Osserman and Weinstein [1980], and Yau [1982].

*The notation $H^1(\Omega)$ stands for the Sobolev space of functions $u \in L^2(\Omega)$ with distributional derivatives grad $u \in L^2(\Omega)$; roughly speaking, the subspace $H_0^1(\Omega)$ is the set of functions in $H^1(\Omega)$ that vanish on $\partial\Omega$. (See Lions and Magenes [1968] and Adams [1975].) If one wants to consider Neumann instead of Dirichlet homogeneous boundary conditions, one must replace $H_0^1(\Omega)$ by $H^1(\Omega)$.

2. Asymptotic Behavior of the Eigenvalues

For $\lambda > 0$, let $N_+(\lambda)$ be the number of positive eigenvalues of (P) less than or equal to λ; similarly, $N_-(\lambda)$ denotes the number of negative eigenvalues λ_{-n} larger than or equal to $-\lambda$. In the following, $|B|$ (resp., Int$B$) stands for the Lebesgue measure (resp., the interior) of $B\subset\mathbf{R}^k$. Here and thereafter, we adopt the convention according to which $a_\pm = b_\pm$ means that $a_+ = b_+$ and $a_- = b_-$; also, n will always be a positive integer.

Our first theorem —which extends a famous result of Hermann Weyl [Weyl, 1911; Courant and Hilbert, 1953: Theorem 14, p. 435; Reed and Simon, 1978: Theorem XIII.78, p. 271] —gives the leading term of the asymptotics of $\lambda_{\pm n}$ as $n\to+\infty$, or, equivalently, of $N_\pm(\lambda)$ as $\lambda\to+\infty$.

THEOREM 2.1. *Assume that* $|\Omega_+\backslash Int\,(\Omega_+)| > 0$ *and* $|\Omega_-\backslash Int(\Omega_-)| > 0$. *Then we have*

$$N_\pm(\lambda)\sim\frac{B_k}{(2\pi)^k}\lambda^{k/2}[\int_\Omega(\rho_\pm)^{k/2}],\ \ as\ \ \lambda\to+\infty \tag{2.1}$$

and

$$|\lambda_{\pm n}|\sim C_k n^{2/k}\|\rho_\pm\|^{-1}_{L^{k/2}(\Omega)},\ \ as\ \ n\to+\infty, \tag{2.2}$$

where B_k is the volume of the unit ball in $\mathbf{R}^k$ and C_k denotes Weyl's constant; recall that $B_k = \pi^{k/2}/\Gamma(1+k/2)$ and $C_k = (2\pi)^2(B_k)^{-2/k}$.

REMARK. Note that when ρ is continuous and changes sign in Ω, Theorem 1.1 implies that (2.1) and (2.2) hold for any nonempty open subset Ω of $\mathbf{R}^k$; for, in this case, Ω_+ and Ω_- are open and the conditions $|\Omega_\pm| > 0$ and $|\Omega_\pm\backslash\mathrm{Int}(\Omega_\pm)| = 0$ are automatically satisfied.

Before giving the proof of Theorem 1.1, we need to recall two simple comparison principles; when necessary, we use the notation $\lambda_n(\rho;\Omega)$ or $N(\lambda;\rho,\Omega)$ to emphasize the dependence on the weight function ρ and the open set Ω.

LEMMA 2.1 (Monotonicity with respect to the weight). *If $\rho_1 \le \rho_2$ a.e. in Ω and if $\lambda_n(\rho_2;\Omega)$ exists, then $\lambda_n(\rho_1;\Omega)$ exists and $\lambda_2(\rho_1;\Omega) \le \lambda_1(\rho_2;\Omega)$.*

LEMMA 2.2 (Monotonicity with respect to the open set). *If $\Omega_1\subset\Omega_2$ and if $\lambda_n(\rho;\Omega_j)$ exist for $j = 1,2$, then $\lambda_n(\rho;\Omega_1) \ge \lambda_n(\rho;\Omega_2)$.*

These principles are "physically obvious" and well known in the case of positive weights [Courant and Hilbert, 1953: Theorem 3, p. 403 and Theorem 7, p. 411; Reed and Simon, 1978, Proposition 4, p. 270]. Mathematically, they follow immediately from the max-min formula (1.1); since the latter still holds, their proof remains unchanged in this context. (See also Weyl [1911, pp. 58-62].)

PROOF OF THEOREM 2.1. The idea of the proof is rather simple and can be put to use in similar situations. Assume that we know the result for positive weights; we then reduce the problem to this case by means of the above monotonicity principles. Indeed, we "trap" $N_+(\lambda)$ between two expressions having the same asymptotic behavior.

In view of (1.2), it suffices to establish (2.1) for $N_+(\lambda)$ since $(-\rho)_+ = \rho_-$; the estimate (2.2) will then follow since $N_+(\lambda_n) = n$.

Fix $\varepsilon > 0$. Since $\rho \le \rho_+ + \varepsilon$, we have by Lemma 2.1

$$N_+(\lambda;\rho,\Omega) \le N_+(\lambda;\rho_+ + \varepsilon,\Omega). \tag{2.3}$$

Set $D = \mathrm{Int}(\Omega_+)$. Since $D \subset \Omega$ and $|D_+| > 0$, it follows from Lemma 2.2 that

$$N_+(\lambda;\rho,D) \subset N_+(\lambda;\rho,\Omega). \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$N_+(\lambda;\rho,D) \le N_+(\lambda) \le N_+(\lambda;\rho_+ + \varepsilon,\Omega). \tag{2.5}$$

Note that ρ (resp., $\rho_+ + \varepsilon$) is positive in D (resp., Ω). By the "classical result" for positive weights, essentially due to Weyl (see Lemma 2.3 below), we have

$$\lim_{\lambda\to+\infty} \frac{N_+(\lambda;\rho,D)}{\lambda^{k/2}} = \frac{B_k}{(2\pi)^k} \int_D \rho^{k/2} \tag{2.6}$$

and

$$\lim_{\lambda\to+\infty} \frac{N_+(\lambda;\rho_+ + \varepsilon,\Omega)}{\lambda^{k/2}} = \frac{B_k}{(2\pi)^k} \int_\Omega (\rho_+ + \varepsilon)^{k/2}. \tag{2.7}$$

Consequently, the equations (2.5) through (2.7) imply

$$\frac{B_k}{(2\pi)^k} \int_D \rho^{k/2} \le \liminf_{\lambda\to+\infty} \frac{N_+(\lambda)}{\lambda^{k/2}}$$

$$\le \limsup_{\lambda\to+\infty} \frac{N_+(\lambda)}{\lambda^{k/2}} \le \frac{B_k}{(2\pi)^k} \int_\Omega (\rho_+ + \varepsilon)^{k/2}. \tag{2.8}$$

It follows from Lebesgue's dominated convergence theorem that $\int_\Omega (\rho_+ + \varepsilon)^{k/2} \to \int_\Omega (\rho_+)^{k/2}$ as $\varepsilon \to 0$; recall that, in particular, $(|\rho| + 1)^p \in L^1(\Omega)$ with $p > k/2$ if $k \ge 2$ and $p = 1$ if $k = 1$.

Now observe that

$$\int_D \rho^{k/2} = \int_{\Omega_+} \rho^{k/2} = \int_{\Omega_+} (\rho_+)^{k/2},$$

for $|\Omega_+ \ D| = 0$ and the zero set of ρ does not contribute to the latter integral.

We thus obtain (2.1) by letting $\varepsilon \to 0$ in (2.8). ///

In the course of the proof of Theorem 2.1, we have used the following lemma.

LEMMA 2.3. *Theorem 2.1 holds for positive weights.*

PROOF. Since this is essentially known, we only outline the main steps. One may argue as follows:

1. Theorem 2.1 holds for any positive continuous function ρ on $\bar{\Omega}$. Indeed, when $\rho = 1$, this follows from Metivier [1977: Theorem 5.12, p. 188]* and, for positive ρ in $C(\bar{\Omega})$, this is obtained by substituting the operator $-\Delta/\rho$ for $-\Delta$.

2. Theorem 2.1 holds for any positive $\rho \in L^p(\Omega)$. One proceeds much as in Reed and Simon [1978: Proof of Theorem XIII.80, p. 274]; one approximates ρ in $L^{k/2}(\Omega)$ [and possibly $L^1(\Omega)$], by continuous functions with compact support in Ω. To do just this, one makes use of an *a priori* estimate of the type

$$N_+(\lambda) \le c\,\lambda^{k/2} \|\rho\|^p_{L^p(\Omega)},$$

*In his memoir Métivier [1977], by refining the method of Courant-Weyl, extended Weyl's theorem to more general open sets (and operators).

where c is a constant depending only on k, as well as of the following inequality, which results easily from (1.1) and Lemma 2.1:

$$N_+(\lambda;\rho_1+\rho_2) \leq N_+(\lambda;|\rho_1+\rho_2|) \leq N_+(\lambda;|\rho_1|) + N_+(\lambda;|\rho_2|).$$

Actually, Lemma 2.3 could be obtained directly by combining Theorem XIII.80 of Reed and Simon [1977: p. 274] and Step (iv) of Li and Yau [1983: pp. 317-318]. ///

Since, in particular, $\lambda_n = O(n^{2/k})$, we derive from Theorem 2.1 the following.

COROLLARY 2.1. *Under the assumptions of Theorem 2.1, the "zeta function"* $\zeta(\alpha) = \sum_{n=1}^{\infty}(\lambda_n)^{-\alpha} + \sum_{n=1}^{\infty}(\lambda_{-n})^{-\alpha}$ *and the "eta function"* $\eta(\alpha) = \sum_{n=1}^{\infty}(\lambda_n)^{-\alpha} - \sum_{n=1}^{\infty}(\lambda_{-n})^{-\alpha}$ *are well defined for all complex numbers* α *with* Reα *sufficiently large; here* $z^{-\alpha}$ *is defined by cutting the complex plane along the negative imaginary axis.*

REMARKS.

2.1. Theorem 2.1 was announced by Lapidus [1984]. From that paper we recover Theorem 1 (p. 266) by recalling that the boundary of a (Jordan) contented set (see, for example, Loomis and Sternberg [1968: Chap. 8, §§ 6-7]; and Reed and Simon [1978: p. 271]) has Lebesgue measure zero [Loomis and Sternberg, 1968: Proposition 6.1, p. 332].

2.2. Note that no assumption has been made about the zero set of ρ in Ω. Moreover, Ω, Ω_+, and Ω_- need not be connected.

2.3. We deduce from Theorem 2.1 that the positive (resp., negative) eigenvalues of (P) have the same asymptotic behavior as the eigenvalues of the elliptic operator $-\Delta/\rho$ in Int(Ω_+) [resp., $-\Delta/\rho$ in Int(Ω_-)].

2.4. The physical intuition that led us to Theorem 2.1 is the following: as is well known, for positive ρ the eigenvalues of (P) represent the fundamental frequencies of a vibrating membrane with mass density ρ; in the present case, the membrane has both positive and negative "frequencies," the large values of which are determined by its positive and negative "mass distributions."

After this work was completed, the author learned of several references where results related to Theorem 2.1 could be found: the article of Pleijel [1942] (treating by variational methods the case of a continuous weight on $\Omega\subset\mathbf{R}^2$) and the paper of Birman and Solomyak [1979] (studying by techniques pertaining to the theory of pseudodifferential operators and differential geometry a more abstract problem which, when specialized to the present setting, corresponds to a C^∞ weight in a smooth domain $\Omega\subset\mathbf{R}^k$). The work of Birman and Solomyak, like that of the author, must have been conducted independently of Pleijel's since it does not cite it. Indeed, its situation at the confluent of many mathematical areas is what makes this subject very interesting.

We now give two instances of application of Theorem 2.1.

Example 2.1. Let Ω be a bounded open subset of $\mathbf{R}^k$. Let Ω_+, Ω_- be two disjoint measurable subsets of Ω of positive measure such that $|\Omega_\pm \setminus \text{Int}(\Omega_\pm)| = 0$. Set $\rho(x) = +1$ if $x\in\Omega_+$, -1 if $x\in\Omega_-$, and 0 otherwise. We then obtain a natural extension of Weyl's formula:

$$\lambda_{\pm n} \sim C_k \left(\frac{n}{|\Omega_\pm|} \right)^{2/k} \text{ as } n \to +\infty. \tag{2.9}$$

Equation (2.9) shows that the spectrum [i.e., the set of eigenvalues of (P)] determines the volume of Ω_+ and Ω_-. How much more information can be obtained from the spectrum? (For the classical isospectral problem, see, for example, Kac [1966] and Yau [1982: pp. 23-24].) More generally, this suggests the following question.

Question. To what extent does the spectrum determine the sign of the weight?

Example 2.2. Let $\Omega = \{x \in \mathbb{R}^k : |x| < 1\}$ and $\rho(x) = |x|^{-2\beta}\operatorname{sign}(|x|-1)$, where $|x|$ denotes the Euclidean length of x and $\beta < 1$; note that ρ may be quite singular. It then follows from Theorem 2.1 that, as $\lambda \to +\infty$,

$$N_+(\lambda) \sim \frac{(B_k)^2}{(2\pi)^k} \frac{1}{1-\beta},$$

$$N_-(\lambda) \sim \frac{(B_k)^2}{(2\pi)^k} \frac{\lambda^{k(1-\beta)}-1}{1-\beta}.$$

3. Remainder Estimates

We now indicate how, by the method developed in the proof of Theorem 2.1, one can obtain further information about the asymptotic behavior of $N_+(\lambda)$. Depending on the assumption made on ρ and Ω, one can arrive in this manner at various kinds of error estimates. We give such an example below.

We make the following hypotheses:

(i) $|\Omega \quad \operatorname{Int}(\Omega_+) \cup \operatorname{Int}(\Omega_-))| = 0$ and $\operatorname{Int}(\Omega_+)$, $\operatorname{Int}(\Omega_+) \cup \operatorname{Int}(\Omega_-)$, have "finite $(k-1)$-dimensional boundary."*

(ii) The restriction of ρ to $\operatorname{Int}(\Omega_+) \cup \operatorname{Int}(\Omega_-)$ is Hölder continuous of order h, $h \in (0,1]$, and is bounded away from zero. Moreover,

$$|\rho(x)| \leq \text{Cst.}[\operatorname{dist}(x, \operatorname{Int}(\Omega_-))]^h, \quad x \in \operatorname{Int}(\Omega_+).$$

Note that ρ need not be continuous in Ω. We now state the following theorem.

THEOREM 3.1. *If (i) and (ii) are satisfied, we have the remainder estimate*

$$N_+(\lambda) = (2\pi)^{-k} B_k \lambda^{k/2} \left[\int_\Omega (\rho_+)^{k/2} \right] + O(\lambda^{(k-r)/2}), \text{ as } \lambda \to +\infty, \tag{3.1}$$

where $r = h/(h+1)$; with the obvious changes, a similar estimate holds for $N_-(\lambda)$.

PROOF. In view of (i), we may assume that $\Omega = \operatorname{Int}(\Omega_+) \cup \operatorname{Int}(\Omega_-)$. Let $0 < \varepsilon \leq 1$. Set $D = \operatorname{Int}(\Omega_+)$ and $\varphi(\lambda) = (2\pi)^{-k} B_k \lambda^{k/2}$. We deduce from (2.5) that

$$N_+(\lambda; \rho, D) - \varphi(\lambda) \|\rho\|_{L^{k/2}(D)}^{k/2} \leq N_+(\lambda) - \varphi(\lambda) \|\rho_+\|_{L^{k/2}(\Omega)}^{k/2}$$

---

*A subset ω of $\mathbb{R}^k$ is said to have "finite $(k-1)$-dimensional boundary" if $\lim_{\delta \to 0_+} \sup \delta^{-1} \omega_\delta < +\infty$, where $\omega_\delta = \{x \in \omega : \operatorname{dist}(x, \partial\omega) < \delta\}$. (See Métivier [1977: p. 194].)

$$\leq N_+(\lambda;\rho_+ + \varepsilon,\Omega) - \varphi(\lambda)\|\rho_+ + \varepsilon\|_{L^{k/2}(\Omega)} \tag{3.2}$$
$$+ \varphi(\lambda)[\|\rho_+ + \varepsilon\|^{k/2}_{L^{k/2}(\Omega)} - \|\rho_+\|^{k/2}_{L^{k/2}(\Omega)}].$$

Note that if a function is bounded away from zero and Hölder continuous of order h, so is its inverse. It then follows from (ii) that $1/\rho$ [resp., $1/(\rho_+ + \varepsilon)$] is Hölder continuous of order h on D (resp., Ω). Hence Theorem 6.1 of Métivier [1977: p.195] yields

$$|N_+(\lambda;\rho,D) - \varphi(\lambda)\|\rho\|^{k/2}_{L^{k/2}(\Omega)}| \leq \psi(\lambda),$$

$$|N_+(\lambda;\rho_+ + \varepsilon,\Omega) - \varphi(\lambda)\|\rho_+ + \varepsilon\|^{k/2}_{L^{k/2}(\Omega)}| \leq \psi(\lambda), \tag{3.3}$$

for all sufficiently large λ; here, $\psi(\lambda) = \text{Cst}.\lambda^{(k-\tau)/2}$, for some constant independent of ε since $\|1/(\rho_+ + \varepsilon)\|_{L^\infty}$ is uniformly bounded in ε. We now obtain (3.1) by fixing λ large enough, inserting (3.3) into (3.2), and letting $\varepsilon \to 0$. ///

REMARK. Under a different set of assumptions, one could deduce error estimates of the form $O(\lambda^{(k-1)/2})$ or $O(\lambda^{(k-1)/2}\log\lambda)$.

4. Lower Bounds for the Eigenvalues

Whereas the last two sections gave results concerning the large eigenvalues of (P), the following theorem — announced by Lapidus [1984] — provides lower estimates for λ_n which hold for all n.

THEOREM 4.1. *Assume that Ω is a bounded domain of $\mathbf{R}^k$ with C^2 boundary and that $k \geq 3$. Set $\gamma_k = (k(k-2)/4e)(kB_k)^{2/k}$ with $e = \exp(1)$. Then, we have for all $n \geq 1$ and $\lambda > 0$:*

$$|\lambda_{\pm n}| \geq \gamma_k n^{2/k} \|\rho_\pm\|^{-1}_{L^{k/2}(\Omega)}, \tag{4.1}$$

and

$$kB_k(k(k-2)/4e)^{k/2}N_\pm(\lambda) \leq \lambda^{k/2}\int_\Omega (\rho_\pm)^{k/2}. \tag{4.2}$$

PROOF. As before, it is enough to establish (3.1) for λ_n. Fix $n \geq 1$ and $\varepsilon > 0$. By Lemma 2.1, $\lambda_n(\rho) \geq \lambda_n(\rho_+ + \varepsilon)$.

Moreover, by Li and Yau [1983: Theorem 2, p. 314] applied to the positive weight $\rho_+ + \varepsilon$,

$$\lambda_n(\rho_+ + \varepsilon) \geq \gamma_k n^{2/k}\|\rho_+ + \varepsilon\|^{-1}_{L^{k/2}(\Omega)}. \tag{4.3}$$

To obtain (4.1) we now let $\varepsilon \to 0$ in (4.3) and apply the dominated convergence theorem. ///

REMARK 4.1. An immediate consequence of Theorem 4.1 is that if $|\Omega_\pm| > 0$, then $|\lambda_{\pm n}|$ exists for all n and tends to $+\infty$ as $n \to +\infty$; for $k \geq 3$, we then obtain an alternative proof of this fact recalled in the introduction. (Compare, for example, the proof of Manes and Micheletti [1973: Proposition 3, p. 290] or de Figueiredo [1982: Proposition 1.11, p. 43].)

The result quoted in the proof of Theorem 4.1 is a beautiful estimate of Li and Yau established for positive weights; in the same paper [Li and Yau, 1983], it was used by these authors to improve the so-called "Cwikel-Lieb-Rozenbljum bound" for the number of "bound states" (that is, negative eigenvalues) of the Schrödinger operator in $\mathbf{R}^k$ ($k \geq 3$). (See also Lieb [1980] and Reed and Simon

[1978: pp. 101-106].)

In the first part of their paper, Li and Yau attempted to solve Polya's conjecture. In the present context, and for Dirichlet boundary conditions, the latter would state that

$$\lambda_n \geq C_k n^{2/k} \|\rho_{\pm}\|_{L^{k/2}(\Omega)}, \quad \text{for all } n. \tag{4.4}$$

From Theorem 2.1 we know that the ratio of the left- and right-hand sides of (4.4) tends to 1 as n gets larger and larger; of course, it can be checked that the constant γ_k occurring in the statement of Theorem 4.1 is smaller that C_k. Even in one dimension, (4.4) does not seem to be known, whereas Polya's conjecture ($\rho = 1$) is easily verified in this case by explicit computation. Needless to say, whoever wants to attempt solving this problem in higher dimensions must be ready to face considerable difficulties. For literature on the classical problem, see Polya [1961] where the analogue of (4.4) was established for "tilting domains" and the recent works of Lieb [1980] and Li and Yau [1983], where partial results were obtained on arbitrary domains.

Finally, we would like to point out a problem about which very little seems to be known. Assume that Ω is unbounded, say, $\Omega = \mathbf{R}^k$; moreover, in the definition of (P), replace $-\Delta$ by the "Schrodinger operator" $-\Delta + V$, where V is a real-valued function defined on Ω and satisfying appropriate regularity conditions; suppose that V is unbounded from below. Consequently, (P) is now both "left" and "right" nondefinite.

Problem. Obtain, in terms of the properties of the "potential" V and the "weight" ρ, a partial classification of the types of spectra that can occur in this situation.

Of course, one must first define precisely what is meant by terms like "essential spectrum" in this setting. This can be done, for instance, by using the concept of approximate eigenfunctions or the language of indefinite inner product spaces (Krein spaces). Naturally, the "spectrum" need not be discrete in general. Actually, a try at a few examples will quickly convince the reader that the "spectrum" could be quite complicated in this case.

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J-SYMMETRIC DIFFERENTIAL SYSTEMS

*Heinz-Dieter Niessen**

Abstract

For differential systems, the notion of formal J-symmetry is defined in such a way that systems arising from formally J-symmetric differential equations are formally J-symmetric in this sense. It is indicated how J-symmetric differential systems give rise to a J-symmetric operator in some suitable Hilbert space. The general theory of J-symmetric operators in Hilbert spaces can then be applied to this operator to obtain analogous results for J-symmetric systems as they are known for J-symmetric differential equations.

1. Introduction

Let T be a linear operator in a Hilbert space H with inner product $(\,,\,)$, and let J be a conjugation operator on H. (For the problems considered here, H can be assumed to be a function-space and J the operator of complex conjugation.) Then T is called J-symmetric if for all y,z in the domain of T

$$(Ty,Jz)=(y,JTz),$$

i.e., if

$$T\subset JT^*J, \tag{1.1}$$

where T^* denotes the ordinary adjoint of T. If equality holds in (1.1), then T is said to be J-selfadjoint.

The theory of J-symmetric operators in Hilbert space has been developed mainly by Glazman [1957], Zhikhar [1959], Galindo [1963], and Knowles [1980 and 1981]. Two of the main results are given below.

THEOREM 1.2 (Extension Theorem). *To every J-symmetric operator T there exists a J-selfadjoint extension $\tilde{T}$. If λ belongs to the regularity field of T, then T can be chosen in such a way that λ belongs to the resolvent set of $\tilde{T}$.*

Here the regularity field — denoted by $\Pi(T)$ — is defined to be the set of all complex λ, for which $T-\lambda$ is boundedly invertible (but not necessarily surjective). The regularity field of a J-symmetric operator may be empty.

THEOREM 1.3 (Modified von Neumann Formula). *Let T be a J-symmetric operator with non-empty regularity field and — according to (1.2) — let $\tilde{T}$ be a J-selfadjoint extension of T with non-empty resolvent set. Then for all λ in the resolvent set of $\tilde{T}$*

$$D_{JT^*J}=D_T\dot{+}(\tilde{T}-\lambda)^{-1}N((T-\lambda)^*)\dot{+}JN((T-\lambda)^*).$$

Here D_A and $N(A)$ denote the domain and the null space of A, respectively.

Now let τ be the (in general) quasi-differential operator defined by

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$$\tau\eta = \sum_{i=0}^{m}(-1)^i(p_{m-i}\eta^{(i)})^{(i)}, \tag{1.4}$$

where the coefficients p_i are complex-valued functions defined on some real interval I. For sufficiently smooth η,ζ with compact support in the interior of I the equation

$$(\tau\eta,J\zeta) = (\eta,J\tau\zeta) \tag{1.5}$$

holds true, where J denotes complex conjugation. Therefore, the minimal operator T_0 in $L^2(I)$ induced by τ is J-symmetric. Thus, the theory of J-symmetric operators can be applied to T_0. For example, by applying the extension theorem and the modified von Neumann formula, all J-selfadjoint extensions can be described by boundary conditions, and their resolvents can be constructed. This application of the general theory of J-symmetric operators has been achieved, for instance, by Glazman [1957], Zhikhar [1959], Knowles [1981], and Race [1980].

The aim of this paper is to generalize the theory of "formally J-symmetric" differential operators to differential systems. As an example we consider the system arising from (1.4) or, more generally, from the equation

$$\tau\eta = \lambda w\eta, \tag{1.6}$$

where w denotes a (for the moment) complex-valued function. Let $y = (\eta^{[0]}, \ldots, \eta^{[2n-1]})^t$, where the $\eta^{[i]}$ are the quasi-derivatives of η. Then (1.6) is equivalent to the system

$$-y' + Ay = \lambda By \tag{1.7}$$

with $(2m,2m)$-matrices A and B defined by

$$A := \left(\begin{array}{cccc|cccc} 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & 0 & \frac{1}{p_0} & & & \\ \hline & & & p_1 & 0 & -1 & & \\ & & \cdot^{\cdot^{\cdot}} & & & \ddots & \ddots & \\ & \cdot^{\cdot^{\cdot}} & & & & & \ddots & -1 \\ p_m & & & & & & & 0 \end{array}\right), \quad B := \left(\begin{array}{ccc} & & \\ & 0 & \\ w & & \end{array}\right)$$

Thus the aim is to define —in a most general way —the concept of "formally J-symmetric" differential systems such that (1.7) in particular becomes a formally J-symmetric system and such that there exists a suitable Hilbert space and a J-symmetric operator in it describing the differential system. Then the general theory of J-symmetric operators can be applied to this operator in order to get results analagous to those for differential equations.

For simplicity we assume all coefficient functions to be continuous, e.g., the p_i and w above.

2. Definition and Algebraic Structure

Let I be an arbitrary real interval with endpoints a and $b\,(-\infty\le a<b\le\infty)$, and let n be a positive integer. Then the set of all locally absolutely continuous functions from I into $\mathbf{C}^n$ will be denoted by $\mathbf{A}$, and $\mathbf{A}_p$ denotes the set of all piecewise locally absolutely continuous functions defined on I with values in $\mathbf{C}^n$. Here a function y is called piecewise locally absolutely continuous if there exists a finite set $A_y\subset I$ such that y is locally absolutely continuous on $I\setminus A_y$. Furthermore, the set of all measurable functions defined almost everywhere on I with values in $\mathbf{C}^n$ will be denoted by $\mathbf{M}$. Obviously, $\mathbf{A}_p$, $\mathbf{A}$, and $\mathbf{M}$ are linear spaces, the last one with equality almost everywhere.

Now for $i = 1,2$ let F_i, G_i, S_i be continuous mappings from I into the set of all complex (n,n)-matrices. Then by

$$Fy := F_1y' + F_2y,\ Gy := G_1y' + G_2y,\ Sy := S_1y' + S_2y,$$

we define differential operators F, G, and S mapping $\mathbf{A}_p$ linearly into $\mathbf{M}$. Furthermore, for $\lambda\in\mathbf{C}$, denote by C_λ the mapping from I into the set of all $(2n,2n)$-matrices defined by

$$C_\lambda := \begin{pmatrix} F_1-\lambda G_1 & F_2-\lambda G_2 \\ S_1 & S_2 \end{pmatrix}.$$

We consider the system $Fy = \lambda Gy$, i.e.,

$$(F_1-\lambda G_1)y' + (F_2-\lambda G_2)y = 0 \tag{2.1}$$

and define it or —more correctly —the pair (F,G) to be formally J-symmetric in the following sense.

DEFINITION 2.2. *(F, G) is "formally J-symmetric with respect to S" if*

$$F_1-\lambda G_1 : I\to GL_n(\mathbf{C}) \tag{2.3}$$

for all $\lambda\in\mathbf{C}$, and there exists a continuously differentiable function $H:I\to GL_n(\mathbf{C})$ such that for all $\lambda\in\mathbf{C}$:

$$C_\lambda^t\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}C_\lambda = \begin{pmatrix} 0 & H \\ H & H' \end{pmatrix}. \tag{2.4}$$

Then H is called "concomitant" of (F,G) with respect to S.

Here ^{t} denotes transposition and E_n is the (n,n)-unit matrix. Condition (2.3) ensures that the Cauchy problem for (2.1) is uniquely solvable; equation (2.4) reflects a commutation property (see Eq. 2.6 below).

Equation 2.4 shows that H is uniquely determined by F, G, and S —but not by F and G alone —and that $H(x)$ is skew-symmetric for each $x\in I$. Since $H(x)$ is supposed to be nonsingular, and since each odd-order skew-symmetric matrix is singular, the order n of the system (2.1) must be even.

The system (1.7) can be shown to be formally J-symmetric with respect to a suitably defined S:

Let $n = 2m$. Then $F_1 = -E_n$, $F_2 = A$, $G_1 = 0$, $G_2 = B$. Defining

$$S_1 := 0,\ S_2 := \begin{pmatrix} 0 & -\hat{E}_m \\ \hat{E}_m & 0 \end{pmatrix},$$

where $\hat{E}_m$ denotes the (m,m)-matrix with 1's in the second diagonal and 0's elsewhere, the pair (F,G) is formally J-symmetric with respect to this S, and the concomitant equals S_2. Thus, the concept of formally J-symmetric systems generalizes that of formally J-symmetric differential operators.

We use the notation

$$[y,z]: = \int_I z^*(x)y(x)\mathrm{d}x$$

for all $y,z \in \mathbf{M}$ for which the integral exists —not only for square-integrable y and z.

Then the following characterization of formally J-symmetric systems can be proved.

THEOREM 2.5. *(F,G) is formally J-symmetric with respect to S iff the following conditions are fulfilled:*

1. For all $x \in I$

a. *$F_1(x)$ is nonsingular.*
b. *$F_1^{-1}(x)G_1(x)$ is nilpotent.*
c. *$S_1^t(x)G_2(x) = G_1^t(x)S_2(x)$.*
d. *$S_1^t(x)F_2(x) - F_1^t(x)S_2(x)$ is nonsingular.*

2. For all $\lambda \in \mathbf{C}$ and for all continuously differentiable $y,z \in \mathbf{A}$ with compact support in the interior of I the equation

$$[(F-\lambda G)y, JSz] = [Sy, J(F-\lambda G)z] \qquad (2.6)$$

holds true.
(Then $H = S_1^t F_2 - F_1^t S_2$).

Conditions (a) and (b) are a restatement of (2.3); equation (2.6) is equivalent to (2.4) with some H, possibly λ-depending and singular valued. Then (c) guarantees that H is independent of λ, and condition (d) ensures that $H(x)$ is nonsingular everywhere.

Comparing the commutation property (2.6) for $\lambda = 0$ with (1.5) indicates once more that we have defined a generalization of the concept of formally J-symmetric differential operators. In (2.6) the differential operator S occurs instead of the identity in (1.5).

Another characterization of formally J-symmetric systems, which gives even more insight into the algebraic structure of the operators F, G, and S is the following.

THEOREM 2.7. *(F,G) is formally J-symmetric with respect to S and with concomitant H iff the following conditions hold:*

1. $F_1(x)$ is nonsingular for all $x \in I$.

2. H is a continuously differentiable solution of $H' = HT + T^t H$ with $T: = F_1^{-1}F_2$ such that $H(x)$ is nonsingular and skew-symmetric for all $x \in I$.

3. There exist continuous functions W, L from I into the set of all (complex) symmetric (n,n)-matrices such that WL is nilpotent everywhere and such that

$$G_i = WS_i \quad (i=1,2) \text{ and} \qquad (2.8)$$

$$S_1 = LF_1,\ S_2 = LF_2 - (F_1^{-1})^t H. \qquad (2.9)$$

Equation 2.9 and Condition 1 show that L is uniquely determined; Equation 2.8 implies the following corollary.

COROLLARY 2.10. $G = WS$.

We shall use the same symbol for a matrix-valued function and for the operator (in $\mathbf{M}$) of multiplication by this function. So in Corollary 2.10, W denotes this operator of multiplication.

Let x_0 be a fixed point of I and let Y_λ be the fundamental matrix of solutions of (2.1) such that $Y_\lambda(x_0) = E_n$. Then we have the following theorem.

THEOREM 2.11. *If (F,G) is formally J-symmetric with respect to S and with concomitant H, then for all $\lambda \in \mathbb{C}$*

1. $Y_\lambda^t H Y_\lambda = H(x_0)$.

2. $(SY_\lambda)^t (F_1 - \lambda G_1) Y_\lambda = H(x_0)$.

The first equation follows by multiplication of (2.4) from the left by $\begin{pmatrix} Y_\lambda' \\ Y_\lambda \end{pmatrix}^t$ and from the right by $\begin{pmatrix} Y_\lambda' \\ Y_\lambda \end{pmatrix}$; the second equation can be proved by multiplying (2.4) from the left by $\begin{pmatrix} Y_\lambda' \\ Y_\lambda \end{pmatrix}^t$, from the right by $\begin{pmatrix} Y_\lambda \\ 0 \end{pmatrix}$ and by using Eq. 1.

The second equation especially implies that $(SY_\lambda)(x)$ is nonsingular for all $x \in I$. In connection with (2.10) this shows that

$$W = (GY_\lambda)(SY_\lambda)^{-1}$$

is uniquely determined by (2.8). For example, in the problem arising from the system (1.7),

$$W = \mathrm{diag}(0,\dots,0,w),\ L = 0.$$

Another consequence of Theorem 2.7 is that (2.1), solved for y', is a polynomial in λ of degree n at most. This follows from (2.8), (2.9), and the fact that WL is nilpotent:

$$(F_1 - \lambda G_1)^{-1} = (F_1 - \lambda WS_1)^{-1} = (F_1 - \lambda WLF_1)^{-1}$$

$$= F_1^{-1}(E_n - \lambda WL)^{-1} = F_1^{-1}\sum_{i=0}^{n-1}(\lambda WL)^i.$$

Suppose from now on that (F,G) is formally J-symmetric with respect to S with concomitant H and W,L as in Theorem 2.7.

In order to obtain a suitable Hilbert space and a J-symmetric operator in it which reflects the properties of the system (2.1), some further restrictions have to be imposed.

3. Right-definite Systems

DEFINITION 3.1. *(F,G) is called "right-semidefinite (with respect to S)" if $W(x)$ is positive semidefinite for all $x \in I$.*

From now on (F,G) will be assumed to be right-semidefinite. Then there exists a uniquely defined continuous function K on I with values in the set of Hermitian (n,n)-matrices such that for all $x \in I$

$$K(x) \geq 0 \text{ and } K^2(x) = W(x). \tag{3.2}$$

COROLLARY 3.3. *For all $x \in J$ $W(x)$ and $K(x)$ are real matrices.*

For W this follows, since $W(x)$ is Hermitian and symmetric by Definition 3.1 and Theorem 2.7, respectively. By the symmetry of $W(x)$, K^t is a solution of (3.2), too. Since this solution is unique, $K(x)$ is symmetric. Being Hermitian, $K(x)$ is real.

In the case of Example (1.7), right-semidefiniteness means that the "weight" function w is real and nonnegative. Then $K = \mathrm{diag}(0,\dots,0,+\sqrt{w})$.

In the general case we define a linear operator U from $\mathbf{A}_p$ into $\mathbf{M}$ by

$$U := KS. \tag{3.4}$$

Then (2.10), (3.2), and (3.4) imply the following corollary.

COROLLARY 3.5. $G = KU$.

We use the following notations.

DEFINITION 3.6.

$L^2 := \{y \in \mathbf{M} \;/\; |y(\cdot)|^2$ *is integrable on* $I\}$,

$\mathbf{E} := U^{-1}L^2 = \{y \in \mathbf{A}_p \;/\; Uy \in L^2\}$,

$\mathbf{D} := \mathbf{A} \cap \mathbf{E} \cap F^{-1}KL^2$,

$\mathbf{E}_\lambda := \{y \in \mathbf{A} \cap \mathbf{E} \;/\; (F-\lambda G)y = 0\}$ for all $\lambda \in \mathbf{C}$,

$(y,z) := [Uy, Uz]$ for all $y,z \in \mathbf{E}$.

REMARKS.

a. The space $\mathbf{E}_\lambda$ of continuous solutions of $(F-\lambda G)y = 0$ belonging to $\mathbf{E}$ is contained in $Y_\lambda \mathbf{C}^n \cap \mathbf{D}$, since $Fy = \lambda Gy = K(\lambda Uy)$ and $Uy \in L^2$.

b. $(y,z) = [Gy, Sz] = [Sy, Gz]$ for all $y,z \in \mathbf{E}$ by (3.4) and (3.5), since $K(x)$ is Hermitian. This shows that the positive-semidefinite inner product $(\,,\,)$ on $\mathbf{E}$ is nothing artificial but that it is given by the problem itself. In the case of (1.7), for example, $(y,z) = \int_I \eta\bar{\zeta} w\,dx$, where η and ζ are the first components of y and z, respectively.

c. Obviously, U is a linear isometry from the semi-prehilbert space $(\mathbf{E},(\,,\,))$ into the Hilbert space $(L^2,[\,,\,])$.

d. The space $\mathbf{D}$ plays a role similar to that of the domain of the maximal operator induced by τ in $L^2(I)$.

GREEN'S FORMULA 3.7. *Let $y,z \in \mathbf{D}$. Then*

1. *$[Fy, JSz]$ and $[Sy, JFz]$ exist;*

2. *$\langle y,z\rangle := (z^t Hy)(b-0) - (z^t Hy)(a+0)$ exists; and*

3. *$[Fy, JSz] - [Sy, JFz] = \langle y,z\rangle$.*

The integrals in (1) exist by the definition of $\mathbf{D}$. Then (2) and (3) follow by (1) on multiplying (2.4) from the left by $\begin{pmatrix} z' \\ z \end{pmatrix}^t$, from the right by $\begin{pmatrix} y' \\ y \end{pmatrix}$, and integrating over I.

We now assume that (F,G) is right definite:

DEFINITION 3.8. *(F,G) is "right-definite (with respect to S)" if for some $\lambda \in \mathbf{C}$ the positive-semidefinite inner product $(\ ,\ )$ is definite on $\mathbf{E}_\lambda$.*

It can be shown that $\{y \in \mathbf{E}_\lambda / (y,y) = 0\}$ is independent of λ. Therefore, if the condition of Definition 3.8 holds for one λ, it holds for all λ. Especially one may assume $(\ ,\ )$ to be definite on $\mathbf{E}_0$. Then a compactness argument gives the following lemma.

LEMMA 3.9. *(F,G) is right definite iff for some compact interval M in the interior of I the integral $\int_M (UY_0)^* UY_0$ is positive definite.*

This lemma can be used to show that for every $y \in \mathbf{D}$ there exists a $u \in \mathbf{D}$ which coincides with y to the left of M and vanishes to the right of M. This gives a decomposition $y = u+v$ of y into elements of $\mathbf{D}$ which vanish in a neighborhood of the right and left endpoint of I, respectively. This decomposition is frequently used in the proofs of the following results.

In the case of Example 1.7 it can be shown that the problem is right definite iff, apart from being nonnegative, the weight function w is not identically zero.

4. An Appropriate Hilbert Space

Since $(\mathbf{E},(\ ,\ ))$ is a semi-prehilbert space, there exists a completion $(\mathbf{G},(\ ,\ ))$ of $(\mathbf{E},(\ ,\ ))$ and a unique continuous extension of $U/_{\mathbf{E}}$ from all of $\mathbf{G}$ into L^2. This extension will be denoted by the same symbol U. Then U is a linear isometry from $(\mathbf{G},(\ ,\ ))$ into $(L^2,[\ ,\ ])$. In view of (3.5), $G/_{\mathbf{E}}$ can be extended to all of $\mathbf{G}$ by

$$G := KU, \tag{4.1}$$

where again the same symbol G is used for this continuation.

Then the following theorem can be proved.

THEOREM 4.2. $L^2 = U\mathbf{G} \oplus N(K)$.

Here $N(K)$ denotes the null space (in L^2) of the operator of multiplication by K.

COROLLARY 4.3.

1. $U\mathbf{G} = N(K)^\perp$,

2. $KL^2 = G\mathbf{G}$,

3. $\mathbf{D} = \mathbf{A} \cap \mathbf{E} \cap F^{-1} G\mathbf{G}$,

4. $J(U\mathbf{G}) = U\mathbf{G}$.

The first — obvious — part of the corollary shows that one knows $U\mathbf{G}$ if K is known; in particular, no completion process is needed in calculating $U\mathbf{G}$. The second equation follows by applying K to the decomposition of Theorem 4.2 and by (4.1). This equation then implies (3). L^2 and — since K is real valued — $N(K)$ are invariant with respect to J. Therefore, (4) follows from Theorem 4.2.

DEFINITION 4.4. $\mathbf{H} := U\mathbf{G}$, $\tilde{K} := K/_{\mathbf{H}}$

Then $(\mathbf{H},[\ ,\ ])$ is a sub-Hilbert space of $(L^2,[\ ,\ ])$; by Corollary 4.3 (4), J is a conjugation on $\mathbf{H}$; U is a linear isometry from $(\mathbf{G},(\ ,\ ))$ onto $(\mathbf{H},[\ ,\ ])$; and $\tilde{K}$ maps $\mathbf{H}$ bijectively onto $G\mathbf{G}$ since $\mathbf{H}\cap N(\tilde{K}) = \{0\}$ by Theorem 4.2. Furthermore, $G = \tilde{K}U$ (compare the right-hand part of the commutative diagram 5.9).

5. The J-Symmetric Operator

By Corollary 4.3 (3), F maps $\mathbf{D}$ into $G\mathbf{G}$. Therefore, the following definition makes sense (compare the upper left-hand part of Diagram 5.9):

DEFINITION 5.1. $\Gamma := \tilde{K}^{-1}F_{/\mathbf{D}}$.

Then $F_{/\mathbf{D}} = \tilde{K}\Gamma = K\Gamma$, and since J and K commute, Green's Formula 3.7 implies the following theorem.

THEOREM 5.2. $[\Gamma y, JUz] - [Uy, J\Gamma z] = \langle y,z \rangle$ *for all* $y,z \in \mathbf{D}$.

Let $\mathbf{D}_0$ denote the orthogonal complement of $\mathbf{D}$ in itself with respect to the bilinear form $\langle\ ,\ \rangle$:

$$\mathbf{D}_0 := \{y \in \mathbf{D} / \langle y,z \rangle = 0 \text{ for all } z \in \mathbf{D}\}, \tag{5.3}$$

and denote by $\mathbf{D}_0'$ the set of all $y \in \mathbf{D}$ with compact support contained in the interior of I. Then $\mathbf{D}_0' \subset \mathbf{D}_0 \subset \mathbf{D}$, and (5.3) and Theorem 5.2 imply the following corollary.

COROLLARY 5.3. $[\Gamma y, JUz] = [Uy, J\Gamma z]$ *for all* $y \in \mathbf{D}_0$ *and all* $z \in \mathbf{D}$.

A converse of this corollary is the following.

THEOREM 5.4. *Let* $f, g \in \mathbf{H}$ *and assume that*

$$[\Gamma y, Jf] = [Uy, Jg] \text{ for all } y \in \mathbf{D}_0'.$$

Then there exists a uniquely defined $z \in \mathbf{D}$ such that $f = Uz$ and $g = \Gamma z$.

As a consequence of Corollary 5.3 and Theorem 5.4, we get the following.

COROLLARY 5.5. $(U\mathbf{D}_0')^{\perp} = J\Gamma(\mathbf{D}\cap N(U))$.

Here $\perp$ denotes the orthogonal complement in $\mathbf{H}$, and the null space $N(U)$ contains all elements of $\mathbf{G}$ with norm zero.

We now make an additional assumption.

ASSUMPTION 5.6. $\mathbf{D}\cap N(U) = \{0\}$.

In terms of U, right-definiteness means that $\mathbf{E}_0\cap N(U) = \{0\}$. Since by Remark (a) following Definition 3.6 $\mathbf{E}_0$ is contained in $\mathbf{D}$, Assumption 5.6 slightly strengthens the assumption of right-definiteness. In most cases this additional hypothesis is fulfilled, especially in the case of the system (1.7) arising from the differential equation (1.6) or in the case of systems arising from differential equations $\tau\eta = \lambda\sigma\eta$, where σ is another formally J-symmetric differential operator.

Assumption 5.6 and Corollary 5.5 immediately imply the following corollary.

COROLLARY 5.7.

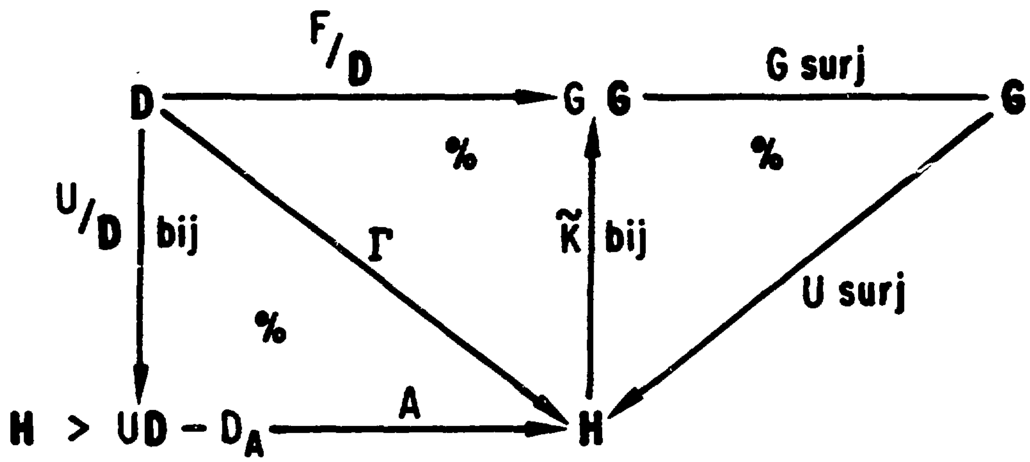
1. $U_{/\mathbf{D}}: \mathbf{D} \to U\mathbf{D}$ *is bijective.*

2. $\mathbf{D}_0'$ *is dense in* G.

Now it is possible to define an appropriate linear operator A from $D_A := U\mathbf{D}$ into $\mathbf{H}$ by the following definition.

DEFINITION 5.8. $A := \Gamma(U_{/\mathbf{D}})^{-1}$.

The following commutative diagram elucidates the situation:



From this diagram the following remark arises.

REMARK 5.10.

$$A = U(G^{-1}F_{/\mathbf{D}})\,(U_{/\mathbf{D}})^{-1};$$

that is, apart from the isometries U and $U_{/\mathbf{D}}$, A is the relation $G^{-1}F_{/\mathbf{D}}$. This shows that the operator A reflects the properties of the differential system (2.1). Now let

$$A_0' := A_{/\mathfrak{D}_0'},\quad A_0 = A_{/\mathfrak{D}_0}.$$

Then $A_0' \subset A_0 \subset A$ and by Corollary 5.7, (2), these operators are densely defined.

The bilinear form $<\,,\,>$ can be transferred to D_A by the definition

$$<f,g> := <(U_{/\mathbf{D}})^{-1}f,\,(U_{/\mathbf{D}})^{-1}g> \text{ for } f,g \in D_A.$$

Then Theorem 5.2 and Corollary 5.3 can be formulated as follows.

THEOREM 5.11.

1. $[Af, Jg] - [f, JAg] = <f,g>$ *for all* $f,g \in D_A$.
2. $[A_0 f, Jg] = [f, JAg]$ *for all* $f \in D_{A_0}$ *and all* $g \in D_A$.

The last assertion shows that A_0 is J-symmetric and that

$$JAJ \subset A_0^*. \tag{5.12}$$

In the same way Theorem 5.4 can be reformulated as follows.

THEOREM 5.13. *Let* $f,g \in \mathbf{H}$, *and assume that*

$$[A_0'h, Jf] = [h, Jg] \text{ for all } h \in D_{A_0'}.$$

Then $f \in D_A$ *and* $g = Af$. *In particular, this implies*

$$(A_0')^* \subset JAJ. \tag{5.14}$$

Since $A_0^* \subset (A_0')^*$, (5.12) and (5.14) imply the following theorem.

THEOREM 5.15. $JAJ = A_0^* = (A_0')^*$.

Consequently, A is closed and $\bar{A}_0 = \bar{A}_0'$. Since by Theorem 5.11 (1), $<,>$ is continuous with respect to the graph norm on D_A, A_0 can easily be shown to be closed, too. Therefore, we have the following corollary.

COROLLARY 5.16. *A is closed, and* $A_0 = \bar{A}_0'$.

Theorem 5.15 also implies that every J-symmetric extension B of A_0 is a restriction of A:

$$B \subset JB^*J \subset JA_0^*J = A.$$

Therefore the domain of B equals $U\mathbf{C}$ for some linear space $\mathbf{C}$ with $\mathbf{D}_0 \subset \mathbf{C} \subset \mathbf{D}$. Define for such a space the space $\mathbf{C}_0$ by the following definition.

DEFINITION 5.17. $\mathbf{C}_0 := \{y \in \mathbf{D} / <y, z> = 0 \textit{ for all } z \in \mathbf{C}\}$.
($\mathbf{D}_0$ in the sense of this definition is the same as the old $\mathbf{D}_0$.)

Then Theorem 5.11 gives the following characterization of J-symmetric and J-selfadjoint extensions of A_0, respectively.

THEOREM 5.18. *Let B be a linear operator with $A_0 \subset B \subset A$, and let* $\mathbf{C} := (U_{/\mathbf{D}})^{-1} D_B$. *Then B is J-symmetric iff $\mathbf{C} \subset \mathbf{C}_0$, and B is J-selfadjoint iff $\mathbf{C} = \mathbf{C}_0$.*

Since A_0 is J-symmetric, the general theory of J-symmetric operators can now be applied to A_0. Together with the special structure of the operators A_0 and A this application yields a lot of new results for our problem. We will sketch only a few of these results.

First of all, the extension theorem (1.2) shows that there exist J-selfadjoint extensions of A_0. Another result of the general theory gives that the "deficiency index in λ", i.e., $m(\lambda) := \dim R_{A_0-\lambda}^{\perp}$ is constant in the regularity field $\Pi(A_0)$. Now an easy calculation shows that

$$R_{A_0-\lambda}^{\perp} = N((A_0-\lambda)^*) = JN(A-\lambda) = JU\mathbf{E}_\lambda. \tag{5.19}$$

Therefore, $m(\lambda) = \dim \mathbf{E}_\lambda$ is constant in $\Pi(A_0)$. In particular, $0 \le m(\lambda) \le n$. It can be proved that $m(\lambda) \ge \frac{n}{2}$, if a or b belongs to I.

For $\lambda \in \Pi(A_0)$ the modified von Neumann formula gives in connection with (5.19) and Theorem 5.15

$$D_A = D_{A_0} \dot{+} (B_\lambda - \lambda)^{-1} JU\mathbf{E}_\lambda \dot{+} U\mathbf{E}_\lambda,$$

where B_λ is a J-selfadjoint extension of A_0 with λ in its resolvent set. Applying $(U_{/\mathbf{D}})^{-1}$ to this equation gives

$$\mathbf{D} = \mathbf{D}_0 \dot{+} R_\lambda JU\mathbf{E}_\lambda \dot{+} \mathbf{E}_\lambda. \tag{5.20}$$

Here $R_\lambda = (U_{/\mathbf{D}})^{-1}(B_\lambda - \lambda)^{-1}: \mathbf{H} \to \mathbf{D}$.

If $\Pi(A_0) \neq \phi$, then (5.20) holds for some $\lambda \in \mathbf{C}$. Therefore, $\dim \mathbf{D}/\mathbf{D}_0 = 2m(\lambda) =: 2m$, and for every linear space $\mathbf{C}$ with $\mathbf{D}_0 \subset \mathbf{C} \subset \mathbf{D}$ there exist

$w_1,\ldots,w_{2m} \in \mathbf{D}$ linearly independent mod$\mathbf{D}_0$ such that for some $l \le 2m$

$$\mathbf{C} = \mathbf{D}_0 \dot{+} [w_1,\ldots,w_l],\ \mathbf{D} = \mathbf{C} \dot{+} [w_{l+1},\ldots,w_{2m}].$$

Then $\mathbf{C} = \mathbf{C}_0$ is equivalent to

$$l = m,\ \langle w_i, w_j\rangle = 0\ (i,j \le m),\ \mathbf{C} = \{y \in \mathbf{D} / \langle y, w_i\rangle = 0\ (i \le m)\}.$$

This is the description of J-selfadjoint extensions of A_0 by boundary conditions:

THEOREM 5.21. *Let* $\Pi(A_0) \ne \phi$, *and let* $m := \frac{1}{2}\dim\mathbf{D}/\mathbf{D}_0$. *Then* $D_B = U_{/_\mathbf{D}}\mathbf{C}$ *is the domain of a J-selfadjoint extension B of* A_0 *iff there exist* $w_1,\ldots,w_m \in \mathbf{D}$ *linearly independent* mod$\mathbf{D}_0$ *such that*

$$\langle w_i, w_j\rangle = 0\ (i,j \le m)$$

and

$$\mathbf{C} = \{y \in \mathbf{D} / \langle y, w_i\rangle = 0\ (i \le m)\}.$$

In contrast to the situation here, in the case of formally symmetric systems there do not always exist selfadjoint extensions; and even if there exist such selfadjoint extensions, it may happen that none of these extensions is generated by separated boundary conditions. In the case of formally J-symmetric systems, the situation is quite different. If $\Pi(A_0) \ne \phi$, then there always exist J-selfadjoint extensions which can be described by separated boundary conditions. We cannot go into details here.

Let us finally consider the "resolvent" $R_\lambda := (U_{/_\mathbf{D}})^{-1}(B-\lambda)^{-1}$ of the J-selfadjoint extension B with λ in its resolvent set. It can be shown that R_λ is an integral operator from $\mathbf{H}$ into $\mathbf{D}$, although $(B-\lambda)^{-1}$ in general is not an integral operator. For simplicity, let $a \in I$ and define $m := m(\lambda)$,

$$Q_1 := \begin{pmatrix} E_m & 0 \\ 0 & 0 \end{pmatrix},\quad Q_2 := \begin{pmatrix} 0 & 0 \\ 0 & E_{n-m} \end{pmatrix}.$$

Then there exists a constant regular (n,n)-matrix P such that the first m solutions of $Z_\lambda := Y_\lambda P$ span $\mathbf{E}_\lambda$. Let $H_0 := Z_\lambda^t(x_0)H(x_0)Z_\lambda(x_0)$. With these assumptions and definitions, the following theorem holds true.

THEOREM 5.22. $Q_2H_0Q_2 = 0$ *and there exists an* (n,n)*-matrix* T *such that for all* $f \in \mathbf{H}$ *and all* $x \in I$

$$(R_\lambda f)(x) = \int_I G(x,s)f(s)ds,$$

where

$$G(x,s) := Z_\lambda(x)\begin{Bmatrix} Q_1H_0^{-1} & +\,Q_1TQ_1 \\ -Q_2H_0^{-1}Q_1 & +\,Q_1TQ_1 \end{Bmatrix}(UZ_\lambda)^t(s)\ \begin{cases}\text{for } s \le x \\ \text{for } s > x\end{cases}.$$

The matrix T can be determined by the boundary conditions describing B. Theorem 5.22 gives the following necessary condition for λ to be in the regularity field of A_0.

COROLLARY 5.23. *If* $\lambda \in \Pi(A_0)$, *then* $Q_2H_0Q = 0$.

This also shows that in the case $a \in I$, $m(\lambda) = m \ge \frac{n}{2}$.

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POINTWISE EQUISUMMABILITY OF ELLIPTIC OPERATORS

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Abstract

We present criteria for determining pointwise equisummability of expansions in eigenfunctions of certain pairs of elliptic operators on general domains of $\mathbf{R}^n$. Applications are given for Sturm-Liouville systems and the heat equation.

In this paper we give criteria for L^∞-equisummability of eigenfunction expansions for certain pairs of elliptic operators on $\mathbf{R}^n$. Namely, L^∞-equisummability of two elliptic operators, or the pointwise convergence of the difference of two summability means, is reduced to showing that the difference of the modified resolvent operators is uniformly bounded. The class of elliptic operators is a generalization of the class of Sturm-Liouville operators. Our elliptic operators have leading terms that are positive and lower-order terms whose coefficients are singular on nowhere dense sets. Our work is motivated by the fact that it is easier to study expansions in eigenfunctions of unperturbed operators, say, Laplacian $\equiv -\Delta$ on $\mathbf{R}^n$, than of perturbed ones, $-\Delta + q(x)$.

The prototypical case for equisummability is found in the study of equiconvergence for differential operators; see Haar [1910], Walsh [1922-23], Birkhoff [1908], and Tamarkin [1912]. That is, the difference between expansions with respect to eigenfunctions of a Sturm-Liouville operator and the ordinary Fourier series tends to zero uniformly in every finite interval. In some one-dimensional cases where equiconvergence fails, one can show (see Stone [1926]; Levitan and Sargsjan [1975]; and Benzinger [1970]) equisummability for Riesz typical means of eigenfunction expansions of differential operators. For equisummability in higher dimensions, Gurarie and Kon [1983] give conditions under which the expansion of an L^p function in eigenfunctions of an elliptic operator is equisummable with the corresponding expansion obtained from its leading term.

We will illustrate our theory with two examples. The first shows that for $f \in L^2(\mathbf{R})$, the generalized Fourier transform of f associated with certain classes of Sturm-Liouville problems is analytic summable to $f(x)$ pointwise if and only if the Fourier transform of f is analytic summable to $f(x)$ pointwise. The second application is to convergence of solutions of the heat and perturbed heat equations to their common initial value.

1. Equisummability for Differential Operators in $L^p(\mathbf{R}^n)$, $1 \le p < \infty$

We begin with basic definitions. Let A be a differential operator on $L^p(\mathbf{R}^n)$, $1 \le p < \infty$. Let $\{u(x,\lambda)\}$ be the set of generalized eigenfunctions associated with the eigenvalue λ belonging to the spectrum $\sigma(A)$. Let $f \in L^p(\mathbf{R}^n)$, and assume that the eigenfunction expansion

$$f(x) \sim \int_{\sigma(A)} F(\lambda) u(x,\lambda) d\rho(\lambda) \qquad (1)$$

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and its generalized Fourier coefficient

$$F(\lambda)\sim\int_{\mathbb{R}^n} f(x)u(x,\lambda)dx$$

exist. Here ~ denotes L^p convergence as the limits of integration become infinite, and ρ is a combination of spectral functions. Let $\varphi(\lambda$ be continuous and $\varphi(0) = 1$. We say that the ***eigenfunction expansion (1) is φ-summable*** in an appropriate topology if the summability means

$$\varphi(\varepsilon A)f = \int_{\sigma(A)} F(\lambda)\varphi(\varepsilon\lambda)u(x,\lambda)d\rho(\lambda)\to f$$

as $|\varepsilon|\to 0$, ε again belonging to some domain D in **C**. When the summator function $\varphi(\varepsilon\lambda) = \frac{1}{1+\varepsilon\lambda} = \frac{1}{1-\frac{\lambda}{z}} = \frac{z}{z-\lambda}$, for $\varepsilon = -\frac{1}{z}$, and $\frac{z}{z-A}f\to f$ as $|z|\to\infty$, z in $D' = \{z \mid -1/z\,\varepsilon D\}$, the eigenfunction expansion is ***resolvent summable*** to f.

Two differential operators A and B on $L^p(\mathbb{R}^n)$, $1\le p<\infty$, are ***φ-equisummable*** from $L^p(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ if

$$\|\varphi(\varepsilon A)f-\varphi(\varepsilon B)f\|_{L^\infty} = \sup_{x\in\mathbb{R}^n}\|\varphi(\varepsilon A)f(x)-\varphi(\varepsilon B)f(x)\|\to 0 \text{ as } |\varepsilon|\to 0$$

for all f in $L^p(\mathbb{R}^n)$.

If summability does not occur with respect to the $L^p(\mathbb{R}^n)$ norm, $1\le p<\infty$, then equisummability from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ becomes interesting. Our first theorem states conditions under which equisummability from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ holds for $1\le p<\infty$.

THEOREM 1. *Let A, B be closed differential operators on $L^p(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$, $1\le p<\infty$. Assume $z(z-A)^{-1}-z(z-B)^{-1}$ is uniformly bounded from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ for z in $D\subset\mathbf{C}$. If $z(z-A)^{-1}$ and $z(z-B)^{-1}$ are uniformly bounded from $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, then A and B are resolvent equisummable from $L^p(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$ in D.*

Our proof depends on the fact that the space of infinitely differentiable functions with compact support, $C_0^\infty(\mathbb{R}^n)$, is dense in $L^p(\mathbb{R}^n)$, $p\neq\infty$; the algebraic identity

$$z(z-A)^{-1}f-z = z(z-A)^{-1}[z^{-1}Af];$$

and a Banach space theorem.

The setting for our principal theroems follows. Let K be a set in **C**. Here ∂K denotes the boundary of K, while K^c denotes the complement of K in **C**. The regions K_i, $(i=1,2,)$ defined in Theorem 2 can be informally thought of as concentric keyholes, the intersection of one of which with the exterior of the other contains the contour Γ. These regions will be used also in Theorem 3.

THEOREM 2 (Kon). *Let A, B be closed differential operators $L^p(\mathbb{R}^n)$ for $1\le p<\infty$. Assume $R_z f = z(z-A)^{-1}f - z(z-B)^{-1}f$ is uniformly bounded from $L^p(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$. Let $D_{r_i} = \{z\in\mathbf{C} \mid |z|\le r_i\}$, and $\Omega_{\vartheta_i} = \{z\in\mathbf{C} \mid |\arg z|\le\vartheta_i\}$, and $K_i = D_{r_i}\cup\Omega_{\vartheta_i}$ for $i=1,2$ be such that $K_2\subset K_1$.*

Let A and B be resolvent equisummable on the complement of K_2. If φ is analytic on K_1 such that $\varphi(0) = 1$, $\varphi(z) = 0(z^{-\delta})$, $(\delta>0)$ z in $K_1\cap K_2^c$, then A and B are φ-equisummable in ε, ε in $D = \{z \mid |\arg z| < \vartheta_1-\vartheta_2\}$ from $L^p(\mathbb{R}^n)\to L^\infty(\mathbb{R}^n)$ for $1\le p<\infty$.

The proof uses the Dunford operator calculus and the Dominated Convergence Theorem.

2. Equisummability of Elliptic Operators on $\mathbb{R}^n$

We now present some notation. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index, and

$$D^\alpha = \frac{1}{i^{|\alpha|}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_1^n \alpha_i .$$

Consider the differential operator

$$A = \sum_{|\alpha| \le m} b_\alpha(x) D^\alpha \equiv A_0 + C,$$

where A_0 contains the leading terms and C is the remainder. We assume that A_0 is constant coefficient positive elliptic [i.e., $A_0 \equiv \sum_{|\alpha|=m} b_\alpha D^\alpha$ and $\sum_{|\alpha|=m} b_\alpha z^\alpha > 0$ for $z = (z_1, \ldots, z_n) \neq 0$]. We assume that the coefficients of C can be expressed as sums of functions in certain $L^p(\mathbb{R}^n)$ spaces, i.e., $b_\alpha(x) \varepsilon L^{r_\alpha} + L^\infty$, $|\alpha| < m$, where $d = \sup_{|\alpha| < m} \{ \frac{n}{r_\alpha} + |\alpha| \} < m$. We choose for the domain the L^p Sobolev space L^p_m. (If p is outside $1 \le p \le \min r_\alpha$, then A may not be densely defined.) The next theorem is an application of the theory in Section 1.

THEOREM 3 (Gurarie-Kon). *Let $A = A_0 + C$ and $B = A_0 + D$ be closed elliptic operators defined as above on the Sobolev space L^p_m, $1 \le p \le \min r_\alpha$, where A_0 is constant coefficient positive elliptic containing the leading terms of order m. Assume the coefficients $b_\alpha(x)$ of A,B are singular on a nowhere dense set. Let K_1 and K_2 be as in Theorem 2 and φ be analytic on K_1, $\varphi(0) = 1$, $\varphi(z) = O(z^{-\delta})$, $\delta > 0$, z in $K_1 \cap K_2^c$. Then A and B are φ-equisummable from $L^p(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ for $p > n/(m-d)$, and ε in $D = \{z \mid |\arg z| < \vartheta_2 - \vartheta_1\}$.*

The key technique of the proof is to analyze kernels of the resolvents end use L^1-radial bounds of the resolvents developed in Gurarie and Kon [1984 and 1983].

We observe that when A is a differential operator, $(z-A)^{-1}$ is L^p-smoothing, so we expect equisummability from $L^p(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$. For pointwise equisummability when A,B are φ-equisummable, then $\varphi(\varepsilon A)f \to f$ at exactly the set of points where $\varphi(\varepsilon B)f \to f$, since $(\varphi(\varepsilon A) - \varphi(\varepsilon B))f$ converges uniformly to 0. Thus we state the following corollary.

COROLLARY. *If A,B are as in Theorem 3, then $\varphi(\varepsilon A)f \to f$ at $x \in \mathbb{R}^n$ if and only if $\varphi(\varepsilon B)f \to f$ at x.*

3. Applications

It follows immediately that under the conditions of Theorem 3, the generalized Fourier expansion associated with certain classes of Sturm-Liouville problems and the ordinary Fourier transform are analytic equisummable.

For example (see Levitan and Sargsjan [1975]), consider the equation

$$\frac{-d^2u(x)}{dx^2} + q(x)u(x) = \lambda u(x) \quad x \in (-\infty, \infty)$$

and $q(x) \in (L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n))$ is real and continuous. Let $u_1(x,\lambda)$ denote the solution of this equation which satisfies the initial conditions

$$u_1(0,\lambda) = 0,\ \frac{d}{dx}u_1(0,\lambda) = -1,$$

and by $u_2(x,\lambda)$ the solution under the initial conditions

$$u_2(0,\lambda) = 1,\ \frac{d}{dx}u_2(0,\lambda) = 0.$$

Let $f \in L^2(R)$ and $F_i(\lambda)$ be the generalized Fourier coefficients of f with respect to $u_i(x,\lambda)$, $i=1,2$. Let $\rho_i(x)$, $i=1,2$ and $\eta(x)$ be the standard limits of step functions where the jumps occur at the eigenvalues of the boundary value problems on a finite interval $[a,b]$. Here, ρ_i are monotone and bounded, while η is of bounded variation in every finite interval. Then we can write

$$f(x)\sim\int_{-\infty}^{\infty}F_1(\lambda)u_1(x,\lambda)d\rho_1(\lambda) + \int_{-\infty}^{\infty}F_1(\lambda)u_2(x,\lambda)d\eta(\lambda)$$

$$+ \int_{-\infty}^{\infty}F_2(\lambda)u_1(x,\lambda)d\eta(\lambda) + \int_{-\infty}^{\infty}F_2(\lambda)u_2(x,\lambda)d\rho_2(\lambda), \qquad (2)$$

where $\sim$ is L^2-convergence as the limits of integration become infinite.

It is well known that in the case $q(x) = 0$, (2) is the Fourier transform, namely,

$$f(x)\sim i\int_{-\infty}^{\infty}(-)\sin\sqrt{\lambda}x(\hat{f}(\sqrt{\lambda}) - \hat{f}(-\sqrt{\lambda}))d\sqrt{\lambda}$$

$$+ \int_{-\infty}^{\infty}\cos\sqrt{\lambda}x(\hat{f}(\sqrt{\lambda}) + \hat{f}(-\sqrt{\lambda}))d\sqrt{\lambda},\ \lambda \geq 0,$$

where $\hat{f}$ denotes the Fourier coefficient of f.

Now when the summator function $\varphi(\lambda)$ is analytic and satisfies the conditions of Theorem 3, and the summability means with respect to φ of each of the integrals is absolutely convergent, then the φ-equisummability of the generalized Fourier expansion and Fourier transform follows, that is,

$$\int_{-\infty}^{\infty}F_1(\lambda)\varphi(\varepsilon\lambda)u_1(x,\lambda)d\rho_1(\lambda) + \int_{-\infty}^{\infty}F_1(\lambda)\varphi(\varepsilon\lambda)u_2(x,\lambda)d\eta(\lambda)$$

$$+ \int_{-\infty}^{\infty}F_2(\lambda)\varphi(\varepsilon\lambda)u_1(x,\lambda)d\eta(\lambda) + \int_{-\infty}^{\infty}F_2(\lambda)\varphi(\varepsilon\lambda)u_2(x,\lambda)d\rho_2(\lambda)\to f(x)$$

pointwise if and only if

$$\int_{-\infty}^{\infty}\varphi(\varepsilon k)\hat{f}(k)e^{-ikx}dk \to f(x),\quad k^2 = \lambda,\ \text{as } \varepsilon\to 0$$

pointwise.

For our final example, consider the heat equation $A_0u = -\Delta u = \frac{-\partial}{\partial t}u$ with the initial condition $u(x,0) = f(x)\in L^p(\mathbb{R}^n)$, $p \geq n$, and an associated perturbed heat equation

$$Au^* = (-\Delta+q(x))u^* = \frac{-\partial}{\partial t}(u^*),\ u^*(x,0) \equiv f(x)\in L^p(\mathbb{R}^n),\ p \geq n.$$

Here $q(x)$ is the sum of $L^n(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ functions. So by Theorem 3, when $\varphi(\varepsilon A) = e^{-\varepsilon A}$, $\|e^{-tA}f - e^{-tA_0}f\|_\infty\to 0$ as $t\to 0$. In other words, the solutions $u(x,t)$

and $u^*(x,t)$ converge to $f(x)$ as $t \to 0$ at the same set of points x in R^n.

Proofs of the theorems will appear elsewhere.

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THE ESSENTIAL SPECTRUM OF A CLASS OF ORDINARY DIFFERENTIAL OPERATORS

Bernd Schultze *

Abstract

The essential spectrum of ordinary differential operators is investigated using the theory of relatively compact perturbations. For several classes of ordinary differential expressions, the essential spectrum can be determined exactly, results that seem to be new even for the even-order symmetric case.

In this paper, we will consider the (in general) non-symmetric differential expression

$$My = \sum_{i=0}^{n} p_i y^{(i)} \tag{1}$$

on $I = [1,\infty)$ and determine the shape of the essential spectrum of closed differential operators $T(M)$ generated by M in $L^2(I)$, i.e., of the set

$$\sigma_e(T(M)) = \{\lambda\in\mathbf{C} \mid R(T(M)-\lambda) \text{ is not closed}\}.$$

There are a large number of important results concerning the essential spectrum of differential operators generated by symmetric expressions M. However, there are only few examples where this set is not empty and entirely known: if n is even, in the case that M is some perturbation of the Euler differential expression or of an expression with constant coefficients, as in Müller-Pfeiffer [1977] and Evans, Kwong, and Zettl [1983], and if n is odd, the limit-point criteria [Hinton, 1978; Schultze, 1984] imply that the essential spectrum must be the whole real line.

In the more general case of expressions of the form (1), the basic theory for the essential spectrum has been developed by Evans, Lewis, and Zettl [1984] and by Rota [1958]. It turns out that as in the symmetric case, the decomposition principle holds and that the essential spectrum is invariant under finite dimensional extensions of the operator. For this reason, we can confine our considerations to $\sigma_e(T_0(M))$, where $T_0(M)$ is the minimal operator generated by M, and define $\sigma_e(M) = \sigma_e(T_0(M))$. In Rota [1958], the following spectral mapping theorem is proven: If p is a polynomial with constant complex coefficients and $p(\sigma_e(M)) \neq \mathbf{C}$, then $p(\sigma_e(M)) = \sigma_e(p(M))$.

The most frequently used method to get information on the essential spectrum consists in determining the numerical range, since it contains the essential spectrum. Sometimes this numerical range coincides with the essential spectrum, but it can also happen that there is a large gap between these two sets. Thus if, for instance, M is a symmetric odd-order expression, then its numerical range is $\mathbf{R}$, but a subclass of such expressions is known having empty essential spectrum.

The method applied here is more direct. Up till now, it has been restricted to have real powers of the variable t as dominating functions for some subscripts i. But with this method the essential spectrum of several classes of expressions can be exactly determined. These results seem to be new, even for

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the even-order symmetric case.

We shall use the theory of relatively compact perturbations developed by Kauffman [1977] for differential operators (slightly generalized in Schultze [1984] and also use the information on the structure of M^+M given by Read [1982].

To be more precise, we first consider special real expressions

$$M_0 y = \sum_{\sigma=0}^{s} p_{\zeta_\sigma} y^{(\zeta_\sigma)}, \tag{2}$$

where $s \in \mathbb{N}_0$, $\zeta_0, \ldots, \zeta_s$ integers with $0 \le \zeta_0 < \ldots < \zeta_s = n$, $p_{\zeta_\sigma}(t) = a_\sigma t^{\alpha_\sigma}$ with $\alpha_\sigma \in \mathbb{R}$ $(\sigma = 0, \ldots, s)$ satisfying for $s \ge 1$ the condition

$$\frac{\alpha_1 - \alpha_0}{\zeta_1 - \zeta_0} < 1 \tag{3}$$

and for $s > 1$

$$\frac{\alpha_{\sigma+1} - \alpha_\sigma}{\zeta_{\sigma+1} - \zeta_\sigma} < \frac{\alpha_\sigma - \alpha_{\sigma-1}}{\zeta_\sigma - \zeta_{\sigma-1}} \quad \text{for } \sigma = 1, \ldots, s-1. \tag{4}$$

For the coefficients $a_\sigma \in \mathbb{R}$ we make the following restrictions:

$$\begin{cases} \operatorname{sgn}(-1)^{\frac{1}{2}\zeta_\sigma} a_\sigma = \text{const} & \text{for } \zeta_\sigma \text{ even} \\ \operatorname{sgn}(-1)^{\frac{1}{2}(\zeta_\sigma+1)} a_\sigma = \text{const} & \text{for } \zeta_\sigma \text{ odd} \end{cases} \tag{5}$$

We want to mention that the conditions (4) and (5) may be weakened without affecting the results. Thus we can admit the "≤" relation in (4), and (5) can be replaced by a condition of the following form: The "sign" of these even-order (odd-order) coefficients must be constant as long as they lie between two consecutive odd (even) kink points. (In this context ζ_0 and ζ_s are defined to be even and odd.) But from now on we always assume that M_0 satisfies conditions (3), (4), and (5). Conditions (3) and (5) mean that the points $(\zeta_\sigma, \alpha_\sigma)$ in $\mathbb{R}^2 (\sigma = 0, \ldots, s)$ form a polynomial path with decreasing slopes smaller than 1.

For such expressions M_0 we are able to determine $M_0^+ M_0$, which is a real symmetric expression of the form

$$M_0^+ M_0 y = \sum_{k=\zeta_0}^{n} (-1)^k (q_k y^{(k)})^{(k)}.$$

The form of the coefficients q_k is given in the next lemma.

LEMMA 1. *For $\zeta_\sigma \le k \le \zeta_{\sigma+1}$ $(\sigma = 0, \ldots, s-1)$, we have*

$$q_k(t) = (c_k + o(1)) t^{\frac{2}{\zeta_{\sigma+1} - \zeta_\sigma}\{(k - \zeta_\sigma)\alpha_{\sigma+1} + (\zeta_{\sigma+1} - k)\alpha_\sigma\}}$$

with $c_k \ge 0$ $(k = \zeta_0, \ldots, n)$ and $c_{\zeta_\sigma} = a_\sigma^2$ for $\sigma = 0, \ldots, s$.

This representation will be used to give lower estimates of

$$\| M_0 f \|_2^2 = (M_0^+ M_0 f, f) = \sum_{k=\zeta_0}^{n} \int q_k \, | f^{(k)} |^2$$

for test functions $f \in C_0^\infty(\overset{0}{I})$. Since not all c_k are positive, the idea is to blur the growth rates between the kink-points ζ_σ and to the left of ζ_0. To do this, we use the following inequalities.

LEMMA 2. *Let $n\in\mathbb{N}$, $0\le i<n$, $\alpha\in\mathbb{R}$. Then we have for all $f\in \overset{0}{C}{}_0^\infty(I)$:*

$$\int_I t^{2\alpha}\,|f^{(n)}|^2 \ge \Big[2^{i-n}\prod_{l=1}^{n-i}(2\alpha-2l+1)\Big]^2 \int_I t^{2(\alpha-n+i)}\,|f^{(i)}|^2.$$

Besides this known inequality, we need the following lemma (cf. Kwong and Zettl [1981: Theorem 9].

LEMMA 3. *Let $n\in\mathbb{N}$, $n\ge 2$, $\alpha,\beta\in\mathbb{R}$. Then for every $\varepsilon>0$ there exist $K(\varepsilon)>0$ and $\eta_\varepsilon\in I$ such that for all $f\in C_0^\infty((\eta_\varepsilon,\alpha))$ we have*

$$\sum_{i=1}^{n-1}\int_I t^{\frac{2}{n}(i\alpha+(n-i)\beta)}\,|f^{(i)}|^2 \le \varepsilon\int_I t^{2\alpha}\,|f^{(n)}|^2 + K(\varepsilon)\int_I t^{2\beta}\,|f|^2.$$

With these tools we are able to prove the following crucial lower estimation.

LEMMA 4. *There are $\eta\in I$, $b_k>0$ ($k=0,\ldots,n$), $K\ge 0$ such that for all $f\in C_0^\infty((\eta,\infty))$ we have*

$$\|M_0 f\|_2^2 \ge \sum_{\sigma=0}^{s-1}\sum_{k=\zeta_\sigma+1}^{\zeta_{\sigma+1}}\int_I b_k\, t^{\frac{2}{\zeta_{\sigma+1}-\zeta_\sigma}\{(k-\zeta_\sigma)\alpha_{\sigma+1}+(\zeta_{\sigma+1}-k)\alpha_\sigma\}}\,|f^{(k)}|^2$$

$$+\sum_{k=0}^{\zeta_0}\int_I b_k\, t^{\xi_k}\,|f^{(k)}|^2 - K\,\|f\|_2^2$$

with

$$\xi_k=\begin{cases}2(\alpha_0-\zeta_0+k) & \text{for } \alpha_0>\zeta_0 \text{ or } \zeta_0=0\\ 2\alpha_0 k\,\zeta_0^{-1} & \text{for } \alpha_0\le 0 \text{ and } \zeta_0>0\end{cases}.$$

In addition, we can choose $K=0$ if $\alpha_0\ge\zeta_0$.

This result enables us to apply perturbation theory in order to get information for more general expressions. As admissible perturbations of M_0 we can take expressions

$$My=\sum_{k=0}^{n} r_k\, y^{(k)} \tag{6}$$

with complex-valued $r_k\in C^k(I)$ $(k=0,\ldots,n)$ satisfying

$$r_k(t)=\begin{cases} o\big(t^{\frac{1}{\zeta_{\sigma+1}-\zeta_\sigma}\{(k-\zeta_\sigma)\alpha_{\sigma+1}+(\zeta_{\sigma+1}-k)\alpha_\sigma\}}\big) & \text{if } \sigma\in\{0,\ldots,s\} \textit{ exists} \text{ with } \zeta_\sigma<k\le\zeta_{\sigma+1}\\ o(t^{\frac{1}{2}\xi_k}) \quad \text{for } k=0,\ldots,\zeta_0\end{cases} \tag{7}$$

Then we have for $f\in \overset{0}{C}{}_0^\infty(I)$

$$\|Mf\|_2^2 \le \Big(\sum_{k=0}^{n}\|r_k f\|_2\Big)^2 \le (n+1)\sum_{k=0}^{n}\|r_k f^{(k)}\|^2$$

$$\le \sum_{\sigma=0}^{s-1}\sum_{k=\zeta_\sigma+1}^{\zeta_{\sigma+1}}\int_I t^{\frac{2}{\zeta_{\sigma+1}-\zeta_\sigma}\{(k-\zeta_\sigma)\alpha_{\sigma+1}+(\zeta_{\sigma+1}-k)\alpha_\sigma\}}\,o(1)\,|f^{(k)}|^2 + \sum_{k=0}^{\zeta_0}\int_I t^{\xi_k}\,o(1)\,|f^{(k)}|^2 .$$

First we consider the case $\alpha_0>\zeta_0$. Here we can take $K=0$ in Lemma 4, and we have the following lemma.

LEMMA 5. *If M is given as in (6) with coefficients satisfying (7), then there is*

$0 < \alpha < 1$ *and* $\eta \in I$ *such that for all* $f \in C_0^\infty((\eta,\infty))$ *we have* $\| Mf \|_2 \leq \alpha \|M_0 f\|_2$.

REMARK. Condition (7) implies that M is even relatively compact with respect to M_0 in this case $\alpha_0 > \zeta_0$ if $r_n = 0$. But using information that the proof of Lemma 4 gives for the constants b_k, the assertion of Lemma 5 can be obtained under weaker conditions than (11).

The main result in this case holds for every bounded perturbation of M_0 with bound smaller than 1.

THEOREM 1. *Let* $\alpha_0 > \zeta_0$ *and* M *be given as in (6) such that there is* $0 < \alpha < 1$ *and* $\eta \in I$ *with* $\|Mf\|_2 \leq \alpha \|M_0 f\|_2$ *for all* $f \in C_0^\infty((\eta,\infty))$. *Then* $\sigma_e(M_0+M) = \phi$.

Next we consider the case $\alpha_0 < \zeta_0$. Here we have to make the further assumptions that all $\zeta\sigma(\sigma=0,\ldots,\sigma)$ are odd or all are even. Let us first assume that all ζ_σ are odd. Then M_0 has the form

$$M_0 y = \sum_{\sigma=0}^{s} a_\sigma t^{\alpha_\sigma} y^{(\zeta_\sigma)} = \sum_{\sigma=0}^{s} (-1)^{\omega_\sigma} \tilde{a}_\sigma t^{\alpha_\sigma} y^{(2\omega_\sigma+1)}$$

with $\omega_\sigma = \frac{1}{2}(\zeta_\sigma - 1)$, $\tilde{a}_\sigma = (-1)^{\omega_\sigma} a_\sigma$. By reason of (5) we have sgn $\tilde{a}_\sigma$ = const, so we can assume $\tilde{a}_\sigma > 0$. Defining now the symmetric odd-order expression

$$N_0 y = \frac{i}{2} \sum_{\sigma=0}^{s} (-1)^{\omega_\sigma} \tilde{a}_\sigma \{ (t^{\alpha_\sigma} y^{(\omega_\sigma)})^{(\omega_\sigma+1)} + (t^{\alpha_\sigma} y^{(\omega_\sigma+1)})^{(\omega_\sigma)} \} ,$$

it can be shown that $iM_0 - N_0$ is relatively compact with respect to $N_0 - i$. Therefore N_0 and iM_0 have the same essential spectrum. But it is shown by Schultze [1984] that N_0 is in the limit point case for $\alpha_0 \leq \zeta_0$; therefore $\sigma_e(N_0) = \mathbf{R}$ since N_0 is of odd order. So we have the following theorem.

THEOREM 2. *Let* $\alpha_0 < \zeta_0$ *and all* $\zeta_\sigma(\sigma=0,\ldots,s)$ *be odd. If* M *is given as in (6) with* $r_n = 0$ *and the other coefficients satisfy (7), then* $\sigma_e(M_0+M) = i\mathbf{R}$

Similar arguments hold in the case that all $\zeta_\sigma(\sigma=0,\ldots,s)$ are even. But here we have only the information $\sigma_e(M_0) \subset [0,\infty)$ if $(-1)^{\frac{1}{2}\zeta_\sigma} a_\sigma > 0$ $(\sigma=0,\ldots,s)$; and in this generality, more precise results cannot be obtained, as examples will show.

THEOREM 3. *Let* $\alpha_0 < \zeta_0$, *all* ζ_σ *be even, and* $(-1)^{\frac{1}{2}\zeta_\sigma} a_\sigma > 0$ $(\sigma=0,\ldots,n)$. *If* M_0 *is given as in (6) with* $r_n = 0$ *and the other coefficients satisfy (7), then* $\sigma_e(M_0+M) \subset [0,\infty)$.

If we again specialize this case to $s = 0$, $\zeta_0 > 0$, then we can write M_0 in the form

$$M_0 y = (-1)^m t^\alpha y^{(2m)}, \tag{8}$$

where $m = \frac{1}{2}\zeta_0$, $\alpha = \alpha_0$. Denoting the corresponding symmetric expression by

$$\tilde{M}_0 y = (-1)^m (t^\alpha y^{(m)})^{(m)}, \tag{9}$$

the spectral mapping theorem and perturbation theory give for perturbations

$$My = \sum_{k=0}^{2m-1} r_k y^{(k)} \text{ with } r_k(t) = o(t^{\frac{\alpha k}{2m}}) \ (k=0,\ldots,2m-1) \tag{10}$$

the following result (if M is symmetric, given in the symmetric form with coeficients $\tilde{r}_k$, differentiation carries it over in the form (10) with analagous conditions on $\tilde{r}_k$ and some of their derivatives).

THEOREM 4. *Let* M_0 *be given as in (8),* $\tilde{M}_0$ *as in (9),* M *as in (10) with* $\alpha < 2m$ $(m \in \mathbf{N})$. *Then*

$$\sigma_e(M_0+M) = \sigma_e(\tilde{M}_0+M) = [0,\infty).$$

For more involved expressions M_0 the essential spectrum may be different from the whole non-negative real line. If we take for M_0 ($\tilde{M}_0$) the two-term expressions with $c \geq 0$:

$$M_0y = t^{2\alpha}y^{(4m)} + (-1)^m ct^{\alpha}y^{(2m)}, \tag{11}$$

resp.,

$$\tilde{M}_0y = (t^{2\alpha}y^{(2m)})^{(2m)} + (-1)^m c(t^{\alpha}y^{(m)})^{(m)}, \tag{12}$$

then again the spectral mapping theorem (for the polynomial $p(x) = x^2+c$) and perturbation theory give for perturbations

$$My = \sum_{k=0}^{4m-1} r_k y^{(k)} \text{ with } r_k(t)=o(t^{\frac{\alpha k}{2m}}) \quad (k=0,\dots,4m-1), \tag{13}$$

the following generalization of Theorem 4.

THEOREM 5. *If M_0 is given as in (11), $\tilde{M}_0$ as in (12), and M as in (13) with $\alpha < 2m$ $(m \in \mathbf{N})$, $c \geq 0$, then*

$$\sigma_e(M_0+M) = \sigma_e(\tilde{M}_0+M) = [c,\infty).$$

Taking other polynomials, we can generate expressions with more and more complicated essential spectrum in the complex plane. This also seems to happen in the case singled out till now: $\alpha_0 = \zeta_0$.

If N is "some" perturbation of the Euler-expression $N_0y = = \sum_{k=0}^{n} b_k t^k y^{(k)}$, then its essential spectrum is shown to be

$$\sigma_e(N) = \{b_0 + \sum_{k=1}^{n} b_k \prod_{j=0}^{k-1}(z-(j+\tfrac{1}{2})) \mid \operatorname{Re} z = 0\};$$

see Goldberg [1966: VI.7 and VI.8]. For one-term expressions $N_0y = i^{\zeta}t^{\zeta}y^{(\zeta)}$ this gives pictures for the essential spectrum, which are only qualitative (see Fig. 1).

For expressions M_0 given as in (2) satisfying (3), (4), and (5) with $\alpha_0 = \zeta_0$ and perturbations M given as in (6) with coefficients satisfying (7), similar methods of proof as for Theorem 1 show the following result:

If $\lambda \in \mathbf{C}$ with $|\lambda| < |a_0| 2^{-\zeta_0} \prod_{l=0}^{\zeta_0-1}(2l+1)$, then $\lambda \notin \sigma_e(M_0+M)$.

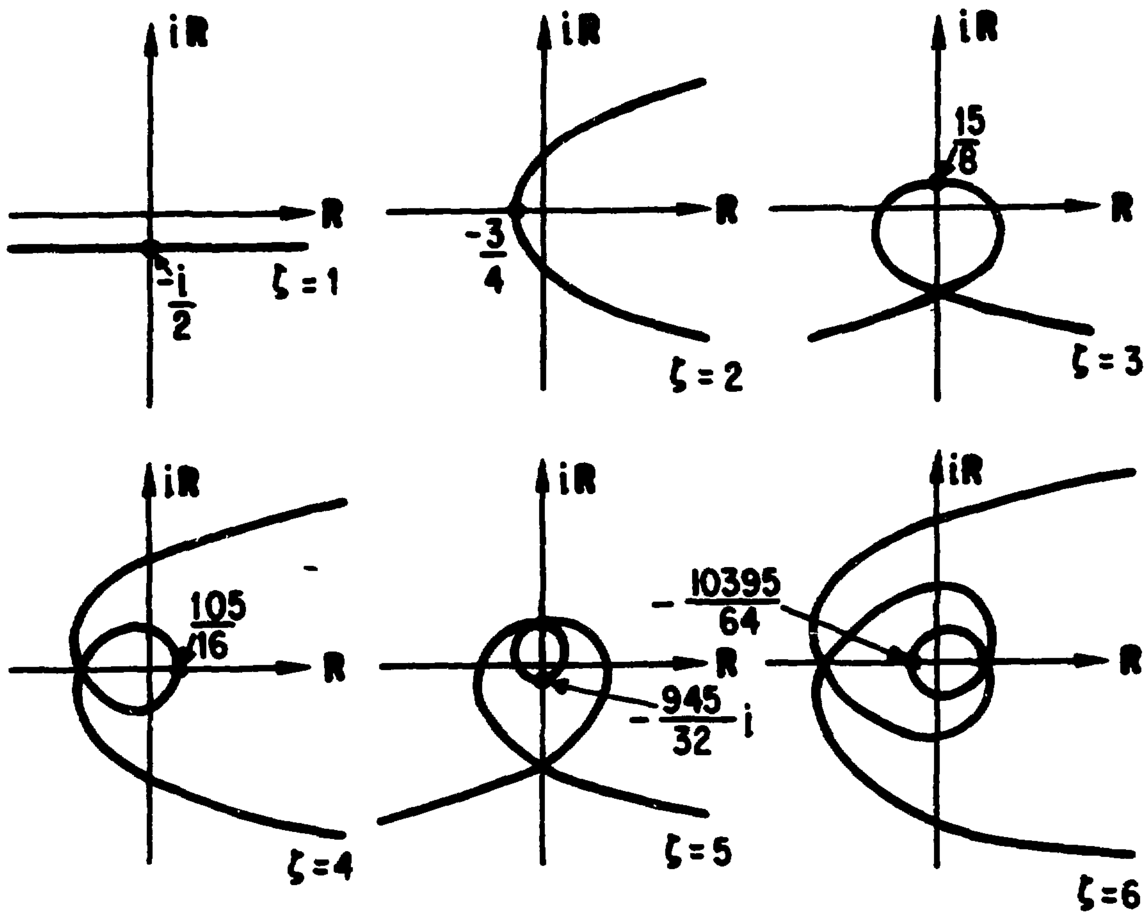

Fig. 1. Essential Spectrum for One-term Expressions
$N_0 y = i^{\zeta} t^{\zeta} y^{(\zeta)}$

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DIRAC SYSTEMS WITH OSCILLATING POTENTIALS AND ABSOLUTELY CONTINUOUS SPECTRA

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Abstract

A Dirac system $y' = [\lambda A(x) + P(x)]y$ is considered on an interval $[a,b)$ of regular points, the equation being singular at $x = b$. Theorems that locate the continuous spectrum are proved, and sufficient conditions that the continuous spectrum be absolutely continuous are given. Then externsions are given for two singular endpoint problems.

1. Introduction

A *Dirac system* is a first-order system of the type

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} p(x) & \lambda\alpha_2(x)+p_2(x) \\ -\lambda\alpha_1(x)-p_1(x) & -p(x) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad a \le x < b \le \infty, \tag{1.1}$$

where the real-valued coefficients p, α_k, and p_k are locally integrable over $[a,b)$, $\alpha_k(x) > 0$, and λ is a complex parameter. Initially, we regard (1.1) as regular at $x = a$ and singular at $x = b$, but see Section 4 for extensions of our results to two singular endpoint problems.

It is well known that the Weyl limit point-limit circle classification holds for systems (1.1) (see Kogan and Rofe-Beketov [1974] and Levitan and Sargsjan [1975]); i.e., the dimension of the space of solutions of (1.1) belonging to $L^2_a[a,b) = \{\mathbf{y} \mid \int_a^b (\alpha_1|y_1|^2+\alpha_2|y_2|^2)dx < \infty,\ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\}$ is either exactly 1 for all nonreal λ or exactly 2 for all nonreal λ. In this paper we will consider only the limit point case (dimension = 1).

For systems of limit point type, we affiliate an operator on $L^2_a[a,b)$ with (1.1) by introducing a boundary condition $B(\mathbf{y}) = \sin\beta y_1(a) + \cos\beta y_2(a) = 0$ and then letting $T: D_{a,\beta} \to L^2_a[a,b)$ be the operator defined by

$$T(\mathbf{y}) = \begin{bmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{bmatrix} \{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{y}' - \begin{bmatrix} p_1 & p \\ p & p_2 \end{bmatrix} \mathbf{y} \}; \tag{1.2}$$

the domain $D_{a,\beta}$ consists of all locally absolutely continuous $\mathbf{y} \in L^2_a[a,b)$ with $B(\mathbf{y}) = 0$ and $T(\mathbf{y}) \in L^2_a[a,b)$. Then T is selfadjoint and thus has a real spectrum $\sigma(T)$ (see Levitan and Sargsjan [1975]).

One of the central problems in differential operator theory is to determine the spectrum and, in particular, to determine where it is discrete and where it is continuous. The present paper takes up this question, together with the problem of finding sufficient conditions that the continuous spectrum be ***absolutely continuous*** (defined below). The absolutely continuous spectrum is important

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for its applications to scattering theory.

One of our objectives will be to extend the work of Titchmarsh [1961 and 1962], Erdelyi [1963], and Weidmann [1971 and 1982] on Dirac systems with Coulomb potential,

$$y' = \begin{bmatrix} -k/x & \lambda+V(x)+c \\ -\lambda-V(x)+c & -k/x \end{bmatrix} y, \; 0<x<\infty, \tag{1.3}$$

where k and c are constants and $V(x)$ is locally integrable on $(0,\infty)$. If $V(x) = Z/x$, Z constant, then (1.3) is the radial wave equation in relativistic quantum mechanics for a particle in a field of potential Z/x. In Titchmarsh [1961 and 1962] and Erdelyi [1963], the spectrum of the two singular endpoint problem (1.3) is considered under assumptions that $V(x) - (\gamma/x)$ be absolutely integrable toward $x = 0$, for some constant γ, and that $V(x)$ should be, in a certain technical sense, either "small" at $x = \infty$ or "large" at $x = \infty$; included in the technical assumptions are $V(x)\to 0$ or $V(x)\to\infty$, respectively, as $x\to\infty$. In the small potential case, the spectrum is continuous in $(-\infty,-c]\cup[c,\infty)$ and discrete in $(-c,c)$; in the large potential case the spectrum continuously covers the whole real line. It is implicit in Titchmarsh and Erdelyi that the continuous spectrum is absolutely continuous, in fact continuously differentiable. Weidmann [1971 and 1982] discusses various subjects concerning oscillation properties and the spectrum of (1.1) for $b = \infty$. In the case where $\alpha_1(x) = \alpha_2(x) = 1$, the potential term $P = \begin{bmatrix} p & p_2 \\ -p_1 & -p \end{bmatrix}$ is broken up into long- and short-range terms $P = P_1+P_2$, where P_1 is of bounded variation, $P_1\to\begin{bmatrix} 0 & \mu_+ \\ -\mu_- & 0 \end{bmatrix}$ $(x\to\infty, \; , \; <\mu_+)$, and $P_2\in L^1(x_0,\infty)$, $x_0>a$. Under these assumptions, the spectrum is absolutely continuous in $(-\infty,\mu_-]\cup[\mu_+,\infty)$.

In the sections that follow, we give generalizations of the results of Titchmarsh and Weidmann by applying a general asymptotic technique and a change of variable developed by Hinton and Shaw [1984]. We allow weight functions, on the one hand, and we permit a decomposition of the potential into long-range, short-range, and oscillatory parts. Specific hypotheses and statements are given in the next section. In Section 3 we outline proofs, and we close the paper with some remarks in Section 4.

2. Statements of Results

The ***resolvent set*** of the operator T in (1.2) consists of all complex λ for which $(T-\lambda I)^{-1}$ is a bounded operator from $L^2_a[a,b)$ into itself [Levitan and Sargsjan, 1975; Hinton and Shaw, 1982]. The ***spectrum*** of T is the complement of the resolvent set.

Let $\vartheta(x,\lambda)$ and $\varphi(x,\lambda)$ be the unique solutions of (1.1) which satisfy for all complex λ the initial conditions

$$[\vartheta(a),\varphi(a)] = \begin{bmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{bmatrix}. \tag{2.1}$$

Then the limit (in the limit point case)

$$m(\lambda) = -\lim_{x\to b} \frac{\vartheta_1(x,\lambda)}{\varphi_1(x,\lambda)} \tag{2.2}$$

exists and defines an analytic function for $\mathrm{Im}(\lambda) \neq 0$ [Levitan and Sargsjan, 1975; Hinton and Shaw, 1981]; $m(\lambda)$ is known as the Titchmarsh-Weyl m-coefficient. The spectral function for T is a nondecreasing and right-continuous function $\rho(\lambda)$ uniquely determined by $\rho(0) = 0$ [Levitan and Sargsjan, 1975; Hinton and Shaw, 1982]. One has the Titchmarsh-Kodaira formula [Hinton and Shaw, 1982]

$$\rho(\mu_2)-\rho(\mu_1) = \lim_{\varepsilon\to 0+} \pi^{-1}\int_{\mu_1}^{\mu_2} \operatorname{Im} m(\mu+i\varepsilon)d\mu \tag{2.3}$$

at points of continuity μ_1 and μ_2 of ρ. The spectrum of T consists of the points of increase of ρ [Levitan and Sargsjan, 1975]: jump discontinuities occur at eigenvalues (both isolated and nonisolated), the continuous spectrum consists of points where ρ is continuous and nonconstant, and an open interval belongs to the absolutely continuous (resp., C^1) spectrum if and only if ρ is absolutely continuous (resp., continuously differentiable) there

We now state the two theorems to be discussed here. More general results are possible; see Hinton and Shaw [1984a]. The following ones illustrate the technique of mixing asymptotics, change of variable, and the Titchmarsh-Weyl coefficient to describe the spectrum of (1.1).

THEOREM 1. *In (1.1), let*

$b = \infty$, $a_1(x) = a_2(x) = 1$,

$p(x) = \Delta_1(x)+\Delta_3(x)$, $p_1(x) = p_{11}(x)+p_{12}(x)+p_{13}(x)$,

$p_2(x) = p_{21}(x)+p_{22}(x)+p_{23}(x)$,

where Δ_1, p_{11}, *and* p_{21} *are differentiable and* $\Delta_1\to 0$, $p_{11}\to -\lambda_1$, $p_{21}\to -\lambda_2 (x\to\infty)$ *where* $\lambda_1<\lambda_2$; *let* p_{k1}', Δ_1', p_{k3}, $\Delta_3\in L^1[x_0,\infty)$, *and suppose the integrals* $P_{k2}(x) = \int_x p_{k2}(s)ds$ *exist (conditionally), and define functions that belong to* $L^1[x_0,\infty)$. *Suppose further that all the functions*

$\int_x^\infty |P_{k2}(s)p_{j1}'(s)|ds$,

$\int_x^\infty |P_{k2}(s)\Delta_1'(s)ds$,

$p_{k2}(x)P_{j2}(x), p_{m2}(x)\int_x^\infty |P_{k2}(s)p_{j1}'(s)|ds$,

and

$p_{m2}(x)\int_x^\infty |P_{k2}(s)\Delta_1'(s)|ds$

lie in $L^1[x_0,\infty)$. *Then (1.1) lies in the limit point case (and thus T is selfadjoint) and the spectrum of T is continuously differentiable in* $(-\infty,\lambda_1)\cup(\lambda_2,\infty)$ *and discrete in* (λ_1,λ_2).

The potentials $p_{k2}(x)$ may be viewed as "oscillatory" terms which must couple with the long-range terms $p_{k1}(x)+\lambda_k$ and $\Delta_1(x)$ in the way prescribed. An example of functions satisfying the hypotheses are $\Delta_1 = x^{-1} = p_{11}$, $p_{21} = -1+x^{-1}$, and $p_{k2}(x) = x^{\alpha}\sin(x^4)$, $\alpha < 2$. Thus we allow a wider range of potentials but draw the same conclusion (see the remarks in Section 4) as Weidmann [1971].

THEOREM 2. *In (1.1) let* $b = \infty$, $p(x) = 0$, $p_k(x) = p_{k1}(x)+p_{k2}(x)$, *where the* $p_{k1}(x)$ *are differentiable*, $p_{11}p_{21} > 0$, *and* $\int_a (p_{11}p_{21})^{1/2} = \infty$; *suppose that either* $\int_a a_1(p_{21}/p_{11})^{1/2} = \infty$ *or* $\int_a a_2(p_{11}/p_{21})^{1/2} = \infty$ *and that the functions* $\eta = (p_{21}/p_{11})^{1/4}$ *and* $\Delta(x) = \eta'/\eta(p_{11}/p_{21})^{1/2}$ *satisfy* $\Delta = \Delta_1+\Delta_3$ *where* $\Delta_1\to 0$ *and* Δ_1', $\Delta_3\in L^1[a,\infty)$. *Suppose that* a_k/p_{k1} *and* p_{k2}/p_{k1} *can be decomposed into long-range, short-range, and oscillating potentials as in Theorem 1. Then (1.1) lies in the limit point case (and thus T is selfadjoint) and the spectrum of T is continuously differentiable in* $(-\infty,\infty)$.

We outline the proofs of the theorems in Section 3. More details and further generalizations may be found in Hinton and Shaw [1984a].

We shall require a characterization of the spectrum in terms of $m(\lambda)$. Hinton and Shaw [1982 and 1984b] extended the classification theorem of Chaudhuri and Everitt [1968] to the setting of Hamiltonian systems, which include (1.1) as a special case. We are concerned here mainly with the fact that $m(\lambda)$ continues analytically into each interval of the resolvent set, and that the isolated eigenvalues of T are (isolated) poles of $m(\lambda)$; see Hinton and Shaw [1982].

Owing to the fact that $\overline{m(\lambda)} = m(\bar{\lambda})$ [Hinton and Shaw, 1982] for nonreal λ, it follows that $m(\lambda)$ continues analytically to a real interval if and only if it has a real limit there as $\mathrm{Im}(\lambda) \to 0$. In particular, if $m(\lambda)$ is real and meromorphic on an interval, then the spectrum is discrete there, with the eigenvalues coinciding with poles of $m(\lambda)$.

To illustrate the connection of $m(\lambda)$ with the theorems stated above, we give two simple examples. First, consider the constant coefficient equation

$$\mathbf{y}' = \begin{bmatrix} 0 & \lambda+c \\ -\lambda+c & 0 \end{bmatrix} \mathbf{y}, \quad 0 \le x < \infty, \tag{2.4}$$

which is a special case of (1.3) and which is equivalent to the scalar equation $y'' + (\lambda^2 - c^2) y = 0$. Computing ϑ and ϕ of (2.1) and evaluating the limit (2.2) for $\mathrm{Im}(\lambda) > 0$, we find for this example that $m(\lambda) = i[(\lambda-c)/(\lambda+c)]^{1/2}$. This expression is real if and only if $-c \le \lambda \le c$, so the spectrum consists of $(-\infty, -c] \cup [c, \infty)$. This is an instance of Theorem 1 with $p_2(x) = c$, $p_1(x) = -c$, $p(x) = 0$. A nice example of Theorem 2 is the one in which $\alpha_1(x) = \alpha_2(x) = \alpha(x)$, $p(x) = 0$, and $p_1(x) = p_2(x) = b(x)$, for the solutions of

$$\mathbf{y}' = \begin{bmatrix} 0 & \lambda\alpha(x)+b(x) \\ -\lambda\alpha(x)-b(x) & 0 \end{bmatrix} \mathbf{y}$$

are available explicitly and have the form $\mathbf{y} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \exp(\pm i \int_0^x (\lambda\alpha + b) ds)$. The hypotheses of Theorem 2 are satisfied if we take $p_{k2} = 0$, $\alpha_k = 1$, and assume that, say, $b(x) \sim x^{-2}$, $x \to \infty$. It is then an easy matter to compute $m(\lambda) = i$ for $\mathrm{Im}(\lambda) > 0$. Thus $m(\lambda) = -i$ in the lower half plane, and this means that $m(\lambda)$ is analytic nowhere on the real line; i.e., the spectrum is continuously differentiable in $(-\infty, \infty)$ with $\rho(\lambda) = \lambda/\pi$ (see Formula 2.3).

3. Outlines of Proofs

The systems of Theorems 1 and 2 can be put into common form by a change of variable in Theorem 2. Let $f(x)$ and $\eta_1(x)$ be differentiable functions on $[a, b)$, with $f(x)$ monotone increasing and unbounded. Let $\eta_2(x) = 1/\eta_1(x)$, and introduce new independent and dependent variables $t = f(x)$ and

$$\mathbf{y}(x) = \begin{bmatrix} \eta_1(x) & 0 \\ 0 & \eta_2(x) \end{bmatrix} \mathbf{z}(t), \quad t_0 = f(a) \le t < \infty.$$

Then (1.1) becomes

$$\dot{\mathbf{z}}(t) = \frac{1}{f'(x)} \begin{bmatrix} (-\eta_1'/\eta_1) & (\eta_2/\eta_1)(\lambda\alpha_2 + p_2) \\ (-\eta_1/\eta_2)(\lambda\alpha_1 + \rho_1) & (-\eta_2'/\eta_2) \end{bmatrix} \mathbf{z}(t), \tag{3.1}$$

where dots represent differentiation with respect to t, and where the expressions in the matrix are evaluated at $x = f^{-1}(t)$. In Theorem 2, recall that we have $p_1 = p_{11} + p_{12}$, $p_2 = p_{21} + p_{22}$. We now make the special choices

$$f(x) = \int_a^x [p_{11}(s)p_{21}(s)]^{1/2}ds, \ \eta_1(x) = [p_{21}(x)/p_{11}(x)]^{1/4}.$$

Then (3.1) becomes

$$z'(t) = \begin{bmatrix} -\eta_1'/(\eta_1 f') & 1+\lambda(\alpha_2/p_{21})+(p_{22}/p_{21}) \\ -1-\lambda(\alpha_1/p_{11})-(p_{12}/p_{11}) & -\eta_2'/(\eta_2 f') \end{bmatrix} z(t). \qquad (3.2)$$

Note that the diagonal members of (3.2) are negatives of each other because $\eta_2 = 1/\eta_1$. Both (3.2), under the assumptions of Theorem 2, and the system of Theorem 1 (with $x = t$) have the general form

$$z'(t) = \begin{bmatrix} \Delta(t) & a(\lambda)+r_2(t,\lambda) \\ -b(\lambda)+r_1(t,\lambda) & -\Delta(t) \end{bmatrix} z(t), \qquad (3.3)$$

where $a(\lambda)$ and $b(\lambda)$ are entire functions of λ which are real for real λ, $\Delta(t) = \Delta_1(t)+\Delta_3(t)$ decomposes into long- and short-range terms as in the hypothesis on $p(x)$ in Theorem 1, and where $r_k(t,\lambda) = r_{k1}(t)+r_{k2}(t)+r_{k3}(t)+\lambda[s_{k1}(t)+s_{k2}(t)+s_{k3}(t)]$ with the r_{k1} and s_{k1} long-range terms, r_{k3} and s_{k3} short-range terms, and r_{k2} and s_{k2} oscillating terms.

Under hypotheses that include Theorems 1 and 2, we studied the asymptotic form of the solutions of (3.3). We state here the main conclusions as they apply to the present paper, and refer to the earlier work [Hinton and Shaw, 1984a] for technical assumptions and proofs. Let us define

$$\mu_0'(t,\lambda) = [(a(\lambda)+r_{21}(t)+\lambda s_{21}(t))(b(\lambda)-r_{11}(t)-\lambda s_{11}(t))-\Delta_1^2(t)]^{1/2},$$

and let U be a compact subset of the complex λ-plane such that $a(\lambda)b(\lambda) \neq 0$ for $\lambda \in U$. Thus for Theorem 1 we have $a(\lambda) = \lambda-\lambda_2$, $b(\lambda) = \lambda-\lambda_1$, and U can be any compact subset of the plane that omits λ_1 and λ_2. In Theorem 2, $a(\lambda) = b(\lambda) = 1$ from (3.2), so U is any compact subset of the plane. We prove [Hinton and Shaw, 1984a] that for each solution z of (3.3) there exists a function $A(\lambda)$, analytic on the interior of U and continuous on U, such that if $\int \text{Im}\,\mu_0 = \infty$,

$$z(t) = A(\lambda)\exp(-i\int_{t_0}^t \mu_0 ds)\begin{bmatrix} a(\lambda)+r_{21}(t)+\lambda s_{21}(t)+O(1) \\ -\mu_0(t,\lambda)-\Delta_1(t)+O(1) \end{bmatrix}, \quad t\to\infty \qquad (3.4)$$

for $\lambda \in U$. In the cases considered here we have $\text{Im}\mu_0(t,\lambda) > 0$ if $\text{Im}(\lambda) > 0$ and t is sufficiently large. Moreover, $\mu_0(t,\lambda) \sim [a(\lambda)b(\lambda)]^{1/2}$; hence the exponential term dominates the asymptotic formula (3.4) for $\text{Im}(\lambda) > 0$.

Let z_θ and z_φ denote the transformed solutions of (3.3) corresponding to the solutions $\vartheta(x,\lambda)$ and $\varphi(x,\lambda)$ with initial values (2.1). Let $A_\theta(\lambda)$ and $A_\varphi(\lambda)$ be the $A(\lambda)$ coefficients appearing in (3.4) for z_θ and z_φ, respectively. Since our basic change of variables has first component $y_1(x) = \eta_1(x)z_1(t)$, then (2.2) and (3.4) imply

$$m(\lambda) = -\lim_{x\to b} \frac{\vartheta_1(x,\lambda)}{\varphi_1(x,\lambda)} = -\lim_{t\to\infty} \frac{(z_\theta)_1(t,\lambda)}{(z_\varphi)_1(t,\lambda)} = -\frac{A_\theta(\lambda)}{A_\varphi(\lambda)}, \quad \text{Im}(\lambda) > 0. \qquad (3.5)$$

We now have to consider the behavior of the $A(\lambda)$ coefficient when λ and $\mu_0(t,\lambda)$ are real, this case not being covered by (3.4). We proved [Hinton and Shaw, 1984a] that if $\lambda \in U$ is real and $\int \text{Im}\mu_0 < \infty$, then

$$\text{Re}A_\theta(\lambda)\text{Im}A_\varphi(\lambda)-\text{Re}A_\varphi(\lambda)\text{Im}A_\theta(\lambda) = [4a(\lambda)(a(\lambda)b(\lambda))^{1/2}]^{-1}. \qquad (3.6)$$

Note that for the values of μ_0 determined by Theorems 1 and 2, $\mu_0(t,\lambda)$ is either purely real or purely imaginary for t sufficiently large and for real $\lambda \in U$.

We are now ready to prove Theorem 1. Let $\lambda\in(\lambda_1,\lambda_2)$, and pick U to be a compact set omitting λ_1 and λ_2 but containing λ. Then $\mu_0(t,\lambda)$ is purely imaginary, and it follows (see Hinton and Shaw [1984]) that the solutions z of (3.3) are real. In particular, $A_\theta(\lambda)$ and $A_\varphi(\lambda)$ are real for all $\lambda\in(\lambda_1,\lambda_2)$. Hence $m(\lambda) = -A_\theta(\lambda)/A_\varphi(\lambda)$ continues analytically through (λ_1,λ_2) save for zeros of $A_\varphi(\lambda)$ which coincide with its poles. That is, $m(\lambda)$ is meromorphic in (λ_1,λ_2), and so the spectrum of T is discrete there. If instead $\lambda > \lambda_1$ or $\lambda < \lambda_2$, then $\mu_0(t,\lambda)$ is real, and we are under case (3.6). Now by (3.6)

$$\operatorname{Im} m(\lambda) = -\operatorname{Im}\frac{A_\theta(\lambda)}{A_\varphi(\lambda)} = \frac{[4a(ab)^{1/2}(\lambda)]^{-1}}{|A_\varphi(\lambda)|^2}, \tag{3.7}$$

which is also finite and nonzero by the right side of (3.6). Since $m(\lambda)$ has a nonreal value on the real line in $(-\infty,\lambda_1)$ and (λ_2,∞), then it cannot be analytic there. Since (3.7) implies $\lim_{\varepsilon\to0+}\varepsilon m(\lambda+i\varepsilon) = 0$, then the intervals $(-\infty,\lambda_1)$ and (λ_2,∞) lie in the continuous spectrum; see Hinton and Shaw [1982]. Continuous differentiability of the spectral function is a consequence of the Titchmarsh-Kodaira formula (2.3). This proves Theorem 1.

The proof of Theorem 2 is identical to the second part of the above proof because $\mu_0(t,\lambda)$ is real for large t and real λ.

4. Remarks and Extensions

1. The hypotheses regarding $\int_a^\infty \alpha_1(p_{21}/p_{11})^{1/2}$ or $\int_a^\infty \alpha_2(p_{11}/p_{21})^{1/2}$ are present in Theorem 2 to ensure that the limit point case prevails. In the limit circle case the spectrum is discrete.

2. Turning to two singular endpoint problems, we indicate briefly here how Theorem 1 may be extended to the case in which the endpoint $x = a$ is singular. Analogous remarks apply to Theorem 2. Thus, suppose we have a system

$$y'(x) = \begin{bmatrix} p(x) & \lambda+p_2(x) \\ -\lambda-p_1(x) & -p(x)\end{bmatrix} y(x),\ a < x < \infty, \tag{4.1}$$

which is singular at $x = a$. If the end $x = a$ is limit point, define the operator T just as in (1.2) except that the condition $B(y) = 0$ is dropped. If $x = a$ is limit circle, we take T to be a selfadjoint extension of the minimal operator associated with (4.1). Denote by $m_a(\lambda)$ the Titchmarsh-Weyl coefficient for (4.1) at $x = a$; i.e.,

$$m_a(\lambda) = -\lim_{x\to a}\frac{\vartheta_1(x,\lambda)}{\varphi_1(x,\lambda)}, \tag{4.2}$$

where ϑ and φ are the solutions of (4.1) taking values (2.1) at some basepoint a_0, $a < a_0 < \infty$, instead of at $x = a$. The limit (4.2) exists and defines an analytic function for $\operatorname{Im}(\lambda) \neq 0$ in the limit point case. If the limit-circle case prevails, the limit in (4.2) must be taken through a subsequence, and so for simplicity we will consider only the limit point case at $x = a$. It is beneficial to consider "half-line operators" T_a and T_∞ associated with (4.1), as restricted to the intervals (a,a_0) and (a_0,∞), respectively. Here T_∞ is defined just as T in Theorem 1 except that a_0 replaces a, and T_a is defined similarly. Then $m(\lambda)$ describes the spectrum of T_∞, and $m_a(\lambda)$ describes that of T_a, in the sense of the proof of Theorem 1.

The whole line operator T has Titchmarsh-Weyl coefficient a 2x2 matrix function

$$M(\lambda) = [m_a(\lambda)-m(\lambda)]^{-1}\begin{bmatrix} 1 & (m_a(\lambda)+m(\lambda))/2 \\ (m_a(\lambda)+m(\lambda))/2 & m_a(\lambda)m(\lambda) \end{bmatrix}; \tag{4.3}$$

see Hinton and Shaw [1984a and 1984b] and Coddington and Levinson [1955]. The spectral function $\tau(\lambda)$ is a 2×2 Hermitian, nondecreasing (in the positive definite sense), right continuous function which bears the same relationship to $M(\lambda)$ as $\rho(\lambda)$ does to $m(\lambda)$ from (2.3). When $M(\lambda)$ has a nonreal and finite limit at the real axis, the spectrum of T is continuous there (see Hinton and Shaw [1984b]) and therefore continuously differentiable by the matrix analogy of (2.3).

It is not difficult to prove (see Hinton and Shaw [1984a]) that if either $m_a(\lambda)$ or $m(\lambda)$ has a nonreal and finite limit on some interval J, then so does $M(\lambda)$, regardless of the behavior of the other m-coefficient. In essence, this is because $\text{Im}\,m(\lambda) > 0$ and $\text{Im}\,m_a(\lambda) < 0$ for $\text{Im}(\lambda) > 0$, and if one of these inequalities preserves its strictness as $\text{Im}(\lambda)\to 0$, then $M(\lambda)$ can be shown to have a nonreal limit. Therefore if λ belongs to the continuous spectrum of either T_∞ or T_a, then it lies in the continuous spectrum of T. Consequently the whole line operator T arising from Theorem 1 has continuously differentiable spectrum in $(-\infty,\lambda_1)\cup(\lambda_2,\infty)$, quite independently of the behavior at the singular end $x = a$. To claim discreteness of the spectrum in (λ_1,λ_2), however, we would need detailed knowledge of the singular behavior at $x = a$; see Theorem 3 in Hinton and Shaw [1984a].

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