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SNAKES, ROTATORS, SERPENTS AND THE OCTAHEDRAL GROUP*

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Summary

Specific configurations of horizontal and vertical bending magnets are given that, when acting on the spin polarization vector of a particle beam, generate a group of 24 operators isomorphic to the group of rotational symmetries of a cube, known as the octahedral group. Some of these configurations have the feature of converting transversely polarized beams to longitudinally polarized beams (or vice versa) at the midpoint of the configuration for, in principle, all beam energies. Since the first order optical transfer matrix for each half of these configurations is nearly that of a drift region, the external geometry remains unchanged and midpoint dispersion is not introduced.

Changing field strengths and/or polarities allows a configuration to serve as either a Snake(1st or 2nd kind) or a Rotator, where in both cases the spin polarization is longitudinal at the midpoint.

In this conceptualization, emphasis has been placed on electron beams and, indeed, for these beams some practical applications can be envisioned. However, due to the relatively high integrated field strengths required, application of these concepts to proton beams may be more promising.

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Introduction

There are several known types of Siberian Snakes[1,2] for manipulation of an electron beam polarization vector.

The Snake of the 1st kind (first proposed by the Soviet authors Y. A. Derbenev and A. M. Kondratenko of Novosibirsk, USSR, hence the name Siberian Snake) has the elegance of not introducing dispersion nor affecting the beam trajectory external to the system. It has a wide range of operating energies but, unfortunately, is not useful by itself in converting transverse polarization into longitudinal polarization. This Snake rotates the polarization vector about the longitudinal axis of the beam (a rotation of 90° for each one-half of a snake).

The Snake of the 2nd kind does rotate transverse polarization into longitudinal polarization, making it more interesting to those doing polarized beam experiments. However, known versions of this snake do introduce midpoint dispersion and may have a limited range of operating energies and/or variable geometry. The Snake of the 2nd kind rotates the polarization vector about the transverse axis (this axis is horizontal in Ref. 1 and, again, the rotation is 90° for one-half of a snake).

Two other novel versions of the Siberian Snake have been proposed.[3] They are the Left and Right Pointed Snakes which represent rotations of 180° about axes lying in the plane containing the horizontal axis and the axis of the beam direction. The direction cosines of this rotation axis have a magnitude of $\pm\frac{\sqrt{2}}{2}$ with respect to the coordinate axes mentioned.

In general then, a Siberian Snake has been defined in Reference 1. to consist of a sequence of magnets that rotate the spin vector by 180° about an arbitrary axis lying in a plane containing the horizontal axis and the axis of the beam direction. This definition requires that a Snake always invert the vertical component of the spin polarization vector.

The importance of Snakes in circular machin[illegible] [illegible]een extensively described but we will refer only to a review ar[illegible] by Montague.[4] One of the uses described in this review and attributed to Derbenev and Kondratenko, applies to electron storage rings. It is explained that two snakes placed in a ring at diametrically opposing positions, one of the 1^{st} kind and the other of the 2^{nd} kind, can be used to achieve a spin tune of 0.5, with the vertical component of the spin vector parallel to the field in one half of the machine and antiparallel in the other half, independent of energy. Such a procedure, according to Montague, reduces "substantially the effects of large energy spread and imperfection resonances at high energies, permitting polarized beams to be obtained up to perhaps 100 GeV." In his review Montague develops an elegant method that uses spinor algebra and unitary transformations for describing spin transformations and calculating spin tune. This method is used in proving that the configuration described above has a spin tune of 0.5 which means, he points out, that "any arbitrary spin vector closes upon itself after two revolutions," around the ring.

There are also systems called Rotators (see Ref. 2). This name has been used to classify systems of magnets which have the property of rotating the vertical component of the polarization vector into the longitudinal direction at the midpoint (Interaction Point) and then restoring the original direction. These systems are useful for polarized beam physics. Montague has shown that, in general, if such a system is constrained to be fully antisymmetric about the midpoint, then the overall spin transformation is the identity, independent of beam energy or the details of the field strengths. This is an important point to which we will again refer. Most Rotators operate at only a specified beam energy or with changing geometry (see Refs. 2 and 4).

One of the configurations of magnets which we will describe is identical in appearance to that of the Snake of the 1^{st} kind. Either Snakes or Rotators can be generated with this configuration. In addition, its function can be easily changed by adjusting field strengths or polarities. Rather than having several names for a single system depending upon which purpose it serves we have chosen to simply use the name, Serpent, when referring to this configuration. We will point out when this configuration is serving as a Snake or Rotator.

Another half Snake or half Rotator we have named the half Up-Down Snake. It shares many of the properties of the Serpent but is sufficiently different that it requires another name. Now, we will describe the Serpent, and return to a description of the Up-Down Snake later.

Figure 1 represents the configuration of horizontal and vertical bend magnets that will generate one-half of a Serpent. This configuration is identical to that of one-half of the Snake of the 1st kind, except that for the Serpent the magnitude of the fields are to be doubled. It retains the nice features of the 1st kind in that it has an extremely wide range of operating energies and acceptance, does not introduce dispersion(at end of half of system), and the beam entering the snake (head) is collinear with the beam exiting (tail) so it can replace a drift region.

For the remainder of this discussion we limit ourselves to only those configurations of magnets that do not introduce dispersion or change external geometry as a function of beam energy.

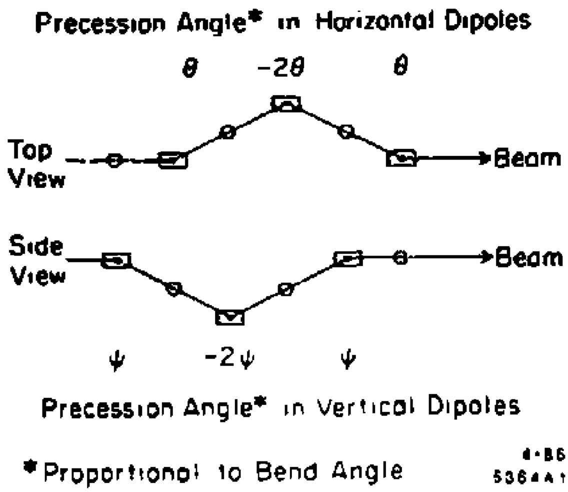

Figure 1.

Before proceeding further, we will comment on notation, write some useful relationships, and define a coordinate system.

One half of a Serpent rotates the spin polarization vector about an axis that can be graphically represented by a vector connecting opposite corners of a cube (the rotation angle is 120° for one half of a Serpent). It may be seen immediately that such an operation can perform an even or cyclic permutation of the coordinates of the polarization vector and thus permute transverse and longitudinal coordinates. Four such non-orthogonal axes are chosen for our representation. They will be defined later as vectors having direction cosines all equal in magnitude but with varying signs. We define directions for these axes and the magnetic fields such that for an electron, a positive rotation about these axes obeys the right hand rule. The reader will note that in general such a rotation applied three times results in an identity. If represented operationally, $A^3(120°) = I$, or $A^2(120°) = A^{-1}(120°) = A(-120°)$. We will choose to write $A^{-1}(120°)$ instead of $A^2(120°)$.

The precession angle of the spin polarization vector is given by the relationship

$$\psi_p = \phi\gamma a_{e-} \quad . \tag{1}$$

In which ϕ is the bending angle of the beam in the transverse magnetic field. ψ_p is the precession angle for the polarization vector of an on-momentum particle about the direction of the field in the coordinate system following the beam (orbit frame). γ is the Lorentz factor, and a_{e^-} is a measure of the electron's anomalous magnetic moment (see Ref. 4. for discussion of the Thomas-BMT equation and detailed references).

$$a_{e^-} = \left(\frac{g-2}{2}\right) = 1.159652 \times 10^{-3} \quad . \tag{2}$$

It is also useful to express ψ_p in terms of the $\int B d\ell$ of the applied field since the energy dependence then factors out:

$$\psi_p = 0.680 \int B d\ell \qquad \text{(radians, T-m)} \quad . \tag{3}$$

Note that an integrated field of 2.31 T-m will precess the spin polarization vector by $\pi/2$ or 90° independent of beam energy. This fact will be referred to later.

At the beam energies of the Stanford Linear Collider ($\sim$ 50 GeV), and higher energies, the bending angle of the beam is small ($< 1^\circ$) in traversing such a field. Hence, the spin precession angle when measured in laboratory coordinates or beam coordinates is nearly the same.

Figure 2 shows the coordinate system following the beam and the definitions of certain vectors.

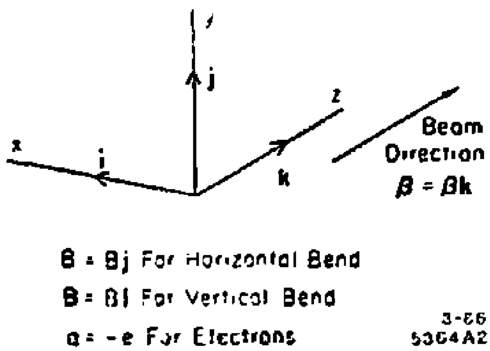

Figure 2.

In this coordinate system the spin polarization vector is expressed in terms of its <u>initial</u> coordinates and it is assumed that its magnitude has been normalized to a value between 0.0 and 1.0. This vector is expressed as

$$\mathbf{p} = H\mathbf{i} + V\mathbf{j} + S\mathbf{k} \quad ,$$

or alternatively as the column vector

$$\mathbf{p} = \begin{pmatrix} H \\ V \\ S \end{pmatrix} .$$

For an electron traversing a horizontal bend magnet, the spin polarization vector will precess through an angle θ, where positive θ is defined by the right hand rule representing rotation about the y axis. The resultant polarization vector will be given by

$$\mathbf{p}' = \mathbf{H}(+\theta)\mathbf{p}$$

where $\mathbf{H}(+\theta)$ is an orthogonal matrix operator defined by

$$\mathbf{p}' = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} H \\ V \\ S \end{pmatrix} . \qquad \text{(Horizontal Bend)} \qquad (4)$$

For a vertical bend magnet the precession angle ψ, again is defined as positive by the right hand rule about the x axis, and

$$\mathbf{p}' = \mathbf{V}(+\psi)\mathbf{p}$$

where the orthogonal matrix $\mathbf{V}(+\psi)$ is defined by

$$\mathbf{p}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} H \\ V \\ S \end{pmatrix} . \qquad \text{(Vertical Bend)} \qquad (5)$$

With these definitions we now represent the effect on the spin polarization vector of the combined vertical and horizontal bend magnets that were shown in

Figure 1 by

$$\mathbf{p}' = \mathbf{H}(+\theta)\mathbf{V}(+\psi)\mathbf{H}(-2\theta)\mathbf{V}(-2\psi)\mathbf{H}(+\theta)\mathbf{V}(+\psi)\mathbf{p} \quad . \tag{6}$$

Note that the beam sees a vertical bending magnet first in this configuration, so the first matrix operating on $\mathbf{p}$ is $\mathbf{V}(+\psi)$, therefore, one reads the matrices from right to left to reconstruct a configuration.

We will represent this configuration by the notation, $\mathbf{V}(+\psi, +\theta)$, which indicates that the first magnet seen by the beam is a vertical bending magnet with a positive precession angle, and the second magnet (horizontal) also has positive precession. Other configurations will follow the same patterns, so $\mathbf{H}(+\psi, -\theta)$, where the arguments are <u>not</u> transposed, would represent a horizontal magnet first with negative precession angle followed by a vertical magnet with positive precession angle.

We write Eq. (6) as

$$\mathbf{p}' = \mathbf{V}(+\psi, +\theta)\mathbf{p} \quad .$$

<u>Serpent</u>

If we now select field strengths such that $\psi = +90^o$ and $\theta = +90^o$ (recalling that the required integrated field is independent of beam energy having a value of 2.3 T-m for 90^o and double that value for 180^o) then

$$\mathbf{p}' = \mathbf{V}(+90, +90)\mathbf{p} \quad .$$

Or in the expanded form of the equation,

$$\mathbf{p}' = \mathbf{H}(+90)\mathbf{V}(+90)\mathbf{H}(-180)\mathbf{V}(-180)\mathbf{H}(+90)\mathbf{V}(+90)\mathbf{p} \quad .$$

Calculation using Eqs. (4) and (5) gives

$$\mathbf{p}' = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} H \\ V \\ S \end{pmatrix}$$

or

$$\mathbf{p}' = \begin{pmatrix} S \\ H \\ V \end{pmatrix} .$$

We see that the vertical component of the initial polarization vector has now been rotated into the longitudinal direction. This configuration performs a cyclic permutation of initial coordinates with no changes of sign. It can be represented by a positive rotation of 120° about the axis, $\mathbf{a}_1$, given by

$$\mathbf{a}_1 = 1/3\ (+\sqrt{3}\ \mathbf{i} + \sqrt{3}\ \mathbf{j} + \sqrt{3}\ \mathbf{k}) \quad .$$

We now define a matrix operator, $\mathbf{A}_1(+120)$ where

$$\mathbf{A}_1(+120) = \mathbf{V}(+90,+90) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

As noted earlier, $\mathbf{A}_1{}^3(+120) = \mathbf{I}$ and

$$\mathbf{A}_1{}^2(+120) = \mathbf{A}_1^{-1}(+120) = \mathbf{A}_1(-120) \quad .$$

Since the rotation angle will be understood to be 120° for this and three following operators, we shorten the notation further to

$$\mathbf{A}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We now select three other axes of rotation given by

$$\mathbf{a}_2 = 1/3\ (+\sqrt{3}\,\mathbf{i} - \sqrt{3}\,\mathbf{j} - \sqrt{3}\,\mathbf{k}) \quad ,$$

$$\mathbf{a}_3 = 1/3\ (-\sqrt{3}\,\mathbf{i} + \sqrt{3}\,\mathbf{j} - \sqrt{3}\,\mathbf{k}) \quad ,$$

and

$$\mathbf{a}_4 = 1/3\ (-\sqrt{3}\,\mathbf{i} - \sqrt{3}\,\mathbf{j} + \sqrt{3}\,\mathbf{k}) \quad .$$

These will be eigenvectors (rotation axes) for corresponding operators

$$\mathbf{A}_2,\ \mathbf{A}_3,\ \mathbf{A}_4 \quad ,$$

and their inverses

$$\mathbf{A}_2^{-1},\ \mathbf{A}_3^{-1},\ \mathbf{A}_4^{-1} \quad .$$

With these definitions we find the following correspondence between configurations and operators for half Serpents:

$$\mathbf{V}(+90,+90) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{A}_1 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} S \\ H \\ V \end{pmatrix}$$

$$\mathbf{V}(+90,-90) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{A}_2 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -S \\ -H \\ V \end{pmatrix}$$

$$\mathbf{V}(-90,+90) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{A}_3 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} S \\ -H \\ -V \end{pmatrix}$$

$$\mathbf{V}(-90,-90) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{A}_4 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -S \\ H \\ -V \end{pmatrix}$$

$$\mathbf{H}(+90,+90) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{A}_4^{-1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} V \\ -S \\ -H \end{pmatrix}$$

$$\mathbf{H}(+90,-90) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{A}_3^{-1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -V \\ -S \\ H \end{pmatrix}$$

$$\mathbf{H}(-90,+90) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{A}_2^{-1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -V \\ S \\ -H \end{pmatrix}$$

$$\mathbf{H}(-90,-90) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{A}_1^{-1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} V \\ S \\ H \end{pmatrix}$$

And, of course,

$$\mathbf{V}(0,0) = \mathbf{H}(0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} H \\ V \\ S \end{pmatrix}.$$

With one-half of a Serpent, there are eight interesting final states where the transverse polarization has been rotated to longitudinal polarization. They are all cyclic (even) permutations of coordinates with some changes of sign. The total number of such permutations and sign changes is 24 (3 even permutations and 8 possible assignments of sign). If both even and odd permutations are included the total is 48. Of that set, a subset of 24 would have a determinant of +1 and represent rotations. Both even and odd permutations can be generated by rotations. The subset of 24 operators having a determinant −1 would include a reflection of right-handed coordinates to left-handed coordinates. The nine operators we have found thus far do not form a group.

Reversing direction of the longitudinal polarization requires that the fields of at least half of the magnets reverse polarity. For electrons the total $\int B d\ell$ required is 18.4 T-m. This high value coupled with the need for low fields to limit synchrotron radiation will require long magnets and hence, large energy dependent beam excursions within the system. Spin depolarization effects may also be enhanced. Applications for electron beams are limited by these considerations. For protons the required integrated field is only slightly less but, shorter magnets can be used. The purpose here is to proceed to investigate other interesting properties of these configurations which, at least in principle, may have applications.

We now seek additional final states by combining two one-half Serpents, end to end. This gives us three new operators, C_1, C_2 and C_3 (see Table I) that can be made in a number of ways. They do not change transverse to longitudinal polarization. In fact, C_1 is the operation defining a Snake of the 1st kind, whereas the operation C_2 is that defining a Snake of the 2nd kind. These two operators and all operators corresponding to a snake will, by definition, invert the vertical component.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{C}_1 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ -V \\ S \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{C}_2 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} H \\ -V \\ -S \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{C}_3 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ V \\ -S \end{pmatrix}$$

Note that $\mathbf{C}_1^2 = \mathbf{C}_2^2 = \mathbf{C}_3^2 = \mathbf{I}$, and $\mathbf{C}_i\mathbf{C}_j = \mathbf{C}_j\mathbf{C}_i = \mathbf{C}_k$ for $i \neq j \neq k$.

The set of 4 operators $\{\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{I}\}$ forms a commutative group with respect to matrix multiplication.

The rotation angle is 180° for operators $\mathbf{C}_1$, $\mathbf{C}_2$, and $\mathbf{C}_3$ about eigenvectors

$$\mathbf{c}_1 = \pm\mathbf{k},$$

$$\mathbf{c}_2 = \pm\mathbf{i},$$

and

$$\mathbf{c}_3 = \pm\mathbf{j}$$

respectively.

TABLE I

Multiplication Table for
combinations of two half Serpents

	A_1	A_2	A_3	A_4	A_4^{-1}	A_3^{-1}	A_2^{-1}	A_1^{-1}
A_1^{-1}	I	C_3	C_1	C_2	A_2	A_4	A_3	A_1
A_2^{-1}	C_3	I	C_2	C_1	A_3	A_1	A_2	A_4
A_3^{-1}	C_1	C_2	I	C_3	A_1	A_3	A_4	A_2
A_4^{-1}	C_2	C_1	C_3	I	A_4	A_2	A_1	A_3
A_4	A_2^{-1}	A_3^{-1}	A_1^{-1}	A_4^{-1}	I	C_1	C_2	C_3
A_3	A_4^{-1}	A_1^{-1}	A_3^{-1}	A_2^{-1}	C_1	I	C_3	C_2
A_2	A_3^{-1}	A_2^{-1}	A_4^{-1}	A_1^{-1}	C_2	C_3	I	C_1
A_1	A_1^{-1}	A_4^{-1}	A_2^{-1}	A_3^{-1}	C_3	C_2	C_1	I

The operators in the top row represent the first half serpent as seen by the beam (or the right hand matrix operator). Those in the left column are for the second half serpent (or the left hand matrix operator). The identity operator has been omitted as a multiplier.

For example:

$$A_4A_1 = A_2^{-1}$$

and

$$A_1A_4 = A_3^{-1} \quad .$$

The set of 12 operators

$$\{A_1, A_2, A_3, A_4, A_4^{-1}, A_3^{-1}, A_2^{-1}, A_1^{-1}, C_1, C_2, C_3, I\}$$

forms a non-commutative group with respect to matrix multiplication. Therefore, combining three or more half Serpents will not generate any additional final states or new operators.

As mentioned, it is shown in TABLE I that either the Snake of the 1st kind or the Snake of the 2nd kind can be generated by the proper combination of half Serpents. There is an important difference, however, between these new combinations and those known earlier. This difference stems from the fact that half Serpents do not rotate about eigenvectors confined to the plane containing the unit vectors $\hat{i}$ and $\hat{k}$. If the first half serpent as seen by the beam corresponds to one of the operators A_1, A_2, A_3 or A_4 there is the bonus of having rotated the vertical component into the longitudinal direction at the interaction region (midpoint). For the first time, we can obtain snakes of either the 1st or 2nd kind with midpoint longitudinal polarization for all energies above a lower limit determined by magnet apertures.

Combinations $A_i^{-1} A_i = I$, where $i = 1, 2, 3, 4$ could be used as Rotators. These Rotators would retain their properties for all beam energies above a lower limit. Notice also that these Rotators are fully antisymmetric with respect to the midpoint, a property that Montague has shown will always result in an identity spin transformation. We can, therefore, reverse the direction of the longitudinal polarization at the midpoint by ramping magnets from one configuration to another while maintaining this antisymmetry.

Example

To illustrate how, in principle, these devices could be used in a circular machine we will emulate the example cited by Montague. We will use in this example a circular machine having four symmetrically placed straight sections or drifts and assume an interaction point (IP) at the midpoint of each. Each drift will be equipped with an identical assemblage of magnets as shown in Figure 3

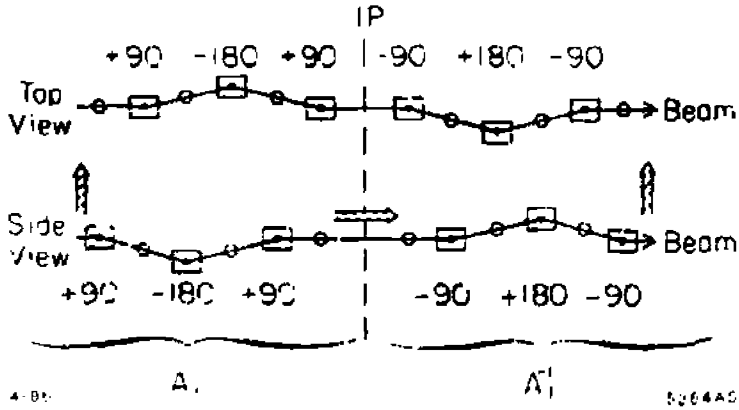

Figure 3.

The configuration chosen for illustration in Figure 3. represents the fully antisymmetric Rotator given by $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{I}$. It can be made apparent by pairing these magnets, starting with the two adjacent to the IP, that the antisymmetry ensures the identity transformation for the spin. Furthermore, if care is taken to preserve the correct field relationships within each of the four triplets of magnets, the overall optical transfer matrix (that of a drift) is preserved for all field strengths. It follows that these magnets (forming a Rotator) can be ramped if this symmetry and field strength relationship are maintained. This will not be true for the snakes that can also be formed by this assemblage as the spin transformation changes during ramping. In this figure the first half serpent as seen by the beam would always be represented by an operator $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ or $\mathbf{I}$ selected by choosing the proper fields and polarities. The first four of these operators would allow longitudinal polarization at the IP. The second half serpent can generate operators $\mathbf{A}_1^{-1}, \mathbf{A}_2^{-1}, \mathbf{A}_3^{-1}, \mathbf{A}_4^{-1}$ or $\mathbf{I}$. Other operators that can be formed by these assemblages are not included in this example and will be discussed later.

Prior to a particular machine running period a selection is made, designating two diametrically opposed interactic regions to serve as snakes. The experiments at these interaction points can still benefit by having longitudinally polarized beams available, but change of direction from parallel to antiparallel can not be readily made.

We number the interaction points 1 through 4 counter clockwise, and arbitrarily select IP 1 and IP 3 to serve as snakes. The fields and polarities of IP 1 are adjusted to perform the operation $\mathbf{A}_4^{-1}\mathbf{A}_1 = \mathbf{C}_2$ (Snake of the 2nd kind) and those of IP 3 to perform the operation $\mathbf{A}_3^{-1}\mathbf{A}_1 = \mathbf{C}_1$ (Snake of the 1st kind). For now, the magnets at IP 2 and IP 4 are left with zeroed fields.

The configuration just describe is shown in Figure 4a. where the direction of a vertically polarized spin vector is indicated in various regions. This vector or an antiparallel vector becomes longitudinal at IP 1 and 3. The initial direction (up or down) of this vector is arbitrary as there is no preferred polarization direction in this configuration. That is because the presence of the snakes will cause any vertical component to be parallel to the bend fields in one half of the machine and antiparallel to the fields in the other half. The Sokolov-Ternov polarizing mechanism (see Ref. 4) is thus turned off. As pointed out by Montague, an alternate polarizing mechanism such as wigglers would have to be provided.

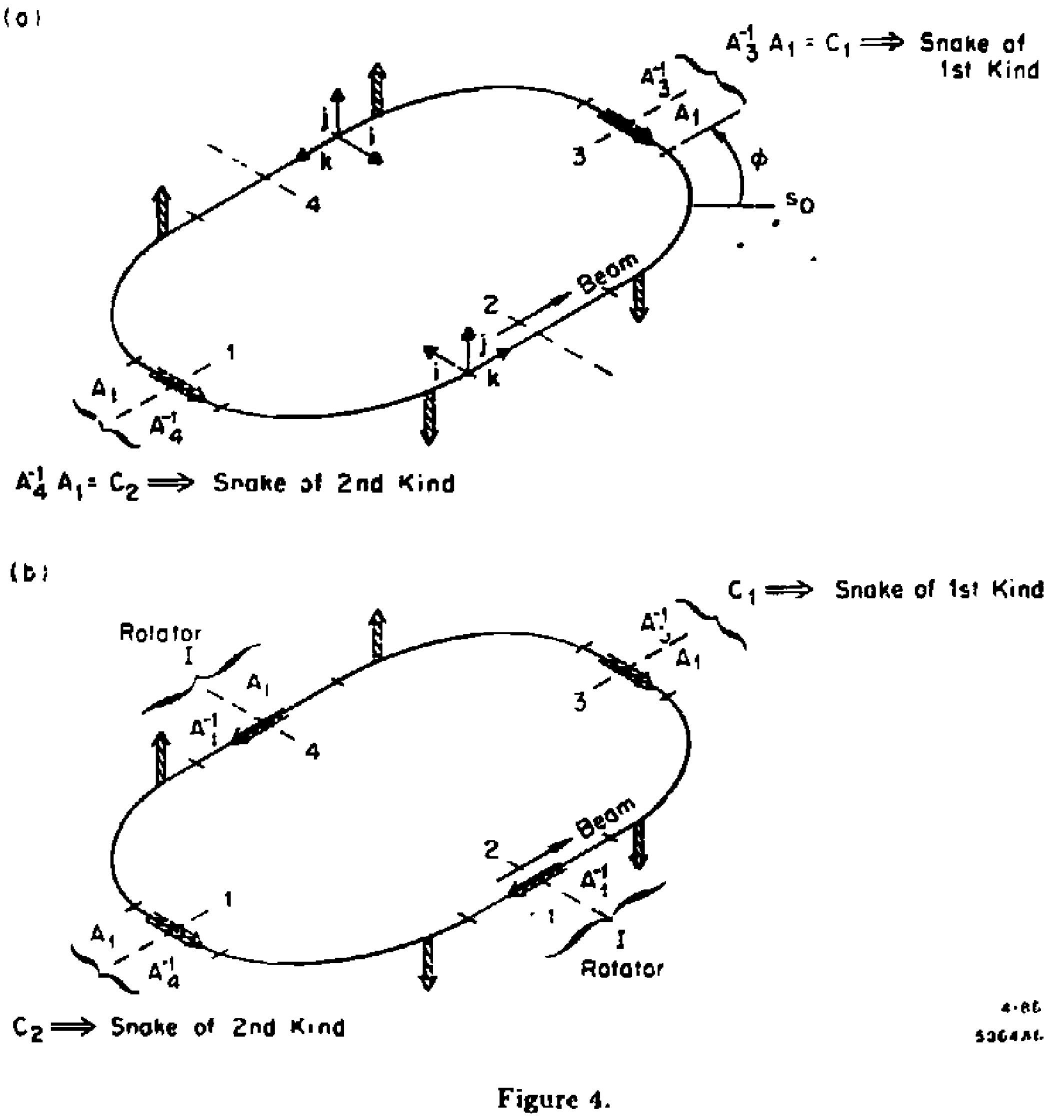
(a)
A₃⁻¹ A₁ = C₁ ⟹ Snake of 1st Kind
A₃⁻¹
A₁
3
φ
s₀
j
i
k
4
2
Beam
1
A₁
A₄⁻¹
j
i
k
A₄⁻¹ A₁ = C₂ ⟹ Snake of 2nd Kind
(b)
C₁ ⟹ Snake of 1st Kind
Rotator
I
A₁
A₁⁻¹
4
3
A₁
2
Beam
A₁⁻¹
1
1
A₁
A₄⁻¹
I
Rotator
C₂ ⟹ Snake of 2nd Kind
4-86
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Figure 4.

In his example, Montague showed in an elegant fashion that this configuration has a spin tune of 0.5. An attempt to depict this in the figure would have an incomprehensible result. We will use a simple but intuitively helpful argument to demonstrate his result. Let us assume that the initial spin tune of this machine, before insertion of the snakes, was given by ν_0. Then the spin precession angle for a beam traversing one half of the machine would be given by $\pi\nu_0$. Now an arbitrary point s_0 (see Fig. 4a) is chosen in one arc. It is then assumed that the beam will bend through an angle ϕ and the spin will precess by angle α ($\alpha = \phi\gamma a_{e-}$) as the beam travels from s_0 to the end of this arc.

Using Eq. (4) we can now represent the spin transformation operation for one turn starting at s_0 and with the snakes now inserted as

$$\mathbf{p}' = \mathbf{H}(\pi\nu_0 - \alpha)\ \mathbf{C}_2\ \mathbf{H}(\pi\nu_0)\ \mathbf{C}_1\ \mathbf{H}(\alpha)\ \mathbf{p} \quad .$$

Since $\mathbf{C}_3{}^2 = \mathbf{I}$, this equation can be rewritten as

$$\mathbf{p}' = \mathbf{H}(\pi\nu_0 - \alpha)\Big[\mathbf{C}_2\ \mathbf{H}(\pi\nu_0)\ \mathbf{C}_1\ \mathbf{H}(\alpha)\ \mathbf{C}_3\Big]\mathbf{C}_3\ \mathbf{p} \quad .$$

The reduction of the expression within the brackets is made simpler by the multiplication rules for $\mathbf{C}_1, \mathbf{C}_2$ and $\mathbf{C}_3$.

$\mathbf{C}_1$ multiplied from the left(right) changes signs of rows(columns) 1 and 2.

$\mathbf{C}_2$ multiplied from the left(right) changes signs of rows(columns) 2 and 3.

$\mathbf{C}_3$ multiplied from the left(right) changes signs of rows(columns) 3 and 1.

After reducing the expression in brackets the equation becomes

$$\mathbf{p}' = \mathbf{H}(\pi\nu_0 - \alpha)\ \tilde{\mathbf{H}}(\pi\nu_0 - \alpha)\ \mathbf{C}_3\ \mathbf{p} \quad .$$

Since $\mathbf{H}(\theta)$ is orthogonal, $\tilde{\mathbf{H}}(\theta) = \mathbf{H}^{-1}(\theta)$, and the equation reduces to

$$\mathbf{p}' = \mathbf{C}_3\ \mathbf{p} \quad .$$

We know $\mathbf{C}_3$ to be the operator representing a precession angle of 180° or π about the rotation axis $\pm\mathbf{j}$. Using the definition of spin tune (precession angle for one turn equals $2\pi\nu$) we calculate a new spin tune ν for this machine with snakes inserted.

$$\pi = 2\pi\nu \qquad \text{or} \qquad \nu = 0.5 \quad .$$

Because $\mathbf{C_3}^2 = \mathbf{I}$, after two revolutions the spin polarization will close upon itself;

$$\mathbf{p}' \quad = \quad \mathbf{C_3}^2\,\mathbf{p} \quad = \quad \mathbf{p} \quad .$$

The new spin tune ν has a fixed value independent of ν_0 or beam energy, whereas the spin tune ν_0 changes with beam energy.

Rotators are now introduced as shown in Figure 4b. The half serpents at IP 2 and IP 4 are ramped up as previously described to act as Rotators. Both interaction regions have been given a configuration identical to that shown in Figure 3. The resulting longitudinal polarization in one region is the reverse of that in the other. When desired, the polarization at either of these regions can be reversed independently of all others. For instance, by ramping only the vertical magnets in IP 2 to the opposite polarity, the operation $\mathbf{A}_1^{-1}\mathbf{A}_1 = \mathbf{I}$ becomes $\mathbf{A}_3^{-1}\mathbf{A}_3 = \mathbf{I}$ and at the interaction point the longitudinal polarization is reversed.

Combinations with half Snakes

Since a given half Serpent can be converted to a half Snake of the 1st kind by just halving the field strength, we may consider configurations where these two are combined end to end. As it turns out, eight new rotation operators are found in this way. They are not cyclic or even permutations. Instead they represent an odd (1 or 3) number of transpositions of initial coordinates; they also change coordinate signs. These operators are interesting in that they also rotate transverse to longitudinal polarization.

First, we calculate the operators $\mathbf{S}_3$ and $\mathbf{S}_3^{-1}$, which are obtained by one half of a snake of the 1st kind (90° rotation about eigenvector $s_3 = \mathbf{k}$).

$$\mathbf{V}(\pm 45, \pm 45) = \mathbf{H}(\mp 45, \pm 45) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{S}_3^{-1} \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} V \\ -H \\ S \end{pmatrix}$$

$$\mathbf{V}(\pm 45, \mp 45) = \mathbf{H}(\pm 45, \pm 45) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{S}_3 \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -V \\ H \\ S \end{pmatrix}$$

As noted earlier,

$$\mathbf{S}_3{}^2 = (\mathbf{S}_3^{-1})^2 = C_1 \quad .$$

DISCLAIMER

Combining S_3 and S_3^{-1} with the operators for one half of a Serpent (A_1, A_2, A_3, A_4 and inverses, see TABLE II), gives the new operators.

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D_1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -S \\ -V \\ -H \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D_2} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} S \\ -V \\ H \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D_3} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} S \\ V \\ -H \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D_3^{-1}} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -S \\ V \\ H \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E_1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ -S \\ -V \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E_2} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ S \\ V \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E_3} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} H \\ -S \\ V \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E_3^{-1}} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} H \\ S \\ -V \end{pmatrix}$$

These operators all rotate transverse to longitudinal polarization. Also note that $\mathbf{D_1}^2 = \mathbf{D_2}^2 = \mathbf{E_1}^2 = \mathbf{E_2}^2 = \mathbf{I}$. $\mathbf{D_1}$ and $\mathbf{D_2}$ invert the vertical component.

TABLE II

Multiplication Table for
combinations of two half snakes of the 1st kind
and/or half Serpents

	A_1	A_2	A_3	A_4	A_4^{-1}	A_3^{-1}	A_2^{-1}	A_1^{-1}	S_3	S_3^{-1}
S_3^{-1}	E_3	E_2	E_1	E_3^{-1}	D_1	D_3^{-1}	D_3	D_2	I	C_1
S_3	E_2	E_3	E_3^{-1}	E_1	D_3	D_2	D_1	D_3^{-1}	C_1	I
A_1^{-1}	I	C_3	C_1	C_2	A_2	A_4	A_3	A_1	E_3^{-1}	E_2
A_2^{-1}	C_3	I	C_2	C_1	A_3	A_1	A_2	A_4	E_2	E_3^{-1}
A_3^{-1}	C_1	C_2	I	C_3	A_1	A_3	A_4	A_2	E_1	E_3
A_4^{-1}	C_2	C_1	C_3	I	A_4	A_2	A_1	A_3	E_3	E_1
A_4	A_2^{-1}	A_3^{-1}	A_1^{-1}	A_4^{-1}	I	C_1	C_2	C_3	D_1	D_3^{-1}
A_3	A_4^{-1}	A_1^{-1}	A_3^{-1}	A_2^{-1}	C_1	I	C_3	C_2	D_3	D_2
A_2	A_3^{-1}	A_2^{-1}	A_4^{-1}	A_1^{-1}	C_2	C_3	I	C_1	D_3^{-1}	D_1
A_1	A_1^{-1}	A_4^{-1}	A_2^{-1}	A_3^{-1}	C_3	C_2	C_1	I	D_2	D_3

The operators in the top row represent the first half serpent or half snake as seen by the beam (the right hand matrix operator). Those in the left column are for the second half serpent or half snake (the left hand matrix operator). The identity operator has been omitted as a multiplier.

The rotation angle is 180° for operators $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{E}_1$, and $\mathbf{E}_2$ about eigenvectors

$$\mathbf{d}_1 = 1/2\,(\pm\sqrt{2}\,\mathbf{i} \mp \sqrt{2}\,\mathbf{k}) \quad ,$$

$$\mathbf{d}_2 = 1/2\,(\pm\sqrt{2}\,\mathbf{i} \pm \sqrt{2}\,\mathbf{k}) \quad ,$$

$$\mathbf{e}_1 = 1/2\,(\pm\sqrt{2}\,\mathbf{j} \mp \sqrt{2}\,\mathbf{k}) \quad ,$$

and

$$\mathbf{e}_2 = 1/2\,(\pm\sqrt{2}\,\mathbf{j} \pm \sqrt{2}\,\mathbf{k}) \quad ,$$

respectively.

Whereas, the rotation angle for $\mathbf{D}_3$ and $\mathbf{E}_3$ is 90° like that of $\mathbf{S}_3$ with their respective eigenvectors given by

$$\mathbf{d}_3 = +\mathbf{j} \quad ,$$

and

$$\mathbf{e}_3 = +\mathbf{i} \quad .$$

Again, it can be shown that either the set of operators

$$\{\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3, \mathbf{D}_3^{-1}, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{I}\}$$

or the set

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}_3^{-1}, \mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3, \mathbf{I}\}$$

will form a non-commutative group with respect to matrix multiplication.

As pointed out, conversion between Snakes of the 1st kind and Serpents is easy, requiring only a change in field strengths. Also reversing polarity of fields generates different configurations. However, it is difficult to convert a magnet from horizontal to vertical bend (mechanical rotation required), but if configurations are constructed using seven magnets as shown in Figure 5, where one magnet is normally turned off, then we can convert from $\mathbf{H}(\psi,\theta)$ to $\mathbf{V}(\psi,\theta)$ configurations if desired.

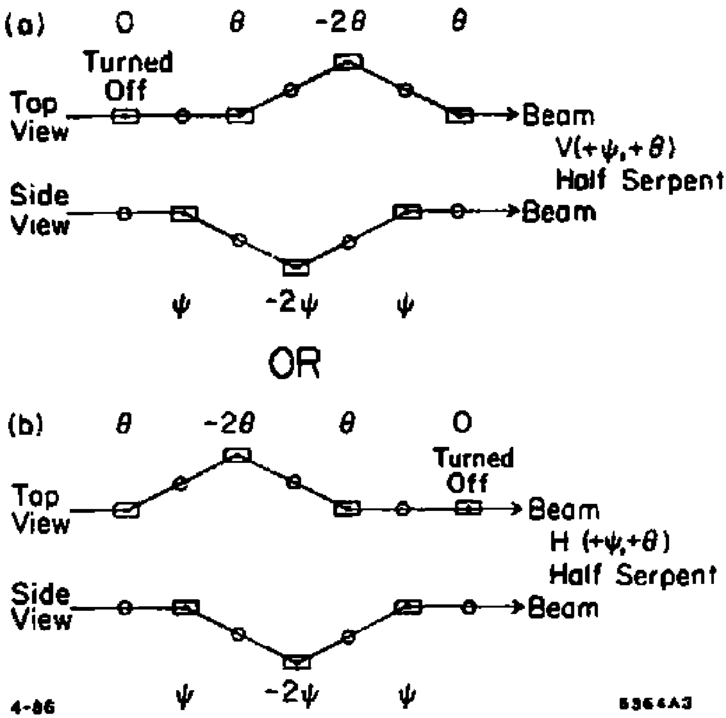


Figure 5.

Left-Right Snakes and Half Up-Down Snakes

It may have been noticed, that the operators $\mathbf{D}_1$ and $\mathbf{D}_2$ correspond to the definitions of Left and Right pointed Snakes. These snakes are shown in Figure 6a and were conceived (see Ref. 3) as a practical means of achieving spin tune of 0.5. They are efficient in that the total $\int B d\ell$ required is only 16.1 T-m whereas for two half Snakes of the 1st kind the $\int B d\ell$ is 18.4 T-m. These Snakes rotate about eigenvectors $\mathbf{d}_1$ or $\mathbf{d}_2$ by 180°, operations which we have shown not only invert the vertical component, but also rotate the horizontal component of the spin polarization vector into the longitudinal direction. This fact immediately suggests that if a Left-Right Snake were to be converted to half an Up-Down Snake (see Figure 6b) by mechanically rotating it about the direction of the beam, then the eigenvectors would become $\mathbf{e}_1$ and $\mathbf{e}_2$ with corresponding operators $\mathbf{E}_1$ and $\mathbf{E}_2$ which we have shown will rotate the vertical component of the spin polarization vector into the longitudinal direction.

We proceed to examine these types of snakes using the same methods as before and similar notations, understanding that the configurations are given in Figure 6.

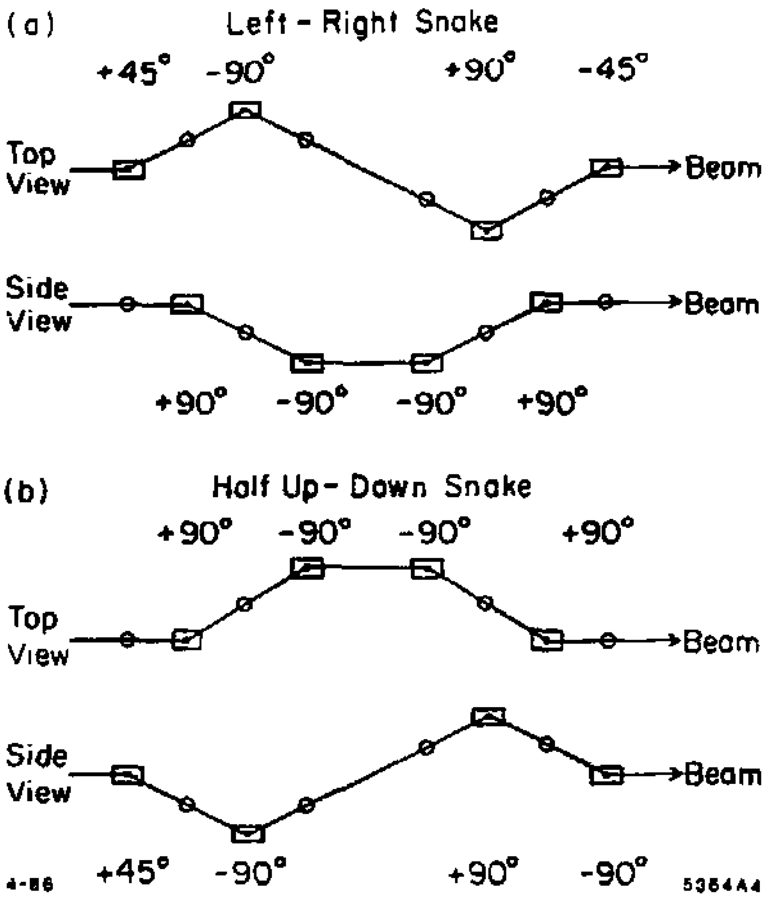
For the Left-Right Snakes we write

$$\mathbf{p}' = \mathbf{LR}(\psi, \theta)\ \mathbf{p} \quad .$$

and for half of an Up-Down Snake we write

$$\mathbf{p}' = \mathbf{UD}(\psi, \theta)\ \mathbf{p} \quad .$$

It will be shown later that snakes can be made by combining two half Up-Down Snakes.



(a)
Left - Right Snake
+45° -90° +90° -45°
Top View
Beam
Side View
Beam
+90° -90° -90° +90°
(b)
Half Up - Down Snake
+90° -90° -90° +90°
Top View
Beam
Side View
Beam
4-86 +45° -90° +90° -90° 5364A4

Figure 6.

With these definitions we find the following correspondence between configurations and operators for L-R Snakes and half U-D Snakes:

$$\mathbf{LR}(+90,+45) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D}_1 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -S \\ -V \\ -H \end{pmatrix}$$

$$\mathbf{LR}(-90,+45) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \mathbf{D}_1 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -S \\ -V \\ -H \end{pmatrix}$$

$$\mathbf{LR}(+90,-45) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D}_2 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} S \\ -V \\ H \end{pmatrix}$$

$$\mathbf{LR}(-90,-45) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{D}_2 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} S \\ -V \\ H \end{pmatrix}$$

$$\mathbf{UD}(+45,+90) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}_2 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ S \\ V \end{pmatrix}$$

$$\mathbf{UD}(+45,-90) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \mathbf{E}_2 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ S \\ V \end{pmatrix}$$

$$\mathbf{UD}(-45,+90) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E}_1 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ -S \\ -V \end{pmatrix}$$

$$\mathbf{UD}(-45,-90) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mathbf{E}_1 \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -H \\ -S \\ -V \end{pmatrix}$$

With L-R Snakes (as expected) or half U-D Snakes we find that they duplicate operations which heretofore required a combination of two half snakes. Again, we inquire about combining these half U-D snakes with others. We limit these combinations to only those in which the vertical component of the polarization vector is rotated to longitudinal at the midpoint. The intention is to summarize such combinations that can be used either as Snakes or Rotators (i.e. operators $\mathbf{C_1, C_2, D_1, D_2}$ or $\mathbf{I}$). Combinations such as these are shown in **TABLE III**.

Again, we find that combining these configurations give two new operators $\mathbf{S_1}$ and $\mathbf{S_2}$. This brings the total number of found operators to 24. All have a determinant of +1 as expected for rotations and represent half of the 48 operators which would include all permutations and all changes of sign.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{S_1} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} V \\ H \\ -S \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \mathbf{S_2} \qquad \begin{pmatrix} H \\ V \\ S \end{pmatrix} \Longrightarrow \begin{pmatrix} -V \\ -H \\ -S \end{pmatrix}$$

Note that $\mathbf{S_1}^2 = \mathbf{S_2}^2 = \mathbf{I}$. Again, it can be shown that the set of operators

$$\{\mathbf{S_1, S_2, S_3, S_3^{-1}, C_1, C_2, C_3, I}\}$$

forms a non-commutative group with respect to matrix multiplication.

The rotation angle is 180° for operators $\mathbf{S_1}$ and $\mathbf{S_2}$ about eigenvectors

$$\mathbf{s_1} = 1/2\ (\pm\sqrt{2}\ \mathbf{i} + \sqrt{2}\ \mathbf{j}) \quad ,$$

and

$$\mathbf{s_2} = 1/2\ (\pm\sqrt{2}\ \mathbf{i} + \sqrt{2}\ \mathbf{j}) \quad .$$

TABLE III

Multiplication Table for combinations of two half serpents or half snakes, where vertical spin becomes longitudinal at midpoint.

	A_1	A_2	A_3	A_4	E_1	E_2
E_2	D_3^{-1}	D_3	D_1	D_2	C_2	I
E_1	D_1	D_2	D_3^{-1}	D_3	I	C_2
A_4	A_2^{-1}	A_3^{-1}	A_1^{-1}	A_4^{-1}	S_3^{-1}	S_2
A_3	A_4^{-1}	A_1^{-1}	A_3^{-1}	A_2^{-1}	S_3	S_1
A_2	A_3^{-1}	A_2^{-1}	A_4^{-1}	A_1^{-1}	S_1	S_3
A_1	A_1^{-1}	A_4^{-1}	A_2^{-1}	A_3^{-1}	S_2	S_3^{-1}
A_4^{-1}	C_2	C_1	C_3	I	D_3^{-1}	D_2
A_3^{-1}	C_1	C_2	I	C_3	D_3	D_1
A_2^{-1}	C_3	I	C_2	C_1	D_2	D_3^{-1}
A_1^{-1}	I	C_3	C_1	C_2	D_1	D_3
D_1	S_2	S_3	S_1	S_3^{-1}	A_1^{-1}	A_3^{-1}
D_2	S_3^{-1}	S_1	S_3	S_2	A_2^{-1}	A_4^{-1}
S_3	E_2	E_3	E_3^{-1}	E_1	A_3	A_2
S_3^{-1}	E_3	E_2	E_1	E_3^{-1}	A_4	A_1

The operators in the top row represent the first half serpent or half snake as seen by the beam. Those in the left column are for the second half serpent or half snake.

Combinations resulting in C_1, C_2, D_1 or D_2 are Snakes. Combinations that result in I are Rotators. For example, $E_1E_2 = C_2$, is combination that forms a Snake of the 2nd kind. Whereas, $E_2E_2 = I$, is a Rotator.

In **TABLE IV** we summarize the 24 operators that have been found by combining half serpents or half snakes of various kinds.

This set of operators forms a group with respect to matrix multiplication. The group is isomorphic to the group of rotational symmetries of a cube, known as the octahedral group.[6]

TABLE IV

Summary of 24 Operators

with respective rotation angles and eigenvectors (rotation axes)

A_1, A_1^{-1}	$+120°, -120°$	$a_1 = 1/3(+\sqrt{3}i + \sqrt{3}j + \sqrt{3}k)$
A_2, A_2^{-1}	$+120°, -120°$	$a_2 = 1/3(+\sqrt{3}i - \sqrt{3}j - \sqrt{3}k)$
A_3, A_3^{-1}	$+120°, -120°$	$a_3 = 1/3(-\sqrt{3}i + \sqrt{3}j - \sqrt{3}k)$
A_4, A_4^{-1}	$+120°, -120°$	$a_4 = 1/3(-\sqrt{3}i - \sqrt{3}j + \sqrt{3}k)$
C_1	$180°$	$c_1 = \pm k$
C_2	$180°$	$c_2 = \pm i$
C_3	$180°$	$c_3 = \pm j$
I	$0°$	
D_1	$180°$	$d_1 = 1/2(\pm\sqrt{2}i \mp \sqrt{2}k)$
D_2	$180°$	$d_2 = 1/2(\pm\sqrt{2}i \pm \sqrt{2}k)$
D_3, D_3^{-1}	$+90°, -90°$	$d_3 = +j$
E_1	$180°$	$e_1 = 1/2(\pm\sqrt{2}j \mp \sqrt{2}k)$
E_2	$180°$	$e_2 = 1/2(\pm\sqrt{2}j \pm \sqrt{2}k)$
E_3, E_3^{-1}	$+90°, -90°$	$e_3 = +i$
S_1	$180°$	$s_1 = 1/2(\pm\sqrt{2}i \pm \sqrt{2}j)$
S_2	$180°$	$s_2 = 1/2(\pm\sqrt{2}i \mp \sqrt{2}j)$
S_3, S_3^{-1}	$+90°, -90°$	$s_3 = +k$

It is interesting to note that the operators D_3, E_3 and S_3 along with their inverses are but special cases of the spin ope. ·tions performed by single magnets: horizontal bending (eq. 4.), vertical bending (eq. 5) or a solenoidal field m.. net, respectively. It is not surprising then that it is simple to show that these operators, representing rotations about the 4-fold symmetry axes, generate all operators of the octahedral group.

In the Introduction, the definition of a snake was cited as the single operation of rotating the spin polarization vector by 180° about an arbitrary axis in the horizontal plane. An alternate way of describing this operation as the product of two operations is also given in Reference 1. This description can be expressed using operators which have been defined here. The first operation represents a rotation of 180° about the axis +i (corresponding to operators E_3^2, $(E_3^{-1})^2$ or C_2 given in TABLE IV) followed by an arbitrary spin precession through an angle α about +j (corresponding to the operator $H(\alpha)$ for a horizontal bend magnet given by eq. 4).

The product defining a snake is then $H(\alpha)\ C_2$.

The arbitrary axis referred to in the initial definition of a snake is defined as the vector a , which is now given by

$$a = \pm \cos\left(\frac{\alpha}{2}\right) i \quad \pm \sin\left(\frac{\alpha}{2}\right) k \quad .$$

For example, the special cases describing L-R snakes are obtained by setting $\alpha = \pm 90°$, since

$$H(+90)\ C_2 = D_3\ C_2 = D_1 = LR(\pm 90, +45)$$

and

$$H(-90)\ C_2 = D_3^{-1}\ C_2 = D_2 = LR(\pm 90, -45).$$

Similarly, $H(0)\ C_2 = C_2$ and $H(180)\ C_2 = C_3\ C_2 = C_1$ define Snakes of the 2nd and 1st kind, respectively.

In TABLE III we show that $\mathbf{D}_1$ or $\mathbf{D}_2$(the L-R Snake operators) can also be obtained by the product of two operators where the first operator rotates the vertical spin component into the longitudinal direction. This suggests that such configurations may be capable of generating snakes with longitudinal polarization at the midpoint and an arbitrary precession angle α. See References 1. and 4. for detailed references and discussion of the advantages of achieving an arbitrary precession angle α .

As an example, we assume that the first operator is given by $\mathbf{A}_1$ which provides midpoint longitudinal polarization and the second operator (to be found) is given by $\mathbf{M}$. Equating their product to the product defining a snake gives

$$\mathbf{M}\ \mathbf{A}_1 = \mathbf{H}(\alpha)\ \mathbf{C}_2$$

or

$$\mathbf{M} = \mathbf{H}(\alpha)\ \mathbf{C}_2\ \mathbf{A}_1^{-1} \quad .$$

Reducing gives

$$\mathbf{M} = \begin{pmatrix} -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & -1 \\ -\cos\alpha & -\sin\alpha & 0 \end{pmatrix} .$$

If we again set $\alpha = +90^\circ$ then $\mathbf{M} = \mathbf{E}_1$ and $\mathbf{M}\ \mathbf{A}_1 = \mathbf{E}_1\ \mathbf{A}_1 = \mathbf{D}_1$ as shown in TABLE III. The operator $\mathbf{E}_1$ is generated by one half of a U-D snake. This suggests that practical solutions for a chosen value of α near 90° may be found by examining $\mathbf{UD}(\psi,\theta)$ configurations for values of ψ and θ that will generate the operator $\mathbf{M}$.

Still assuming that the first operator is given by $\mathbf{A}_1$, similar considerations for α near 0° would indicate examination of the configuration corresponding to operator $\mathbf{A}_4^{-1}$ and for α near 180° the configuration corresponding to the operator $\mathbf{A}_3^{-1}$.

CONCLUSIONS

By combining two half serpents or half snakes of various kinds, we have found a group of 24 rotation operators. It was noted that five different subsets of these operators form subgroups. The isomorphism to the octahedral group suggests that efficient means of obtaining arbitrary rotations for particular applications are achievable.

Several combinations, as shown in Tables II and Tables III, result in longitudinal polarization at the midpoint (interaction point) and also invert the vertical component. In principle, these or other combinations would be useful in circular machines.

For electrons the total $\int B d\ell$ required for these combinations ranges from 32.2 T-m to 36.8 T-m. These high values may limit the usefulness of these configurations in some applications due to synchrotron radiation, which implies long magnets and large beam excursions. In proton machines shorter magnets can be used.

One half of a serpent could be used to rotate longitudinal to vertical polarization for transporting and injecting electrons into a high energy ($\approx$ 10 GeV) damping ring of a Super Linear Collider. Another half serpent would restore the longitudinal polarization in the damped beam. The need for solenoidal fields the strength of which scale as γa_{e^-} would be eliminated.

At the Stanford Linear Collider (SLC) presently under construction, complications due to the precession of a longitudinally polarized beam in the mile long Arcs could be avoided if the polarization was made to be parallel to those fields. The desired longitudinal polarization at the interaction point then could be achieved by introducing the proper half serpent. Unfortunately, the required drift space (10 to 20 m) is not presently available.

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