CONF-920540--10

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TIME OPTIMAL PATHS FOR A CONSTANT SPEED UNICYCLE*

David B. Reister
Center for Engineering Systems Advanced Research
Oak Ridge National Laboratory
P.O. Box 2008
Building 6025, MS-6364
Oak Ridge, TN 37831-6364

To be published in the Proceedings of the 1992 IEEE International Conference on Robotics and Automation, May 10–15, 1992, Nice, France

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* Research sponsored by the Engineering Research Program of the Office of Basic Energy Sciences and the Office of Nuclear Energy, Office of Technology Support Programs, of the U.S. Department of Energy, under contract No. DE-AC05-84OR21400 with Martin Marietta Energy Systems, Inc.

Time Optimal Paths for a Constant Speed Unicycle

D. B. Reister

ABSTRACT

This paper uses the Pontryagin maximum principle to find time optimal paths for a constant speed unicycle. The time optimal paths consist of sequences of arcs of circles and straight lines. The maximum principle introduces concepts (dual variables, bang-bang solutions, singular solutions, and transversality conditions) that provide important insight into the nature of the time optimal paths.

1. INTRODUCTION

In this paper, we consider a constant speed unicycle that has one degree of freedom (the steering angle). We will use the Pontryagin Maximum Principle to find time optimal paths for the platform. We will find that the optimal paths consist of sequences of arcs and lines. The maximum principle introduces concepts (dual variables, bang-bang solutions, singular solutions, and transversality conditions) that provide important insight into the nature of minimum time paths. We will not consider path planning algorithms for complex environments containing obstacles.

A constant speed unicycle is interesting because it is the most simple nonholonomic system. While the constant speed assumption is unrealistic (all platforms start and stop), the assumption is appropriate for platforms with one or more steerable drive wheels that steer together and are moving at a constant speed (for example, Cybermotion [1], Denning [2], Nomadic [3], and HERMIES-III [4]). Thus, time optimal paths for a constant speed unicycle can be the appropriate paths for a real platform with steerable wheels that is required to perform high speed obstacle avoidance.

Time optimal paths for a constant speed platform are similar to minimum length paths. Several recent papers ([5], [6], and [7]) have explored minimum length paths for mobile robots that have a minimum turning radius. Like the time optimal paths, the minimum length paths consist of sequences of arcs of circles and straight lines. The authors have developed path planning algorithms for complex environments containing obstacles.

The next section will use the maximum principle to derive the conditions for time optimal paths. The third section will explore the features of the optimal paths, while the fourth section presents our conclusions.

2. CONDITIONS FOR TIME OPTIMAL PATHS

The basic equations of motion for a single wheel are:

$$\dot{x} = v \cos \phi \tag{1}$$

$$\dot{y} = v \sin \phi \tag{2}$$

where the Cartesian coordinates (x,y) locate the point of contact between the wheel and the floor, $\dot{x}$ is the x component of the wheel velocity (v), and ϕ is the orientation of the plane of the wheel with respect to the x axis. We assume that the velocity of the wheel orientation is the control variable:

$$\dot{\phi} = u \tag{3}$$

where the magnitude of the orientation velocity is bounded: $|u| \leq a$. While the control input could be the acceleration of the wheel orientation, the velocity is the control variable that we use for the HERMIES-III platform [8].

To apply the Pontryagin Maximum Principle. we introduce three state variables: $x = (x,y,\phi)$. The three state variables define the configuration (position and orientation) of the platform. In vector notation, the equations of motion for the state vector x are:

$$\dot{x} = f(x,u) \tag{4}$$

The components of the equations of motion are:

$$\dot{x}_1 = f_1(x,u) = v \cos x_3 \tag{5}$$

$$\dot{x}_2 = f_2(x,u) = v \sin x_3 \tag{6}$$

$$\dot{x}_3 = f_3(x,u) = u \tag{7}$$

The optimization problem is to find a control vector [u] that will move the system from the initial configuration x^0 to the final configuration x^1 and minimize the transition time.

Pontryagin introduces a system of dual variables [ψ] that satisfy:

$$\dot{\psi}_i = -\sum_{j=1}^{3} \frac{\partial f_j(x,u)}{\partial x_i} \psi_j , \qquad i = 1, 2, 3. \tag{8}$$

The equations of motion for the dual variables are:

$$\dot{\psi}_1 = 0 \tag{9}$$

$$\dot{\psi}_2 = 0 \tag{10}$$

$$\dot{\psi}_3 = \psi_1 v \sin x_3 - \psi_2 v \cos x_3 \tag{11}$$

The initial conditions are:

$$\psi_i(t_0) = \mu_i \qquad i = 1, 2, 3. \tag{12}$$

Equations (8) are linear and homogeneous and have a unique solution for any initial conditions.

Pontryagin combines the equations of motion and the dual variables into a single Hamiltonian [H]:

$$H(\psi,x,u) = \sum_{j=1}^{3} \psi_j f_j(x,u) \quad . \tag{13}$$

$$H(\psi,x,u) = \psi_1 f_1(x) + \psi_2 f_2(x) + \psi_3 u \tag{14}$$

In Theorem 2 of his book [9], Pontryagin proves that the optimal set of control variables maximizes the Hamiltonian. Since the Hamiltonian is linear in the control variable u, the optimal solution is bang-bang or singular. When ψ_3 is positive (negative), the optimal control is bang-bang (at its upper (lower) limit). When ψ_3 is zero for an interval, the optimal solution is singular.

We can show that when ψ_3 is zero for an interval, the path is a line segment. Thus, ϕ is a constant and $u = 0$. Using Eqs. (5) and (6), Eq. (11) may be written:

$$\dot{\psi}_3 = \psi_1 \dot{x}_2 - \psi_2 \dot{x}_1 \tag{15}$$

For any arbitrary initial configuration, we can choose the coordinate system to make the initial values of the state variables equal (0,0,0). Eq. (15) can be integrated to yield:

$$\psi_3 = \mu_1 y - \mu_2 x + \mu_3 \tag{16}$$

Since ψ_1 and ψ_2 are constants, we have replaced them with their initial values (μ_1 and μ_2). When ψ_3 is zero, Eq. (16) defines a line. A line can be defined by the inner product of a position vector (r) and a vector (a) that is perpendicular to the line:

$$r \cdot a = c \tag{17}$$

The constant (c) is proportional to the distance from the line to the origin (see Section 1.4 of [10]). For Eq. (16), the vectors are: $r = (x,y)$ and $a = (-\mu_2,\mu_1)$. The parameter (μ_3) is proportional to the distance from the line where ψ_3 is zero to the origin and the value of the dual variable (ψ_3) is proportional to the distance from the point (x,y) to the line where ψ_3 is zero. In the next section, we will find that the line determines the optimum solution. The optimal control steers the wheel to the line, follows the line, and then steers to the goal.

Consider a portion of an optimal trajectory when the control is at its limit. We shall show that the path is an arc of a circle. When the control is constant initially, Eq. (3) can be integrated to yield: $\phi = u\,t$. Eqs. (1) and (2) can be integrated to find the Cartesian path:

$$x = (v / u) \sin \phi \tag{18}$$

$$y = (v / u)(1 - \cos \phi) \tag{19}$$

Recall that the initial values of the state variables are equal to (0,0,0). The path is an arc of a circle. The radius of curvature is the ratio of the wheel velocity and the steering velocity [$R = |(v / u)|$].

The optimal solution is bang-bang or singular. When ψ_3 is not zero, the optimal control is bang-bang and the path is an arc of a circle. When ψ_3 is zero for an interval, the optimal solution is singular and the path is a line segment.

We have assumed that all three state variables are specified at the end of the path. We could be interested in paths with some free boundary conditions. We might want to reach a point (x,y) at an arbitrary orientation or we might want to reach an orientation at an arbitrary point. When an optimal path has free boundary conditions, Pontryagin's Theorem 3 determines the optimal solution. We assume that the goal x^1 is a point in a smooth manifold S. Let T be the tangent plane to S at the goal. The dual solution satisfies the transversality condition if it is orthogonal to T. Theorem 3 requires that the dual solution must satisfy the transversality condition at the goal. If the goal is to reach a point at an arbitrary orientation, the tangent plane is defined by the vector (0, 0, 1) and the transversality condition requires that $\psi_3 = 0$ at the goal. Thus, the last segment of the path to the goal is a line. If the goal is to reach an orientation at an arbitrary point, $\psi_1 = \psi_2 = 0$ and the path to the goal is a circle.

3. TIME OPTIMAL PATHS

In this section, we will use graphs to explore the optimal paths. We begin by considering the case where both position and orientation are specified at the goal. There are three possible paths from the initial point (0, 0, 0), steer right, steer left, or go straight. Similarly, there are three possible paths into the goal (x, y, ϕ). In general, the optimal path follows an arc to a line segment and finishes with an arc.

Figures 1 to 4 illustrate optimal paths to a point for eight different orientations. In Figure 1, the orientation is either 0 or 180 degrees. The path to 0 degrees starts with a left arc and ends with a right arc. The path to 180 degrees starts and ends with a left arc. In Figure 2, the orientation is either 45 or 225 degrees. The path to 45 degrees ends in a line segment. In Figure 3, the orientation is either 90 or 270 degrees. Compared to Figure 1, the concluding arcs have switched circles. The path to 90 degrees starts and ends with a left arc. The path to 270 degrees starts with a left arc and ends with a right arc. In Figure 4, the orientation is either 135 or 315 degrees.

For the paths in Figures 1 to 4, we could choose a consistent set of values for the initial conditions of the dual variables and verify that the paths are optimal. The parameters (μ_1,μ_2) define the slope of the line segment, we will define μ_1 and μ_2 by: $\mu_1 = \cos\theta$ and $\mu_2 = \sin\theta$, where θ is the slope of the line segment. The distance from (0,0) to the line and the sign of the optimal control determine μ_3. For example in Figure 4, the path to 135 degrees starts and ends with a positive control (an arc to the left). Thus, μ_3 is the positive distance from (0,0) to the line. Both the initial point and the goal are on the left side of the line. In Figure 4, the path to 315 degrees starts with a positive control and ends with a negative control. The initial point is on the left side of the line while the goal is on the right side of the line.

A constant speed unicycle cannot make tight maneuvers. Figure 5 shows the path required to turn around [the goal is (0, 0, 180)]. The path consists of three arcs. For this case, the line ($\psi_3 = 0$) is vertical; the optimal control is negative on the left of the line and positive on the right (or vice versa {the path could be traveled in either direction}).

While time optimal paths at constant velocity are similar to minimum length paths, they have one significant difference. A minimum length path can reverse direction at a point. Minimum time paths cannot have discontinuities in slope. If a platform needs to perform tight maneuvers, it should not move at constant speed.

If the goal is to reach a position at an arbitrary orientation, the last segment of the path to the goal is a line. Paths that can reach a point are explored in Figure 6. Figure 6 displays the paths when the steering velocity is positive. A similar set of paths could be obtained if the steering velocity is negative [the point (x,y) is mapped to (x,-y)]. For goals outside the two circles defined by the radius of curvature, we have truncated the lengths of the line segments to remain within the region where the paths are optimal [in the upper half plane (y > 0)]. The paths displayed in Figure 6 are optimal to reach any point in the upper half plane except the points inside the circle. Furthermore, they are optimal to reach any point within the circle for the lower half plane. The paths with negative steering velocity are optimal for the complementary regions (inside the upper circle and outside the lower circle).

4. CONCLUSIONS

In this paper, we have used the Pontryagin Maximum Principle to find time optimal paths for a constant speed unicycle. The time optimal paths are produced by control trajectories that are either bang-bang or singular. The bang-bang controls lead to path segments that are arcs of circles, while the singular controls produce line segments.

A mobile platform can move from a configuration (position and orientation) to a configuration or to a position. When the platform moves to a position, the final orientation is not specified and the transversality conditions of the maximum principle require that the last segment of the path is a line segment.

For practical applications, the line segments cover more distance than the arcs. The paths are best for high speed obstacle avoidance and are not appropriate for maneuvering in complex environments. For example, the rotation path in Figure 5 required the platform to turn 420 degrees (+60, -300, +60) to accomplish a 180 degree change in orientation. This maneuver should be performed when the platform is at rest.

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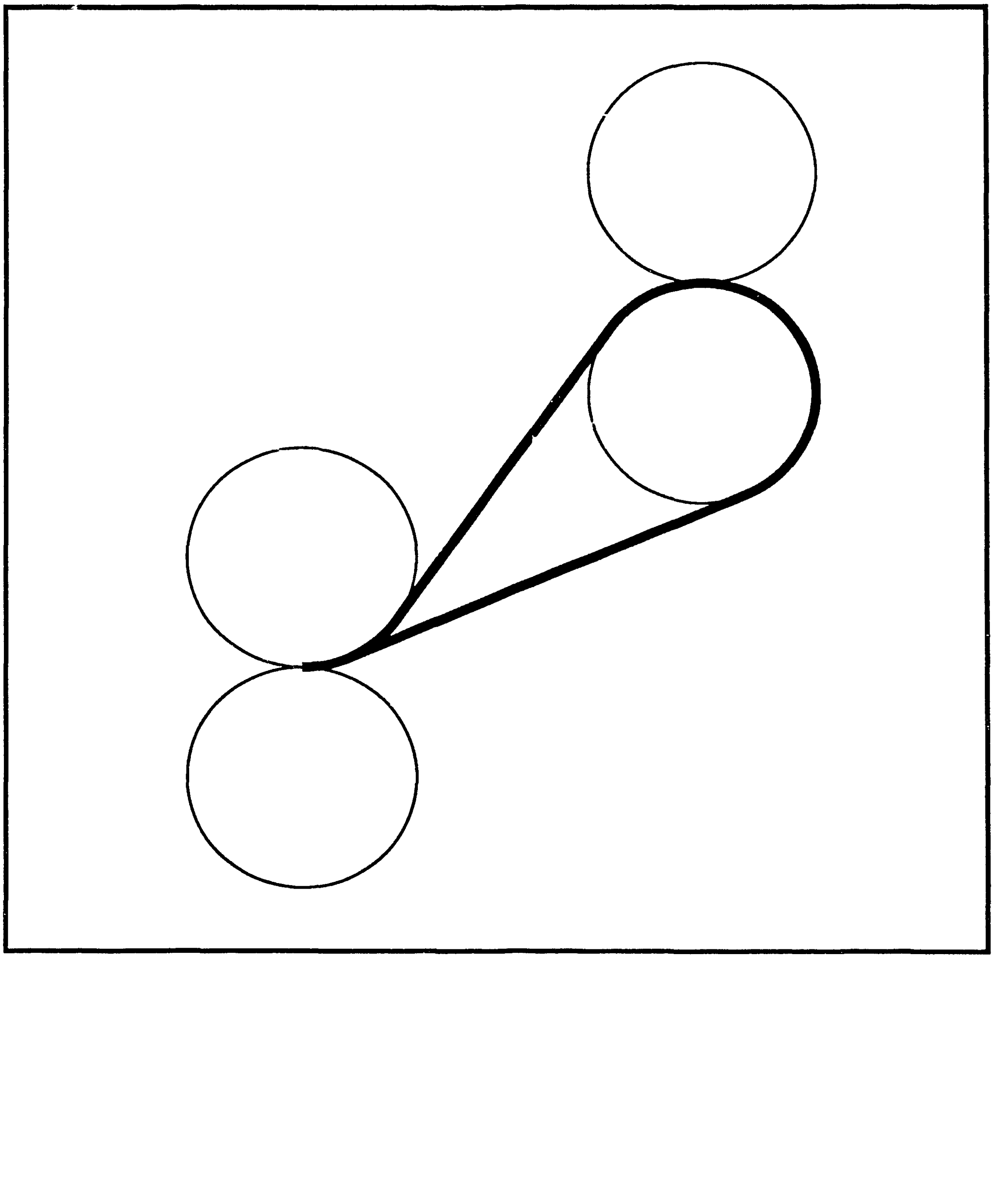

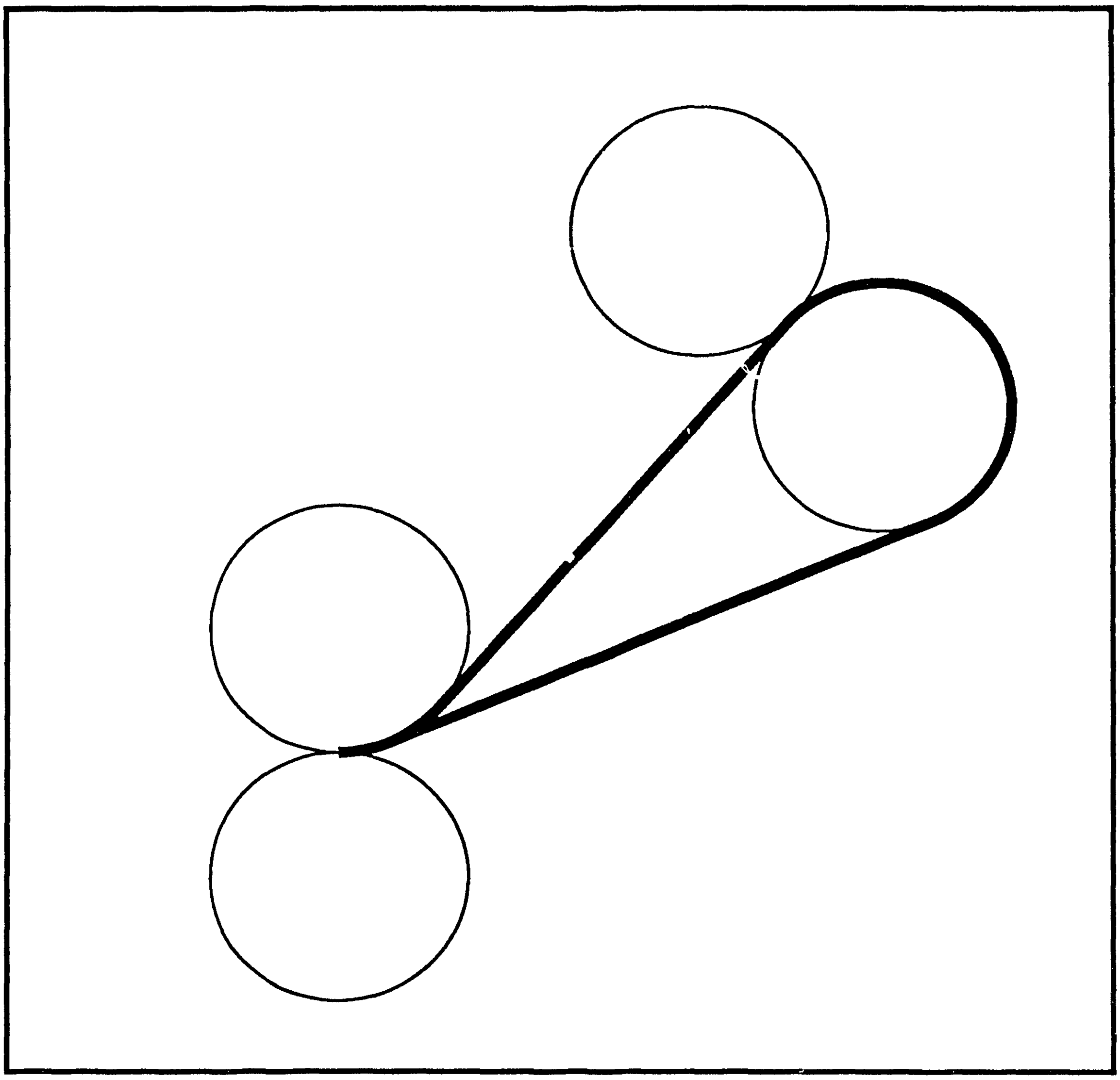

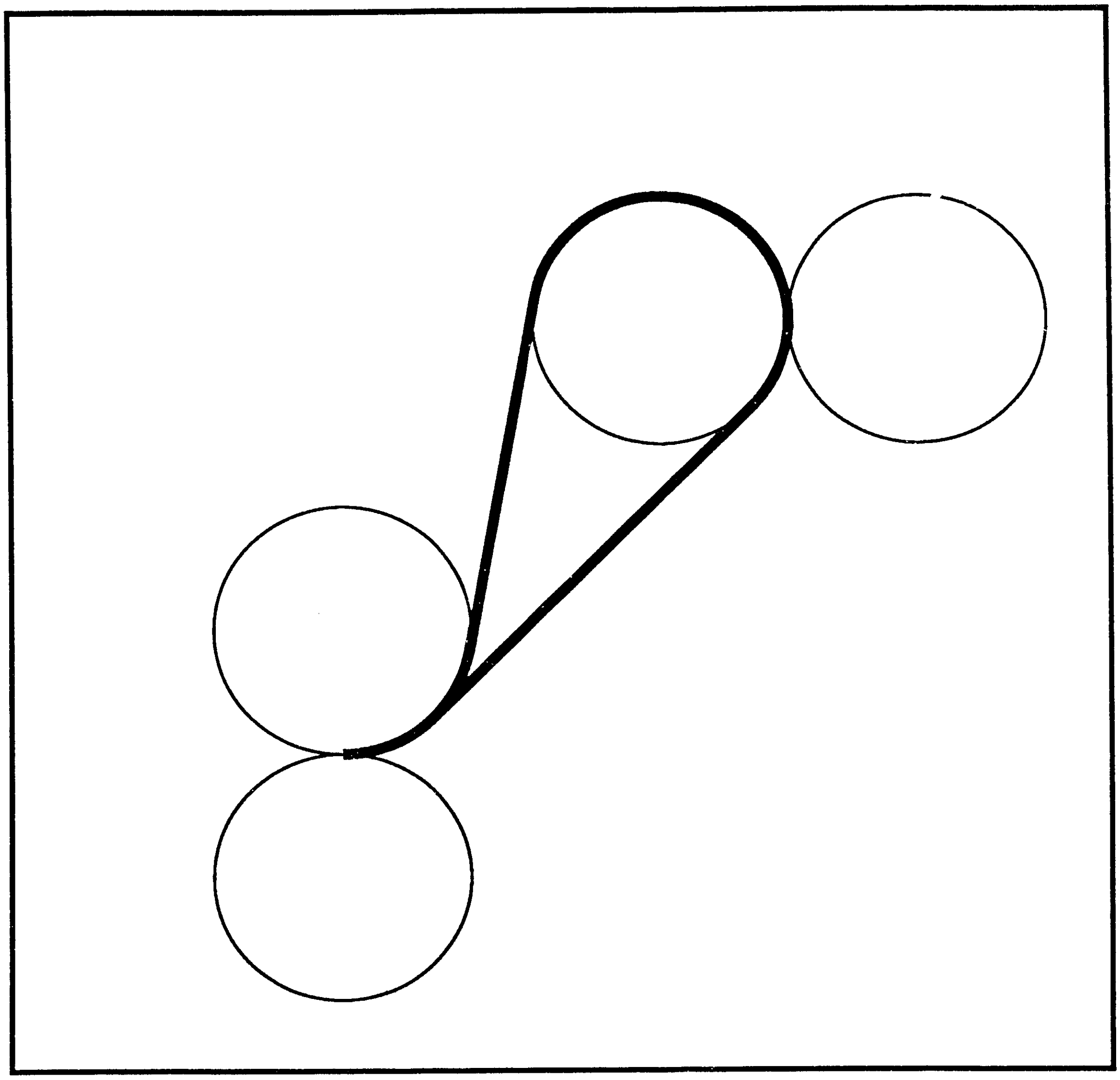

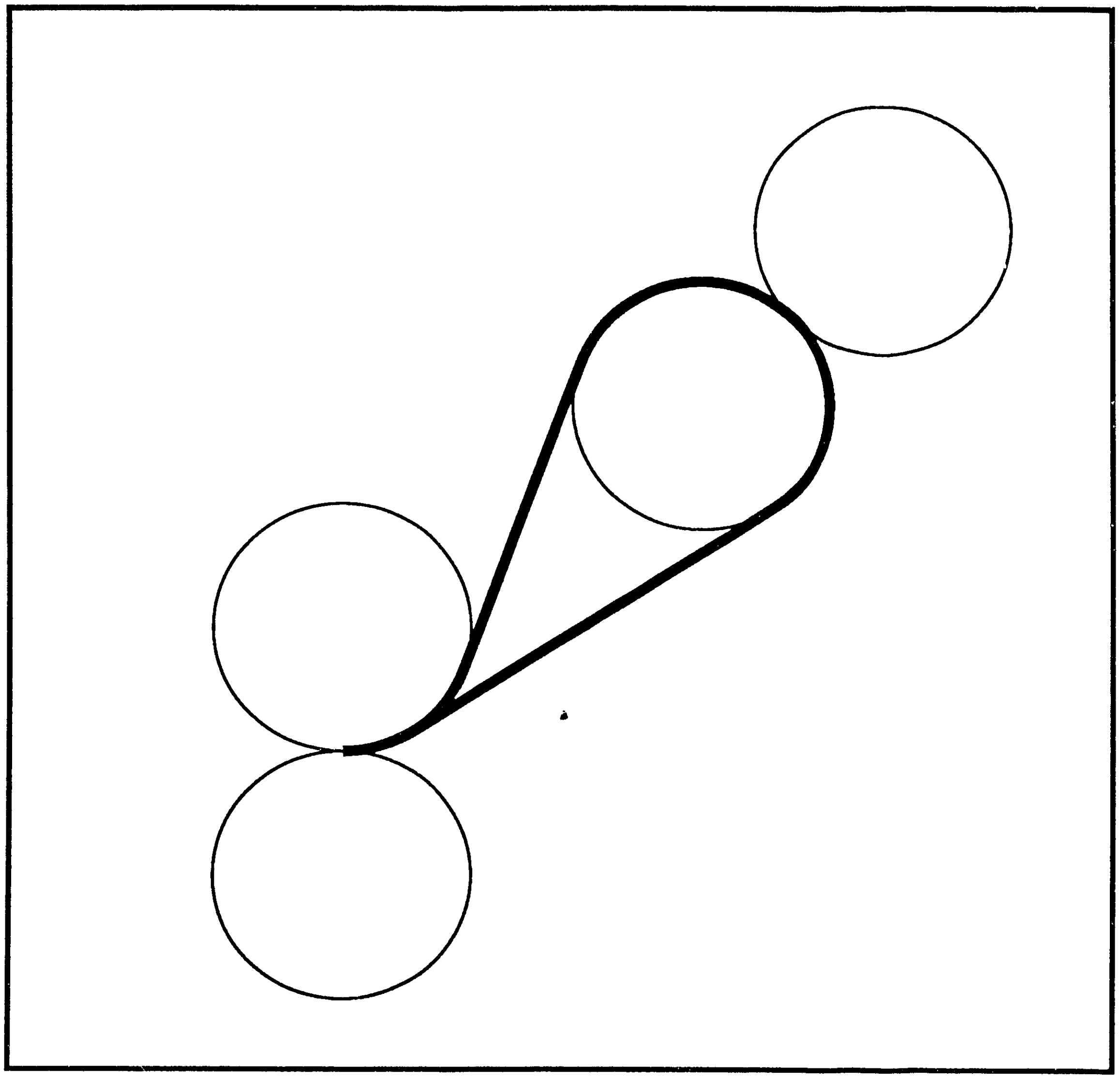

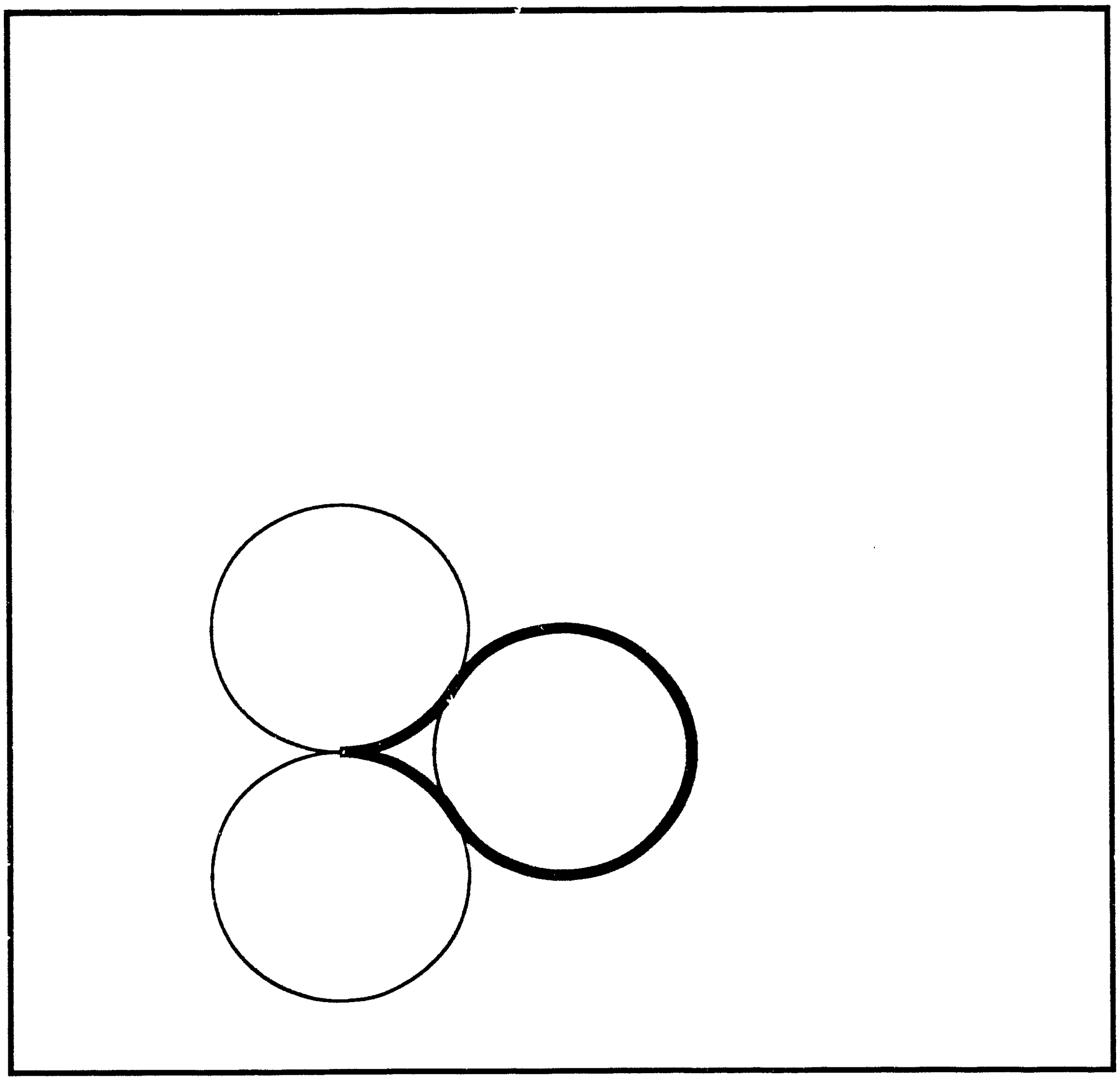

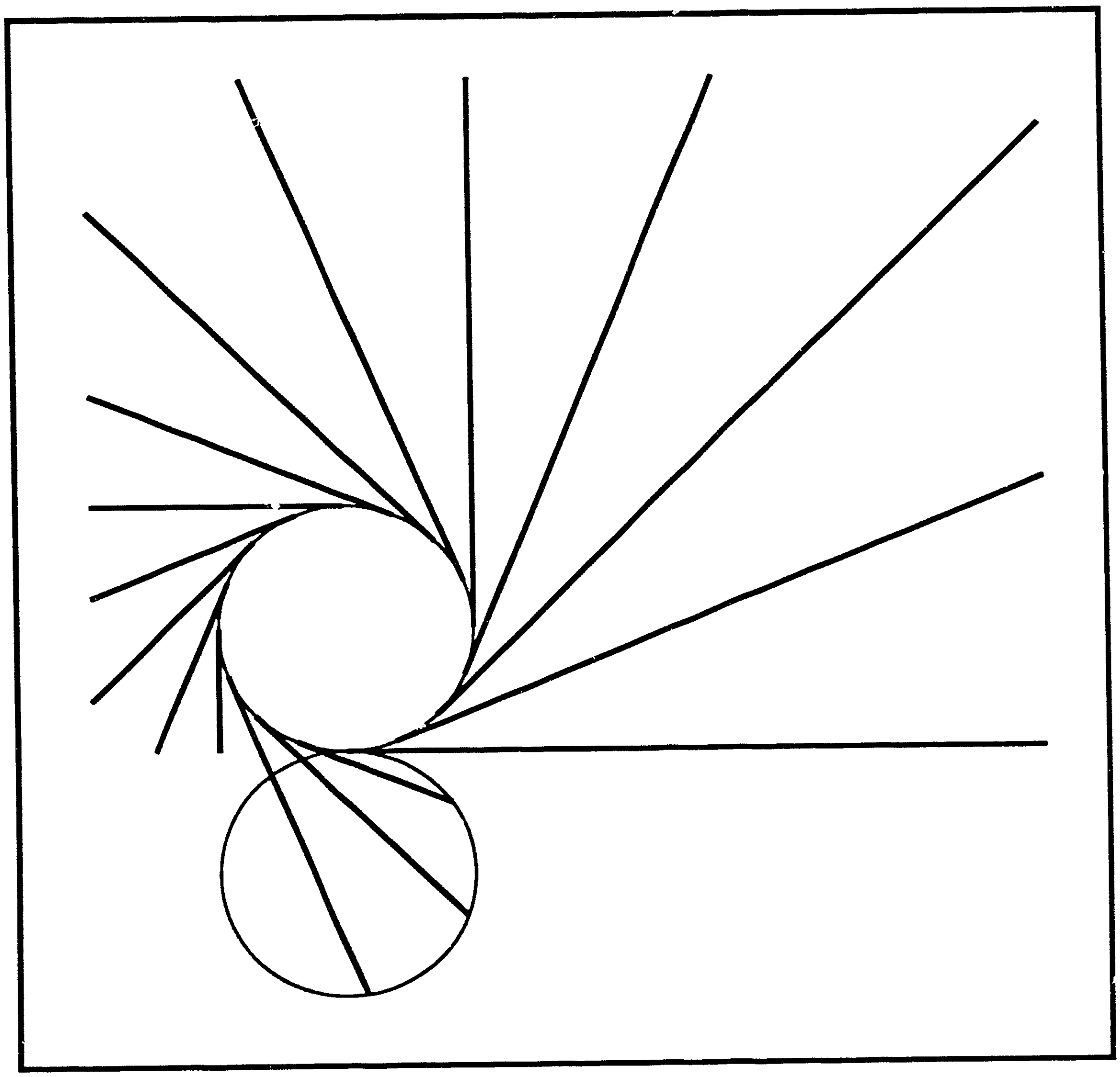

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