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A COMPARISON OF MAXIMUS/BOUNDING AND BAYES/MONTE CARLO FOR FAULT TREE UNCERTAINTY ANALYSIS

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ABSTRACT

A comparison of two methodologies for the analysis of uncertainty in risk analyses is presented. One methodology combines approximate methods for confidence interval estimation of system reliability with a bounding approach for information derived from expert opinion. The other method employs Bayesian arguments to construct probability distributions for component reliabilities using data from experiments and observation and expert opinion. The system reliability distribution is then derived using a conventional Monte Carlo analysis.

An extensive discussion of the differences between confidence intervals and Bayesian probability intervals precedes the comparison. The comparison is made using a trial problem from the Arkansas Nuclear One-Unit 1 Nuclear Power Plant. It is concluded that the Maximus/Bounding methodology tends to produce somewhat longer intervals than the Bayes/Monte Carlo method, although this finding is based on comparisons made under nonidentical assumptions regarding the treatment of operator recovery rates. The Bayes/Monte Carlo method is shown to produce useful information regarding the importance of uncertainty about each component's reliability in determining overall uncertainty.

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1. INTRODUCTION

The recognition of the existence of major uncertainties in the performance of probabilistic risk assessments (PRAs) has led various NRC-sponsored programs to assess the impact of uncertainties. Examples of two such programs are the Risk Methods Integration and Evaluation Program (RMIEP) and the Phenomenology and Risk Uncertainty Evaluation Program (PRUEP), which will provide a PRA for the La Salle Plant. Methods for performing uncertainty analysis are being developed in the PRA Methods Improvement Program and PRUEP. Recently, Spencer and Easterling (1984b) presented a demonstration of a statistical methodology for fault-tree uncertainty analysis referred to as "Maximus/bounding." A second report by Easterling, Spencer and Diegert (1985) provides directions for RMIEP with respect to estimation, uncertainty analysis and sensitivity analysis. Within this second report it is noted that other statistical methods could be used for "calculating approximate statistical confidence limits for general functions of multiple parameters." One of these methods is an "uninformative" Bayesian analysis. The purpose of this report is to compare the results of the statistically-based Maximus/bounding methodology and an alternative statistically based methodology referred to as Bayes/Monte Carlo. The trial problem in Spencer and Easterling (1984b) is used as a vehicle for this comparison. The techniques are compared on the basis of range of applicability or generality, interpretability of the output, ease of applicability, and sensitivity/uncertainty analysis capability.

The analysis of risks from technological hazards requires estimation of types of risks, their magnitudes, and their likelihoods. Further, a comprehensive analysis will quantify the uncertainties in the risk estimates. The analysis of the precision of the risk estimates is known as an uncertainty analysis.

UNCERTAINTY ANALYSIS	The analysis of the precision of the risk estimates is known as an uncertainty analysis.

The objectives for an uncertainty analysis methodology are:

1. Quantify the uncertainties in the various measures of risk.

2. Delineate sensitivities of the uncertainty estimates to subjective and objective information and key assumptions about such information.

3. Identify the relative contribution (importance) of uncertainties regarding individual parameters to the overall uncertainty.

This report details a methodology for conducting an uncertainty analysis that is an alternative to the Maximus/bounding methodology. In this study, uncertainties about model specification and uncertainty about omitted factors that are perhaps unrecognized are not considered. Instead, only uncertainties about the failure rates of components are considered. The alternative methodology employs Bayesian and Monte Carlo methods to obtain probability distributions for possible risks.

A review of the Maximus/bounding methodology will be given before the presentation of the Bayes/Monte Carlo methodology. However, a review of confidence intervals for a single component and for a system will precede the review of the Maximus/bounding methodology to aid the reader in interpreting the results of the methodology. Likewise, reviews of Bayes' theorem, noninformative priors for a single component, Bayesian analysis of parallel systems, and subjective distributions precede the discussion of the Bayes/Monte Carlo methodology to aid the reader with the terminology and interpretation of the results. Martz and Duran (1985) have compared the maximus, Bayes/Monte Carlo and bootstrap methods of obtaining lower uncertainty limits on system reliability. Their study centers on simulation results obtained from the analysis of 20 simple systems. In contrast, this study concentrates on a single system from Arkansas Number One Unit 1. The emphasis in this study is placed on interpreting the output from each of the techniques.

2. CONFIDENCE INTERVALS FOR A SINGLE COMPONENT

A confidence interval is a statistical statement about a parameter that provides an interval of values that is apt to include the value of the parameter. A level of confidence is associated with the interval. The level of confidence is the minimum expected relative frequency of the interval containing the true parameter's value if new random samples were repeatedly taken from the same population and a confidence interval were to be calculated from each sample.

CONFIDENCE INTERVAL AND LEVEL OF CONFIDENCE	A confidence interval is constructed from a random sample and provides an interval of possible values of a parameter. Associated with the interval is a level of confidence that is the minimum expected relative frequency with which similarly constructed intervals will contain the parameters that they estimate.

When constructing a confidence interval from binomial data, the upper $(1-\alpha)\cdot 100\%$ limit when t failures are observed, is θ_u defined through the equation

$$P(\text{number of failures} \leq t | \theta_u) = \alpha.$$

For example, suppose that a random sample of $n = 50$ trials is taken in order to determine the failure rate θ of a component. If the 50 trials result in two failures, the 95% upper confidence interval for θ includes all values of θ between 0 and θ_u where θ_u, the upper 95% confidence limit, is the solution to the equation

$$P(\text{number of failures} \leq 2|\theta_u) = \sum_{i=0}^{2} \binom{50}{i}\theta_u^i(1-\theta_u)^{50-i}$$

$$= (1-.95) .$$

More generally, if an experiment with n trials results in r failures, the upper $(1-\alpha)\cdot 100\%$ confidence limit, θ_u, is the solution to the equation

$$\sum_{i=0}^{r} \binom{n}{i}\theta_u^i(1-\theta_u)^{n-i} = \alpha . \qquad (1)$$

In order to understand how this formula works, assume again that $n = 50$, $\alpha = .05$ and, in addition, $\theta = .10$. Notice that the true value of θ has been specified but not the outcome of the experiment, r. Now, when $\theta = .10$ and $n = 50$, the probabilities of the various values of r are easily calculated using the binomial probability model. These probabilities are shown below along with the value of θ_u calculated from equation (1) using the appropriate r. Since $\theta = .10$ by assumption in this example, the upper confidence interval will contain θ only when $\theta_u > .10$. Note that this occurs only if $r > 1$. But $r > 1$ with probability $1-(.005+.029) = .966$, so that the probability of a correct interval $(0 \leq \theta \leq \theta_u)$ is .966 when $\theta = .10$.

r	Binomial Probability	θ_u
0	.005	.058
1	.029	.091
2	.078	.121
3	.139	.148
4	.181	.174
5	.185	.199
6	.154	.223
7	.108	.246
8	.064	.270

The actual probability of a correct interval is known as the coverage. The coverage varies as the assumed value of θ varies, but is never less than the level of confidence. The reader may verify that the coverage equals the level of confidence whenever $\theta = \theta_u$. Otherwise, the coverage exceeds the level of confidence.

COVERAGE	Coverage is the expected relative frequency with which confidence intervals for a parameter, computed from independent random samples, will contain the value of the parameter. The coverage depends upon the particular value of the parameter.

In practice, the value of θ is unknown, and the observed value of r is used to construct the interval. Once r is observed, the interval either contains the true value of θ or it does not contain the true value. That is, the probability that the confidence interval is correct is either 1 or 0. This is why the term level of confidence is used rather than probability. The notion of probability cannot be applied once the experiment is conducted. This idea is illustrated in Figure 1 where confidence intervals are shown for a number of different samples of size n = 50, with θ = .10. Each interval is either correct or not correct. In this illustration, all intervals are correct except the interval obtained from sample number 3.

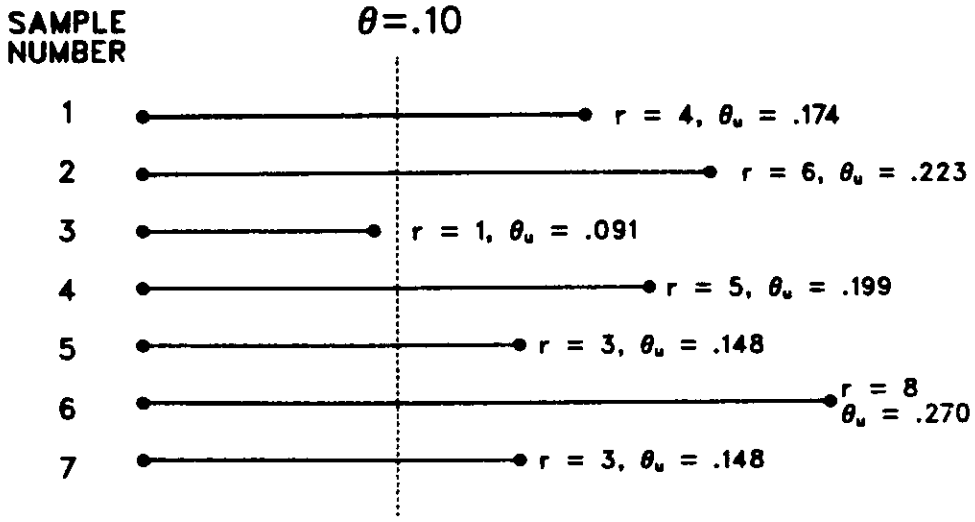

Figure 1. Confidence Intervals for a Binomial Parameter

3. CONFIDENCE INTERVALS FOR A SYSTEM

Confidence interval estimation for a system is more difficult than confidence interval estimation for a single parameter. The difficulty is due to the fact that the failure rate for a system

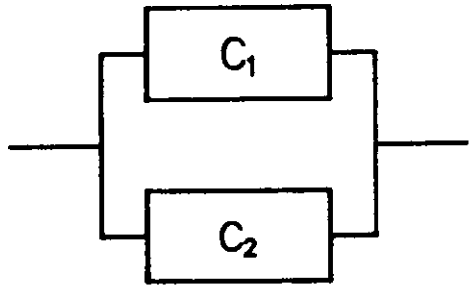

Figure 2. Simple Parallel System

is a composite of failure rates of many components, and a single failure rate of a system may be the result of many different combinations of component failure rates. For example, in the simple parallel system given in Figure 2, the system failure rate is $f(\theta_1,\theta_2) = \theta_1\theta_2$ where θ_1 and θ_2 are the failure rates of the components C_1 and C_2. If $f(\theta_1,\theta_2) = .25$, then any combination of θ_1 and θ_2 such that $.25 \leq \theta_1 \leq 1$ and $\theta_2 = .25/\theta_1$, is possible. More generally, if the component failure rates are $\theta = (\theta_1, \ldots,\theta_k)$ in a k component system, there are many possible combinations of failure rates that are consistent with the system failure rate $\theta_s = f(\theta)$.

Construction of a confidence interval for a system will be illustrated using the parallel system with two components. So that the example remains manageable, the sample sizes will be kept small. Suppose that n_1 trials are performed with component C_1 and n_2 trials with component C_2. If $n_1 = 16$ and $n_2 = 8$, then there are $(16 + 1)(8 + 1) = 159$ possible outcomes to the experiments corresponding to the possible combinations of $r_1 = 0,\ldots,16$ failures of C_1 with $r_2 = 0,\ldots,8$ failures of C_2.

When a confidence interval for a single component failure rate, θ, is constructed, there is a natural ordering of the values of r from 0 to n. In the case of a system, some ordering of pairs of values (r_1,r_2) must be used. One method of ordering the points is according to

$$\hat{\theta}_s = [(r_1+1)/(n_1+2)][(r_2+1)/(n_2+2)] .$$

Another method of ordering the points is to use $\hat{\theta}_s = (r_1/n_1)(r_2/n_2)$. This second method will result in $\hat{\theta}_s=0$ whenever $r_1=0$ regardless of the values of n_1, n_2 and the other r_i. There is no universal agreement on a best method to use (see Chapter 10 of Mann, Schafer and Singpurwalla, 1974 for a more detailed discussion of this point).

We are now prepared to determine whether or not a particular outcome of (r_1,r_2) should result in a particular system failure

rate, θ_s, being included in the confidence interval. Suppose that $t = [(r_1+1)(r_2+1)]/[(n_1+2)(n_2+2)]$. The upper $(1-\alpha)\cdot 100\%$ confidence limit, denoted by θ_u, is the largest value of the system failure rate such that there are θ_1 and θ_2 which satisfy both

$$P(\hat{\theta}_s \leq t | \theta_1, \theta_2) = \alpha ,$$

and

$$\theta_u = f(\theta_1, \theta_2)$$
$$= \theta_1 \cdot \theta_2 .$$

Continuing the example, if $r_1 = 10$ and $r_2 = 1$, then $t = (11)(2)/[(18)(10)] = 11/90$. When $\theta_1 = 1.0$ and $\theta_2 = .25$,

$$P(\hat{\theta}_s \leq 11/90 | \theta_1 = 1.0, \theta_2 = .25) = .10$$

and

$$\theta_s = (1.0)(.25)$$
$$= .25.$$

By computation with binomial probabilities, one may verify that $\theta_s = .25$ is the largest value so that $\theta_s \leq 11/90$ has no more than a .10 chance for all values of θ_1 and θ_2 satisfying $\theta_1\theta_2 = .25$. The idea behind this procedure is that for any value of θ_s greater than .25, the data are inconsistent (too unlikely to have occurred) given the possible values of θ_1 and θ_2. Thus, $\theta_u = .25$ is taken as the upper 90% confidence limit.

Now consider a particular value of θ_s such as $\theta_s = .25$. The coverage of this value is the frequency with which the upper confidence limit exceeds .25, where the frequency is calculated using some assumed values θ_1 and θ_2 such that $\theta_1\theta_2 = .25$. However, the coverage will depend upon the particular choice of θ_1 and $\theta_2 = .25/\theta_1$. When $\theta_1 = 1$ and $\theta_2 = .25$, the coverage for a 90% confidence interval is .9, while at $\theta_1 = .25$ and $\theta_2 = 1$, the coverage is close to .94. Figure 3 shows the coverage of $\theta_s = .25$ for all possible values of θ_1 and $\theta_2 = .25/\theta_1$. Figures 4 and 5 show similar graphs of the coverage of $\theta_s = .10$ and $\theta_s = .40$.

It is important to note that the coverage is always equal to or greater than the level of confidence; and therefore, the level

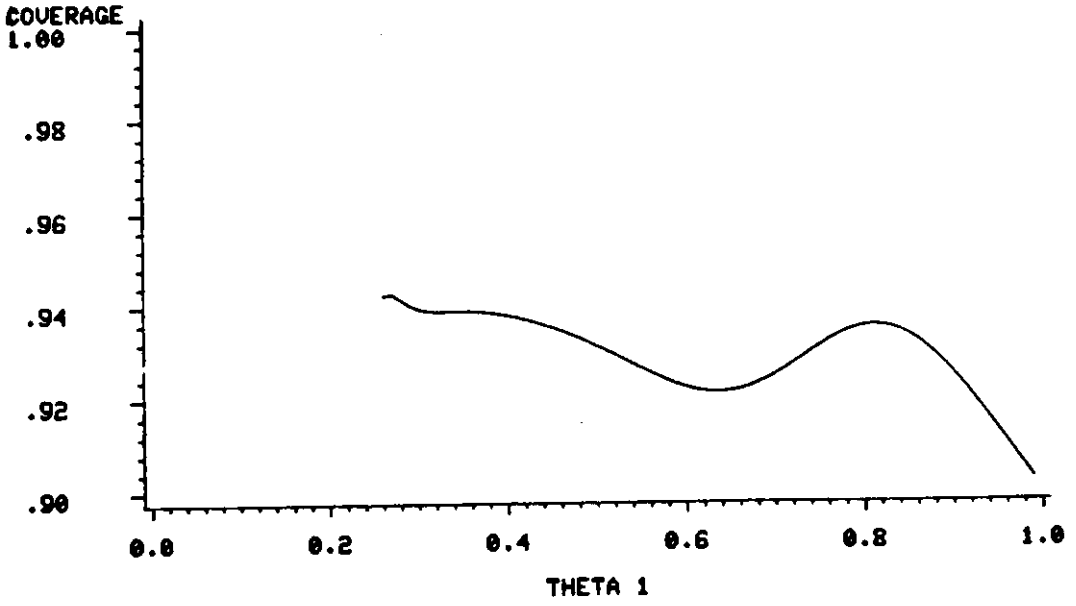

Figure 3. Coverage for System Failure Rate with θ_s = .25 and α = .10

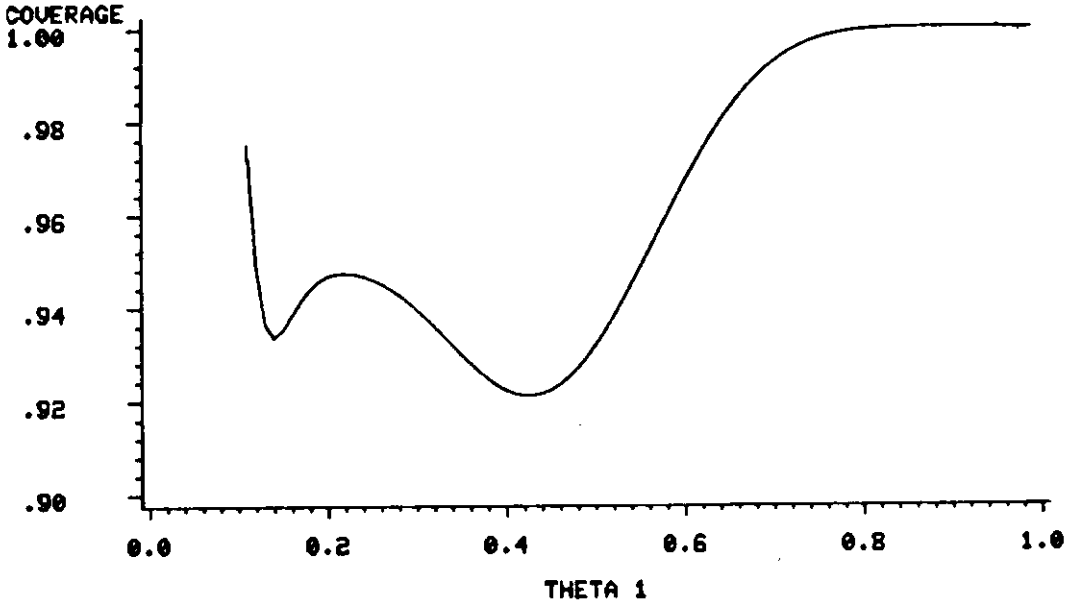

Figure 4. Coverage for System Failure Rate with θ_s = .10 and α = .10

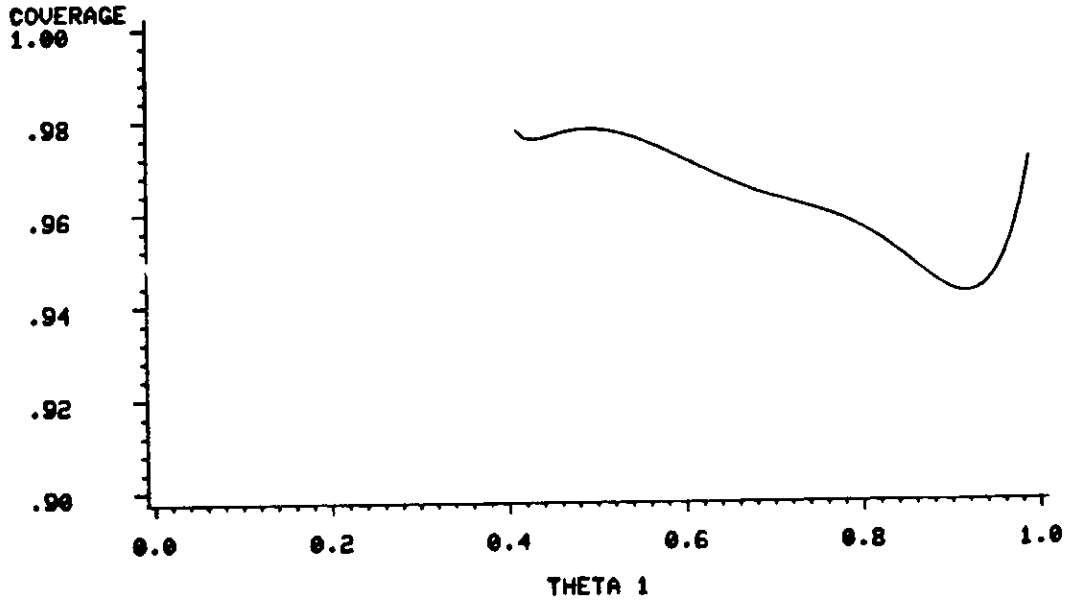

Figure 5. Coverage for System Failure Rate with θ_S = .40 and α = .10

of confidence is correctly interpreted as the minimum level of coverage.

4. REVIEW OF THE MAXIMUS/BOUNDING ANALYSIS

The Spencer and Easterling (1984b) methodology has two components: confidence interval estimation and a bounding analysis. Confidence interval estimation is implemented using the Maximus methodology. This procedure is only used in conjunction with data derived from designed experiments, quasi-designed experiments, or simple observation without a designed experiment. Such data are termed objective in this comparison. The bounding analysis is used with any data that do not fall in one of the specified categories. Such data are termed subjective in this analysis. In Easterling, Spencer and Diegert (1985), the terms data and data-free are used for these two types of information, respectively. The central difference between the data types, objective and subjective, is the extent to which the process that produces the data satisfies the assumptions associated with a probability model. For example, when a component is observed to have two failures in 200 years of operation, one may assume temporal independence and therefore that the process satisfies the assumptions of a Poisson probability model in order to relate the data to the model's parameters. On the other hand,

when an expert estimates that a component will fail an average of once every 100 years, or perhaps, that the component's failure rate is between .005/year and 0.2/year, the association of a probability model with the expert's estimate is far less justified. The position taken in the Maximus/bounding methodology is that any such an association is unjustified.

The second component of the Spencer and Easterling analysis is bounding for subjectively-estimated, system-model parameters, i.e., those parameters that do not have an associated probability model and data. Each of these subjectively estimated parameters, ω_i for $i = 1,\ldots,m$ is given a range of values, say L_i to U_i, that represent a plausible "ballpark" interval arrived at from a consensus of informed opinion. There is no attempt to associate ω_i with probabilities of obtaining L_i and U_i.

The bounding analysis is performed by holding the objectively estimated parameters fixed at some value denoted by θ^* and allowing $\omega' = (\omega_1,\ldots,\omega_m)$ to vary so that the modeled system reliability is minimized and maximized with respect to ω, conditional on fixed θ^*. The nominal point θ^* must be derived apart from the confidence interval analysis, since the confidence interval does not imply that any point inside the interval is more or less likely than any other point. Nor does the confidence interval imply that points inside the interval are equally likely. It just does not address this idea.

The combined confidence interval and bounding analyses generate three intervals. The first of these displays the role of subjective information in the uncertainty estimate conditional on the objectively estimated quantities being held constant. The second interval displays the role of the objective information with the subjectively estimated quantities being held constant, while the third interval combines both types of information to present the overall uncertainty.

4.1 Bounds Using Subjective Information (Subjective Uncertainty)

The system model expresses the system failure rate as a function of both objectively and subjectively estimated parameters

$$f = f(\underset{\sim}{\theta}, \underset{\sim}{\omega}) \ .$$

Fixing $\underset{\sim}{\theta}$ at a nominal point $\underset{\sim}{\theta}^*$ and calculating bounds on f implied by bounds on $\underset{\sim}{\omega}$ gives

$$[\min_{\underset{\sim}{\omega}} f(\underset{\sim}{\theta}^*, \underset{\sim}{\omega}), \max_{\underset{\sim}{\omega}} f(\underset{\sim}{\theta}^*, \underset{\sim}{\omega})] \ , \qquad (2)$$

where the min and max are taken over $\bigcup_{i=1}^{m} [L_i, U_i]$. These bounds express the conditional uncertainty because of ambiguity about the subjectively estimated parameters. The interval is conditional because $\underset{\sim}{\theta} = \underset{\sim}{\theta}^*$ is assumed. There is no level of coverage associated with these bounds.

4.2 Intervals Using Objective Information (Statistical Uncertainty)

A second interval displays the system failure uncertainty attributable to ambiguity about the objectively estimated parameters given that the subjective estimates are held constant. The calculation of this interval requires that a single nominal point $\underset{\sim}{\omega}^*$ for the subjectively estimated parameters be specified. The system confidence interval is constructed using Maximus with the conditional system model

$$f^*(\underset{\sim}{\theta}) = f(\underset{\sim}{\theta}, \underset{\sim}{\omega}^*).$$

The calculation of the interval depends in a complex manner on the chosen $\underset{\sim}{\omega}^*$. That is, there is no simple relationship between Maximus solutions at different values $\underset{\sim}{\omega}^*$. The Maximus confidence interval methodology is used with $f^*(\underset{\sim}{\theta})$ and the data about θ, x, to obtain the conditional interval

$$[c_1(x, \underset{\sim}{\omega}^*), c_2(x, \underset{\sim}{\omega}^*)] . \qquad (3)$$

4.3 Overall Uncertainty Bounds

The third interval is used to characterize the overall uncertainty about f and is given as

$$[\min_{\underset{\sim}{\omega}} c_1(x, \underset{\sim}{\omega}), \max_{\underset{\sim}{\omega}} c_2(x, \underset{\sim}{\omega})]. \qquad (4)$$

This interval is conceptually constructed by calculating confidence intervals for all possible values of ω_i within the bounds L_i and U_i. The smallest lower bound, and the largest upper bound among all these conditional confidence intervals produce the lower and upper overall uncertainty bounds. When f is monotone in $\underset{\sim}{\omega}$, $\min_{\omega} c_1(x, \underset{\sim}{\omega})$ is calculated by applying Maximus

with $\underset{\sim}{\omega}' = (L_1,\ldots,L_m)$. Likewise, $\max_{\omega} c_2(x, \underset{\sim}{\omega})$ is calculated by applying Maximus with $\underset{\sim}{\omega}' = (U_1,\ldots,U_m)$. Thus, under the assumption of monotonicity, only two applications of Maximus are required. If monotonicity is not satisfied, the calculation of the interval may be formidable and involve significant iteration between Maximus and a search algorithm for extrema.

The overall uncertainty bounds are the smallest $(1-\alpha)$ 100% lower confidence bound and largest $(1-\alpha)$ 100% upper confidence bound, obtainable over the entire range of the bounded quantities. However, there is no measure of coverage available for the interval formed by these bounds.

To these three intervals is added a point estimate $f(\theta^*, \omega^*)$ that is determined apart from the confidence interval and bounding methodologies. In the problems addressed in PRA uncertainty analyses, this estimate will almost always be within all three intervals. We next provide a demonstration of the three intervals in a simplified context.

4.4 Example of Maximus/Bounding Intervals

The three types of intervals associated with the Maximus/bounding methodology can be demonstrated using a simple model with two identical components in series, one of which is recoverable by operator action. The block diagram of this simple system is given in Figure 6 where A_1 and A_2 are the identical components and R_2 is the event that A_2 is recovered. Denoting the failure probability of A_1 and A_2 by θ and the probability of nonrecovery by ω, gives a system failure probability of

$$f(\theta, \omega) = 1 - (1-\theta)(1-\theta\omega).$$

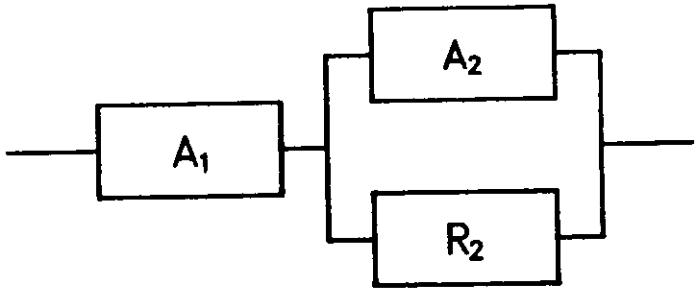

Figure 6. Block Diagram of a Simple System

The data available for estimating θ are 2 failures in 100 trials while ω is subjectively estimated as .3 with bounds of .1 and .5. The nominal point estimate using the maximum likelihood estimate (mle) for θ, is then

$$f(\theta^*, \omega^*) = 1 - [1 - (2/100)][1 - (2/100)(.3)] = .0259,$$

and the subjective uncertainty interval with $\theta = \theta^*$, the nominal estimate, has lower and upper limits of

$$\min_{\omega} f(\theta^*, \omega) = 1 - [1-(2/100)][1 - (2/100)(.1)] = .0220,$$

$$\max_{\omega} f(\theta^*, \omega) = 1 - [1 - (2/100)][1 - (2/100)(.5)] = .0298$$

These bounds are shown in Figure 7.

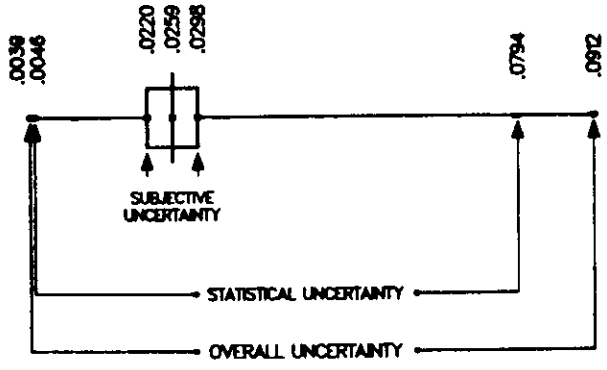

Figure 7. Maximus/Bounding Intervals

Next, the limits for the statistical uncertainty are found using the Maximus approximation. The intervals that are formed by allowing θ to vary from θ^* are more difficult to calculate. This is because the data are split or unpooled between A_1 and A_2 in a manner that depends upon the assumed value of ω. When $\omega = \omega^*$, the unpooling suggested in Spencer and Easterling (1984a) gives effective data of 77 trials for A_1 and 23 trials for A_2, and the system equivalent data would be 2 failures in 77 trials. The system equivalent binomial data of 2 failures in 77 trials yields a lower 95% limit of

$$c_1(x, \omega^*) = .0046$$

and an upper 95% limit of

$$c_2(x, \omega^*) = .0794.$$

These limits, as shown in Figure 7, are calculated from the system equivalent data using standard procedures for confidence intervals for a proportion (failure rate) estimated from success-failure data.

The reason for unpooling is that Maximus cannot directly solve for a system's reliability confidence interval when redundancies of components occur across subsystems. Unpooling is a conservative procedure and therefore unpooling is done in a manner that minimizes the conservatism. This is why unpooling is accomplished in a manner that maximizes the effective sample size for the system.

Maximus is an approximate procedure that can be used in complex systems where exact confidence intervals cannot be found. This case is simple enough for exact confidence intervals to be found. Without going into details, a variation of the procedure illustrated earlier is used to find exact confidence limits. In this case, they are .0046 and .0790 for ω=.3.

The interval that combines both data and subjective uncertainty is also computed by unpooling data. Using the lower bound for ω, a lower 95% confidence interval limit is found, while the upper 95% confidence interval limit is evaluated with ω fixed at its upper bound. If $f(\theta, \omega)$ were not monotone with respect to ω for all values of θ, the calculation of this third interval could become very involved.

The unpooling with ω = .1 gives effective data of 91 trials for A_1 and 9 for A_2. The system equivalent data is 2.00 failures in 91 trials giving a lower bound of

$$\min_{\omega} c_1(x, \omega) = .0039$$

using Maximus. The exact lower bound is also .0039. Similarly, the upper bound is found by using ω = .5, unpooling to obtain 66.6 and 33.3 equivalent trials for A_1 and A_2. The system equivalent data of 1.98 failures among 66.6 trials give a bound of

$$\max_{\omega} c_2(x, \omega) = .0912$$

using Maximus. This is slightly larger than the exact bound, which is .0905. These bounds are also shown in Figure 7. The nominal point $f(\underset{\sim}{\theta}^*, \underset{\sim}{\omega}^*)$ has been added to Figure 7. Thus, Figure 7 is the seven-point representation used by Spencer and Easterling.

5. THE CONSTRUCTION OF PROBABILITY DISTRIBUTIONS FOR SYSTEM PARAMETERS FROM DATA--BAYES' THEOREM

The Bayes/Monte Carlo methodology assesses the reliability of a system by associating probability distributions with the parameters (failure rates) of the system's components. These probability distributions fall under two headings: prior distributions and posterior distributions.

PRIOR DISTRIBUTION	The prior distribution for a system parameter θ is denoted by $p(\theta)$ and represents the best information about the parameter without taking into account the available data on the current system.

The prior distribution depends upon the information that is available about the parameter of interest. If data are available from other plants, they may provide a basis for forming the prior distribution. When data from other plants are not available, it may be necessary to use expert opinion to construct the prior distribution.

POSTERIOR DISTRIBUTION	The posterior distribution for a system parameter θ is denoted by $p(\theta \mid x)$ and represents a revision of the prior distribution based on available data, x.

The posterior distribution is used to represent the distribution of the system parameter θ for the current plant based on reweighting or revising the prior distribution on the basis of data from the current plant.

The prior and posterior distributions for the system parameter θ are related to each other through a third distribution that serves as a conditional probability model for the available data, x. The binomial distribution is an important example of such a model.

CONDITIONAL PROBABILITY MODEL FOR THE DATA	The available data on a system with a given failure rate θ is described by a conditional probability model denoted by $f(x\|\theta)$. A frequently assumed conditional probability model is the binomial distribution.

The prior distribution and the conditional probability model are used to calculate the posterior distribution through Bayes' Theorem.

BAYES' THEOREM	$$p(\theta\|x) = p(\theta)f(x\|\theta) \Big/ \int_{\text{all } \theta} p(\theta)f(x\|\theta)\, d\theta$$

Of these three probability distributions, $p(\theta)$ or the prior distribution is frequently subjectively based. The conditional probability model $f(x|\theta)$ is usually evaluated using objective data. Thus, the posterior distribution $p(\theta|x)$ is frequently a combination of subjective and objective information. That is, it represents a reweighting of the prior distribution (or subjective information) by the objective information. The following example illustrates the above definitions.

5.1 Example of Bayes' Theorem

To illustrate the mechanics of Bayes' theorem, we will perform a short analysis. Suppose that the task is to develop a posterior probability distribution for a parameter θ that is defined to be the relative frequency of a tack landing point up when randomly tossed.

Suppose at the outset we consider every value of θ between 0 to 1 to be equally probable. The prior distribution for the parameter θ would then be a uniform probability distribution on the interval [0,1]. Thus

$$p(\theta) = 1 \quad 0 \leq \theta \leq 1$$

$$= 0 \text{ elsewhere.}$$

Next, suppose an experiment is conducted to obtain information about θ. The experiment consists of randomly tossing the tack 50 times and observing the number of times the tack lands with its point up. Denote by r the observed number of times the point is up. The conditional probability model for the data is binomial,

$$b(50,r) = f(r|\theta,50) = \binom{50}{r} \theta^r (1-\theta)^{50-r}.$$

Combining $p(\theta)$ and $f(r|\theta,50)$ by Bayes' Theorem gives the posterior distribution for θ as,

$$p(\theta|r) = \binom{50}{r} \theta^r (1-\theta)^{50-r} \Big/ \int_0^1 \binom{50}{r} \theta^r (1-\theta)^{50-r} d\theta$$

$$= (50+1)\binom{50}{r} \theta^r (1-\theta)^{50-r}.$$

Now, $p(\theta|r)$ can be used to make probability statements about θ. For example, if $r=20$ one can compute the probability that θ is equal to or less than 1/2 by

$$\int_0^{1/2} p(\theta|r = 20)d\theta = .9196.$$

Likewise, one can say that the expected value of θ (or the mean of the distribution of θ) is

$$\int_0^1 \theta\, p(\theta|r = 20)d\theta = 21/52$$
$$= .4038,$$

which compares with an expected value of .5 for θ from the prior distribution. Thus, the impact of the data has been to weight the mean downward toward the point estimate $r/n = 20/50 = .4000$. This ends the example.

The role of the prior distribution in the preceding example is to serve as a vehicle for developing a density for θ conditional on the data, that is $p(\theta|r)$. The selected prior distribution on θ is uniform and has a density whose graph appears in Figure 8. After weighting the prior distribution with the data, the posterior distribution for θ has a density whose graph appears in Figure 9. Figures 8 and 9 make it clear that the posterior distribution for θ does not closely resemble the prior distribution for θ. Thus, the data have played a major role in shaping the posterior distribution. In fact, the posterior distribution is exactly proportional to the conditional binomial probability model with r fixed at 20.

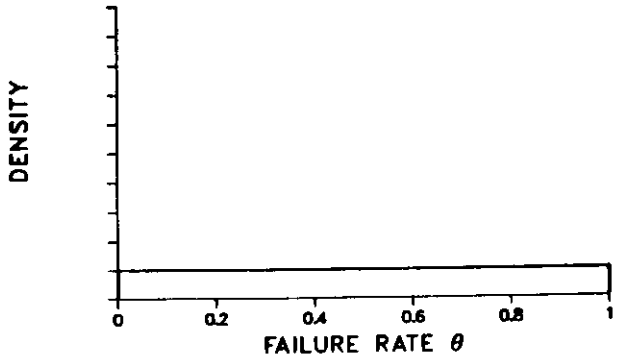

Figure 8. Beta Prior with a = 1 and b = 1.

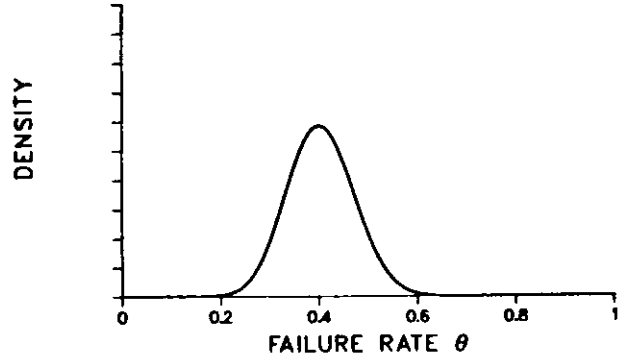

Figure 9. Beta Posterior with a = 21 and b = 31

The role of the prior distribution in forming the revised distribution can be minimized by selecting a distribution that permits the conditional probability model of the data, and the data, to play the dominant role in determining the shape of the posterior distribution. While there exist various criteria for selecting a prior with minimum influence (see Good, 1965), there is no single prior for a failure rate that satisfies all criteria. One must then choose from several candidate priors. The members of the set of candidate priors will be called noninformative priors recognizing that no member is noninformative according to all criteria. The impact of the selection of the prior distribution for θ on the posterior distribution for θ is examined in detail next.

NONINFORMATIVE PRIOR DISTRIBUTION	A noninformative prior distribution allows the data to take a dominant role in shaping the posterior distribution. Thus, its role is not to introduce subjective information but to serve as a vehicle to allow the parameter θ to be treated as a random variable via Bayes' Theorem.

6. NONINFORMATIVE DISTRIBUTIONS AND BINOMIAL DATA

Data in the form of r failures observed in n trials or demands is normally employed for estimating a system's reliability. The conditional probability model for the data is the binomial model,

$$f(r|\theta,n) = \binom{n}{r}\theta^{r}(1-\theta)^{n-r} \quad , \quad 0 \leq \theta \leq 1 \qquad r=0,1,\ldots,n.$$

The data and the model will be denoted by b(n,r).

Among the possible noninformative prior distributions for θ are several members of the beta family of distributions. The generic form of a beta distribution is

$$p(\theta|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ \theta^{a-1}(1-\theta)^{b-1} \quad , \quad 0 < \theta \leq 1 \qquad a,b > 0,$$

where Γ(x) is the gamma function. The beta distribution with parameters a and b will be denoted by ß(a,b).

The beta distribution is used in this study to represent the prior distribution on θ for several reasons.

1. The range of support is the interval [0,1], which is consistent with failure rates.

2. The beta distribution is very flexible and, with the proper choice of a and b, can be used to represent most shapes of distributions from symmetric to highly skewed to U-shaped or J-shaped.

3. The beta distribution is very tractable from a mathematical standpoint; and when combined with b(n,r), the resulting posterior distribution for θ is ß(a+r,b+n-r). This result can be verified by a simple application of Bayes' Theorem.

Among the choices of ß(a,b) that are noninformative, there are four densities that will be given particular attention. A more complete discussion of noninformative prior densities is provided in Good (1965). The first of these is ß(1,1), which is uniform on the interval [0,1]. Figure 10 shows the prior and posterior densities for θ when ß(1,1) is combined with b(50,2).

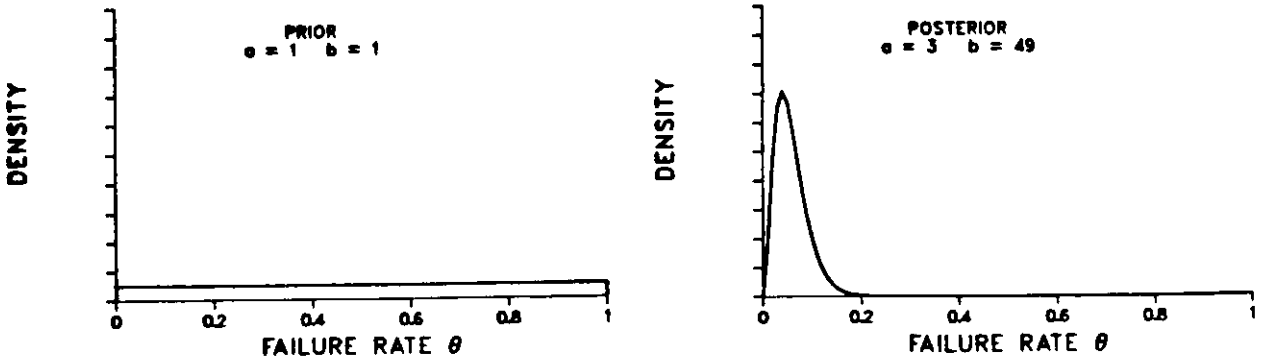

Figure 10. Prior and Posterior Densities for θ when ß(1,1) is combined with b(50,2).

The next three noninformative densities are termed improper distributions because they each violate the restriction a,b>0 needed with the beta density. Thus, these distributions are improper because they cannot be integrated to obtain unity. However, as long as at least one success and one failure are observed, the resulting posterior distribution for θ will be a proper probability distribution. The first improper noninformative distribution is ß(0,1). As long as r>0, the posterior will be a proper probability distribution. Figure 11 shows the prior and posterior densities for θ when ß(0,1) is combined with b(50,2).

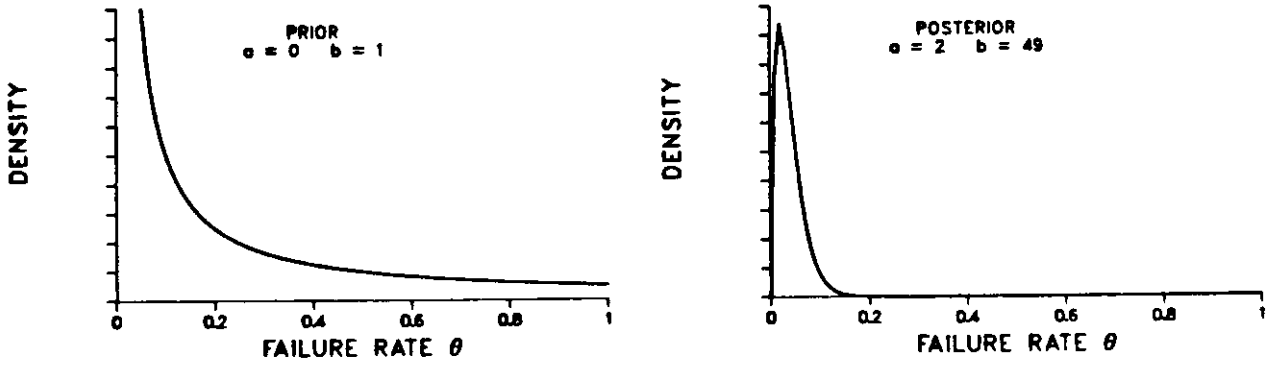

Figure 11. Prior and Posterior Densitites for θ when ß(0,1) is combined with b(50,2)

The next improper noninformative prior distribution to be considered is ß(1,0). As long as n-r>0, the posterior will be a proper probability distribution. Figure 12 shows the prior and posterior densities for θ when ß(1,0) is combined with b(50,2).

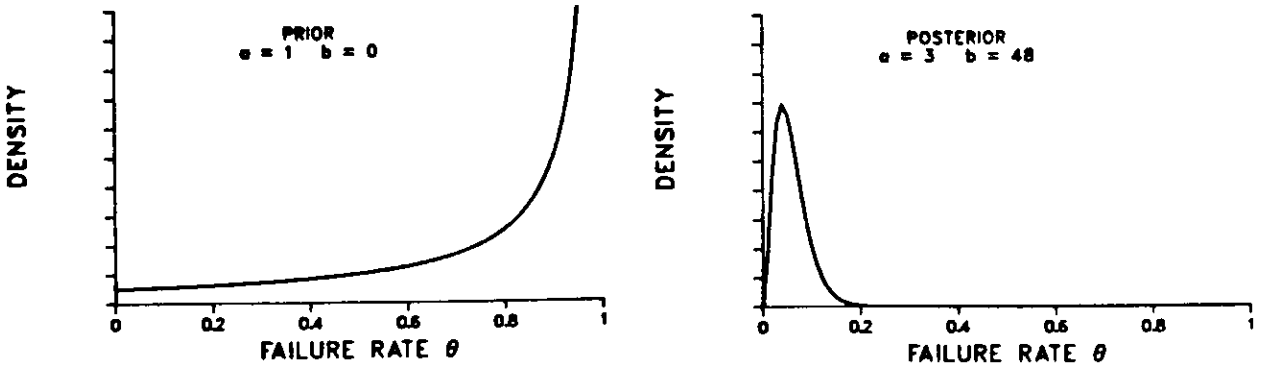

Figure 12. Prior and Posterior Densities for θ when ß(1,0) is combined with b(50,2)

The last noninformative prior distribution to be considered is ß(0,0). As long as r>0 and n-r>0, the posterior will be a proper probability distribution. Figure 13 shows the prior and posterior densities for θ when ß(0,0) is combined with b(50,2).

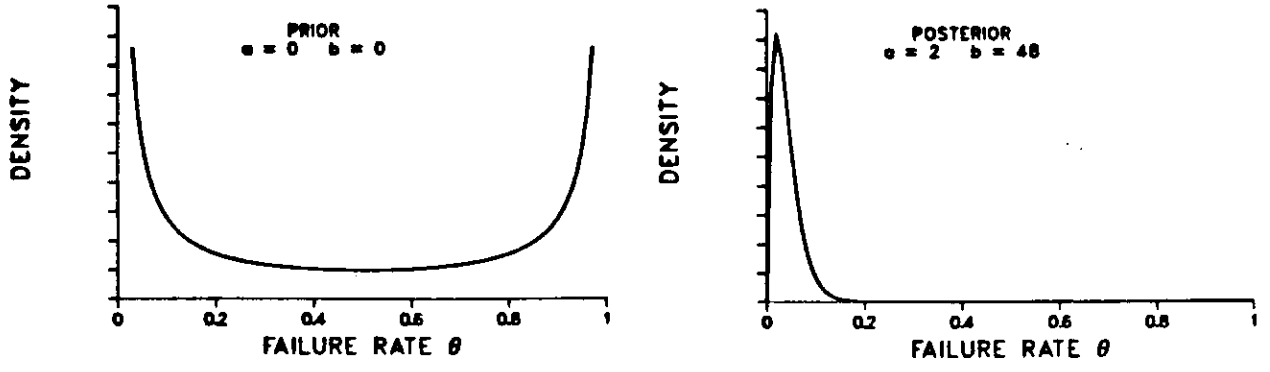

Figure 13. Prior and Posterior Densities for r when ß(0,0) is combined with b(50,2)

The posterior distributions in Figures 10 to 13 show a great deal of similarity in their shape and thus a great deal of resilience against widely differing priors. Part of this resilience is due to the value of n. If n were smaller, the posteriors would appear less similar in shape, while increasing n would increase the similarity.

Each of the above posterior distributions will now be compared on the basis of sampling theory estimates of θ that are derived from b(n,r). The maximum likelihood and centroid likelihood estimates are sampling theory estimates of θ that will be used for comparison. The maximum likelihood estimate (mle) is the value of θ that maximizes $f(r|\theta,n)$ evaluated at the observed data r. The mle is just r/n. The centroid likelihood estimate (cle) is

$$cle = \int_0^1 \theta f(r|\theta,n)d\theta \Big/ \int_0^1 f(r|\theta,n)d\theta \quad .$$

$$= (r+1)/(n+2).$$

The mle is a "modal" value while the cle is an "average" value. The posterior distributions of θ have the following means and modes:

Prior	Posterior	Posterior Mean	Posterior Mode
$\beta(1,1)$	$\beta(r+1,n-r+1)$	$(r+1)/(n+2)$**	r/n*
$\beta(0,1)$	$\beta(r,n-r+1)$	$r/(n+1)$	$(r-1)/(n-1)$
$\beta(1,0)$	$\beta(r+1,n-r)$	$(r+1)/(n+1)$	$r/(n-1)$
$\beta(0,0)$	$\beta(r,n-r)$	r/n*	$(r-1)/(n-2)$

*Same as maximum likelihood estimate from b(n,r).
**Same as centroid likelihood estimate from b(n,r).

Thus, $\beta(0,0)$ yields a mean equal to the mle, while $\beta(1,1)$ has a mean equal to the cle and a mode equal to the mle. The prior $\beta(1,0)$ is of interest because the value of θ having the cumulative posterior probability of $1-\alpha$ is the same value as the $(1-\alpha)\cdot 100\%$ upper confidence limit. Likewise, the prior $\beta(0,1)$ is of interest because the value of θ having the cumulative posterior probability of α is the same value as the $(1-\alpha)\cdot 100\%$ lower confidence limit. Another useful choice for a prior is $\beta(.5, .5)$, but the reasons for choosing this prior are based on complex theoretical arguments that are beyond the scope of this discussion.

There is no clear-cut choice which of the above prior distributions is best. We will now use each of the four prior distributions to obtain an interval for the uncertain parameter θ using the confidence interval approach of Section 2. Again, consider the sample information (n = 50, r = 2). The four 5%

and four 95% probability limits and the lower and upper 95% confidence limits are as follows:

Prior	Posterior	Lower 5% Limit	Upper 95% Limit
ß(1,1)	ß(3,49)	.016	.118
ß(0,1)	ß(2,49)	.007*	.091
ß(1,0)	ß(3,48)	.017	.121**
ß(0,0)	ß(2,48)	.007	.093

*This lower limit is identical to the lower 95% confidence limit.
**This upper limit is identical to the upper 95% confidence limit.

The reader will find that the resulting intervals are similar to each other and similar to the confidence interval found earlier. There is, however, a very substantial difference in the interpretation of the intervals. The quantitative measure of the reliability of the interval is a probability when the interval is estimated using Bayesian methods. In contrast, the quantitative measure of the reliability of a confidence interval is a level of confidence explained earlier.

6.1 Bayesian Analysis of a System's Failure Rate

The Bayesian methodology for estimating the failure rate of a system has three stages:

1. Prior distributions for the component failure rates are constructed.

2. The data and probability models for the data are used to revise the prior information. Posterior distributions are thus obtained for each component using Bayes' Theorem.

3. The probability distribution of the system's failure rate is obtained from the component posterior failure rate distributions. The calculation of the distribution may be accomplished analytically or by a numerical procedure such as Monte Carlo simulation.

An illustration of these steps will be given using the two component parallel system shown in Figure 2 in the illustration of a confidence interval for a system's failure rate. Recall that n_1=16 and n_2=8 observations are to be made on the reliabilities of each of the two components.

The first step is the construction of prior distributions for θ_1 and θ_2, the failure rates of the two components. Assuming that either no prior information is available or that prior subjective information is to be excluded from the analysis, we use noninformative prior distributions for θ_1 and θ_2. The choice of the $\beta(1,1)$ distribution will ensure that the posterior distribution is proper even if $r_1=0$, $r_1=n_1$, $r_2=0$, or $r_2=n_2$. The $\beta(0,0)$, $\beta(1,0)$, and $\beta(0,1)$ distributions will fail to produce a proper posterior distribution under some of these circumstances.

The application of Bayes' theorem to the $\beta(1,1)$ priors yields posterior distributions of $\beta(r_1+1, n_1-r_1+1)$ and $\beta(r_2+1, n_2-r_2+1)$, respectively. Since the system failure rate θ_s is just the product of θ_1 and θ_2, the complementary cumulative probability of θ_s can be found analytically by

$$P(\theta_s \geq y|\theta_1,r_2) = \int_y^1 \int_{y/\theta_1}^1 p(\theta_1|r_1+1,n_1-r_1+1)p(\theta_2|r_2+1,n_2-r_2+1)d\theta_2 d\theta_1$$

where $p(\theta|a,b)$ is the beta density. The region of integration $y \leq \theta_1 \leq 1$, $y/\theta_1 \leq \theta_2 \leq 1$ includes all values of $\theta_1\theta_2$ such that $\theta_s \geq y$. Note that $P(\theta_s \geq y|r_1,r_2)$ is conditional on the particular experimental outcome of r_1 and r_2.

For given r_1 and r_2, the lower and upper $(1-\alpha)$ 100% probability limits are the values y_ℓ and y_u that satisfy the equations

$$P(\theta_s \geq y_\ell|r_1,r_2) = 1-\alpha$$
$$P(\theta_s \geq y_u|r_1,r_2) = \alpha .$$

Note that α and $1-\alpha$ are probabilities even when the values of r_1 and r_2 are specified. Thus, it is not necessary to use a term such as level of confidence to distinguish α from a probability.

Next, we will investigate the properties of the intervals formed by 0 and y_u. We say that $[0,y_u]$ covers θ_s if $0 \leq \theta_s \leq y_u$. Now, suppose that the values of θ_1 and θ_2 are t_1 and t_2, respectively. What is the probability that the actual system failure rate $\theta_s=t_1t_2$ is covered by $[0,y_u]$? To answer this question, we must first determine all those experimental outcomes that result in intervals $[0,y_u]$, which cover θ_s, and then find the total probability of obtaining one of these outcomes given that $\theta_1=t_1$ and $\theta_2=t_2$.

Now, consider the case when $n_1=16$ and $n_2=8$. In order to examine the coverage of $\theta_s=.25$, when $\alpha=.10$, we must first determine the values of r_1 and r_2 that lead to intervals $[0,y_u]$ that cover

θ_s. Evaluating $P(\theta_s \geq .25|r_1,r_2)$ at all values of r_1, r_2 reveals that $P(\theta_s \geq .25|r_1,r_2) \geq .10$ and thus $y_u > .25$ except when $r_1 \leq 1$ regardless of r_2; $r_2=0$ regardless of r_1; $r_1=2$ and $r_2 \leq 7$; $r_1=3$ and $r_2 \leq 5$; $r_1=4$ and $r_2 \leq 4$; $r_1=5$ and $r_2 \leq 3$; $6 \leq r_1 \leq 7$ and $r_2 \leq 2$; or $8 \leq r_1 \leq 11$ and $r_2 \leq 1$. Denote by R_0 the set of values of r_1, r_2 that result in $y_u < .25$, that is, no coverage of $\theta_s=.25$. The probability of obtaining one of the outcomes in R_0 can then be computed for various values of t_1 and t_2 such that $t_1t_2=.25$. The complement of this probability, $1-P(R_0|t_1,t_2)$ is then the coverage of $\theta_s = .25$ conditional on t_1 and t_2. The coverage for various $.25 \leq t_1 \leq 1.0$ and $t_2=.25/t_1$ is shown in Figure 14. Similar graphs of the coverage are given in this figure for $\theta_s=.10$ and $\theta_s=.40$. Clearly, coverage varies with the particular value of t_1 and t_2 that is selected. For $\theta_s = .25$, the coverage ranges from .88 to .92, while for $\theta_s=.10$ and $\theta_s=.40$, the coverage ranges from .88 to 1.0 and from .82 to .92, respectively.

For given values of $.25 \leq t_1 < 1.0$ and $t_2 = .25/t_1$, the coverage may be computed from

$$1 - \sum_{r_1,r_2,\varepsilon R_0} \binom{n_1}{r_1} t_1^{r_1} (1-t_1)^{n_1-r_1} \binom{n_2}{r_2} t_2^{r_2} (1-t_2)^{n_2-r_2} .$$

This is simply the complement of the sum of the binomial probabilities of the outcomes that are in R_0.

For the Bayesian intervals, the level α is a weighted average level of coverage where the parameter values are weighted by the prior distributions on θ_1 and θ_2. When uniform priors are used, as in this example, the simple average coverage across all combinations of θ_1 and θ_2 is precisely .90. When nonuniform priors are used, the weighted average coverage is also precisely equal to .90. It is an analytic property of Bayesian intervals that the weighted average coverage is equal to $1-\alpha$.

In contrast, the confidence interval approach provides a minimum level of coverage rather than an average level of coverage. A comparison of the graphs in Figures 3 to 5 to those in Figure 14 illustrates the difference in coverage. because the typical levels of coverage for confidence intervals are usually larger and cannot be smaller than the nominal level of coverage, confidence intervals are most often wider than their Bayesian counterparts.

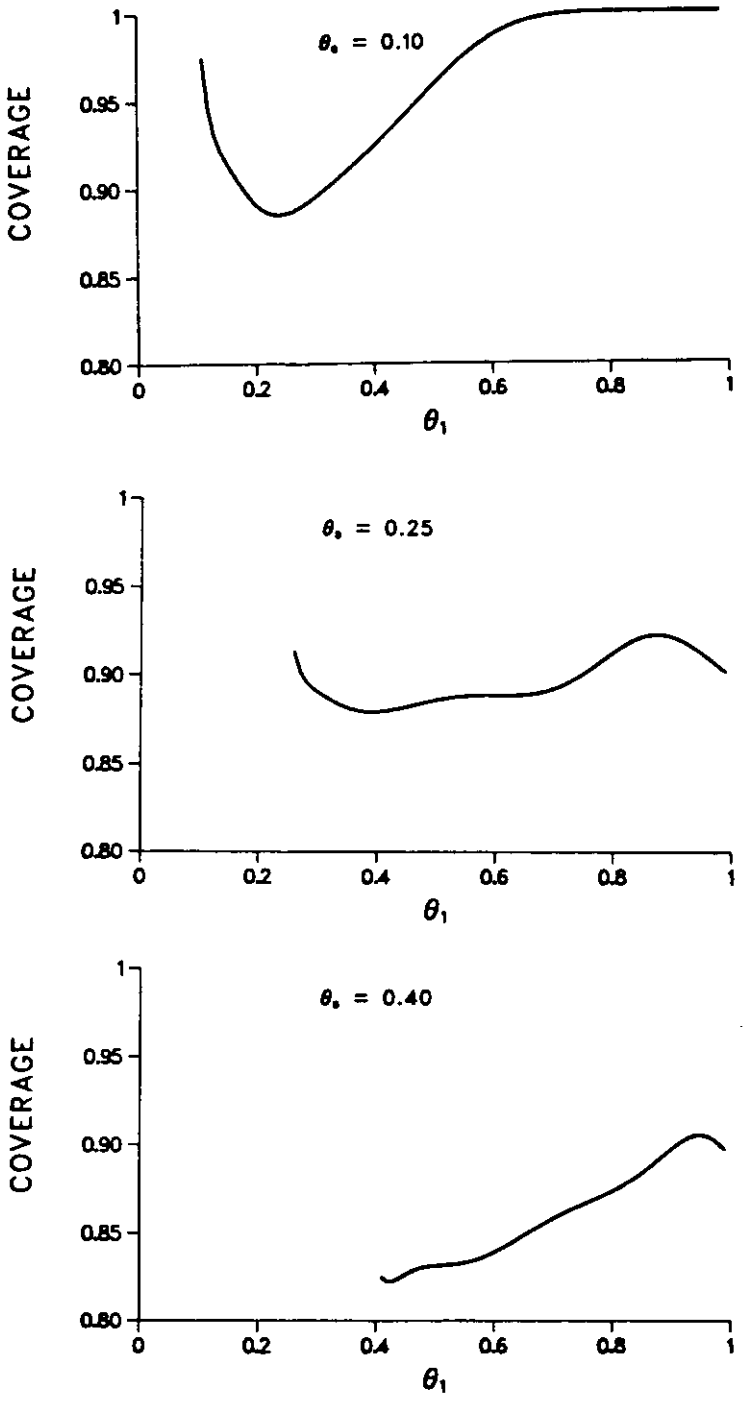

Figure 14. Coverage for θ_s = .10, .25 and .40.

7. SUBJECTIVE DISTRIBUTIONS

Whenever sufficient data are available, the construction of a probability distribution for a parameter can proceed using the noninformative distributions described in the preceding section. When data are available, one need not bother with using subjective information. However, alternative procedures are needed when no data are available, an appropriate probability model is unavailable, or the data are too sparse to provide a useful level of information about the parameter.

Under these circumstances, a subjective probability distribution may usually be constructed using data sources that do not have an associated objective probability model. The subjective distribution can serve by itself or be revised by the model of the sparse data if appropriate. The primary sources of these subjective data include similar components at other installations, engineering and manufacturing specifications and test results, and informed expert opinion. The tools available for the actual construction of subjective probability distributions include empirical Bayes' procedures, elicitation techniques, calibration procedures, and theories of pooling subjective information.

Empirical Bayes' procedures employ data from a number of similar, but not identical, situations such as similar components or plants. The similar values are treated as realizations from the same prior distribution. These realizations are then used to estimate the prior distribution. Traditionally, the population variability curve is adopted as the prior for the similar component. A discussion of empirical Bayes' procedures applicable in reliability is provided by Martz and Waller (1982).

Subjective elicitation, calibration, and pooling procedures are used to form a subjective distribution when the data arise from expert judgments. Under some circumstances, the subjective information may not be sufficient for estimating a probability distribution. For example, if only a point estimate and a range are available, there are numerous probability distributions that will satisfy the available information. A usable probability distribution can be created by choosing a distribution that maximizes some criteria such as the variance, subject to a data derived set of constraints. Constrained maximum entropy distributions (Unwin, 1985) are examples of distributions derived from such arguments. The basis for using this type of procedure is that by choosing a distribution that maximizes the criteria subject to the constraints, one uses the available knowledge without understating the uncertainty about the parameter.

Thus far, we have discussed situations where sufficient objective data are available and situations where no objective

data are available. Intermediate situations occur where some, but insufficient, objective data are available. When this situation arises, the distribution based on subjective data may be used as a prior distribution that is then revised in the light of the sparse objective data and the related probability model. The purpose of combining the sources of information is to obtain a distribution that contains enough information for a reasonable analysis to proceed.

There are alternatives to employing the subjective distributions described here. One alternative is the bounding analysis that is used in conjunction with Maximus. Another is to perform many conditional analyses, each analysis being performed at some important set of parameter values. Neither of these alternatives makes full use of the available information. We feel that it is better to proceed, cautiously, using the subjective data to develop probability distributions for parameters and following up with sensitivity analyses of the influence of subjective information on the risk uncertainty.

Possible sensitivity analyses include:

1. Replacing informative subjective distributions with noninformative distributions and then measuring the resulting change in uncertainty about risk.

2. Replacing all informative subjective distributions with subjective distributions that have maximum variance or entropy (Unwin (1985)), subject to a set of constraints and then measuring the resulting change in uncertainty about risk.

3. Measuring the expected reduction in uncertainty that would be obtained if the true value of a subjectively estimated parameter could be determined.

Analyses 1 and 2 are designed to measure the total impact of introducing subjective information into the analysis. Analysis 3 has the objective of pinpointing subjective distributions that are important determinants of the overall risk estimates. If a subjective distribution is unimportant, then the method of estimation and the source of the data are inconsequential. Conversely, if a subjective distribution is found to be an important determinant of uncertainty, effort should be directed towards obtaining suitable sources of objective information, or subjective information of highest quality, if objective data cannot be acquired. In the event that no data of high quality can be obtained, conditional analyses can be performed across a suitable range of the important parameter.

8. THE BAYES/MONTE CARLO ANALYSIS

An alternative approach to the Maximus/bounding methodology for assessing the uncertainty about risk estimates is to consider the various system parameter vectors, θ, to have different relative likelihoods. These likelihoods are determined by the data concerning those parameters. Probabilistic models identical to those used in the confidence interval analysis are employed with the objectively determined data. Similarly, probabilistic models of the subjective data are employed in place of the bounding analysis to represent information about ω. This requires the analyst to specify various parameter values to be more likely, less likely, or equally likely relative to other values.

The objectives of the Bayes/Monte Carlo methodology are threefold:

Objective 1 - to provide a reasonable, coherent and implementable method for quantifying the uncertainty in the various measures of risk.

Objective 2 - to permit the separation of the influence of subjective and objective information.

Objective 3 - to provide for identification of the relative contribution (importance) of uncertainties in individual parameters to the overall uncertainty.

Consider again the parameters $\underset{\sim}{\theta}$ and $\underset{\sim}{\omega}$. In the trial problem of Spencer and Easterling (1984b), the parameter space for $\underset{\sim}{\theta}$ and $\underset{\sim}{\omega}$ is such that $0 \leq \theta_i \leq 1$, $0 \leq \omega_j \leq 1$ for $i=1,\ldots,\ell$ and $j=1,\ldots,m$ where ℓ and m are the number of elements in $\underset{\sim}{\theta}$ and $\underset{\sim}{\omega}$, respectively. The Bayes/Monte Carlo analysis proceeds in the following manner:

1. The quantities $\underset{\sim}{\theta}$ for which objective data exist are initially assigned a noninformative prior distribution over the ℓ dimensional space of $\underset{\sim}{\theta}$. This distribution is revised according to the data and the assumed probability models for the data. The noninformative prior distribution is selected so that the data will have a dominant effect on the revised distribution and the effect of the prior distribution is minimized.

2. The quantities $\underset{\sim}{\omega}$, for which only subjective estimates are available, are assigned a subjective probability distribution. This distribution may be formed directly from expert judgments, or it may be formed by assuming an

appropriate probability model for the experts' judgments and then used to revise a noninformative prior distribution yielding the subjective distribution. To assess the influence of subjective information on the risk estimate and the sensitivity of uncertainty to the subjective information, the derived subjective distributions can be replaced with noninformative counterparts.

3. The distribution of $f(\underset{\sim}{\theta}, \underset{\sim}{\omega})$ induced by the distribution of $\underset{\sim}{\theta}$ and $\underset{\sim}{\omega}$ is calculated. Because the system model is apt to be complex, Monte Carlo simulation provides an economical and reasonable approach to calculating the distribution of f.

4. The distribution of $f(\underset{\sim}{\theta}, \underset{\sim}{\omega})$ is calculated under alternative assumptions about the subjective knowledge for ω. Comparison of the distributions permits analysis of the influence and sensitivity associated with the subjective information. Partial analyses may also be conducted for subsets of the subjectively estimated parameters.

5. The importance of the uncertainty about each individual parameter is measured by considering the expected reduction in uncertainty that would be obtained if the actual value of the parameter could be determined.

Objective 1 of the uncertainty analysis is satisfied through steps 1 through 3 of the Bayes/Monte Carlo methodology. Step 4 provides the information to satisfy Objective 2. Such information is useful at the policy level and is primarily interpretive in nature. In contrast, step 5 is designed to provide the analyst with insights about specific uncertain parameters and thus is related to Objective 3.

The output of the analysis is in the form of probability distributions and quantities derived from probability distributions. This output provides:

1. Quantitative measures of the likelihood of intervals.

2. A relative weighting of possible risks.

3. Measures of influence and sensitivity.

There are fundamental differences between confidence intervals derived from the sampling distribution of a statistic given a parameter and intervals derived from a probability distribution of a parameter. The latter type of interval, known as a credible interval, has an associated quantitative measure which is a probability. The confidence interval has an associated measure known as a level of confidence which is the minimum coverage.

CREDIBLE INTERVAL	A credible interval for a parameter θ is derived from the posterior distribution for θ and has an associated level which is a probability. In contrast, a confidence interval is derived from the sampling distribution of a statistic when the parameter θ is fixed and has an associated minimum level of coverage.

The confidence interval provides a minimum level of coverage, while the credible interval provides an average level of coverage. The needs of the end user must be given due consideration in the choice of a method of analysis and presentation.

9. TRIAL PROBLEM

The analysis of the trial problem can be broken into several stages. These are:

1. Analysis of component level reliabilities.
2. Synthesis of subsystem and system reliabilities.
3. Analysis of recovery probabilities and probabilities of the initiating event.
4. Interpretation of the output.

The Spencer and Easterling (1984b) analysis treats three versions of the trial problem. The first version is a demonstration of Maximus without the bounding for subjective information. The second version injects recovery probabilities as known constants, and the third version treats the recovery probabilities and some of the objective information as subjective. The analysis presented here will closely parallel these three cases so that direct comparisons can be made.

9.1 Component Level Reliabilities

The trial problem analyzed in Spencer and Easterling (1984b) is the sequence $B(1.2)D_1C$ of the Arkansas Nuclear One Unit 1 Nuclear Power Plant. At the component level there exist 15 population types that appear in the minimal cut sets for the $B(1.2)D_1C$ system. A population type is a grouping of similar components for which a single set of data is available, or for which several sets of data are applicable for all members of the population. Members of a population type have similar failure rates except that, under some circumstances, the failure rate in one application may be a known multiple of the failure rate in

another application. This occurs when a similar component must operate over different periods of time, or when different numbers of demands are made on the components.

Table 1 is taken from Table A3 of Spencer and Easterling (1984b) and shows the data for each of the fifteen component types, the number of occurrences of that component type in the system, and the subsystems in which the components occur. The point estimates and error factors are used to derive the binomial equivalent data by equating the point estimate to r/n where r is the equivalent number of failures and n is the equivalent number of trials, and equating the upper bound of a 95% confidence interval to the product of the error factor and the point estimate. The conversion of data is necessary in the trial problem since Maximus deals exclusively with binomial data.

ERROR FACTOR	An error factor is the ratio of two points, one a central point and the other a limit. The ratio is stated in a form so that the error factor is always greater than one. The limits are often 95% credible limits or 95% confidence limits.

A similar conversion must be made for the Bayes/Monte Carlo analysis in order to obtain a probability distribution for the component failure rates. This is accomplished by using a noninformative prior for the parameter, the information contained in the data, and the corresponding probability model. Bayes' theorem is used to combine the noninformative prior and the data into the desired probability distribution. To ensure comparability between the Maximus/bounding results and the Bayes/Monte Carlo results, the same r, n binomial data have been used. For the trial problem, it is assumed that

1. The component failure rates are a priori independent.

2. Prior information about an individual failure rate may be expressed through a noninformative prior distribution.

Assumption 1 is equivalent to the requirement that information about one failure rate provides no information about any other failure rate. Assumption 2 is met in the trial problem by employing the $\beta(0,0)$ "improper" prior density function discussed earlier.

The choice of this density ensures that

$$E(X|r,n) = r/n \ ,$$

Table 1

Table A3. "Population Type" Data

Pop. Type	File No.	Point Estimate	Error Factor	Equiv. Failures	Equiv. Tests	Total Number of Occurrences	Blocks (# Occurrences)
*a	1	3.3(-3)	3	2.20	667	15	A,B,C,D,F,G,I,J,K,L,M+N(3),0(2)
b	2	2.0(-3)	3	2.20	1100	16	A,B,C,D,F,G,I,J,K,L,M+N(4),0(2)
c	3	1.0(-4)	3	2.20	22000	7	A(2),C(2),F,M+N,0
d	4	5.4(-3)	10	.37	69	2	B,D
e	5	1.0(-4)	3	2.20	22000	29	B(8),D(8),E,J(3),L(3),M+N(3),0(3)
f	6	1.0(-4)	3	2.20	22000	5	B,D,M+N(2),0
g	8	4.1(-3)	3	2.20	357	10	A,B(2),C,D(2),F,G,M+N,0
j	11	1.0(-3)	3	2.20	2200	14	A,B,C,D,F,G,I,J,K,L,M+N(2),0(2)
k	12	5.4(-4)	3	2.20	4074	2	J,L
*l	13	3.7(-3)	3	2.20	595	6	B,D,I,K,M+N,0
*o	26	4.3(-4)	3	2.20	5116	6	B,D,I,K,M+N,0
p	28	1.0(-3)	3	2.20	2200	1	M+N
t	40	8.0(-3)	10	.37	46	1	M+N
*u	41	1.8(-3)	10	.37	206	3	B,D,M+N
v	47	3.0(-3)	3	2.20	733	1	M+N

* These types have individual point estimates within the type that differ.

so that the point estimate will be identical to that employed in Maximus. If the centroid likelihood estimate had been used as the nominal point in the Maximus/bounding analysis (i.e., mle), the $\beta(1,1)$ prior would have produced comparable nominal values for components.

The choice of the prior distribution is usually inconsequential. In those cases where the system reliability is strongly influenced by a component for which little data is available, the choice of the prior distribution may have a strong impact on the analysis.

9.2 Calculations of Subsystem and System Reliabilities

Figure 15 is taken from Spencer and Easterling's (1984b) Figure 2 and is a series-parallel representation of D_1C. Each of the blocks labeled by a capital letter is a subsystem comprised of a series of components. The failure rate of a subsystem can then be written as a function of the component failure rates. For example, the failure rate of subsystem A as found in Table A2 of Spencer and Easterling is

$$f_A(\theta) = 1 - (1-\tfrac{1}{3}\theta_a)(1-\theta_b)(1-\theta_c)(1-\theta_c)(1-\theta_g)(1-\theta_j)$$

$$= \tfrac{1}{3}\theta_a + \theta_b + 2\theta_c + \theta_g + \theta_j$$

$$+ O[(\max\{\theta_a, \theta_b, \theta_c, \theta_g, \theta_j\})^2]$$

where $O(x)$ means order not greater than x. For small values of the θs, the last term is negligible.

The coefficient of θ_a is 1/3 in subsystem A because of the associated time factor. The term $2\theta_c$ appears in f_A because two components of type c appear in subsystem A. The recommended procedures for Maximus do not treat this term's contribution to $f_A(\theta)$ as $2\theta_c$, but instead as $\theta_{c'} + \theta_{c''}$ where $\theta_{c'}$ and $\theta_{c''}$ are separately estimated failure rates, each having one-half of the effective data available for θ_c. This is an unpooling scheme that is used for similar components within and across systems. The essence of unpooling is that similar components are treated as dissimilar and the data are divided among the components so that the point estimate of the system's failure rate remains unchanged, and the length of the confidence interval for the system's failure rate is minimized. The details of the unpooling algorithm are complex and the reader is referred to Spencer and Easterling (1984a) and (1984b). It is unclear whether unpooling will be attempted across dominant accident sequences in large-scale applications of the Maximus/bounding methodology.

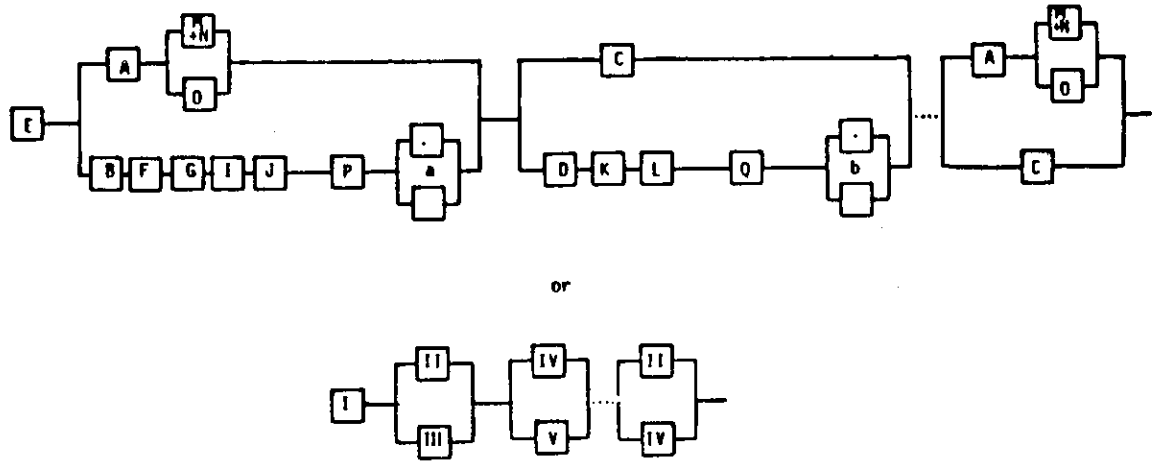

Figure 15. Series-parallel arrangement for B(1.2)D1C. Each block is one or more basic events in series. Those blocks labeled the same represent the same event.

The Bayes/Monte Carlo alternative to the Maximus approach is to use the distribution of θ to obtain the joint distribution of the subsystems' failure rates. Since the subsystems' failure rates are approximately linear functions of the component failure rates, the vector of subsystem failure rates f_s is approximately

$$f_s(\underset{\sim}{\theta}) = \underset{\sim}{s\theta} \ ,$$

where in the trial problem, $\underset{\sim}{s}$ is 13x15 with the elements shown in Table 2. The use of linear functions is not required but is convenient for the trial problem and introduced no significant error. The denominators of fractional terms in Table 2 represent time adjustments, while numerators greater than 1 represent multiple components.

9.3 Version 1 - No Recovery

Version 1 of Spencer and Easterling (1984b) is the D_1C system (Figure 15) analyzed without recovery probabilities and with an initiating event that has a constant (known) probability of either 1.0 or .2. The representation of D_1C in Figure 15 has a third parallel subsystem added, which is not present in the physical system. This subsystem's displayed connection to the remainder of the system is by a dotted line. A requirement of Maximus is that subsystems appear in a series-parallel structure.

Table 2
Subsystem Matrix

Sub-System	a	b	c	d	e	f	g	j	k	l	o	p	t	u	v
A	1/3	1	2				1	1							
B	1/3	1		1	8	1	2	1		1	1			1/9	
C	1/3	1	2				1	1							
D	1/3	1		1	8	1	2	1		1	1			1	
E					1										
F	1	1	1				1	1							
G	1	1					1	1							
I	1/33	1						1		1/2.2	1/1.95				
J	1/3	1			3			1	1						
K	1/33	1						1		1/2.2	1/1.95				
L	1/3	1			3			1	1						
M+N	3/3	4	1		3	2	1	2		1/3.7	1/1.95	1	1	1/9	1
O	1/2	1	1		1		1	1		1/3.7	1/2.15				

Since the system for the trial analysis could not be put in a series parallel representation, the additional subsystem was added to account for a dominant cut set that would otherwise not have been included. (See Spencer and Easterling (1984b) for details.)

The calculation of the approximate system confidence interval, as described in Spencer and Easterling (1984b) appendix, is accomplished by iterative application of the Maximus method and the unpooling algorithm. The result is equivalent data for the system of roughly 1.3 failures in 1430 tests. The point estimate, upper 95% confidence limit, and lower 95% confidence limit are given by 9.2(-4), 3.7(-3), and 8.3(-5). In contrast, the Bayes/Monte Carlo method using 100 simulations gives a point estimate (mean) of 1.0(-3) and upper and lower 95% points of 2.3(-3) and 3.2(-4). These results are repeated in Table 3 and displayed as the upper two intervals in Figure 16. The resulting density function and cumulative distribution function (CDF) are shown in Figures 17 and 18 for the Bayes/Monte Carlo method.

While the nominal (point) estimates of risk are nearly identical, the width of the Bayes/Monte Carlo interval is shorter than the Maximus confidence interval. If the output density in Figure 17 were plotted on a linear scale, substantial skewness in the positive direction would become apparent. Such skewness indicates that the actual system failure probability could be substantially above the upper 95% limit, although the probability of this occurring is small.

Version 1 of the trial problem has also been analyzed with the assumption of an initiating event probability being a known constant of .02. For the Maximus analysis, the equivalent data can be found by using the same number of failures as before, and dividing the number of trials by .02. This gives binomial equivalent data of 1.3 failures in 71,400 trials (or reactor years where each year is a trial). The resulting interval is 1.7(-6) to 7.3(-5) and the nominal estimate is 1.8(-5). The Bayesian interval and nominal estimates are just .02 times their counterparts when the initiating event is not included. This occurs only because the rate of the initiating event is being treated as a constant. These intervals are displayed as the third and fourth intervals in Figure 16.

A third analysis of Version 1 involves treating the initiating event as uncertain. The equivalent data is taken to be two occurrences in 100 years. The initiating event is "in parallel" with the system so that both must occur for a failure to occur. The Maximus procedure for parallel systems yields equivalent data of .61 failures in 33,200 years and resulting confidence limits of 1.9(-7) and 1.2(-04).

Table 3
COMPARATIVE INTERVALS

Version		Maximus/bounding			Bayes/Monte Carlo		
		5%	Nominal	95%	5%	Mean	95%
1	Without B(1.2)	8.3(-5),	9.2(-4),	3.71(-3),	3.2(-4),	1.0(-3),	2.3(-3)
	P[B(1.2)]=.02	1.7(-6),	1.8(-5),	7.3(-5),	6.4(-6),	2.0(-5),	4.6(-5)
	B(1.2)~(2,100)	1.9(-7),	1.8(-5),	1.2(-4),	2.5(-6),	2.0(-5),	4.9(-5)
2	Without B(1.2)				2.6(-5),	1.1(-4),	2.2(-4)
	P[B(1.2)]=0.2	3.7(-7),	4.0(-6),	1.6(-5),	5.1(-7),	2.2(-6),	4.5(-6)
	B(1.2)~(2,100)	4.1(-8),	4.0(-6),	2.7(-5),	1.9(-7),	2.5(-6),	6.6(-6)
3	Without B(1.2)	4.8(-6),	2.0(-4),	1.5(-3)	3.7(-5),	1.2(-4),	2.6(-4)
	P[B(1.2)]=.02	9.6(-8),	4.1(-6),	3.0(-5),	7.5(-7),	2.4(-6),	5.2(-6)
	B(1.2)~(2,100)	7.5(-9),	4.1(-6),	5.4(-5),	2.7(-7),	2.7(-6),	7.2(-6)
	Correlated Recovery Probs.				2.2(-7),	3.0(-6),	7.3(-6)
	Noninformative Uniform Distributions				6.7(-7)	7.0(-6)	1.9(-5)
	Relatively Noninformative Distributions				1.9(-7),	2.7(-6),	7.6(-6)

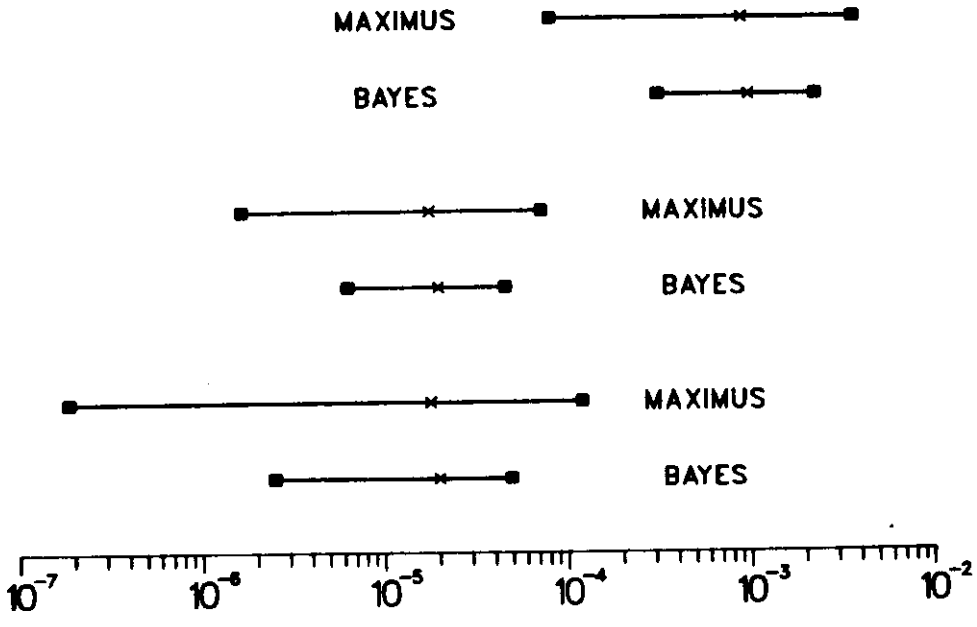

Figure 16. Comparative Intervals Version 1

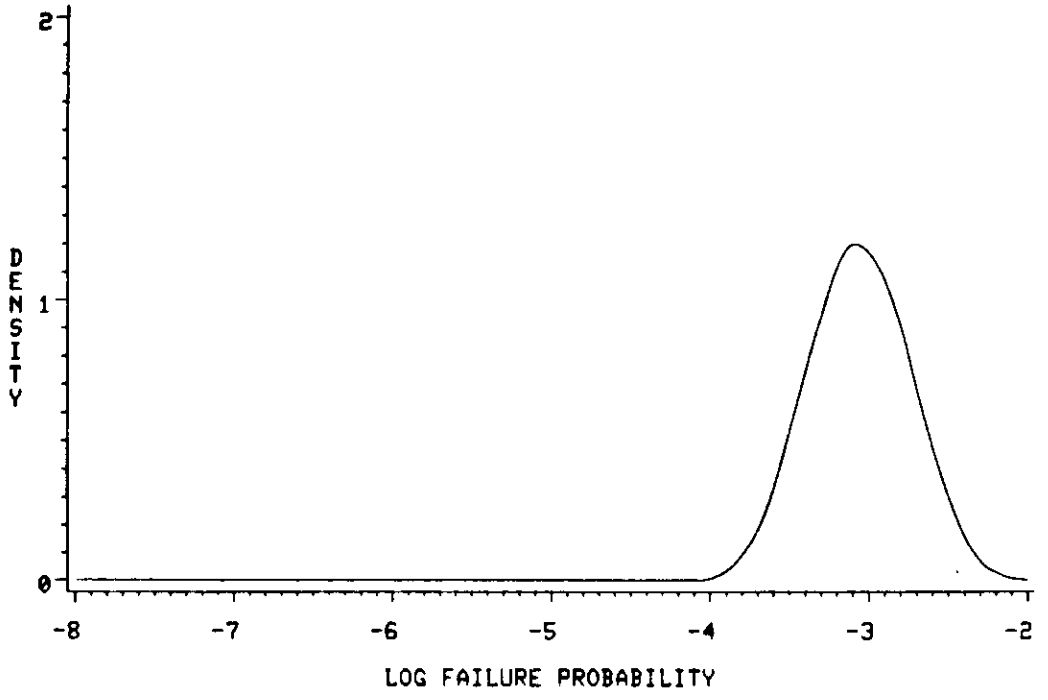

Figure 17. Density for Version 1

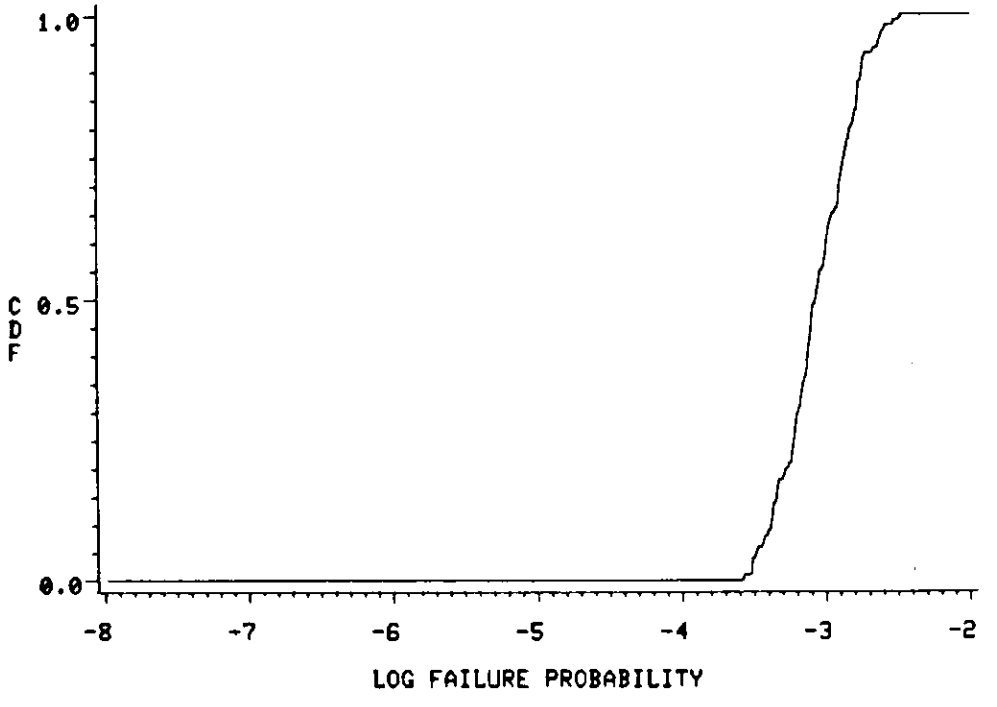

Figure 18. Cumulative Probability for Version 1

The Bayes/Monte Carlo procedure requires that the initiating event rate be treated as a random variable. As was shown earlier, combining a ß(0,0) prior with a binomial probability model and data with r=2 and n=100 results in a ß(2,98) posterior distribution for the initiating event. The density and CDF in Figures 19 and 20 are produced by using the ß(2,98) distribution for the initiating event as input to the system model. The resulting point estimate is 2.0(-5), and the 95% lower and upper limits are 2.5(-6) and 4.9(-5), respectively. These intervals are the fifth and sixth intervals in Figure 16.

The Maximus intervals and the Bayes/Monte Carlo intervals for Version 1 of the trial problem are centered on approximately the same value. While the width of the Maximus intervals are somewhat larger than the corresponding Bayes/Monte Carlo intervals, the upper limits differ by less than a factor of 2.5. It should be remembered that although both intervals employ 95% limits, the interpretation of the intervals is different. The Maximus intervals provide a minimum coverage with typical coverage larger than the stated coverage. The Bayes/Monte Carlo intervals provide an average coverage. Thus, it is not surprising that the Maximus intervals are wider.

Next, we consider the importance of the uncertainty about each component in shaping the overall uncertainty for the system. The procedure used to assess the uncertainty importance (UI) of each variable is to calculate the expected reduction in the variance of the system's failure rate that would be obtained through definitive information about a component's failure rate. The UI is the square root of this quantity. If the system's failure rate is Y and the failure rate of the ith component is X_i, then

$$UI_i^2 = \mathrm{var}(Y) - E_{X_i}\,\mathrm{var}(Y|X_i) \ ,$$

where $\mathrm{var}(Y|X_i)$ is the variance of Y with X_i held constant. The UI_i^2 is then the average of all possible reductions in variance that could be obtained by determining that X_i has a specific value. The average is computed using the probability distribution of X_i. A more detailed discussion of UIs, and how they may be calculated, is given in the Appendix.

The UIs for D_1C without considering the initiating event are given in Figure 21. Note that the greatest expected reduction in uncertainty would be obtained if the true value of the "g" component's failure probability were determined.

The UIs in Figure 21 are conditional on the probability of recovery being zero in all cases. As will be shown later, the

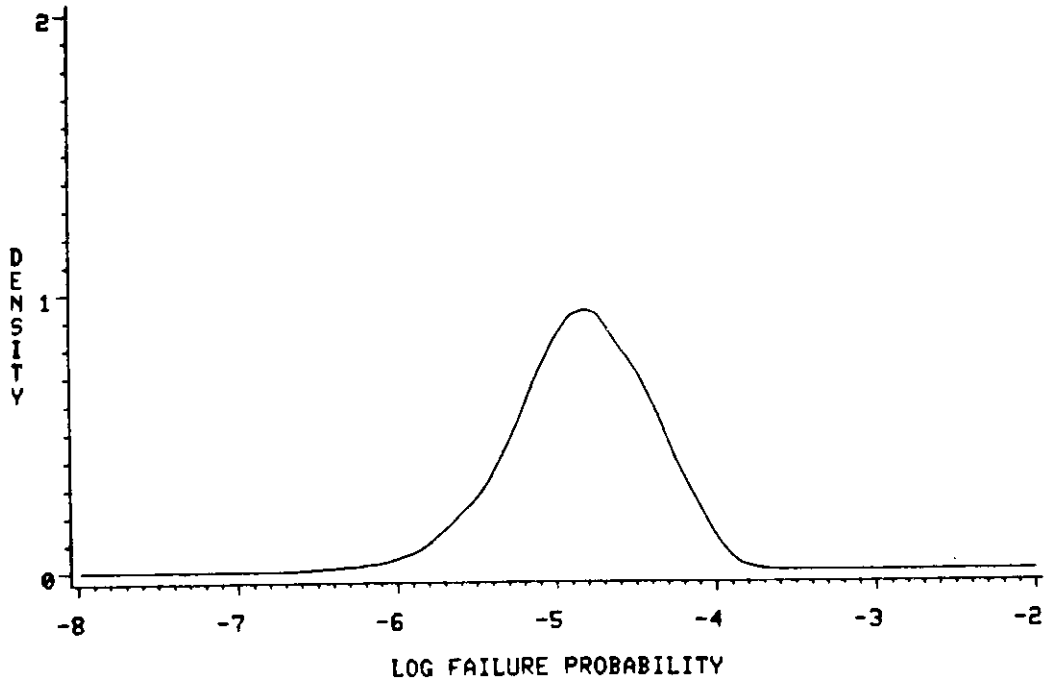

Figure 19. Density for Version 1 - Initiating Event Uncertain

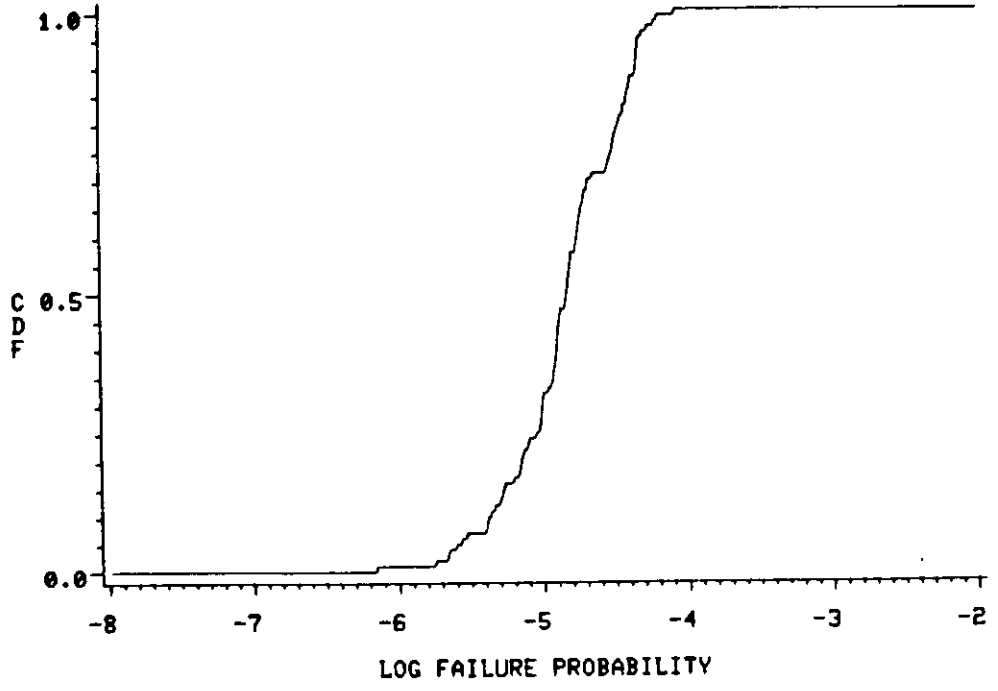

Figure 20. Cumulative Probability for Version 1 - Initiating Event Uncertain

relative magnitude of the UIs will be dramatically altered when recovery probabilities are added. This is reasonable because, if a component has a large recovery probability, the impact of the uncertainty about the component will be diminished.

The calculation of UIs provide a method of meeting Objective 3 of the uncertainty analysis. Spencer and Easterling (1984b) provided no comparable output from the Maximus methodology.

9.4 Version 2 - Recovery Probabilities Certain

The second version of the trial problem analyzed in Spencer and Easterling treats the D_1C system as recoverable and the recovery probabilities as known constants. The trial problem analysis of Spencer and Easterling is implemented by calculating a universal recovery rate for D_1C, which is applicable regardless of the source of failure. As explained in Spencer and Easterling, this constant is found by calculating the probability of nonrecovery for each subtree in a cut set and then taking the probability of nonrecovery for a cut set to be the minimum probability of nonrecovery among the subtrees. It is recommended in Spencer and Easterling that this procedure be followed except that the original fault trees would be employed rather than the cut sets. The recovery value used by Spencer and Easterling is .22 so that the effective data in Version 2 of the trial problem is 1/.22 times the effective number of tests in Version 1.

The Bayes/Monte Carlo analysis presented here incorporates recovery probabilities at the subsystem level rather than at the top level as done in the Spencer and Easterling analysis. This is appropriate where recovery rates differ from subsystem to subsystem, and it is necessary in order to assess accurately the uncertainty importance of the components. The drawback to introducing recovery probabilities at a lower level is that the information about one recovery probability may provide information about another recovery probability; that is, recovery probabilities may be dependent subjective random variables. Introduction of independent recovery rates at too low a level will understate uncertainty by precluding these dependencies. In the trial problem, the subsystems are cut sets and thus the recovery probabilities are introduced at the cut set level in the Bayes/Monte Carlo analysis and at the top level in the Maximus/bounding analysis.

Correlations can be induced among the input values in order to ascertain the effect of dependence on the uncertainty estimates. This is demonstrated in the third version of the trial problem where the recovery probabilities are treated as uncertain quantities. It is found that presence or absence of correlation is inconsequential for this example.

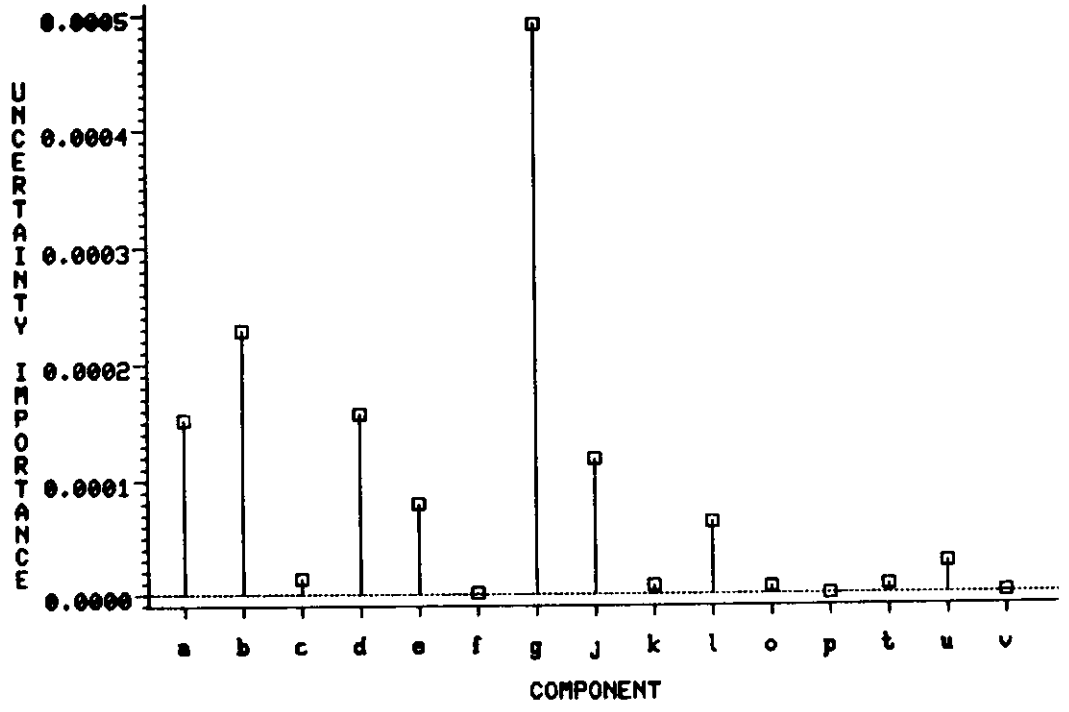

Figure 21. Uncertainty Importances Without Recovery

Version 2 of the trial problem has been analyzed (a) with the probability of the initiating event held constant at .02, and (b) with information about the probability of the uncertain initiating event as 2 failures in 100 trials. The resulting intervals are given in Table 3 and Figure 22. The means in the Bayes/Monte Carlo analysis presented here are shifted to the left relative to the nominal point in the intervals given by Spencer and Easterling (1984b) because of the difference in the method of incorporating recovery probabilities. In the Spencer and Easterling analysis minima of recovery probabilities are used as noted earlier. The Bayes/Monte Carlo intervals are also substantially shorter.

Although Version 2 of the trial problem is not a case of practical interest, it is convenient for examining the impact of the recovery probabilities on the uncertainty importances. The recalculated UIs are shown in Figure 23. It is now the case that component e is the dominant contributor to the uncertainty of the system assuming, of course, that the recovery probabilities are fixed. This is reasonable since subsystem E in Figure 15 is in series with the remainder of the system, subsystem E consists of a single component e, and that component is not recoverable.

Treating the recovery probabilities at the system level, rather than the subsystem level, eliminates the interdependencies between failure rates and recovery probabilities and precludes

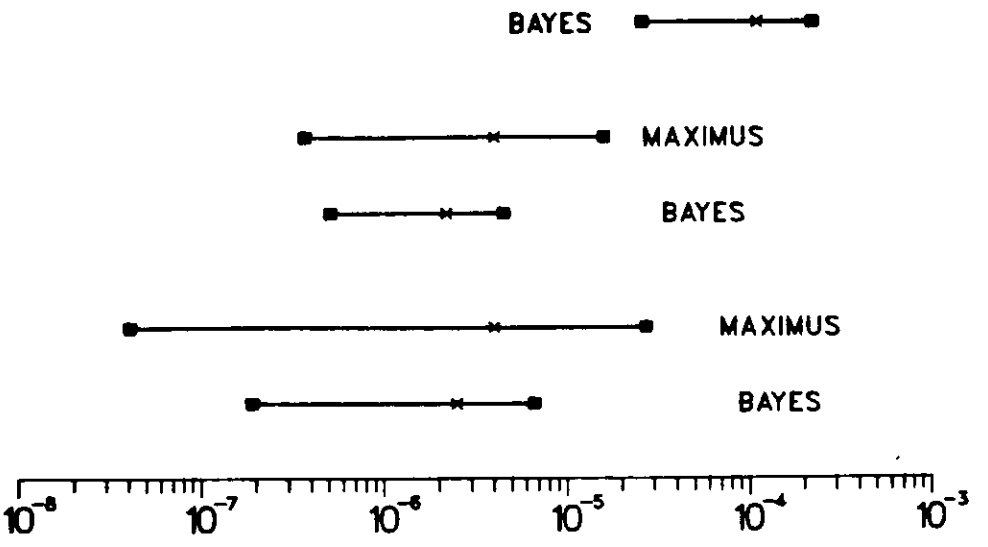

Figure 22. Comparative Intervals for Version 2

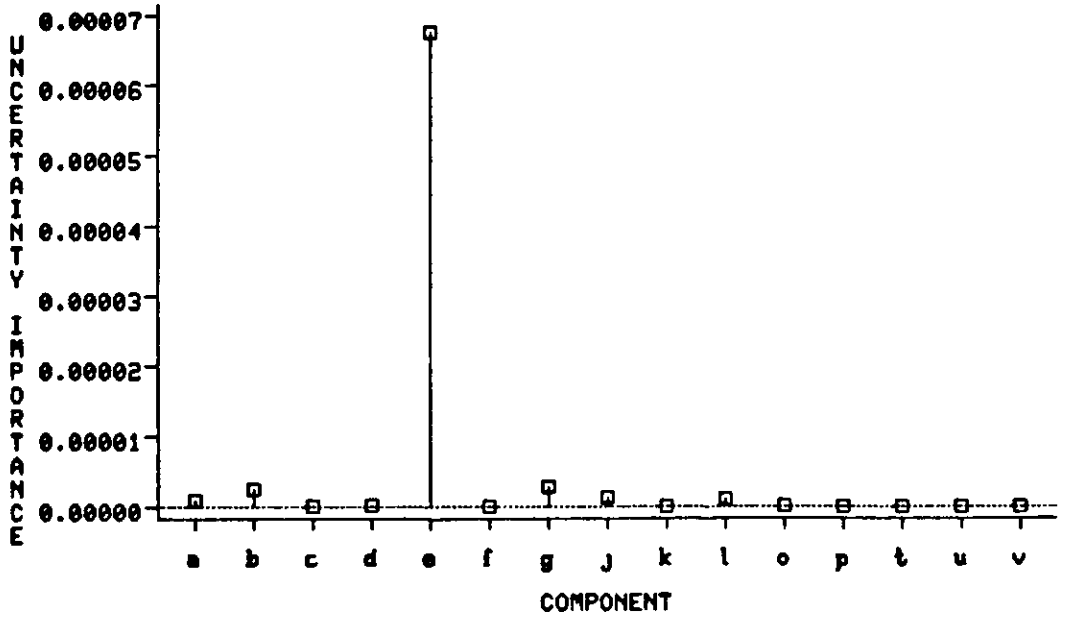

Figure 23. Uncertainty Importances with Recovery

the accurate assessment of the importances of the elements. When a single recovery rate is applied uniformly to all elements (subsystems or components) of a system, the relative importances of the elements are unchanged. On the other hand, differential recovery rates change these importances. For example, an element with a large recovery rate will become relatively less important than one with a small recovery rate. Information relevant to the diagnosis and possible reduction of uncertainty should be part of the findings of a well-considered uncertainty analysis.

9.5 Version 3 - Subjective Recovery Probabilities

In the last version of the trial problem considered by Spencer and Easterling (1984b), the recovery probabilities and the failure rates for components a, f, j, l and o are treated as subjective, contrary to what was done in Versions 1 and 2. Thus, Version 3 provides the most realistic situation for accident sequences.

Implementation of the Bayes/Monte Carlo procedure requires that the subjective information about recovery rates be represented in probability distributions. Since there are no experimental data from which to construct a probability distribution, the distributions are constructed using expert opinion. The Spencer and Easterling report does not contain sufficient information to specify a family of distributions or the parameters of any distribution. As a convenience, the subjective marginal distributions for the recovery rates were assumed to be members of the beta family with means equal to the subjective point estimates and coefficients of variation (ratio of the standard deviation to the mean) equal to unity.* Thus the standard deviation is equal to the mean. There is no justification for this procedure and it is not recommended as a useful method of forming subjective probability distributions. Its use here is tolerated as a vehicle to allow the analysis to proceed with the information at hand. Normally, one would have at least an interval and point estimate from which a subjective probability distribution can be constructed using the maximum variance or maximum entropy approach discussed earlier in the section on subjective probability.

Thus, all of the input variables are assumed to have beta distributions. These distributions are given in Table 4.

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* The estimated recovery rate for subsystem K is .44. Since the procedure used for the other variables would give rise to a bimodal distribution, the distribution with mean .44 and maximum variance subject to being unimodal was used instead.

In the primary analysis of Version 3 of the trial problem, the subjective recovery rates were treated as independent. The implication of this assumption will be examined later.

The results obtained from applying the Maximus/bounding methodology to $B(1.2)D_1C$ are displayed in Figure 24 and Table 3. Figure 24 shows the seven-point analysis (three intervals and a point estimate) for the cases where the initiating event probability is treated as a constant .02 and where it is uncertain and evidence of 2 occurrences in 100 years is available. The box gives the bounds using subjective uncertainty only, the points next to the extreme points give the bounds with statistical uncertainty only, and the extreme points give the total range of uncertainty.

The comparable results from the Bayes/Monte Carlo analysis are presented in Figures 26 and 27 for the case with P[B(1.2)] = .02 fixed, and in Figure 28 and 29 with P[B(1.2)] treated as a random variable. The derived intervals are given in Table 3 and Figure 25 along with the comparable Maximus/bounding overall uncertainty intervals. The pairs of intervals in Figure 25 are the Maximus/bounding interval and the Bayes/Monte Carlo interval without B(1.2), with B(1.2) having a known probability of .02, and with binomial equivalent data of 2 failures in 100 trials for B(1.2). The point estimates obtained from the Maximus/bounding analysis are slightly higher than those obtained from the Bayes/Monte Carlo analysis because of the procedure used to calculate the universal system recovery probability. More important, however, are the differences in widths of the intervals. The Bayes/Monte Carlo intervals cover between one and two orders of magnitude while the Maximus/bounding intervals cover between two and three orders of magnitude. The error factors (the ratio of the upper 95% point to the point estimate) for the second and third intervals are 7.3 and 13.2 for the Maximus/bounding procedure intervals and 2.2 and 2.7 for the Bayes/Monte Carlo procedure.

Figure 30 displays the uncertainty importances for the recovery probabilities. The UIs for the component failure rates, Figure 23, remain nearly the same as in Version 2 of the trial problem. The uncertainty about component e is dominant.

9.6 Sensitivity to Correlation

In order to assess the impact of introducing correlation among the subjectively estimated quantities, a secondary analysis was made using a rank correlation of .9 between all pairs of recovery probabilities. Version 3 with the probability of B(1.2) uncertain was examined. The resulting interval is 2.2(-7) to 7.3(-6) compared to an interval of 2.7(-7) to 7.2(-6) obtained with uncorrelated inputs. These values appear in Table 3 and Figure 31. The change in the width of the interval is relatively

Table 4

Parameters p and q of the Beta Probability Distributions

Failure Rates

Component	p	q
a	2.20	664.8
b	2.20	1097.8
c	2.20	21997.2
d	.37	68.63
e	2.20	21997.8
f	2.20	21997.8
g	2.20	534.8
j	2.20	2197.8
l	2.20	592.8
o	2.20	2197.8
p	2.20	2197.8
t	.37	45.63
u	.37	205.63
v	2.20	730.8

Nonrecovery Probabilities

Subsystem	p	q
A	.54	1.808
B	.98	97.02
C	.54	1.808
D	.98	97.02
E	Not applicable	
F	.82	8.29
G	.82	8.29
I	.82	8.29
J	.98	97.02
K	.78571	1.0
L	.98	97.02
M+N	.76	5.573
O	.53	1.808

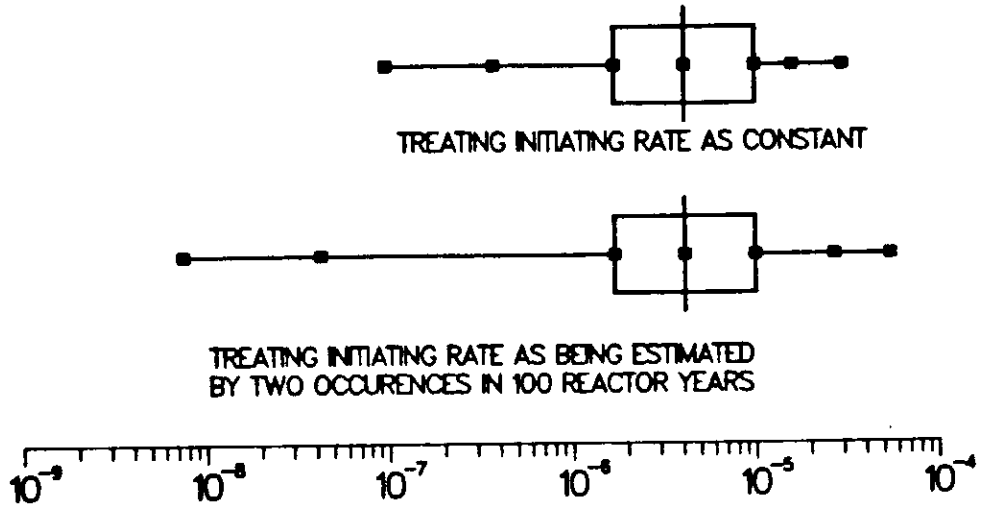

Figure 24. Total Uncertainty Presentation for $B(1.2)D_2C$ (Including Recovery)

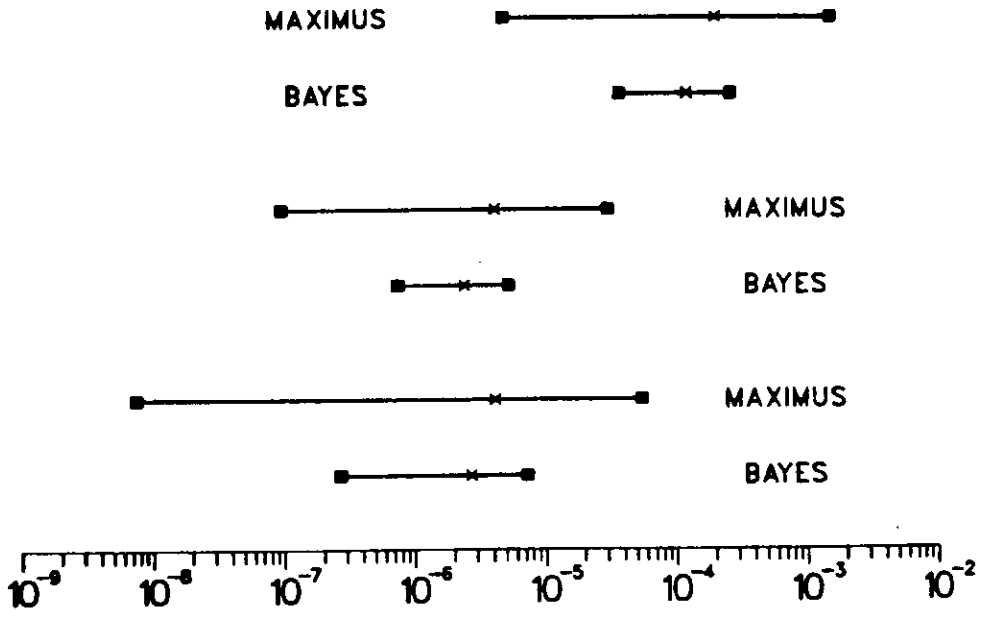

Figure 25. Comparative Intervals for Version 3

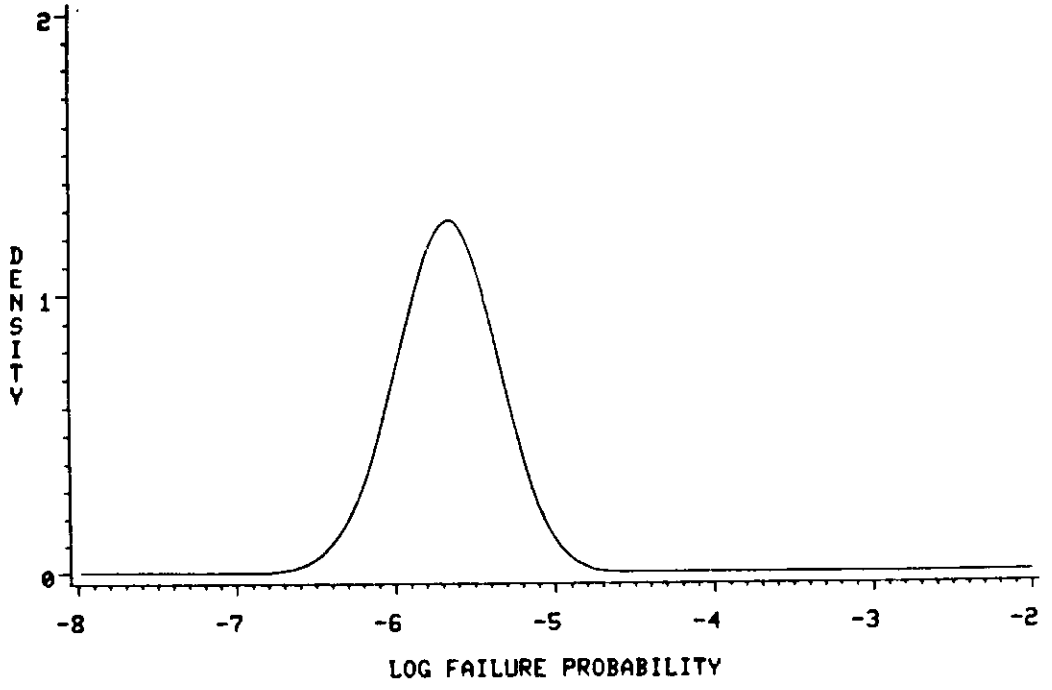

Figure 26. Density for Version 3 with P[B(1.2)] = .02

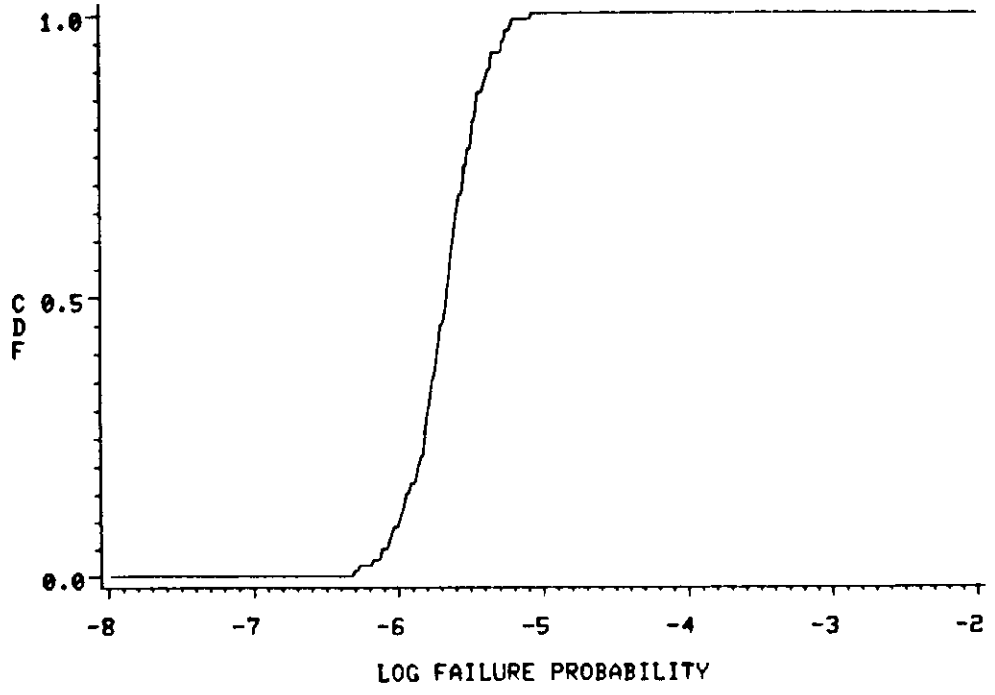

Figure 27. Cumulative Probability for Version 3 with P[B(1.2)]=.02

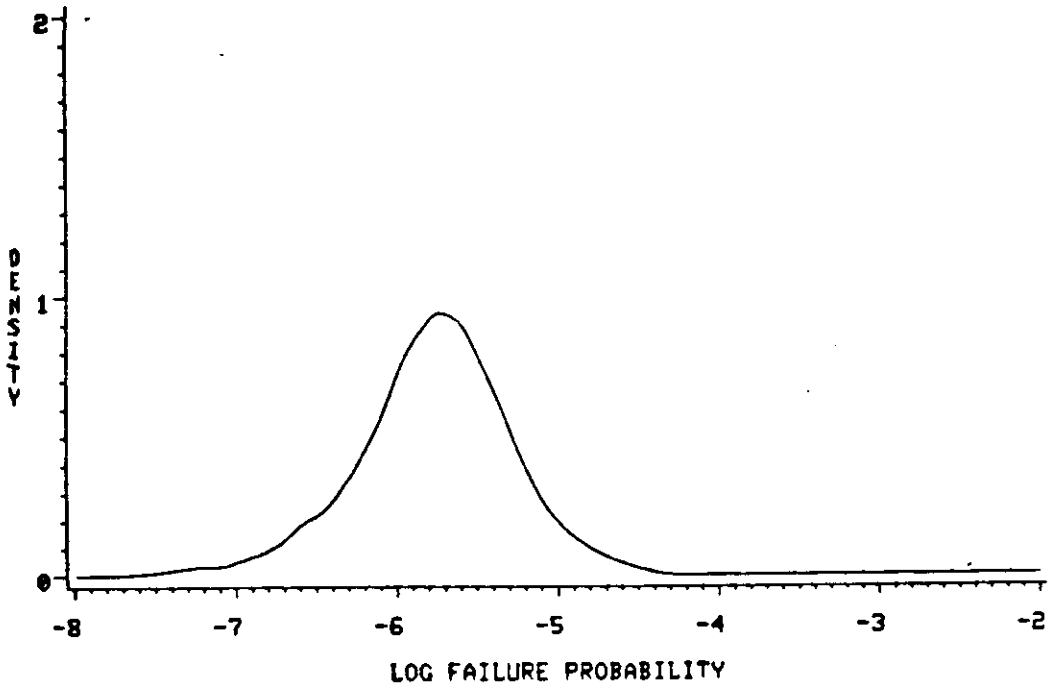

Figure 28 . Density for Version 3 with P[B(1.2)] Random

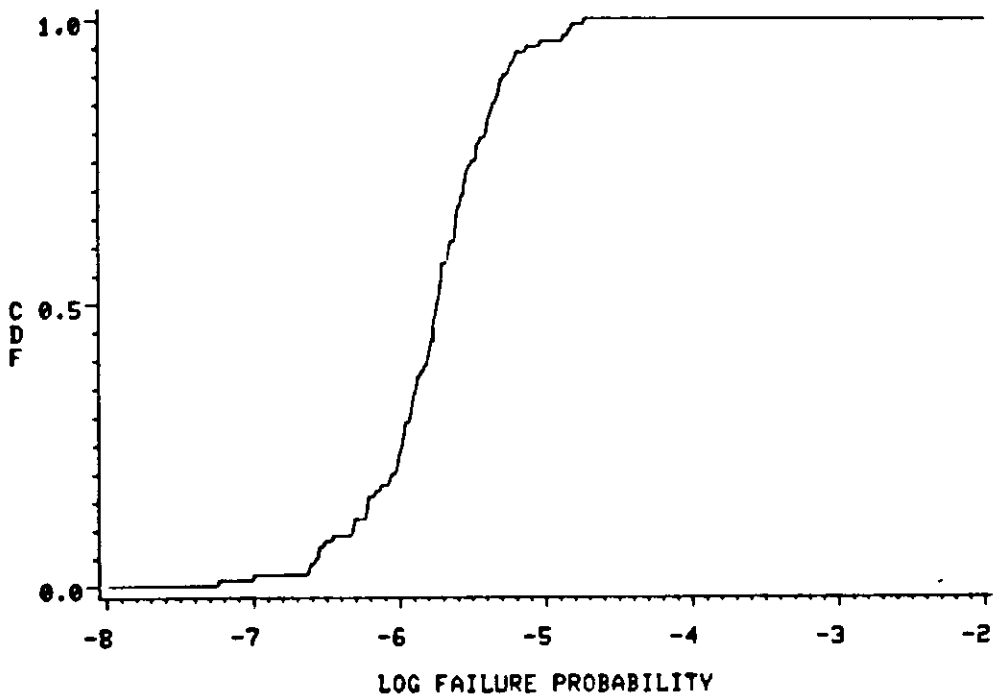

Figure 29. Cumulative Probability for Version 3 with P[B(1.2)] Random

Table 5

Subjective Distributions for Recovery Probabilities

Subsystem	Recovery Probabilities Mean	p	q
A	.23	.2987	1
B	.01	.0101	1
C	.23	.2987	1
D	.01	.0101	1
E	not recoverable		
F	.09	.0989	1
G	.09	.0989	1
I	.09	.0989	1
J	.01	.0101	1
K	.44	.78571	1
L	.01	.0101	1
M+N	.12	.1354	1
O	.23	.2987	1

There also exist differences between the two methodologies with respect to dependencies among the parameters. Because subsystems share components from common population types, there exist dependencies among estimators of the subsystems' reliabilities. The Bayes/Monte Carlo methodology models these dependencies directly. The Maximus/bounding methodology recognizes common components as dependent, but allocates the data to the individual components by unpooling in order that they can be considered independent. The Maximus/bounding methodology maximizes the system's failure rate simultaneously with respect to all the subjective parameters. This is, in a sense, a worst case analysis. The Bayes/Monte Carlo methodology allows for the introduction of arbitrary levels of dependence among the subjectively estimated quantities. However, in the absence of data related to the appropriate level of these dependencies, all that can be done is a sensitivity analysis to examine the impact of various levels of dependence.

The question of implementation in a large-scale analysis has not been addressed in the trial analysis. However, Maximus is a block-diagram oriented procedure and has not been shown to be extendable to other types of analyses such as those dealing with the magnitudes of releases or consequences. The equivalent data form of the output from Maximus is easily adapted as input to another form of analysis, but it is not clear how the bounding information would be adapted. There are still open questions about the ease of implementing Maximus/bounding in the fault tree portion of the analysis. These questions relate to:

1. The need to express systems in a series-parallel form.

2. Unpooling across systems.

3. Bounding with nonmonotonic subjectively estimated parameters.

4. The ability to automate the calculations.

Implementation of the Bayes/Monte Carlo methodology appears to be straightforward. However, in order for either form of analysis to be worthwhile, additional work must be done in order to ensure that the subjective information, whether bounds or probability distributions, is obtained using procedures that are suited to both the problem at hand and the experts whose knowledge is used.

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APPENDIX

THE UNCERTAINTY IMPORTANCE OF INDIVIDUAL VARIABLES

The diagnosis of uncertainty and the eventual reduction or resolution of uncertainty require that the analyst be able to ascertain the importance of various factors contributing to uncertainty. In this appendix, technical details of the computation of uncertainty importance are discussed. The subjective probability approach to uncertainty analysis facilitates the separation of uncertainties so that the effect of resolving the uncertainty about any input variable can be measured. These measurements provide direction for further efforts in reducing uncertainty.

The Method

Suppose that the input to a model is a k-dimensional vector X, having the joint probability density p(x). Denote the model by

$$Y = g(X)$$

where Y is a p-dimensional vector of output variables. Consider a single element of Y, say $Y_i = g_i(X)$. One measure of the uncertainty about Y_i is its variance defined as

$$\text{var}(Y_i) = \int_{R_k} [g_i(x) - \bar{g}_i]^2 \, p(x)dx$$

where $\bar{g}_i = \int_{R_k} g_i(x)p(x)dx$ and R_k is a k-dimensional space.

The variance of Y_i could be reduced if the value of a variable, say X_j, was known with certainty. The conditional variance of Y_i given X_j, denoted by $\text{var}(Y_i|X_j)$, is this reduced variance and thus $\text{var}(Y_i) - \text{var}(Y_i|X_j)$ is the conditional reduction in variability attributable to ascertaining the true value of the input X_j. The troublesome feature of the conditional reduction of variance is that it usually depends upon the value of X_j. A solution to determining the reduction in variance is to calculate the expected reduction given by

$$UI(i,j) = \text{var}(Y_i) - E_{X_j}[\text{var}(Y_i|X_j)].$$

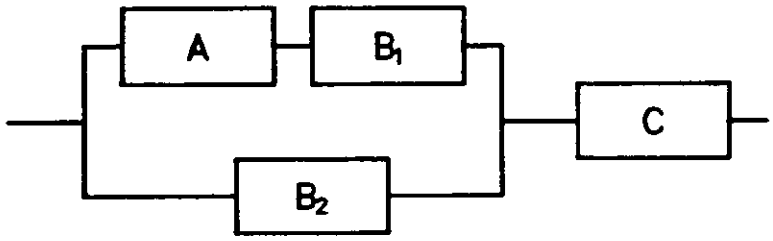
The right side of the preceding equation may also be written as $\mathrm{var}_{X_j}[E(Y_i|X_j)]$ which is normally a simpler quantity to calculate

The quantity UI(i,j) is termed the uncertainty importance of input variable j for output variable i. In the following analyses we will invariably use the simpler computational version of UI(i,j) given by

$$UI(i,j) = \mathrm{var}_{X_j}[E(Y_i|X_j)].$$

An Example

Consider the simple system



where A, B_1, B_2, and C are distinct components, but B_1 and B_2 are identical. Let X_1 and X_3 be the uncertain failure rates of components A and C, and let X_2 be the common uncertain failure rate of B_1 and B_2. The system failure rate is then the uncertain quantity

$$Y = X_3 + X_1X_2 + X_2^2 - X_1X_2^2 - X_1X_2X_3 - X_2^2X_3 + X_1X_2^2X_3.$$

The conditional expectation of Y given X_1 is

$$E(Y|X_1) = E(X_3 + X_2^2 - X_2^2X_3|X_1) + X_1E(X_2 - X_2^2 - X_2X_3 + X_2^2\,X_3|X_1).$$

When the input variables X_1, X_2, and X_3 are independent, the conditional expectations in the above expression become equal to their unconditional counterparts. The uncertainty importance of X_1 can then be calculated as

$$\text{var}[E(Y|X_1)] = a^2\text{var}(X_1)$$

where $a = E(X_2) - E(X_2^2) - E(X_2)E(X_3) + E(X_2^2)E(X_3)$. The uncertainty importances of the remaining two variables may be calculated in a similar fashion. Similar expressions for X_2 and X_3 are

$$\text{var}[E(Y|X_2)] = \text{var}(b_1X_2 + b_2X_2^2) ,$$

and

$$\text{var}[E(Y|X_3)] = c^2\text{var}(X_3) ,$$

where $b_1 = E(X_1) - E(X_1)E(X_3)$

$b_2 = 1 - E(X_1) - E(X_3) + E(X_1)E(X_3)$

and

$c = 1 - E(X_1)E(X_2) - E(X_2^2) + E(X_1)E(X_2^2)$.

Continuing with the example, suppose that the input distributions are members of the beta family as shown in the following table.

INPUT DISTRIBUTIONS

VARIABLE	DISTRIBUTION	E(X)	VAR(X)
X_1	ß(10,90)	.1	.00089
X_2	ß(5,95)	.05	.00047
X_3	ß(1,99)	.01	.00010

Additionally, from the assumed distribution for X_2^3, $E(X_2)$=.000204 and $E(X_2^4)$=.0000158. From the table and the conditional expectations we obtain a =.0466, b_1=.099, b_2=.8910, and c =.992327. The resulting UIs are

$$UI(Y,1) = 1.92 \times 10^{-6}.$$

$$UI(Y,2) = 1.99 \times 10^{-5}.$$

$$UI(Y,3) = 9.85 \times 10^{-5}.$$

The expected reduction in variance is greatest for X_3 and least for X_1.

Computation of the Conditional Expectations

The computation of the variance of the conditional mean requires that the analyst be able to compute the conditional mean for numerous values of X_i. For models of modest size and complexity, this task is straightforward. For larger and more complex models, however, the cost may be prohibitive. We are aware of two methods of simplifying the calculation of the conditional mean.

First, $E[g_i(X)|X_j]$ may be approximated using statistical estimation procedures such as regression, monotone and rank regression, and response surface methods. If $g(x)$ is approximated using an estimate, say $h(x)$, then $h(x)$ may be used to obtain estimates of the UIs by substitution of h for g. When the function $h(x)$ is linear and the input variables are uncorrelated, the UIs will have the same ordering as the simple correlations of the Y with the Xs.

A second kind of simplification can be evoked when the input variables are independent and small in magnitude as in the case of fault trees. In this situation the response function evaluated at the marginal expectations will behave very much like the conditional expectation of the function. Using this simplification in the example given in the preceding section gives

$$g(W_1) = .002475 + .047025\ X_1$$

where $W_1 = (X_1, E(X_2), E(X_3))$

$$g(W_2) = .01 + .099\ X_2 + .891\ X_2^2$$

where $W_2 = (E(X_1), X_2, E(X_3))$

$$g(W_3) = .00725 + .99275\ X_3$$

where $W_3 = (E(X_1), E(X_2), X_3)$

which yield the approximate UIs

$$UI(Y,1) = 1.97 \times 10^{-6}.$$

$$UI(Y,2) = 1.99 \times 10^{-5}.$$

$$UI(Y,3) = 9.86 \times 10^{-5}.$$

This simplification is not applicable when the input variables are dependent. However, under dependence, a linearization of the dependencies may be useful. For example, the analyst may approximate X_ℓ, $j \neq \ell$, by

$$E(X_\ell | X_j) = t_{\ell j} + s_{\ell j}\ X_j\ ,$$

and then use these representations in g(x) or h(x). This procedure is untried as of this date.

The uncertainty importances also may be calculated in a black box mode. Controlling one variable at a time, the other variables can be generated according an appropriate conditional distribution. The conditional mean can then be estimated at several points; and in combination with the marginal density of the controlled variable, an estimate of the variance of the conditional mean can be made. This procedure will likely require greater resources than the previously given simplifications. The advantage of this method is in its generality.

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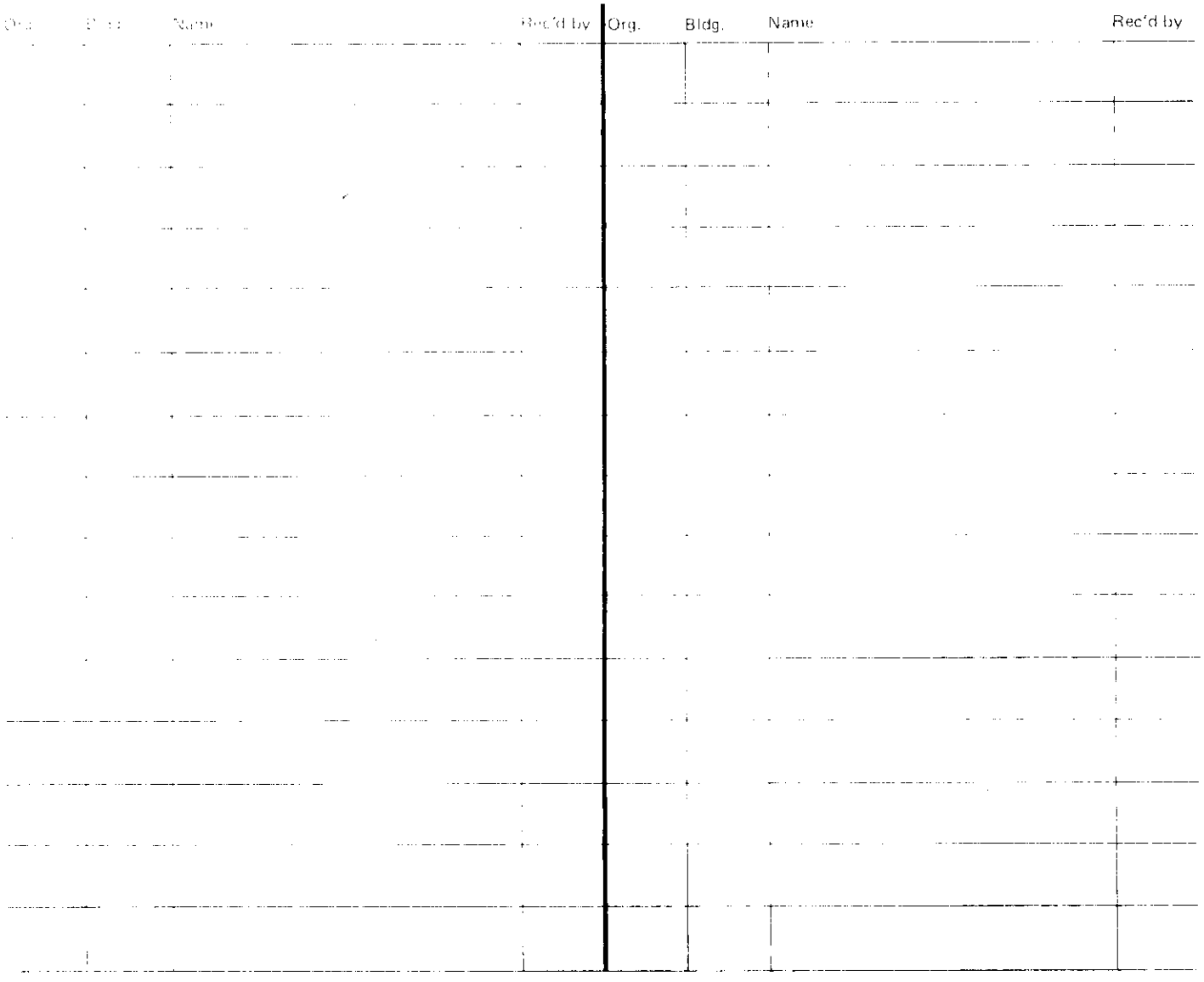

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