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PROGRESS IN LATTICE GAUGE THEORY

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ABSTRACT

These lectures first provide an overview of the current status of lattice gauge theory calculations. They then review some technical points on group integration, gauge fixing, and order parameters. Various Monte Carlo algorithms are discussed. Finally, alternatives to the Wilson action are considered in the context of universality for the continuum limit.

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I. OVERVIEW

This is an exciting time for particle physics. In addition to the recent experimental discoveries we now have a rather successful candidate theory for the strong interactions. Indeed, we have no major puzzles at presently accessible energies. This is in part responsible for the recent enthusiasm for more speculative topics, such as supersymmetry, supergravity, and Kaluza-Klein type theories.

Nevertheless, reliable calculations of low energy phenomena in the Yang-Mills theory of quarks and gluons remain frustratingly elusive. Here nonperturbative effects play a key role, requiring implementation of new techniques. In the last few years, the most popular framework for the study of such phenomena has been the lattice formulation of Wilson[1]. This scaffolding serves primarily as an ultraviolet cutoff rendering the theory well defined and amenable to numerical and analytical work. As with any cutoff, one must ultimately consider its removal; thus, for particle physics applications one must discuss the continuum limit of vanishing lattice spacing.

Let me begin by reviewing the parameters on which the gauge theory of the strong interactions depends. The quark masses presumably arise through a grand unification of the interactions and are thus generally regarded as uncalculable when the strong interactions are considered in isolation. These masses are intimately tied to the pseudoscalar meson masses, which would vanish in a chirally symmetric world of massless quarks. The remarkable feature of the strong interactions is that these are the only parameters. Once the quark masses have been determined, all dimensionless observables, such as the ratio of the rho meson mass to that of the nucleon, are fixed. This applies not only to mass ratios, but also to three point vertices such as the pion nucleon coupling.

But shouldn't the gauge theory coupling constant be regarded as parameter? Indeed, it is not; in the continuum limit the coupling drops out of physical quantities via the phenomenon of dimensional transmutation[2], which I will now discuss in lattice language. With this cutoff in place, it is natural to consider measuring some particular mass in units of the lattice spacing. In statistical mechanics this dimensionless combination represents an inverse correlation length.

$$ma = \xi^{-1} \qquad (1)$$

where m is the mass in question and a is the lattice spacing. A continuum limit requires taking a to zero while holding m at its physical value. In this limit the correlation length diverges. Thus, we have the statement that one must go to a critical point of a statistical system to have a continuum field theory model.

The non-Abelian gauge theory of the strong interactions is asymptotically free[3]. This means that we know something about how the coupling varies for the continuum limit. As the bare coupling is an effective coupling at the scale of the cutoff, standard renormalization group arguments show that it decreases logarithmically with the cutoff

$$g_0^2 = (\gamma_0 \ln(a^{-2}\Lambda_0^{-2}) + (\gamma_1/\gamma_0)\ln\ln(a^{-2}\Lambda_0^{-2}) + (g_0^2))^{-1} \qquad (2)$$

Here γ_0 and γ_1 are the first two coefficients in a weak coupling expansion of the Gell-Mann-Low renormalization group function[4-5]. The parameter Λ_0 is an integration constant of the renormalization group equation. Its value sets the scale for the theory and will cancel from any dimensionless ratio. We now take equation (2), solve it for a as a function of the bare coupling g_0, and put the result in equation (1). This gives the explicit form of the divergence of the correlation length as we approach the critical point at vanishing bare coupling

$$\xi^{-1} = \frac{m}{\Lambda_0}\,(\gamma_0 g_0^2)^{-\gamma_1/2\gamma_0^2}\,\exp\left(\frac{-1}{2\gamma_0 g_0^2}\right) \times (1 + 0)(g_0^2)] \qquad (3)$$

Note that the coefficient of the divergence is the ratio of the mass in question to the integration constant Λ_0.

The important point here is that Λ_0 is independent of the particular mass being measured. One could consider the correlation between operators with say rho quantum numbers and also the correlation between operators with nucleon quantum numbers. The ratio of the coefficients of corresponding weak coupling divergences gives the ratio of the rho to proton mass with no parameters to adjust beyond the bare quark masses, which have presumably been already determined via current algebra and the pseudoscalar meson masses.

To illustrate the idea, consider the long range linear potential between quark-like sources in the pure SU(3) gauge theory. In figure (1) I show Monte Carlo measurements of the effective force between such sources at various separations on a 12^4 site lattice[6]. The points form an envelope representing the strength of the constant long range force K in units of the lattice spacing squared. Plotting this force against $\beta = 6/g_0^2$ makes the weak coupling exponential behavior of equation (3) appear as a nearly straight line on this logarithmic graph. The normalization gives the string tension in units of the square of the parameter Λ_0. The band plotted in the figure represents

$$\Lambda_0 = (8 + 1) \times 10^{-3}\sqrt{K} \qquad (4)$$

Correcting for renormalization scheme dependence, and putting in the phenomenological value $\sqrt{K}$ = 400 MeV, this corresponds to the more conventional Λ_{MOM} being about 270 MeV.

Let me now turn to a brief review of Wilson's formulation of gauge theories on a lattice[1]. A lattice cutoff is a highly non-unique concept.

With the cutoff in place, one can add to the Lagrangian terms which will not contribute in the continuum limit. Using this freedom, Wilson presented a particularly elegant discrete action which keeps an exact local gauge invariance.

Lattice gauge theory is based on the concept of a gauge theory as a theory of non-integrable phase factors[7]. When a charged particle traverses a world line C in space time, its interaction with the electromagnetic field can be described by saying that its wave function picks up a phase

$$\psi \to \psi \exp\left(ig_0 \int_C A_\mu dx_\mu\right) \tag{5}$$

This is easily understood in the particle's rest frame, where the wave function picks up additional time oscillation proportional to the scalar potential. This increase in energy when transformed to an arbitrary frame gives equation (5). To put this concept on a lattice, Wilson approximates an arbitrary path with a sequence of nearest neighbor steps on a hypercubical lattice. The phase factor in equation (5) then becomes a product of elementary phases associated with the lattice links making up C. One nice feature of this formulation is that the generalization to a non-Abelian model is straightforward; one merely replaces the phases with matrices from the internal symmetry group. Thus, our variables are elements U_{ij} of the gauge group associated with every nearest neighbor pair of sites (i,j) of the lattice. For the strong interactions, these elements are of the group SU(3).

In terms of these variables, we need an action which will reduce to the ordinary gauge theory action in the continuum limit. Wilson proposed the particularly simple form

$$S = \sum_P \left(1 - N^{-1} \mathrm{ReTr} \prod_{ij \in P} U_{ij}\right) \tag{6}$$

Here the sum is over all elementary squares or "plaquettes" of the lattice, and the product is an ordered group product of the elements surrounding the given square. For convenience, we choose the normalization factor N as the dimension of the group matrices, i.e. 3 for SU(3).

Given our variables and action, we use the Feynman path integral to obtain a quantum theory. Thus we integrate the exponential of the action over all link variables

$$Z = \int du\, e^{-\beta S(U)} \tag{7}$$

Here β is proportional to the inverse bare coupling constant; for SU(N) we have

$$\beta = 2Ng_0^{-2} \tag{8}$$

Note that the path integral is equivalent to the partition function for the statistical mechanics of a four dimensional system of spins belonging to the gauge group. Their interaction is through the four spin coupling in Eq. (6).

Numerous techniques have been developed or borrowed from statistical mechanics for the study of this system. As the lattice is just a cutoff, one could proceed with ordinary perturbation theory. In condensed matter language, this is a spin wave expansion. This should reproduce all the standard results obtained with other regulators. The lattice propagators are, however, rather complicated, and thus only a few one loop calculations have been done[8]. As other schemes for perturbative analysis are rather highly developed, the value of the lattice lies elsewhere.

Ordinary perturbation theory is a weak coupling expansion. The opposite extreme is the strong coupling series proposed for lattice gauge theory in Wilson's original work on the subject[1]. In the statistical analog, this series is the high temperature expansion, a well developed technique in condensed

matter physics. In Wilson's formulation, this series is particularly simple and readily shows confinement. Indeed, the theory reduces to one of quarks on the ends of strings of gluonic flux. These strings have a finite energy per unit length and thus we have a linear interquark potential. Unfortunately, the strong coupling limit is not the continuum limit, and one must worry as to whether this confinement phenomenon survives as we reduce the coupling to the asymptotically free fixed point at zero coupling, as discussed above. In the statistical picture, in going from strong to weak coupling we reduce the temperature from infinity to zero, and one must consider possible phase transitions between regimes with qualitatively distinct behaviors.

Another tool of the solid state theorist is mean field theory. This technique becomes exact as the space time dimensionality of the system becomes large. In this approximation a deconfining phase transition is predicted to occur and be first order[9]. Such a transition is also known to exist for certain toy models based on discrete groups, where duality arguments locate the transition temperature exactly. Thus the utility of the lattice approach depends on four dimensions being sufficiently small that the mean field theory treatment breaks down at weak coupling for non-Abelian gauge groups.

The Migdal-Kadanoff recursion relations provide an analytic approximation to the renormalization group function of a general statistical system[10]. Before the Monte Carlo evidence appeared, these relations provided the strongest arguments for confinement in Yang-Mills theories. This technique indicates a close analogy between spin models in two dimensions and gauge theories in four. From this point of view, the absence of a deconfining phase transition in a four dimensional gauge theory corresponds directly to the absence of ferromagnetism in a corresponding two dimensional model. Although

this approximation to the renormalization group function does appear to correctly predict the critical dimensionality and the existence of some transitions, it can misidentify their nature. Tomboulis has recently argued that this approach can give a lower bound to the interquark potential and thus may be a first step towards a rigorous proof of confinement[11].

Currently the most popular tool for investigating lattice gauge theory is Monte Carlo simulation. This is an old technique of the solid state theorist which the particle physicist has recently realized can be used to numerically evaluate path integrals. The method converges reasonably well for bulk properties properties in all domains of coupling. As all field values are stored, in principle any desired correlation function is available. In a sense one is directly solving an interacting field theory.

Nevertheless, Monte Carlo methods do have inherent practical limitations. As we live in a four dimensional world, the lattices are necessarily rather limited in linear dimension, typically being of order 10 sites on a side. This means that both finite volume and finite lattice spacing effects must be carefully monitored and compromised against each other. This may not be as bad as it at first seems because precocious scaling in deep inelastic lepton scattering shows that short and long distance phenomena are not widely separated in nature's solution of the theory. The other inherent limitation in Monte Carlo approaches is statistical. Such errors decrease only with the square root of the computer time. This is a severe problem when signals are comparable to noise, as in the case of glueball masses.

The first observable number to be extracted from Monte Carlo analysis of non-Abelian gauge theory was the ratio of the string tension to the asymptotic freedom scale parameter indicated in Eq. (4). The second reasonably

uncontroversial number measured numerically was the physical temperature of a phase transition where confinement loses meaning because the vacuum becomes a soup of gluonic flux[12]. This concept of a physical temperature must not be confused with the abstract coupling-temperature analog between the Feynman path integral and the partition function for a four dimensional statistical system. The physical deconfinement transition with gauge group SU(3) appears to be first order, with the vacuum going abruptly from a dilute thermal gas of glueballs to a nearly black body spectrum of gluons. This may tie in well with the success of the bag model[13], which would suggest this transition occurring when the pressure of a free gluon gas equals the bag constant. The estimated deconfinement transition temperature of 180 MeV then gives a bag constant of

$$B = \frac{8}{3}\sigma T_c^4 = (8\pi^2 T_c^4/45) \approx (210\ \text{MeV})^4 \quad . \tag{9}$$

Note that the recent uncertainties in the ratio of string tension to asymptotic freedom scale are unlikely to dramatically change the estimate of the deconfinement transition. This numerical value is normalized to the string tension, and this ratio of two physical dimensional numbers should not have higher order perturbative contributions.

Another observable that has received considerable attention is the glueball mass. Indeed, a whole spectrum of states is under investigation, primarily by the groups in Ref. 14. In these calculations a $J^{PC} = 0^{++}$ state is reasonably clear in the range 700-1000 MeV. The higher states are more controversial, but all indications are for a rich spectrum below 2 GeV.

Much of the recent work in this field has been on including quark fields in the simulations. How to do this efficiently remains an active and unsettled question. From an analytic point of view, a fermionic path integral is

perfectly well defined[15]. Nevertheless it is not an integral in the classical sense and it is unclear how to use importance sampling techniques for numerical work. This problem can be circumvented, perhaps foolishly, by integrating out the anticommuting variables analytically. As the action is a quadratic form in the quark variables, this integration merely gives a determinant which can then be included in the Monte Carlo weighting procedure for the gauge fields. The difficulty with this line of attack is that the determinant is of an extremely large matrix, the number of rows being the product of the number of lattice sites with the ranges of the spinor, gauge symmetry, and flavor indices. Also, this determinant introduces effective long range couplings between the gauge variables.

The actual situation is not quite so hopeless because of various tricks. The fermionic action itself is local, and thus the matrix of which we need the determinant has an enormous number of zero elements. Furthermore, on changing a link variable only a few elements of this matrix are altered. Various schemes exploiting these features are being developed. I will briefly describe a version of Ref. 16 as formulated in Ref. 17. What we actually need in deciding whether to accept a change in a gauge field is the ratio of two determinants $|M'|/|M|$ where M' differs from M only in a few elements. This ratio can be determined from a ratio of path integrals over auxiliary scalar fields, referred to as "pseudofermions"

$$|M'|/|M| = \left(\int d\phi d\phi^* \, e^{-\phi^* M \phi}\right) / \left(\int d\phi d\phi^* \, e^{-\phi^* M' \phi}\right) \tag{10}$$

Adding and subtracting M' from the exponential in the numerator gives

$$|M'|/|M| = \langle e^{-\phi^*(M-M')\phi} \rangle_{M'} \tag{11}$$

where the expectation value is over fields ϕ weighted by the exponential in the denominator of Eq. (10). Now as M'-M is nonvanishing in only a local region, this expectation is readily evaluated with standard Monte Carlo methods. This method in principle becomes exact as the latter evaluation improves. It remains to be demonstrated that this can be done in a reasonable amount of computer time. Nevertheless, I regard the concept of a Monte Carlo inside of a Monte Carlo as esthetically distasteful.

Postponing these problems, an interesting approximation has been reasonably successful in approximately reproducing the hadronic spectrum[18]. The "valence" or "quenched" approximation consists of evaluating the propagators for quarks in a gauge field configuration obtained from a Monte Carlo simulation without any feedback from virtual quark loops. The experimental fact that a valence quark picture works fairly well suggests that the approximation may not be unreasonable. The results support the existence of chiral symmetry breaking and the consequent massless pion when the quark masses are zero. Other than this, the results are essentially those of a simple quark model, although the nucleon masses are coming out a few hundred MeV too heavy.

II. GROUP INTEGRATION, GAUGE INVARIANCE, AND ORDER PARAMETERS

Lattice gauge theory is defined by the partition function in Eq. (7). Here we are instructed to integrate over all possible values for the variables, which are matrices from the gauge group. For a compact Lie groups such as considered here, there exists a rather natural invariant measure to use in this summation. I will begin this lecture with a brief review of the meaning of this invariant measure, and then discuss the gauge invariance of this system and its implication for the definition of order parameters to distinguish various phases of the analog statistical system.

The invariant group measure is uniquely determined up to normalization by the requirement of left invariance

$$\int dg\, f(g) = \int dg\, f(g'g) \quad , \tag{12}$$

where f is an arbitrary function over the group and g' is an arbitrary group element. For compact groups as considered here we normalize such that

$$\int dg\, 1 = 1 \quad . \tag{13}$$

The measure is easily constructed by taking a small volume near the identity element and translating it to an arbitrary point in the group. To see this explicitly, consider some parametrization of the group in terms of a set of parameters α_i where the index runs from 1 to n, the dimension of the group manifold. Assume that as the parameters run over some domain D of R^n, the corresponding group element runs once over the group

$$G = \{g(\alpha) \,|\, \alpha \in D\} \quad . \tag{14}$$

The group multiplication law is represented by a function $\alpha(\beta,\gamma)$ satisfying

$$g(\alpha(\beta,\gamma)) = g(\beta)g(\gamma) \quad , \tag{15}$$

where α,β, and γ are in the region D. We now wish to express the invariant measure as an ordinary integral over our parameters

$$\int dg\, f(g) = \int d\alpha_1 \cdots d\alpha_n\, K(\alpha) f(g(\alpha)) \quad . \tag{16}$$

Using this notation, the group invariance property reads

$$\int d\beta J(\beta) f(g(\beta)) = \int d\beta J(\beta) f(g(\alpha(\gamma,\beta))) \quad , \tag{17}$$

where γ parametrizes the factor g' of Eq. (12). We now change variables on the right hand side of this equation to $\alpha(\gamma,\beta)$ with the result

$$\int d\beta J(\beta) f(g(\beta)) = \int d\alpha \left|\frac{\partial\alpha}{\partial\beta}\right|^{-1} J(\beta) f(g(\alpha)) \quad , \tag{18}$$

where $|\partial\alpha/\gamma\beta|$ represents the Jacobian determinant for the change of variables. Since f is arbitrary, we conclude

$$J(\alpha) = |\partial\alpha/\partial\beta|^{-1} J(\beta) \quad . \tag{19}$$

Taking β to the identity, denoted by e, we find

$$J(\gamma) = K\left|\partial(\alpha(\beta,\gamma))/\partial\beta\right|^{-1}\Big|_{\beta=e'} \tag{20}$$

where $K = J(e)$ is a normalization factor, determined in magnitude via eq. (13). We conclude that the invariant measure is simply a Jacobian factor. Using associativity of the group operation, one can show that this measure factor indeed works and that it is unique. A further straightforward argument shows that the left invariant measure is also right invariant.

Knowing of its existence may not be useful if the group combination law is complicated. A more explicit formula for the measure for groups of matrices follows from a natural definition of a metric tensor on the group

$$M_{ij} = \mathrm{Tr}(g^{-1}(\partial_i g) g^{-1}(\partial_j g)), \tag{21}$$

where the derivatives are with respect to the parameters

$$\partial_i g = \partial/\partial\alpha_i g(\alpha). \tag{22}$$

A standard formula of differential geometry then gives the integration measure

$$\int dg f(g) = K\int d\alpha \left|\det(M)\right|^{1/2} f(g(\alpha)), \tag{23}$$

where the factor K is again a normalization.

For most theoretical purposes an explicit form for the measure is unnecessary. In Monte Carlo simulation with the Metropolis algorithm, a random walk around the group automatically generates the correct weight. For analytic work, symmetry arguments give many integrals directly. A group integral selects the singlet part of any function over the group. For example, we have the relation

$$\int dg \chi_{R_1}(g) \ldots \chi_{R_k}(g) = n_s(R_1 \otimes \ldots \otimes R_k), \tag{24}$$

where the character $\chi_R(g)$ denotes the trace of the matrix corresponding to g in representation R and $n_s(R_1 \otimes \ldots)$ is the number of times the singlet representation occurs in the direct product of the representations R_1 to R_k.

I will now change the subject and discuss order parameters for lattice gauge theory. Note how the Wilson formulation emphasizes the analogy with a model of magnetism in statistical mechanics. The link variables U_{ij} are much like spins located on the bonds of a four dimensional hypercubical crystal. These variables interact through the four-spin coupling of the Wilson action. It is then natural to ask whether a lattice gauge theory can develop a spontaneous magnetization wherein these spins acquire an expectation value

$$\langle U_{ij} \rangle \neq 0. \tag{25}$$

However, due to gauge invariance, this is impossible in the Wilson theory[19].

The Wilson formulation maintains an exact local symmetry of the action under the substitution

$$U_{ij} \to g_i U_{ij} g_j^{-1} \tag{26}$$

Because this is a local symmetry, thermal fluctuations will induce such rotations and ultimately average over all gauges. Thus, the average value of any single link variable is necessarily zero, and the magnetization vanishes in pure lattice gauge theory.

This is an unfortunate result because in spin models the magnetization is a useful parameter for distinguishing phases. At high temperatures the system is disordered and the magnetization vanishes. If at lower temperatures the spins do acquire an expectation value, then we know that somewhere in between there must be a phase transition. In lattice gauge theory the expectation value of U_{ij} always vanishes and therefore cannot be used to monitor phase changes.

As thermal fluctuations will result in an averaging over gauges, it is natural to look for a gauge-invariant order parameter. The simplest quantity invariant under the transformation of equation (26) is the trace of the product of four links around a plaquette. The expectation value of this is the average action or the internal energy of the corresponding thermal system and it is obtained by a derivative of the partition function

$$P = \langle 1 - N^{-1} \mathrm{Tr} U \rangle = (1/6)\, \partial/\partial\beta \log Z \qquad (27)$$

The average plaquette P is an order parameter in the sense that it will exhibit the thermodynamic singularities of the bulk system. However, it lacks a useful property of a magnetization in that it never vanishes identically except at exactly zero temperature. We cannot distinguish phases by P vanishing in one and not the other. Indeed, gauge invariance precludes any local order parameter from having this property[20]. Despite this shortcoming, the average plaquette has played a major role in numerical work where it is the simplest variable to evaluate; indeed, many transitions are easily seen as jumps or singularities in P regarded as a function of the coupling.

Although local order parameters do not work for lattice gauge theory, there are some interesting non-local ones. An unconfined phase in a gauge theory based on a continuous group should contain massless gauge bosons. In contrast, in a confining phase the spectrum should consist of massive glueballs. This suggests that we can use the mass gap as an order parameter expected to vanish in one phase and not the other. The mass gap is a non-local order parameter because to determine it one must study the asymptotic behavior of correlation functions. The mass of a particle determines how it propagates over long distances. A slight complication with the use of the mass gap to study confinement occurs in the full theory containing quarks if some of the quark species happen to be massless. In this case one also expects massless Goldstone bosons associated with the breaking of chiral symmetry. A discussion of confinement in terms of the mass gap would then require a spin analysis of the massless quanta.

For the pure gluon theory without quarks the most popular order parameter in lattice gauge theory is the area law coefficient of Wilson[1]. The trace of a product of links around a closed loop is a gauge invariant construction called the Wilson loop

$$W(C) = \langle \mathrm{Tr} \prod_{ij \in C} U_{ij} \rangle \qquad (28)$$

Here C denotes the loop in question and the group elements are ordered as encountered in a circumnavigation of the contour. Wilson has argued that if widely separated quark-like sources experience a linear potential, then asymptotically large loops should give an expectation value decreasing exponentially with the area of the loop

$$W(C) \propto e^{-KA(C)} \qquad (29)$$

where A(C) is the minimal surface area enclosed by the loop C. Furthermore, as this exponential decay is associated with the action of the world sheet swept out by a gluonic flux tube ending on the contour C, the coefficient K should be precisely the coefficient of the long range linear interquark force

$$E(R) \underset{R\to\infty}{\to} KR \tag{30}$$

The coefficient K of the area law provides order another parameter for lattice gauge theory. It vanishes identically in unconfined phases while remaining non-zero whenever the linear long range force is present. It is similar to the mass gap in that it represents a non-local quantity, being defined in terms of the asymptotic behavior of a correlation function. Nevertheless, the area law criterion for confinement loses its value when quarks are introduced as dynamical variables. In this case widely separated sources will reduce their energy by creating a pair of quarks from the vacuum fluctuations and screening the long range gauge fields. A large Wilson loop will then effectively measure the potential between two mesons rather than simple bare quarks.

I will now leave the question of order parameters and turn to the question of gauge fixing in the lattice theory. In Wilson's formulation the integrals over the link variables are over compact domains and thus there are no divergences associated with integrating over gauges. This contrasts with usual continuum formulations where the volume of the gauge orbits is infinite and some sort of gauge fixing is necessary. Nevertheless, the gauge invariance of the Wilson action still permits working within a fixed gauge without affecting expectations of gauge invariant operators. I will now discuss a particularly simple class of gauges for lattice gauge theory[21].

Let P(U) be some polynomial in the link variables which is invariant under the transformation of equation (26). Associated with this polynomial is a Green's function

$$G(P) = Z^{-1}\int(dU)e^{-\beta S(U)}P(U) \tag{31}$$

Consider now one of the lattice links, say from site i to j. Suppose that in evaluating eq. (31) we forget to integrate over that particular variable. Remarkably, the result for G(P) will be unaffected. To see this formally, note that fixing U_{ij} at some element g replaces eq. (31) with

$$I(P,g)=Z^{-1}\int(dU)\delta(U_{ij},g)e^{-S(U)}P(U). \tag{32}$$

where $\delta(u_{ij},g)$ represents a Dirac delta function on group space. Clearly if we integrate over g we return to eq. (31). Now consider the gauge transformation of eq. (26). Because the action, P(U), and the integration measure are all invariant under this transformation, we can change variables to obtain

$$I(P,g) = I(P,g_i^{-1}gg_j). \tag{33}$$

Since g_i and g_j are arbitrary, we conclude that I(P,g) is actually independent of g and we have

$$I(P,g) = G(P), \tag{34}$$

The above process can be repeated to fix more link variables. The final result is that we can continue to fix links in any set which contains no closed loops. The fixed links should form a tree, which may be disconnected. The gauge is completely fixed when we have a maximal tree, a tree to which the addition of any more links would create a closed loop.

A particularly simple gauge is obtained when all links pointing in a particular direction are set to unity. This corresponds to an axial gauge where one component of the vector potential vanishes. Note that in this

situation those plaquettes parallel to the fixed axis represent a simple two spin coupling of the unfixed variables. The theory reduces into a set of one-dimensional spin chains interacting with each other via the four spin coupling of the remaining plaquettes. In two dimensions there is no interchain coupling and the pure gauge theory is equivalent to the exactly solvable one-dimensional spin system.

The gauges discussed above are simple extensions of the axial gauge and are all ghost free. One can introduce an arbitrary gauge fixing function f(U) into the path integral at the expense of introducing the necessary Fadeev-Popov correction factor[22]. Thus, we can write

$$G(P) = Z^{-1}\int(dU)(f(U)/\phi(U))e^{-S}P(U), \tag{35}$$

where $\phi(U)$ is an integral of the function f over all gauges

$$\phi(U) = \int(\prod_i dg_i)f(g_iU_{ij}g_j^{-1}). \tag{36}$$

MONTE CARLO ALGORITHMS

In this lecture I review the basic idea of a Monte Carlo simulation of a statistical system and discuss several alternative algorithms. A Monte Carlo program begins with some initial configuration of the fields stored in the computer memory. Pseudo-random changes are then induced on the variables, thus generating a Markov chain of configurations. The procedure is designed so that the ultimate probability for encountering any given configuration is proportional to the Boltzmann factor

$$P(C) = e^{-\beta S(C)} \tag{37}$$

where S(C) is the action associated with the given configuration.

To discuss Monte Carlo algorithms, we need a criterion for determining when some procedure will bring an ensemble of configurations closer to equilibrium. Each step of a Markov chain is specified by a probability distribution P(C',C) for taking configuration C into C'. An obvious necessary condition is that P leave an equilibrium ensemble in equilibrium. Thus, the Boltzmann weights should be an eigenvector of P

$$\sum_{C'} P(C,C')e^{-\beta S(C')} = e^{-\beta S(C)} \tag{38}$$

I will now show that if the algorithm also has eventual access to any configuration, this provides a sufficient condition for any ensemble to ultimately approach the equilibrium distribution. For this purpose, I need a notion of distance between two ensembles. Suppose we have two ensembles E and E', each of many configurations. Denote the probability for configuration C in E or E', by p(C) or p'(C), respectively. The distance between E and E' is then defined to be

$$\|E-E'\| = \sum_C |p(C)-p'(C)| \tag{39}$$

where the sum is over all possible configurations. Now suppose that E' resulted from the application of a Monte Carlo algorithm satisfying eq. (38) to ensemble E. This means that

$$p'(C) = \sum_{C'} P(C,C')p(C') \tag{40}$$

As P(C',C) is a probability, it satisfies

$$P(C',C) > 0 \tag{41}$$

$$\sum_{C'} P(C',C) = 1 \tag{42}$$

Comparing the distance of E' from equilibrium with the distance of E from equilibrium gives

$$\|E'-E_{eq}\| = \sum_C \Big| \sum_{C'} P(C,C')\big(p(C') - p_{eq}(C')\big)\Big|$$

$$\leq \sum_{C,C'} P(C,C')\Big|p'(C')-p_{eq}(C')\Big| = \|E-E_{eq}\|. \qquad (43)$$

We conclude that the algorithm brings an ensemble closer to equilibrium. Note that if P(C',C) never vanishes, i.e. if any configuration can be reached from any other, then the inequality is strict unless we are already in equilibrium.

To ensure that an algorithm has the equilibrium distribution as an eigenvector, most algorithms in practice are based on a condition of detailed balance

$$P(C',C)\, e^{-\beta S(C)} = P(C,C')\, e^{-\beta S(C')} \qquad (44)$$

Summing over the index C' and using eq. (42) immediately gives eq. (38). This condition, which is sufficient but not necessary for the approach to equilibrium, far from uniquely specifies P(C',C). The most intuitive Monte Carlo procedure effectively consists of taking a heat bath at inverse temperature β and touching it sequentially to the variables of the system[23]. A real thermal source in contact wih a link variable would cause that element to fluctuate throughout the group manifold. When the source is removed, the link would be left in any of its allowed states with a probability proportional to the Boltzmann weight associated with the interaction with its neighbors. A heat bath program, then, consists of calculating Boltzmann factors and generating correspondingly weighted group elements. This automatically satisfies the detailed balance condition because the two probabilities in eq. (44) are directly proportional to the Boltzmann weights in the same equation.

The main disadvantage of the heat bath algorithm is that when the group manifold is intricate the required selection procedure may be rather difficult or time consuming. The Metropolis et al. algorithm[24] uses the detailed balance criterion to give another procedure which, because of its calculational simplicity, has become the most popular in practice. For the gauge theory, the method begins with the selection of a trial U' as a tentative replacement for some link variable U. The test variable is selected with a distribution $P_T(U,U')$ which is usually taken as symmetric in U and U'

$$P_T(U,U') = P_T(U',U) \qquad (45)$$

Once U' is chosen, we evaluate the tentative new action S(U') for comparison with its old value S(U). If the action is lowered, that is, if the new configuration has a larger Boltzmann weight, then the change is accepted. The detailed balance condition then determines the remainder of the algorithm; if the action is raised, the change is accepted with conditional probability $\exp\big(-\beta(S(U')-S(U)\big)$.

The Metropolis procedure depends essentially on two parameters. First is the trial distribution P(U,U'), which is normally given a weighting toward small changes. This peaking can be adjusted to optimize convergence and should be more extreme at low temperature where thermal fluctuations decrease. A second parameter of the algorithm is the number of trial changes attempted on any given link before proceeding to the next. In statistical mechanics this is usually taken to be one; however, for a gauge theory the interaction is rather complicated and considerable arithmetic must be done with an element's neighbors in the process of evaluating the old and new actions. With several trial changes much of this arithmetic need only be done once and thus it can be

quite beneficial to do as good a job as possible before moving on. Note that as the number of tries, or "hits," increases, the Metropolis algorithm approaches the heat bath. This is because repeating the procedure on one link will ultimately bring that link into thermal equilibrium with its neighbors. This is what the heat bath *does* in one step. For SU(3) typically 5-20 hits per link seems to work well.

A recent resurrection of the Langevin stochastic differential equation in the context of quantum field theory has provided an interesting variation on the more conventional Monte Carlo methods[25]. I will describe the procedure for a system of one degree of freedom x, for which we want to obtain an ensemble of values distributed with a Boltzmann probability distribution

$$P(x) \propto \exp(-\beta\, S(x)) \tag{46}$$

For this purpose we introduce a new "Monte Carlo" time variable τ and study the evolution of some particular x under the differential equation

$$dx/d\tau = -\,\partial S/\partial x + \eta(\tau) \tag{47}$$

Here $\eta(\tau)$ represents a random "thermal" noise satisfying

$$\langle \eta(\tau)\, \eta(\tau') \rangle = (2/\beta)\delta(\tau-\tau') \tag{48}$$

To make numerical sense of this equation, we do the discreet thing and discretize time by introducing the infinitesmal time step $dt = \varepsilon$. Going from one time step to the next, x changes to

$$x' = x - \varepsilon\, \partial S/\partial x + \varepsilon\eta \tag{49}$$

At each time η is independently obtained from a random number generator and is distributed with some probability distribution $\rho(\eta)$. Equation (48) tells us that we want

$$\begin{aligned} &\int d\eta\, \rho(\eta) = 1 \\ &\int d\eta\, \eta\, \rho(\eta) = 0 \\ &\int d\eta\, \eta^2 \rho(\eta) = 2/\beta\varepsilon \end{aligned} \tag{50}$$

Higher moments of ρ are irrelevant to our discussion here, although it is popular to consider a Gaussian distribution.

We now have the machinery necessary to discuss the Langevin equation in the context of the evolution of ensembles. If we have an ensemble with some distribution P(x), in one time step it will evolve into a new distribution

$$P'(x) = \int dx'\, P(x')\, \rho\left((x-x' + \varepsilon\partial S/\partial x')/\varepsilon\right) \tag{51}$$

Changing variables to the argument of ρ and expanding in powers of ε shows that to order ε^2 the equilibrium distribution of Eq. (46) is an eigenvector of the procedure. Thus the earlier discussion applies and iterating this equation must bring one closer to equilibrium.

It is interesting to contrast this method with that of Metropolis. In both cases one makes random steps in the variables. To maintain the peaking of the equilibrium distribution towards lower actions, the Metropolis procedure rejects some changes to higher action. The Langevin approach, on the other hand, always accepts changes, but selects the new variables in a domain shifted towards lower action by the "force" term $\partial S/\partial x$.

A different approach also using a differential evolution to simulate lattice gauge theory has been recently pursued in Ref. (26). This method is analog of molecular dynamics techniques in statistical physics, and differs from Monte Carlo simulations in that no random numbers are involved in the evolution of the system. For the U(1) gauge group, in addition to the fields U_{ij} on the lattice links, the authors introduce conjugate momenta E_{ij} and a new time coordinate τ. They then study the classical evolution of the four dimensional Euclidean lattice under the dynamics governed by the Hamiltonian

$$H = S(U) + \frac{1}{2}\sum_{ij} \ell_{ij}^2 \quad . \tag{52}$$

This results in a meandering through phase space on the surface of constant total "energy". If the behavior is ergodic, this will sample the microcanonical ensemble. Note that in addition to not involving random numbers, the procedure does not depend on a temperature parameter. The randomness arises from the complexity of the system itself, and the temperature can be measured from the average kinetic energy

$$\langle \tfrac{1}{2} \ell^2 \rangle = \tfrac{1}{2} kT = 1/(2\beta) \quad . \tag{53}$$

An interesting hybrid of microcanonical and Monte Carlo methods consists of a random walk through configurations of constant total energy. In simple cases this can be done without the introduction auxiliary variables such as the momenta in Eq. (52). Consider, for example, SU(2) gauge theory. When looking at a given link, the action would be unchanged if that link were replaced by another SU(2) element on a two dimensional submanifold of the gauge group. With Wilson's action, this submanifold is a simple sphere from which it is quite easy to select a random element. This suggests the simulation algorithm wherein one sweeps through the lattice and replaces each group element with another from the corresponding sphere. This has the advantage over the differential equation approach that it allows large steps. Empirically, the algorithm seems to converge comparably to either a heat bath or a well optimized Metropolis algorithm. It has the disadvantage that extracting the temperature is not straightforward. This is not a particular problem for particle physics applications because the bare coupling is not a physical observable. To test asymptotic freedom scaling laws, one can use any

quantity which perturbatively begins as the coupling squared. The simplest such parameter is the internal energy, which is exactly constant in this algorithm.

This procedure takes special advantage of the simplicity of the SU(2) manifold. I will now present a microcanonical simulation technique which is easily applied to any type of field variable, including discrete ones[27]. A single extra degree of freedom, a "demon", travels around the system, transferring energy as he changes the dynamical variables. This demon is analogous to the extra kinetic energy terms in the conventional molecular dynamics method, except that he is not associated with any particular lattice variable. In his travels he carries a sack of energy with non-negative content. Upon visiting a variable, he first attempts to change it using some distribution as in Eq. (45). Instead of accepting or rejecting the change based on random numbers, he makes the change only if he has sufficient energy in his sack. The latter is then adjusted so that the demon and the lattice together maintain a constant total energy.

The demon might be regarded as a very small heat reservoir. From his average load, one can obtain the temperature

$$\langle E_D \rangle = T = 1/\beta \tag{54}$$

If instead of a single demon one releases a number comparable to the number of lattice variables, then they collectively can contain an appreciable amount of energy. When the number of demons goes to infinity, the algorithm reduces to that of Metropolis et al.

The procesure has some advantages. First, the demon has no need for transcendental functions; his energy becomes automatically exponentially distributed. Secondly, he is rather lenient in his demands on the random

number generator. Indeed, with an Ising model no random numbers are needed at all. Third, for discrete groups all arithmetic can be done with small integers. This raises the possibility of very fast programs involving many demons riding on a few computer words and all acting simultaneously on different parts of the lattice. Finally, the method does not treat the Boltzmann weight as a probability. Hopefully this will point the way to future algorithms which do not even treat the action as a number, which it is not when fermionic fields are present.

VARIANTS ON THE WILSON THEORY

Although rather elegant, Wilson's formulation of gauge theories on a lattice is somewhat arbitrary. It is only on the removal of the cutoff that one should obtain unique answers for the continuum renormalizable field theory. The fact that physical results are independent of cutoff scheme has been investigated perturbatively for many years. Monte Carlo methods give us a chance to study this universality of the continuum limit in a non-perturbative manner. In this lecture I will review results on alternatives to the Wilson action.

One simple alternative is to merely place a vector potential A_μ on each site of a lattice and define an action by replacing derivatives in the continuum Yang-Mills Lagrangian with nearest neighbor differences. This procedure does not keep local gauge invariance as an exact symmetry. An interesting unresolved point is whether the gauge non-invariant pieces of the action go to zero sufficiently rapidly in the continuum limit to overcome any new ultraviolet divergences associated with these terms. In this formulation the integral over gauges is not compact, and thus a gauge fixing is necessary at the outset. In Ref. (28) preliminary Monte Carlo simulations were done on this model. The authors did not observe the area law behavior for Wilson

loops, but this is probably due to a renormalization of the bare charge making the string tension unobservably small except at very strong coupling.

Closer in spirit to the Wilson model, Ref. (29) considers replacing the simple plaquette with the 2 by 1 Wilson loop as the fundamental term in the action. Simulations with this "fenetre" action show a strong first order transition. This means that the absence of a phase transition is not a universal property of a gauge model. This transition is presumably not deconfining and is an artifact of the cutoff scheme when the lattice spacing is not small.

Keeping the action a function of plaquette variables, it is still possible to use other class functions than the trace. Manton[30] presented a particularly simple alternative, taking for the action of a single plaquette

$$S_p(U) = d^2(U,I) \tag{55}$$

where $d(U,I)$ is the minimal distance in the group manifold between the element U and the identity. The metric for defining this distance is that of Eq. (21). The Manton action is convenient for analytic work in the weak coupling limit but is singular for those elements lying at a maximal distance from the identity. This singularity has the annoying property of giving a theory which does not satisfy reflection positivity[31]. A closely related action which avoids this problem is the "heat kernel" form[32]. Using the metric tensor of Eq.(21), one can define a Laplace operator on the group

$$\nabla^2 = (\det M)^{-1/2} (\partial/\partial\alpha_i) (\det M)^{1/2} M^{-1}_{ij} \partial/\partial\alpha_j \ . \tag{56}$$

Introducing a new time parameter t, consider the heat diffusion equation

$$\nabla^2 K(t,g) = - d/dt\ K(t,g) \ . \tag{57}$$

For an initial condition, take

$$K(0,g) = \delta(g,I) \ . \tag{58}$$

The heat kernel action is directly identified with the solution of this equation at a "time" given by the coupling constant

$$\exp(S_p(\beta,U)) = K(1/\beta,U) \quad . \tag{59}$$

Both the Manton and heat kernel actions have been subjected to Monte Carlo analysis with the gauge group SU(2)[33]. The string tension was obtained as discussed earlier in these lectures. For comparison with the usual Wilson action results, the scheme dependence of the integration constant Λ_0 has been calculated perturbatively. The results show deviations of 20-40% from the theoretical calculations assuming a universal string tension. This should be regarded as the systematic uncertainty due to the practical fact that the lattice spacing must be kept fairly large and higher terms in the renormalization group function can be important.

Another variation of the theory is to replace the trace of the plaquette variables with the trace in some other group representation than the fundamental. For SU(2) if we use the trace in the adjoint representation, a clear first order transition appears[34]. One possible explanation of this transition is in terms of Z_2 monopole excitations. These arise because the adjoint or SO(3) representation of SU(2) does not see the Z_2 center of the group. A plaquette near -1 has the same energy as one near 1. This can be used to define a Dirac string as a sequence of plaquettes near -1. Several closely related schemes have been presented for making this concept precise[35].

Including both fundamental and adjoint traces in the action gives a two parameter action which interpolates between the SU(2) and the SO(3) Wilson theories[36]

$$S_p(U) = \tfrac{1}{2}\,\beta\ \mathrm{Tr}\ U + \tfrac{1}{3}\,\beta_A\ \mathrm{Tr}_A\ U \quad . \tag{60}$$

Here Tr_A denotes the trace or character in the adjoint representation. In addition to giving the above theories when either β_A or β is zero, this theory reduces to a Z_2 lattice gauge theory when β_A goes to infinity. This forces all plaquettes to lie in the group center, a situation which is gauge equivalent to all links lying in this same set.

Monte Carlo simulations have studied the evolution of the Z_2 and SO(3) transitions into this two coupling phase plane. The resulting phase diagram is shown in Fig. (2). Note that these transitions enter the diagram and meet at a triple point. The third line emanating from this point is directed towards the region of rapid crossover from strong to weak coupling behavior in the Wilson theory.

This system provides a nice place to test universality. The connection between the bare charge and the parameters is

$$g_0^{-2} = \beta/4 + 2\beta_A/3 \quad . \tag{61}$$

A continuum limit requires taking g_0^2 to zero; however, this can be done along numerous paths. Note that some of these paths could cross a first order transition line. Nevertheless, we can continue around this transition in the two coupling plane and thus the encountered transition is not deconfining. If the continuum limit is indeed independent of cutoff scheme, then physical observables should approach the same values along any of these paths. This has been tested for the string tension, ratios of Wilson loops, and the deconfinement transition.[36-37]. As always with Monte Carlo methods, statistical errors remain, but in the vicinity of the Wilson theory along the axis these analyses are consistent with universality. However, if one keeps the lattice spacing fairly large and goes near the critical endpoint in this phase diagram, then new physics associated with the lattice comes into play.

As the parameter β_A increases relative to β, the extremum of the action at plaquette variables near -1 changes from a maximum to a minimum upon crossing the line $\beta_A = 3\beta/8$. The critical endpoint in Fig. (2) lies just above this line. Investigations with SU(3) indicate a similar critical endpoint in the analogous two coupling plane near the line where new minima of the plaquette action appear for $U_p = \exp(\pm 2\pi i/3)$. As the N of SU(N) increases beyond four, those elements of the group center nearest the identity become minima of the action even for the conventional Wilson action[38]. This correlates well with the observation of first order transitions in SU(N) lattice gauge theory with the Wilson action when N is four or larger[39].

As finite lattice spacing effects do depend on formulation, it is interesting to look for an action which minimizes them[40]. This will permit more accurate calculations on a fixed size lattice at the expense of a more complicated action. Preliminary results with this approach are quite promising[41].

To conclude, Monte Carlo simulation has become a powerful tool for non-perturbative studies in field theory. Nevertheless, in several areas the techniques are approaching technological limits. The glueball and fermionic calculations are using hundreds of hours on state-of-the-art computers. Relative to this, analytic techniques for lattice gauge theory are being somewhat neglected. Presumably new hybrid approaches could be developed. In any case, as I stated at the beginning of these lectures, the progress in the last few years has been dramatic and exciting. This will hopefully continue in new and unpredictable directions.

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FIGURE CAPTIONS

Fig. 1 The effective interquark force at various separations. The precise definition of X(I,J) is in Ref. 6.

Fig. 2. The phase diagram resulting from the action in Eq. (60).

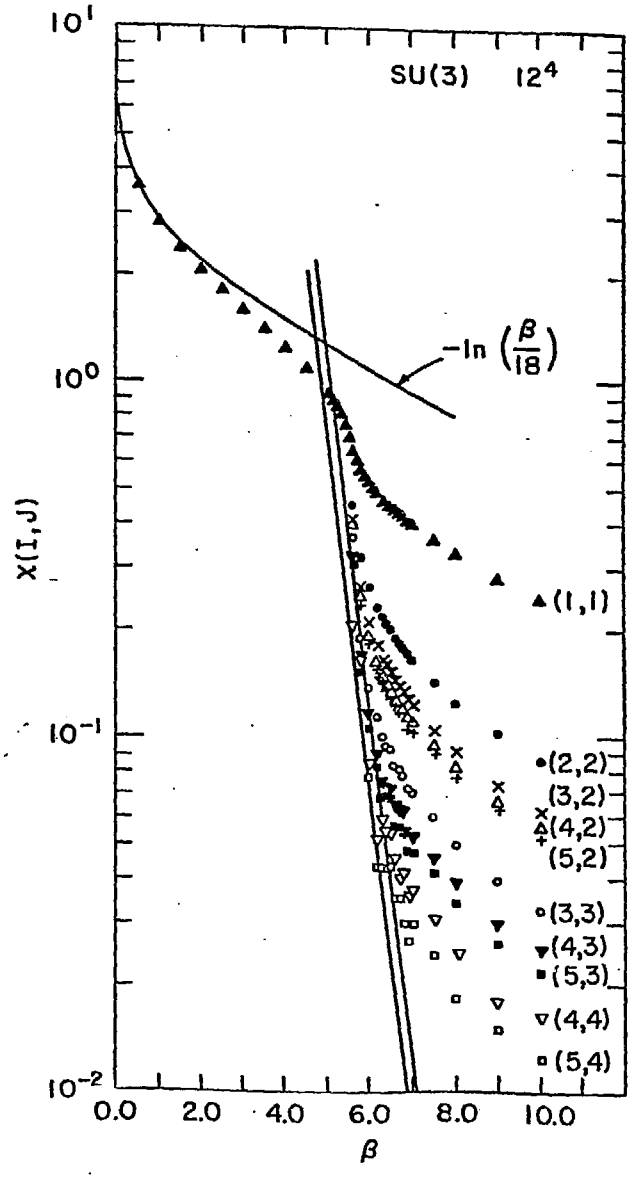


Fig. 1a

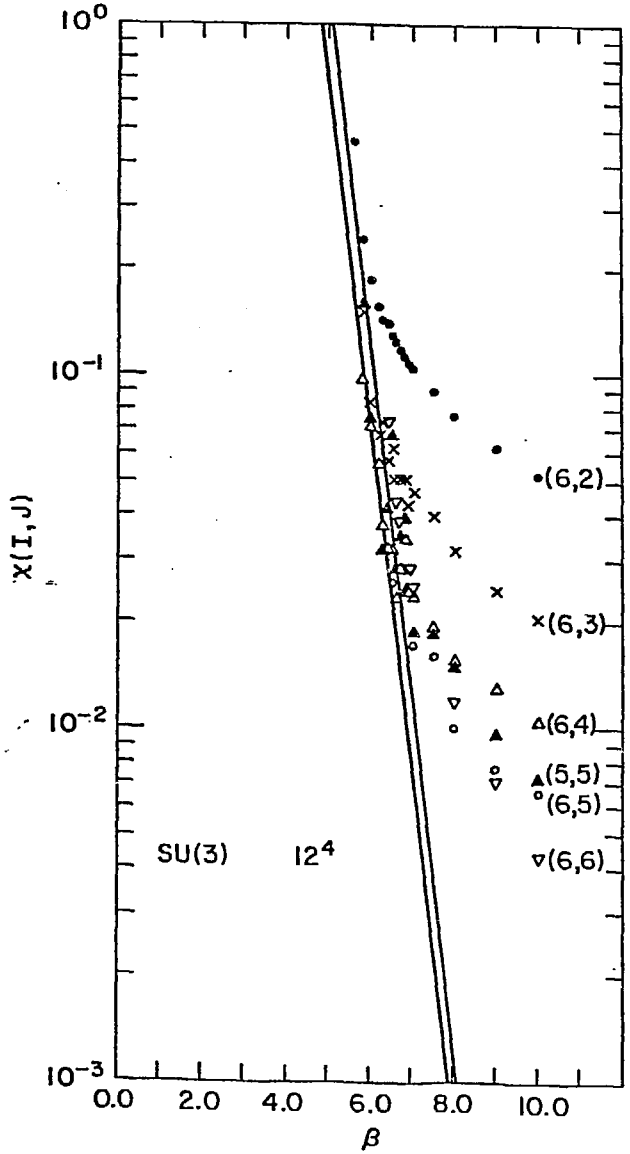
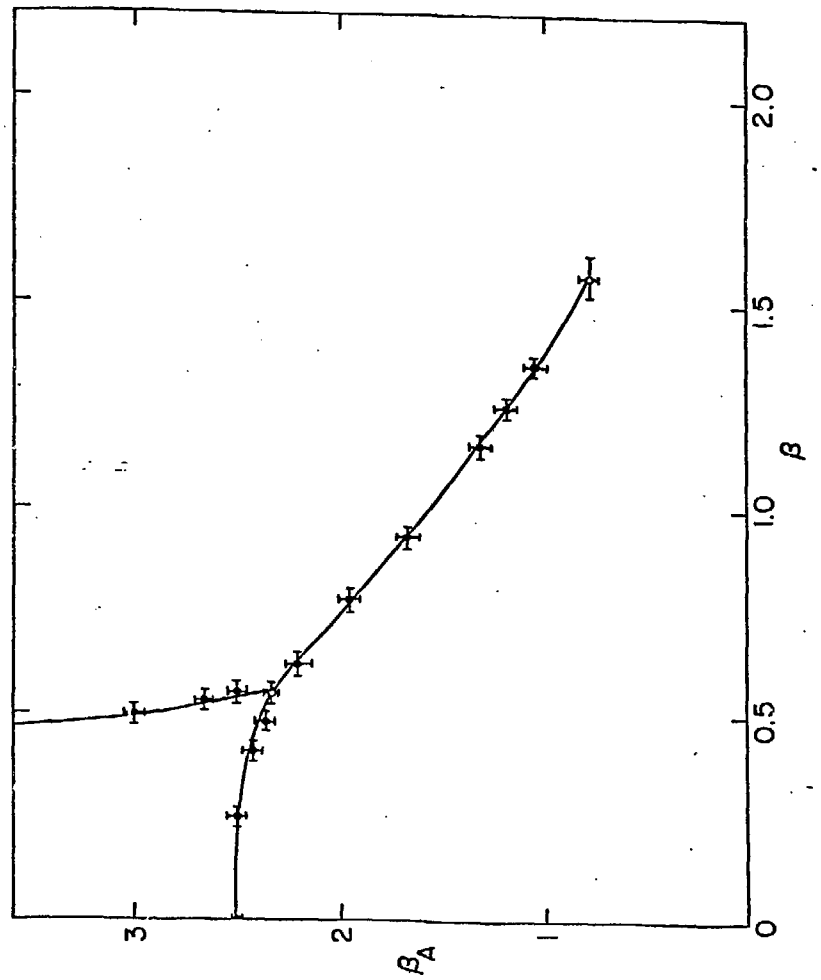

10^0
10^{-1}
10^{-2}
10^{-3}
$\chi(I,J)$
SU(3) 12^4
•(6,2)
×(6,3)
△(6,4)
▲(5,5)
°(6,5)
▽(6,6)
0.0 2.0 4.0 6.0 8.0 10.0
β

Fig. 1b


3 2 1
β_A
0 0.5 1.0 1.5 2.0
β

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