SANDIA REPORT SAND86—0403 • Unlimited Release
Printed March 1986

Derivation of Kane's Equations of Motion and the Formulation of an Initial Value Problem for the Polaris A3 Missile

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Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550
for the United States Department of Energy
under Contract DE-AC04-76DP00789

SF2900Q(8-81)

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

Printed in the United States of America
Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

NTIS price codes
Printed copy: A03
Microfiche copy: A01

DERIVATION OF KANE'S EQUATIONS OF MOTION AND THE FORMULATION OF AN INITIAL-VALUE PROBLEM FOR THE POLARIS A3 MISSILE

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SAND--86-0403

DE86 009784

ABSTRACT

Kane's Equations of Motion for the Polaris A3 missile are derived, and an initial-value problem which can be used in a time simulation of the missile's flight is developed. A brief overview of Kane's method for dynamical analysis is included to assist readers who may not be familiar with the techniques employed. The model assumes rigid body dynamics, and accommodates the earth's geometry and gravity field, the aerodynamic forces, and the missile's thrust vector control.

ACKNOWLEDGMENT

I wish to thank R. D. Tucker for his knowledgeable advice, suggested improvements, and reference materials. I also appreciate the assistance rendered by J. E. White, who reviewed the report and provided constructive criticism and suggestions.

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I. Introduction

This report consists primarily of the derivation of the equations of motion for the Polaris A3 missile. The derivation of the dynamical equations is based on the method developed by T. R. Kane of Stanford University, and employs the various elements which characterize the method, such as generalized coordinates, generalized speeds, partial velocities, and finally, Kane's equations of motion. A brief overview of the method is included to assist readers who are not familiar with the technique.

In addition to the equations of motion, an initial value problem is developed which can be incorporated in a simulation of the missile's flight to provide velocity and position predictions as a function of time.

This analysis deals with a specific application; however, the approach was intentionally made as general as possible to enhance its application to similar problems.

II. Definition of Reference Frames and Generalized Coordinates

A. Reference Frame Orientations

The reference frame geometries used in this analysis are shown in Figures 1a and 1b. Reference frame I is assumed to be an inertial reference frame and is represented by the set of mutually perpendicular unit vectors $\underline{i}_1$, $\underline{i}_2$, and $\underline{i}_3$ that are fixed in I with their origin at C, the geometric center of the earth, E. The earth is represented by the unit vectors $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$, fixed in E, whose origin is also located at C.

The mass center of the missile M is located at O. Point O is also the origin of the unit vector sets $\underline{c}_1$, $\underline{c}_2$, $\underline{c}_3$ and $\underline{N}$, $\underline{E}$, $\underline{D}$ which represent the Geocentric and Geographic frames, respectively. The $\underline{c}_3$ vector of the Geocentric frame lies in the direction opposite to the geocentric position vector, $\underline{r}$. The $\underline{c}_2$ vector points east, and the $\underline{c}_1$ vector, which lies in the local meridian plane, completes the right-handed orthogonal vector set. Vector $\underline{E}$ of the Geographic frame coincides with $\underline{c}_2$, but as seen in Figure 1b, $\underline{D}$ is defined to be normal to the reference ellipsoid which represents the earth's shape[1]. The reference ellipsoid is a solid of revolution that is symmetrical about the polar axis. Vector $\underline{N}$ completes the right-handed orthogonal set.

B. Generalized Coordinates

Specification of the configuration of the system is accomplished through the use of generalized coordinates. By choosing a set of independent coordinates, the derivation of the equations of motion is usually simplified. One such set for this problem, which would completely and independently specify the missile's six degrees of freedom, would be the polar coordinates λ, L_c, and r, and the three Euler angles of the body relative to the Geocentric frame. Unfortunately, practical navigation

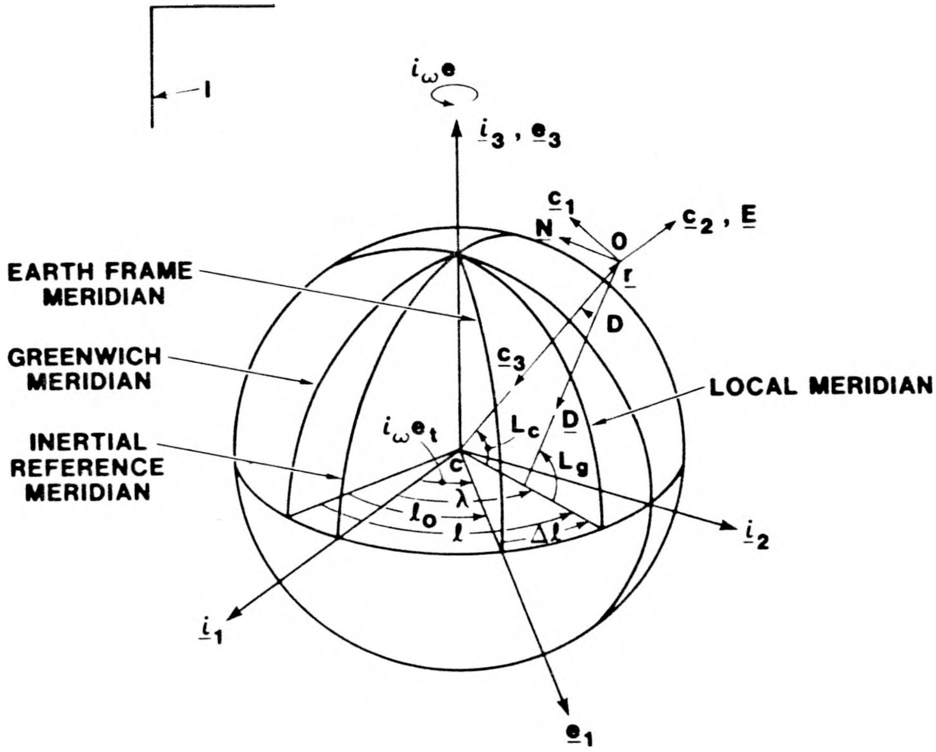

FIGURE 1a. Coordinate Frame Geometry.

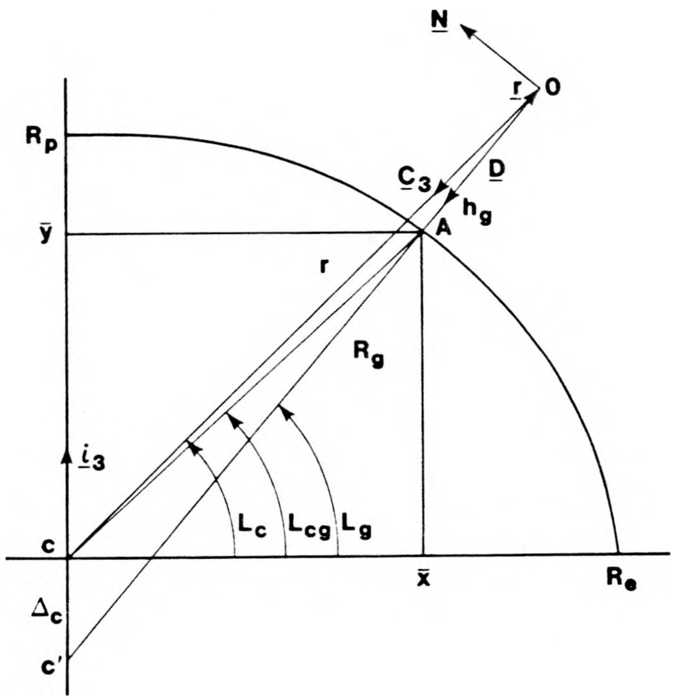

FIGURE 1b. Coordinate Relationships.

problems commonly utilize the geographic latitude angle L_g rather than L_c, and the geographic altitude h_g rather than the geocentric radius r.

If L_g and h_g are used as coordinates, it is no longer apparent that the missile's six degrees of freedom can be specified with six independent generalized coordinates. Referring again to Figures 1a and 1b, one possible set of coordinates would be the angles λ and L_g, the lengths Δ_c, R_g and h_g, and finally, the three Euler angles of roll, pitch, and yaw of the missile relative to the Geographic frame $\underline{N}$, $\underline{E}$, $\underline{D}$. However, only six of these eight generalized coordinates can be independent. As will be seen later, Δ_c and R_g can be expressed in terms of L_g, and as a result, the final dynamical equations will not involve Δ_c or R_g.

C. Missile Orientation

Figure 2 shows the vector sets which represent the missile and its nozzles. The missile's vector set is located at O, with $\underline{X}$ pointed forward along the longitudinal, or roll axis of the missile. Vector $\underline{Y}$ coincides with the pitch axis, and $\underline{Z}$ coincides with the yaw axis. The orientation of M is determined using a "body-three, 3-2-1" rotation sequence[6]. That is, M is brought into general orientation by first aligning $\underline{X}$, $\underline{Y}$, and $\underline{Z}$ with $\underline{N}$, $\underline{E}$, and $\underline{D}$, respectively, and then performing successive right-handed rotations of ψ about $\underline{Z}$, θ about $\underline{Y}$, and ϕ about $\underline{X}$. Other rotation sequences can be chosen, but this particular sequence is perhaps the most intuitive since the first rotation brings $\underline{X}$ into its final azimuth; the second rotation brings $\underline{X}$ into its final elevation; and the final rotation brings the missile into its final roll position. If the rotations are performed in a different sequence, or about only two axes, their effects on heading, elevation, and roll are not independent.

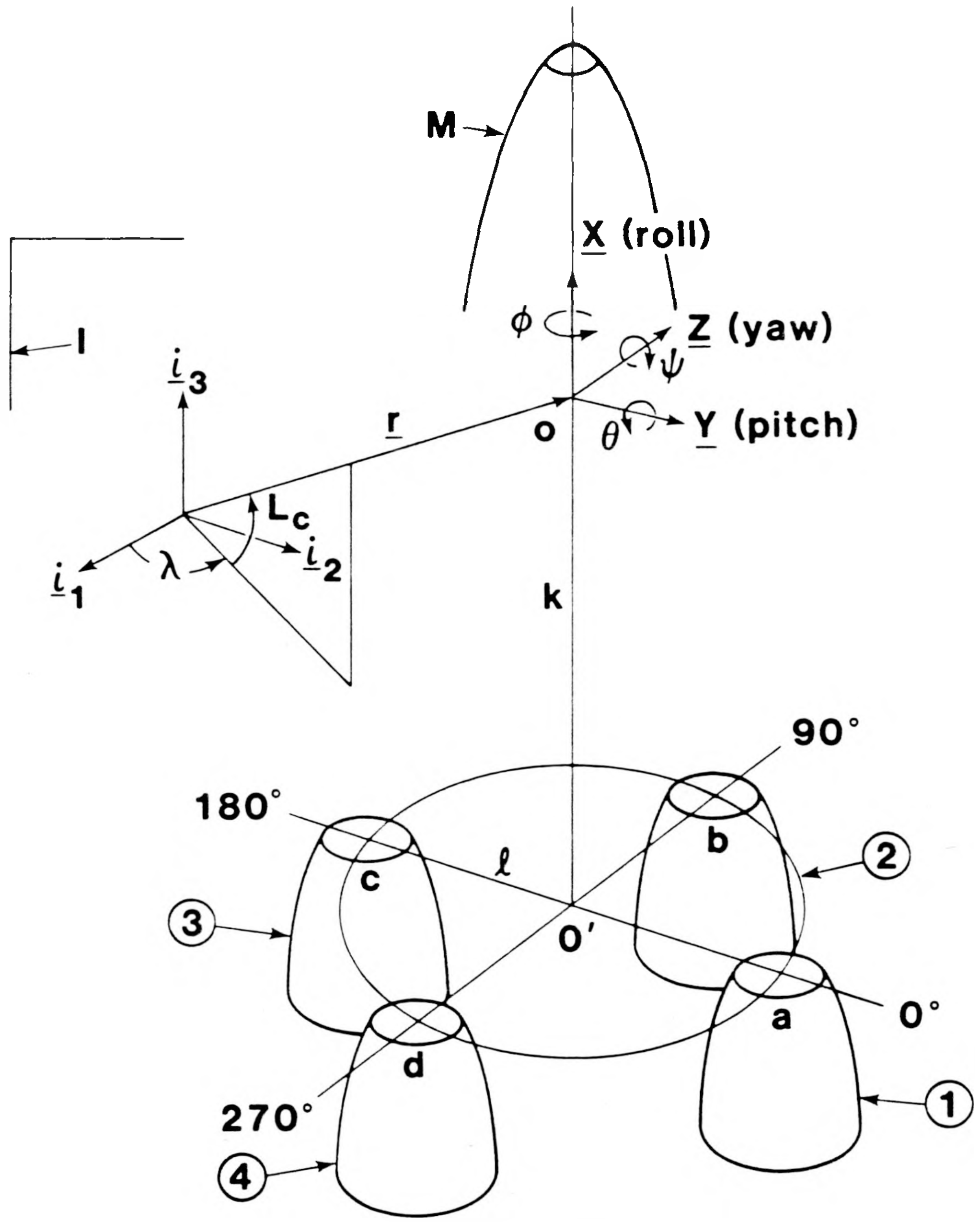

FIGURE 2. Missile Frame and Nozzle Locations.

D. Nozzle Orientation

Four nozzles are located at the base of the missile in such a way that the direction of their thrust vectors can be changed by rotating each nozzle on a bearing whose axis is not parallel to the axis of symmetry of the nozzle. In Figure 2, points a, b, c, and d represent the points where the four bearing axes and the four nozzle axes are presumed to coincide, and where each nozzle thrust vector is assumed to be acting. These points lie on a circle of radius l whose origin O' is on the missile centerline. The distance between O and O' is represented by k. The azimuthal locations of a, b, c, and d are at 0, 90, 180, and 270 degrees respectively, measured clockwise, looking forward, and starting at the positive pitch axis. Nozzles 1, 2, 3, and 4 are mounted at a, b, c, and d, respectively.

Four unit vector vector sets, which are fixed with respect to the nozzles, have their origins at a, b, c, and d. Each nozzle is brought into a general orientation using a "body-two" rotation sequence[6]. Figure 3 shows the vector system used to determine the orientation of the centerline of Nozzle 1. The vectors $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$ are fixed relative to the nozzle mounted at point a, with $\underline{a}_1$ coinciding with the nozzle's centerline and its assumed thrust axis. The orientation of the nozzle is determined by first aligning $\underline{a}_i$, i = 1,2,3, with $\underline{X}$, $\underline{Y}$, and $\underline{Z}$ respectively, and then performing successive right-handed rotations of the nozzle, of $-\beta$, the bearing cant angle, about $\underline{a}_3$, and then of δ_1, the nozzle rotation angle, about $\underline{a}_1$, and finally of γ, the nozzle cant angle, again about $\underline{a}_3$. Likewise, the orientations of nozzles 2, 3, and 4 are determined by first aligning vector sets $\underline{b}_i$, $\underline{c}_i$, and $\underline{d}_i$, i = 1,2,3, with $\underline{X}$, $\underline{Y}$, and $\underline{Z}$, and performing successive right-handed rotations in a manner similar to that for Nozzle 1. Control of the net thrust vector is achieved by adjusting the nozzle rotation angles δ_i, i = 1-4. For the

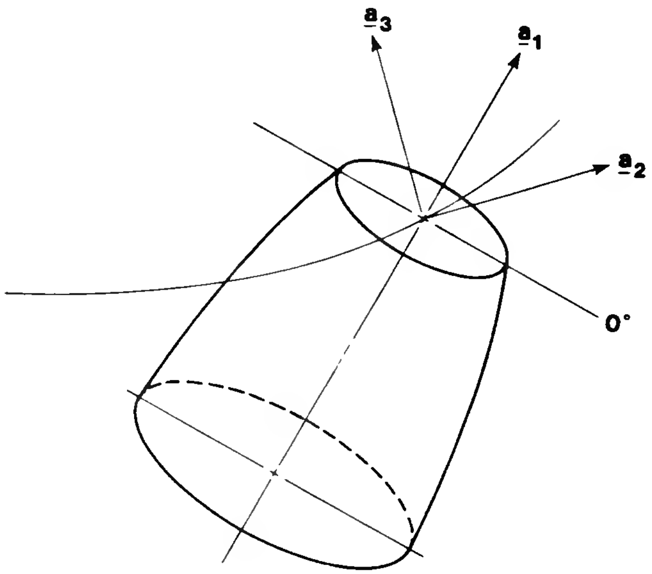

FIGURE 3. Nozzle Frame.

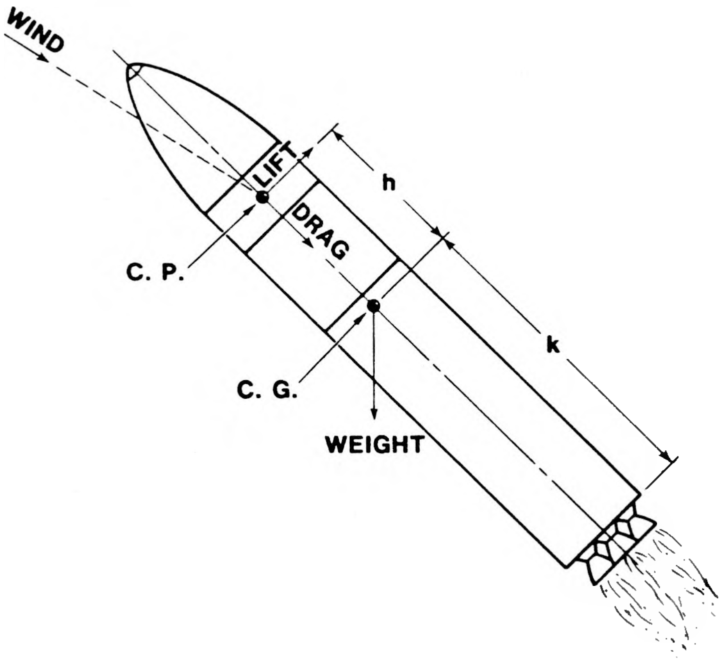

FIGURE 4. Body Forces.

Polaris system, positive control system commands produce positive rotations of nozzles 1 and 2, and negative rotations of nozzles 3 and 4.

The thrust vector control for the second stage motor can be handled in a manner similar to that of the first stage motor by using the same vector system, but with different angles and rotations. The second stage motor nozzles do not physically move, but instead, their exhaust is deflected by injecting freon into the sides of the nozzles. The thrust deflection can be represented by setting the bearing and nozzle cant angles, β and γ, to zero and performing nozzle "rotations" δ_i, i = 1-4, about the vectors which point in the radial directions, $\underline{a}_2$, $\underline{b}_3$, $\underline{c}_2$, and $\underline{d}_3$.

E. Body Forces

In addition to the motor thrust, the missile also experiences wind and gravity body forces during its flight. These distributed forces can be replaced with forces acting as shown in Figure 4. The aerodynamic forces are replaced with an equivalent force located at the missile's center of pressure, and the gravity forces are replaced with the weight of the missile located at it's center of gravity. The locations of the center of pressure and the center of gravity, as well as the orientations and magnitudes of the forces are all assumed to be variable during the flight of the missile.

F. Rotation Matrices

The reference frames and vector systems described above and shown in Figures 1-4 define the position, orientation, and dimensions of the missile as well as the points of application and directions of the various forces used in deriving the equations of motion. Throughout the analysis, it will be necessary to express certain quantities in terms of a set of vectors that are oriented differently from those in which the quantities are currently written. Rotation matrices will be used to describe the orientation of one set of vectors relative to another. For example, suppose a vector has been expressed in terms of the vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ and is denoted as $\underline{v}^e$. The same vector expressed in terms of $\underline{i}_1$, $\underline{i}_2$, $\underline{i}_3$, or $\underline{v}^i$, is given by:

$$\underline{v}^i = \underline{R}_e^i \, \underline{v}^e$$

Also note that the columns of each matrix are the components of the unit vector shown above the column written in terms of the unit vectors shown to the left the matrix. Similarly, the rows are the components of the unit vector shown to the left of the row written in terms of the unit vectors shown above the columns.

Inertial-Earth[1]

$$\underline{R}_e^i = \begin{array}{c} \\ \underline{i}_1 \\ \underline{i}_2 \\ \underline{i}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{array} \\ \begin{bmatrix} \cos {}^i\omega^e t & -\sin {}^i\omega^e t & 0 \\ \sin {}^i\omega^e t & \cos {}^i\omega^e t & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (1)$$

Inertial-Geocentric[1]

$$\underline{R}_c^i = \begin{array}{c} \\ \underline{i}_1 \\ \underline{i}_2 \\ \underline{i}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \underline{c}_1 & \underline{c}_2 & \underline{c}_3 \end{array} \\ \begin{bmatrix} -\sin L_c \cos \lambda & -\sin \lambda & -\cos L_c \cos \lambda \\ -\sin L_c \sin \lambda & \cos \lambda & -\cos L_c \sin \lambda \\ \cos L_c & 0 & -\sin L_c \end{bmatrix} \end{array} \qquad (2)$$

Inertial-Geographic[1]

$$
\underline{R}_n^i = \begin{matrix} \underline{i}_1 \\ \underline{i}_2 \\ \underline{i}_3 \end{matrix}
\overset{\begin{matrix}\underline{N} & \underline{E} & \underline{D}\end{matrix}}{\begin{bmatrix} -\sin L_g \cos\lambda & -\sin\lambda & -\cos L_g \cos\lambda \\ -\sin L_g \sin\lambda & \cos\lambda & -\cos L_g \sin\lambda \\ \cos L_g & 0 & -\sin L_g \end{bmatrix}} \tag{3}
$$

Geographic-Geocentric[1]

$$
\underline{R}_c^n = \begin{matrix} \underline{N} \\ \underline{E} \\ \underline{D} \end{matrix}
\overset{\begin{matrix}\underline{c}_1 & \underline{c}_2 & \underline{c}_3\end{matrix}}{\begin{bmatrix} \cos D & 0 & \sin D \\ 0 & 1 & 0 \\ -\sin D & 0 & \cos D \end{bmatrix}} \tag{4}
$$

Earth-Geographic[1]

$$
\underline{R}_n^e = \begin{matrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{matrix}
\overset{\begin{matrix}\underline{N} & \underline{E} & \underline{D}\end{matrix}}{\begin{bmatrix} -\sin L_g \cos\Delta l & -\sin\Delta l & -\cos L_g \cos\Delta l \\ -\sin L_g \sin\Delta l & \cos\Delta l & -\cos L_g \sin\Delta l \\ \cos L_g & 0 & -\sin L_g \end{bmatrix}} \tag{5}
$$

Geographic-Body[6] (Body 3: 3-2-1)

This rotation matrix is based on the following rotation sequence:

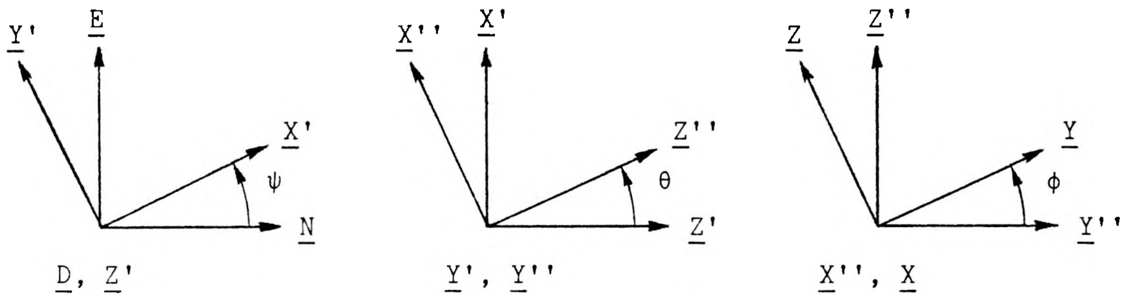

Figure 5. Rotation Sequence: $\psi \underline{Z}'$, $\theta \underline{Y}''$, $\phi \underline{X}$

(Abbreviate "sin ψ cos θ" as "sψcθ", etc.)

$$\underline{R}_m^n = \begin{array}{c} \\ \underline{N} \\ \underline{E} \\ \underline{D} \end{array} \begin{array}{c} \begin{array}{ccc} \underline{X} & \underline{Y} & \underline{Z} \end{array} \\ \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - c\phi s\psi & c\psi s\theta c\phi + s\phi s\psi \\ s\psi c\theta & s\psi s\theta s\phi + c\phi c\psi & s\psi s\theta c\phi - s\phi c\psi \\ - s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \end{array} \quad (6)$$

Redefine (6) as $\underline{C}$, the direction cosine matrix:

$$\underline{C} \triangleq \begin{array}{c} \\ \underline{N} \\ \underline{E} \\ \underline{D} \end{array} \begin{array}{c} \begin{array}{ccc} \underline{X} & \underline{Y} & \underline{Z} \end{array} \\ \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \end{array} \quad (7)$$

Nozzle-Body[6] (Refer to Sec II.D)

Define: β = bearing cant angle

δ_i = nozzle rotation for nozzle i, i = 1-4

γ = nozzle cant angle

Nozzle 1: δ_1 = nozzle rotation

Rotation sequence: Body 2, 3-1-3; $-\beta\underline{a}_3,\ \delta_1\underline{a}_1,\ \gamma\underline{a}_3$

$$
\underline{R}_a^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix}
\overset{\begin{matrix} \underline{a}_1 & \qquad\qquad\qquad & \underline{a}_2 & \qquad\qquad & \underline{a}_3 \end{matrix}}{
\begin{bmatrix}
c\beta c\gamma + s\beta s\gamma c\delta_1 & -c\beta s\gamma + s\beta c\gamma c\delta_1 & -s\beta s\delta_1 \\
-s\beta c\gamma + c\beta s\gamma c\delta_1 & s\beta s\gamma + c\beta c\gamma c\delta_1 & -c\beta s\delta_1 \\
s\gamma s\delta_1 & c\gamma s\delta_1 & c\delta_1
\end{bmatrix}} \tag{8}
$$

Similarly,

Nozzle 2: δ_2 = nozzle rotation

Rotation sequence: Body 2, 2-1-2; $\beta\underline{b}_2,\ \delta_2\underline{b}_1,\ -\gamma\underline{b}_2$

$$
\underline{R}_b^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix}
\overset{\begin{matrix} \underline{b}_1 & \qquad\qquad & \underline{b}_2 & \qquad\qquad & \underline{b}_3 \end{matrix}}{
\begin{bmatrix}
c\beta c\gamma + s\beta s\gamma c\delta_2 & s\beta s\delta_2 & -c\beta s\gamma + s\beta c\gamma c\delta_2 \\
-s\gamma s\delta_2 & c\delta_2 & -c\gamma s\delta_2 \\
-s\beta c\gamma + c\beta s\gamma c\delta_2 & c\beta s\delta_2 & s\beta s\gamma + c\beta c\gamma c\delta_2
\end{bmatrix}} \tag{9}
$$

For nozzles 3 and 4, positive control system commands produce counterclockwise rotations of the nozzles, viewed looking forward.

Nozzle 3: δ_3 = nozzle rotation

Rotation sequence: Body 2, 3-1-3; $\beta \underline{c}_3$, $-\delta_3 \underline{c}_1$, $-\gamma \underline{c}_3$

$$\underline{R}_c^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix} \overset{\begin{matrix} \underline{c}_1 & \qquad\qquad\qquad \underline{c}_2 & \qquad\qquad\qquad \underline{c}_3 \end{matrix}}{\begin{bmatrix} c\beta c\gamma + s\beta s\gamma c\delta_3 & c\beta s\gamma - s\beta c\gamma c\delta_3 & -s\beta s\delta_3 \\ s\beta c\gamma - c\beta s\gamma c\delta_3 & s\beta s\gamma + c\beta c\gamma c\delta_3 & c\beta s\delta_3 \\ s\gamma s\delta_3 & -c\gamma s\delta_3 & c\delta_3 \end{bmatrix}} \qquad (10)$$

Nozzle 4: δ_4 = nozzle rotation

Rotation sequence: Body 2, 2-1-2; $-\beta \underline{d}_2$, $-\delta_4 \underline{d}_1$, $\gamma \underline{d}_2$

$$\underline{R}_d^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix} \overset{\begin{matrix} \underline{d}_1 & \qquad\qquad\qquad \underline{d}_2 & \qquad\qquad\qquad \underline{d}_3 \end{matrix}}{\begin{bmatrix} c\beta c\gamma + s\beta s\gamma c\delta_4 & s\beta s\delta_4 & c\beta s\gamma - s\beta c\gamma c\delta_4 \\ -s\gamma s\delta_4 & c\delta_4 & c\gamma s\delta_4 \\ s\beta c\gamma - c\beta s\gamma c\delta_4 & -c\beta s\delta_4 & s\beta s\gamma + c\beta c\gamma c\delta_4 \end{bmatrix}} \qquad (11)$$

Since the angles γ and β remain constant, define:

$$
\begin{aligned}
K_1 &= c\beta c\gamma & K_5 &= s\gamma \\
K_2 &= s\beta s\gamma & K_6 &= c\gamma \\
K_3 &= s\beta c\gamma & K_7 &= s\beta \\
K_4 &= c\beta s\gamma & K_8 &= c\beta
\end{aligned}
\qquad (12)
$$

Using (12), Eqs. (8)-(11) become:

Nozzle 1:

$$
\underline{R}_a^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix}
\overset{\begin{matrix} \underline{a}_1 & \qquad\qquad \underline{a}_2 & \qquad\qquad \underline{a}_3 \end{matrix}}{\begin{bmatrix}
K_1 + K_2 \cos\delta_1 & -K_4 + K_3 \cos\delta_1 & -K_7 \sin\delta_1 \\
-K_3 + K_4 \cos\delta_1 & K_2 + K_1 \cos\delta_1 & -K_8 \sin\delta_1 \\
K_5 \sin\delta_1 & K_6 \sin\delta_1 & \cos\delta_1
\end{bmatrix}} \qquad (13)
$$

Nozzle 2:

$$
\underline{R}_b^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix}
\overset{\begin{matrix} \underline{b}_1 & \qquad\qquad \underline{b}_2 & \qquad\qquad \underline{b}_3 \end{matrix}}{\begin{bmatrix}
K_1 + K_2 \cos\delta_2 & K_7 \sin\delta_2 & -K_4 + K_3 \cos\delta_2 \\
-K_5 \sin\delta_2 & \cos\delta_2 & -K_6 \sin\delta_2 \\
-K_3 + K_4 \cos\delta_2 & K_8 \sin\delta_2 & K_2 + K_1 \cos\delta_2
\end{bmatrix}} \qquad (14)
$$

Nozzle 3:

$$\underline{R}_c^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix} \overset{\begin{matrix} \underline{c}_1 & \underline{c}_2 & \underline{c}_3 \end{matrix}}{\begin{bmatrix} K_1 + K_2\cos\delta_3 & K_4 - K_3\cos\delta_3 & -K_7\sin\delta_3 \\ K_3 - K_4\cos\delta_3 & K_2 + K_1\cos\delta_3 & K_8\sin\delta_3 \\ K_5\sin\delta_3 & -K_6\sin\delta_3 & \cos\delta_3 \end{bmatrix}} \tag{15}$$

Nozzle 4:

$$\underline{R}_d^m = \begin{matrix} \underline{X} \\ \underline{Y} \\ \underline{Z} \end{matrix} \overset{\begin{matrix} \underline{d}_1 & \underline{d}_2 & \underline{d}_3 \end{matrix}}{\begin{bmatrix} K_1 + K_2\cos\delta_4 & K_7\sin\delta_4 & K_4 - K_3\cos\delta_4 \\ -K_5\sin\delta_4 & \cos\delta_4 & K_6\sin\delta_4 \\ K_3 - K_4\cos\delta_4 & -K_8\sin\delta_4 & K_2 + K_1\cos\delta_4 \end{bmatrix}} \tag{16}$$

III. Overview of Kane's Dynamical Equations

A. Equations of Motion

The Polaris missile M possesses six degrees of freedom in I, all motions of which are governed by the equations

$$F_r + F_r^* = 0 \qquad (r = 1,\ldots,6) \tag{17}$$

These equations are called Kane's dynamical equations.

B. Derivation

Briefly, the derivation of Kane's equations begins with D'Alembert's principle, which in effect reduces a dynamics problem to a statics problem by writing Newton's second law of motion for a particle P in the form[4]

$$\underline{F} - m\underline{a} = 0 \tag{18}$$

If the term $-m\underline{a}$ is viewed as another force, specifically an inertial force defined as

$$\underline{F}^* = - m\underline{a} \tag{19}$$

then (18) can be written as

$$\underline{F} + \underline{F}^* = 0 \tag{20}$$

Eq. (20) now has the appearance of a statics problem which states that the vector sum of all forces, both applied and inertial, is zero.

Now, dot-multiplying (20) with the velocity $\underline{v}$ of the particle P,

$$\underline{v} \cdot \underline{F} + \underline{v} \cdot \underline{F}^* = 0 \tag{21}$$

and defining two scalars A and A^* as

$$\begin{aligned} A &= \underline{v} \cdot \underline{F} \\ A^* &= \underline{v} \cdot \underline{F}^* \end{aligned} \tag{22}$$

leads to the scalar equation

$$A + A^* = 0 \tag{23}$$

The terms A and A^* are referred to as the <u>activity</u> of the force $\underline{F}$ and the <u>activity</u> of the force $\underline{F}^*$, respectively, and (23) is a statement of the <u>activity principle</u>[5].

If P has more than one degree of freedom, (23) does not provide sufficient information for the solution of the problem. In this sense, (20) provides more information, because it is equivalent to three scalar equations. However, there is a potential advantage of (23) over (20) which is important, and it forms the basis of one of the major strengths of Kane's equations. If $\underline{F}$ contains any contributions from unknown constraint forces, they will certainly affect (20), but the dot multiplication which led to (23) will frequently eliminate these unknown forces.

To arrive at a formulation which contains sufficient information for a multiple degree of freedom problem, yet still automatically eliminates unknown constraint forces, Kane's method replaces (21) with

$$\underline{v}_r \cdot \underline{F} + \underline{v}_r \cdot \underline{F}^* = 0 \qquad (r = 1,\ldots,n) \tag{24}$$

where n is equal to the number of degrees of freedom for P and $\underline{v}_r$ is the $r^{\underline{th}}$ partial velocity of P. These partial velocities will be defined in Sec. IV.B.

Now, if F_r and F_r^* are defined in a manner similar to (22) as

$$F_r \triangleq \underline{v}_r \cdot \underline{F} \qquad (r = 1,\ldots,n)$$

$$F_r^* \triangleq \underline{v}_r \cdot \underline{F}^* \qquad (r = 1,\ldots,n) \qquad (25)$$

then (24) becomes

$$F_r + F_r^* = 0 \qquad (r = 1,\ldots,n) \qquad (26)$$

Thus, (26) is like a "generalized" form of the activity principle, but now there are a sufficient number of equations to solve the problem.

The scalar terms F_r and F_r^* are referred to as the generalized active force and the generalized inertia force, respectively. These generalized forces, and the generalized coordinates that are associated with them, are analogous to those employed in the derivation of Lagrange's equations[4], and as a matter of fact, Lagrange's equations of motion can easily be derived from Kane's equations. However, Kane's equations offer an important advantage over Lagrange's equations when a system is subjected to motion constraints. In these cases, Lagrange's equations require the use of Lagrange multipliers to solve for the constraint forces[4]. The effort required to derive the equations of motion when Lagrange multipliers are needed is often increased significantly, yet ironically, neither the Lagrange multipliers or the constraint forces appear in the final equations of motion. Kane's method uses constraint equations to define the position or motion constraints which are applied to the system. The need for multipliers and the determination of constraint forces is eliminated.

The derivation of (26) above was for a single particle P. More generally, for a system of N particles $P_1, \ldots, P_N$, the generalized active forces and the generalized inertia forces are defined as:

$$F_r \triangleq \sum_{i=1}^{N} \underline{v}_r^{P_i} \cdot \underline{F}_i \qquad (r = 1, \ldots, n)$$

and (27)

$$F_r^* \triangleq \sum_{i=1}^{N} \underline{v}_r^{P_i} \cdot \underline{F}_i^* \qquad (r = 1, \ldots, n)$$

Also, when a dynamics problem involves the motion of rigid bodies, the set of contact and/or body forces acting on particles $P_1, \ldots, P_N$, of a rigid body B can be replaced with a couple of torque $\underline{T}$ together with a force $\underline{F}$ applied at a point Q such that

$$\underline{F} = \sum_{i=1}^{N} \underline{F}_i$$

and (28)

$$\underline{T} = \sum_{i=1}^{N} \underline{r}_i \times \underline{F}_i$$

where $\underline{r}_i$ is the position vector of P_i relative to Q. The generalized active force for B can then be written in terms of $\underline{F}$ and $\underline{T}$ as

$$F_r = \underline{v}_r \cdot \underline{F} + \underline{\omega}_r \cdot \underline{T} \qquad (29)$$

where $\underline{v}_r$ is the partial velocity of point Q, and $\underline{\omega}_r$ is the partial angular velocity of B. If the point Q is chosen to be the mass center of B, denoted as B^*, then in a manner similar to (29), the generalized inertia force for B is given by

$$F_r^* = \underline{v}_r \cdot \underline{F}^* + \underline{\omega}_r \cdot \underline{T}^* \tag{30}$$

$\underline{F}^*$ is called the inertia force for B, and is defined as the product of the mass m of B and the acceleration $\underline{a}^*$ of the mass center of B:

$$\underline{F}^* = - m\underline{a}^* \tag{31}$$

$\underline{T}^*$ is called the inertia torque for B and is defined as

$$\underline{T}^* = - \sum_{i=1}^{N} m_i \underline{r}_i \times \underline{a}_i \tag{32}$$

where m_i is the mass of particle P_i of B, N is the number of particles, $\underline{r}_i$ is the position vector of P_i relative to the mass center of B, and $\underline{a}_i$ is the acceleration of P_i.

For a body such as a missile, (32) has little use except for deriving more practical (and more familiar) expressions for $\underline{T}^*$. For example, if $\underline{n}_1$, $\underline{n}_2$, $\underline{n}_3$ are mutually perpendicular unit vectors, each parallel to a principal axis of inertia of B for B^*; I_1, I_2, I_3 are the associated principal moments of inertia of B for B^*; and the angular velocity and angular acceleration of B are expressed as

$$\underline{\omega}^B = \omega_1\underline{n}_1 + \omega_2\underline{n}_2 + \omega_3\underline{n}_3$$

and (33)

$$\underline{\alpha}^B = \alpha_1\underline{n}_1 + \alpha_2\underline{n}_2 + \alpha_3\underline{n}_3$$

then the inertia torque is given as

$$\underline{T}^* = [\omega_2\omega_3(I_2 - I_3) - \alpha_1 I_1]\underline{n}_1 + [\omega_3\omega_1(I_3 - I_1) - \alpha_2 I_2]\underline{n}_2 + [\omega_1\omega_2(I_1 - I_2) - \alpha_3 I_3]\underline{n}_3 \quad (34)$$

The three right hand components of (34) are commonly known as <u>Euler's equations of motion</u>[4] and are used frequently in solving for the rotational motion of a rigid body.

C. Nomenclature

Now that a brief discussion of Kane's method has been completed, the derivation of the Polaris equations of motion can be initiated. The previous equations show that the derivation will require expressions for the velocity $\underline{v}^*$ and the acceleration $\underline{a}^*$ of the missile's mass center; the angular velocity $\underline{\omega}$ and angular acceleration $\underline{\alpha}$; and the applied forces acting on the missile.

The nomenclature used in the following analysis is consistent with that used in Kane's textbooks on dynamics[5,6,7]. The following equation provides an example:

$$^i\underline{\omega}^n \underset{37}{=} \dot{\lambda} \cos L_g \underline{N} - \dot{L}\underline{E} - \dot{\lambda} \sin L_g \underline{D}$$

In this equation, $^i\underline{\omega}^n$ is read as "the angular velocity of reference frame n relative to reference frame i". A dot above a variable represents it's time derivative. The numbers written under the "=" sign are references to equations which are used in the derivation of the current expression. Unless needed, the superscript "i" may be omitted to simplify notation.

IV. Derivation of the Equations of Motion

A. Kinematics

1. Angular Velocity, $\underline{\omega}$

$$\underline{\omega}^{m} = \underline{\omega}^{n} + {}^{n}\underline{\omega}^{m} \tag{35}$$

From Figure 1:

$$\underline{\omega}^{n} = \dot{\lambda}\underline{i}_3 - \dot{L}_g\underline{E} \tag{36}$$

$$\underset{3}{=} \dot{\lambda}(\cos L_g\underline{N} - \sin L_g\underline{D}) - \dot{L}_g\underline{E} \tag{37}$$

$$\underline{\omega}^{n} \underset{37}{=} \dot{\lambda} \cos L_g\underline{N} - \dot{L}_g\underline{E} - \dot{\lambda} \sin L_g\underline{D} \tag{38}$$

or,

$$\begin{aligned} \underline{\omega}^{n} \underset{7}{=} & (\dot{\lambda} \cos L_g C_{11} - \dot{L}_g C_{21} - \dot{\lambda} \sin L_g C_{31})\underline{X} \\ & + (\dot{\lambda} \cos L_g C_{12} - \dot{L}_g C_{22} - \dot{\lambda} \sin L_g C_{32})\underline{Y} \\ & + (\dot{\lambda} \cos L_g C_{13} - \dot{L}_g C_{23} - \dot{\lambda} \sin L_g C_{33})\underline{Z} \end{aligned} \tag{39}$$

From Figure 5:

$${}^{n}\underline{\omega}^{m} = \dot{\psi}\underline{D} + \dot{\theta}\underline{Y}' + \dot{\phi}\underline{X} \tag{40}$$

or,

$$\begin{aligned} {}^{n}\underline{\omega}^{m} = & (-\dot{\psi} \sin \theta + \dot{\phi})\underline{X} + (\dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi)\underline{Y} \\ & + (\dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi)\underline{Z} \end{aligned} \tag{41}$$

Finally, combining (39) and (41) gives:

$$\begin{aligned} \underline{\omega}^{m} = & (-\dot{\psi} \sin \theta + \dot{\phi} + \dot{\lambda} \cos L_g C_{11} - \dot{L}_g C_{21} - \dot{\lambda} \sin L_g C_{31})\underline{X} \\ & + (\dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi + \dot{\lambda} \cos L_g C_{12} - \dot{L}_g C_{22} - \dot{\lambda} \sin L_g C_{32})\underline{Y} \\ & + (\dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi + \dot{\lambda} \cos L_g C_{13} - \dot{L}_g C_{23} - \dot{\lambda} \sin L_g C_{33})\underline{Z} \end{aligned} \tag{42}$$

In certain situations, it may be useful to have an expression for the angular velocity of the missile relative to the earth, rather than inertial space:

$$ {}^{e}\underline{\omega}^{m} = {}^{e}\underline{\omega}^{n} + {}^{n}\underline{\omega}^{m} \tag{43} $$

Similar to (38),

$$ {}^{e}\underline{\omega}^{n} = \dot{i} \cos L_g \underline{N} - \dot{L}_g \underline{E} - \dot{i} \sin L_g \underline{D} \tag{44} $$

Thus,

$$ \begin{aligned} {}^{e}\underline{\omega}^{m} = & (- \dot{\psi} \sin \theta + \dot{\phi} + \dot{i} \cos L_g C_{11} - \dot{L}_g C_{21} - \dot{i} \sin L_g C_{31})\underline{X} \\ & + (\dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi + \dot{i} \cos L_g C_{12} - \dot{L}_g C_{22} - \dot{i} \sin L_g C_{32})\underline{Y} \\ & + (\dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi + \dot{i} \cos L_g C_{13} - \dot{L}_g C_{23} - \dot{i} \sin L_g C_{33})\underline{Z} \end{aligned} \tag{45} $$

2. <u>Velocity</u>, $\underline{v}^*$

$$ \underline{v}^* = \frac{d\underline{r}}{dt} \tag{46} $$

From Figure 1b:

$$ \underline{r} = - r\underline{c}_3 = - \Delta_c \underline{i}_3 - (R_g + h_g)\underline{D} \tag{47} $$

$$ \underset{3}{=} - \Delta_c(\cos L_g \underline{N} - \sin L_g \underline{D}) - (R_g + h_g)\underline{D} \tag{48} $$

$$ \underline{r} \underset{48}{=} - \Delta_c \cos L_g \underline{N} - (R_g + h_g - \Delta_c \sin L_g)\underline{D} \tag{49} $$

From the Law of Sines and Figure 1b:

$$ \frac{\Delta_c}{\sin(L_g - L_{cg})} = \frac{R_g}{\sin(L_{cg} + \pi/2)} = \frac{R_g}{\cos L_{cg}} \tag{50} $$

Thus,

$$\Delta_c \underset{50}{=} \frac{R_g \sin(L_g - L_{cg})}{\cos L_{cg}}$$

$$= \frac{R_g(\sin L_g \cos L_{cg} - \cos L_g \sin L_{cg})}{\cos L_{cg}} \tag{51}$$

$$\Delta_c \underset{51}{=} R_g \sin L_g (1 - \frac{\sin L_{cg} \cos L_g}{\cos L_{cg} \sin L_g}) \tag{52}$$

$$\Delta_c \underset{52}{=} R_g \sin L_g (1 - \frac{\tan L_{cg}}{\tan L_g}) \tag{53}$$

From Figure 1b:

$$\tan L_{cg} = \bar{y}/\bar{x} \tag{54}$$

The tangent of L_g is the slope of the normal to the ellipse at point A, which is given by the negative reciprocal of the slope of the ellipse at A. The equation of the ellipse,

$$x^2/R_e^2 + y^2/R_p^2 = 1 \tag{55}$$

can be used to obtain the derivative, or slope of the ellipse at A:

$$\left.\frac{dy}{dx}\right|_{\bar{x},\bar{y}} \underset{55}{=} - (\bar{x}/\bar{y})(R_p^2/R_e^2) \tag{56}$$

Thus,

$$\tan L_g = - \left.\frac{1}{(dy/dx)}\right|_{\bar{x},\bar{y}} \underset{56}{=} (\bar{y}/\bar{x})(R_e^2/R_p^2) \tag{57}$$

From (53), (54), and (57):

$$\Delta_c = R_g \sin L_g (1 - R_p^2/R_e^2) \tag{58}$$

If the eccentricity of the reference ellipsoid is defined as[1]

$$\varepsilon = (1 - R_p^2/R_e^2)^{1/2} \tag{59}$$

then (58) becomes

$$\Delta_c = \varepsilon^2 R_g \sin L_g \tag{60}$$

Thus, $\underline{r}$ can now be expressed as

$$\underline{r} \underset{49}{=} - \varepsilon^2 R_g \sin L_g \cos L_g \underline{N} - (R_g + h_g - \varepsilon^2 R_g \sin^2 L_g)\underline{D} \tag{61}$$

Eq. (46) can be expanded as follows:

$$\underline{v}^* \underset{46}{=} \frac{d\underline{r}}{dt} = \frac{{}^n d\underline{r}}{dt} + \underline{\omega}^n \times \underline{r} \tag{62}$$

$$\begin{aligned} \frac{{}^n d\underline{r}}{dt} \underset{61}{=} & - \varepsilon^2 [\dot{R}_g \sin L_g \cos L_g + \dot{L}_g R_g (1 - 2 \sin^2 L_g)]\underline{N} \\ & - (\dot{R}_g + \dot{h}_g - \dot{R}_g \varepsilon^2 \sin^2 L_g - 2\dot{L}_g R_g \varepsilon^2 \sin L_g \cos L_g)\underline{D} \end{aligned} \tag{63}$$

From (38) and (61):

$$\begin{aligned} \underline{\omega}^n \times \underline{r} = & \dot{L}_g (R_g + h_g - \varepsilon^2 R_g \sin^2 L_g)\underline{N} \\ & + [\dot{\lambda} \varepsilon^2 R_g \sin^2 L_g \cos L_g + \dot{\lambda} \cos L_g (R_g + h_g - \varepsilon^2 R_g \sin^2 L_g)]\underline{E} \\ & - \dot{L}_g \varepsilon^2 R_g \sin L_g \cos L_g \underline{D} \end{aligned} \tag{64}$$

Eqs. (63) and (64) can now be combined and simplified to obtain:

$$\underline{v}^* = [\dot{L}_g(R_g + h_g - R_g\varepsilon^2\cos^2 L_g) - \dot{R}_g\varepsilon^2 R_g \sin L_g \cos L_g]\underline{N}$$
$$+ \dot{\lambda}(R_g + h_g)\cos L_g\underline{E}$$
$$+ (- \dot{R}_g - \dot{h}_g + \dot{L}_g R_g\varepsilon^2 \sin L_g \cos L_g + \dot{R}_g\varepsilon^2\sin^2 L_g)\underline{D} \qquad (65)$$

An expression for R_g as a function of L_g can now be obtained by noting from Figure 1b that

$$\cos L_g = \bar{x}/R_g \qquad (66)$$

Thus,

$$R_g \underset{66}{=} \bar{x}/\cos L_g \qquad (67)$$

Now, by squaring (67),

$$R_g^2 \underset{67}{=} \bar{x}^2/\cos^2 L_g \qquad (68)$$

and since

$$x^2/R_e^2 + y^2/R_p^2 \underset{55}{=} 1 \qquad (69)$$

then

$$R_g^2 \underset{68}{=} R_e^2(1 - \bar{y}^2/R_p^2)/\cos^2 L_g \qquad (70)$$

However, from Figure 1b:

$$\sin L_g = (\bar{y} + \Delta_c)/R_g \qquad (71)$$

If (71) is solved for $\bar{y}$ and and combined with (60), then

$$\bar{y} = R_g \sin L_g(1 - \varepsilon^2) \qquad (72)$$

Eq. (72) can then be squared and substituted into (70):

$$R_g^{\,2} = \frac{R_e^{\,2}}{\cos^2 L_g[1 + (R_e/R_p)^2 \tan^2 L_g(1 - \varepsilon^2)^2]} \tag{73}$$

Now, (73) is simplified using

$$1 - \varepsilon^2 \underset{59}{=} R_p^{\,2}/R_e^{\,2} \tag{74}$$

to obtain

$$R_g^{\,2} \underset{73}{=} \frac{R_e^{\,2}}{1 - \varepsilon^2 \sin^2 L_g} \tag{75}$$

Thus,

$$R_g \underset{75}{=} R_e(1 - \varepsilon^2 \sin^2 L_g)^{-1/2} \tag{76}$$

The derivative of (76) gives

$$\dot{R}_g = \dot{L}_g R_e \varepsilon^2 \sin L_g \cos L_g (1 - \varepsilon^2 \sin^2 L_g)^{-3/2} \tag{77}$$

Finally, R_g and $\dot{R}_g$ are substituted into (65) and simplified to obtain

$$\begin{aligned} \underline{v}^* &= \dot{L}_g[h_g + R_e(1 - \varepsilon^2)(1 - \varepsilon^2 \sin^2 L_g)^{-3/2}]\underline{N} \\ &\quad + \dot{\lambda} \cos L_g[h_g + R_e(1 - \varepsilon^2 \sin^2 L_g)^{-1/2}]\underline{E} - \dot{h}_g \underline{D} \end{aligned} \tag{78}$$

Eq. (78) gives the velocity of the center of mass of the missile in terms of geographical coordinates. Since it is frequently more useful to express velocity in body coordinates, the rotation matrix given by (7) can be used to obtain:

$$\underline{v}^* \underset{78}{=} \{\dot{L}_g[h_g + R_e(1 - \varepsilon^2)(1 - \varepsilon^2 \sin^2 L_g)^{-3/2}]C_{11}$$

$$+ \dot{\lambda} \cos L_g[h_g + R_e(1 - \varepsilon^2 \sin^2 L_g)^{-1/2}]C_{21} - \dot{h}_g C_{31}\}\underline{X}$$

$$+ \{\dot{L}_g[h_g + R_e(1 - \varepsilon^2)(1 - \varepsilon^2 \sin^2 L_g)^{-3/2}]C_{12}$$

$$+ \dot{\lambda} \cos L_g[h_g + R_e(1 - \varepsilon^2 \sin^2 L_g)^{-1/2}]C_{22} - \dot{h}_g C_{32}\}\underline{Y}$$

$$+ \{\dot{L}_g[h_g + R_e(1 - \varepsilon^2)(1 - \varepsilon^2 \sin^2 L_g)^{-3/2}]C_{13}$$

$$+ \dot{\lambda} \cos L_g[h_g + R_e(1 - \varepsilon^2 \sin^2 L_g)^{-1/2}]C_{23} - \dot{h}_g C_{33}\}\underline{Z} \qquad (79)$$

And if needed, an expression for the velocity of the missile relative to the earth, ${}^e\underline{v}^*$, can be obtained by replacing $\dot{\lambda}$ with $\dot{l}$ in (78) and (79).

3. Generalized Speeds

If "generalized speeds" are defined as follows[6]:

$$u_1 = \underline{\omega}^m \cdot \underline{X} \qquad u_4 = \underline{v}^* \cdot \underline{X}$$
$$u_2 = \underline{\omega}^m \cdot \underline{Y} \qquad u_5 = \underline{v}^* \cdot \underline{Y} \tag{80}$$
$$u_3 = \underline{\omega}^m \cdot \underline{Z} \qquad u_6 = \underline{v}^* \cdot \underline{Z}$$

then the expressions derived earlier for the angular velocity and the velocity of the mass center of the missile are simplified:

$$\underline{\omega}^m \underset{42}{=} u_1\underline{X} + u_2\underline{Y} + u_3\underline{Z} \tag{81}$$

$$\underline{v}^* \underset{79}{=} u_4\underline{X} + u_5\underline{Y} + u_6\underline{Z} \tag{82}$$

There are two reasons for taking this step. First, in analyzing the motion of a system, there is often more interest in the velocity components of the body, rather than the coordinates which measure its position. In this problem, all of the quantities are of interest since it is also a navigation problem; but the velocities are still the primary variables from which the secondary quantities of longitude, latitude and altitude will be computed.

The second reason is another important aspect of Kane's method. The introduction of generalized speeds can dramatically simplify the kinematics (as can be seen by comparing the different expressions for the velocities earlier), and lead to particularly effective formulations for the generalized forces which enable the analyst, with a minimum amount of effort, to construct the simplest possible form of the equations of motion.

It should be noted that if there were no requirement to solve for L_g, λ, or h_g, then the lengthy derivations of (42) and (79) can be omitted. The motion of the vehicle is then determined by starting with (81) and (82), thus saving all the effort leading up to them. In effect, that is what will be done now by continuing the analysis with (81) and (82), while setting (79) aside to be used later in solutions for L_g, λ, and h_g.

4. Angular Acceleration, $\underline{\alpha}$

$$\underline{\alpha}^m = \frac{d\underline{\omega}^m}{dt} \tag{83}$$

Since $\underline{X}$, $\underline{Y}$ and $\underline{Z}$ are fixed in M, (81) and (83) give

$$\underline{\alpha}^m = \dot{u}_1\underline{X} + \dot{u}_2\underline{Y} + \dot{u}_3\underline{Z} \tag{84}$$

5. Acceleration, $\underline{a}^*$

$$\underline{a}^* = \frac{d\underline{v}^*}{dt} = \frac{{}^m d\underline{v}^*}{dt} + \underline{\omega}^m \text{ X } \underline{v}^* \tag{85}$$

$$\frac{{}^m d\underline{v}^*}{dt} \underset{82}{=} \dot{u}_4\underline{X} + \dot{u}_5\underline{Y} + \dot{u}_6\underline{Z} \tag{86}$$

From (81) and (82):

$$\underline{\omega}^m \text{ X } \underline{v}^* = (u_2u_6 - u_3u_5)\underline{X} + (u_3u_4 - u_1u_6)\underline{Y} + (u_1u_5 - u_2u_4)\underline{Z} \tag{87}$$

Thus, (86) and (87) combine to give

$$\underline{a}^* = (\dot{u}_4 + u_2u_6 - u_3u_5)\underline{X} + (\dot{u}_5 + u_3u_4 - u_1u_6)\underline{Y} + (\dot{u}_6 + u_1u_5 - u_2u_4)\underline{Z} \tag{88}$$

B. Partial Velocities

The partial angular velocities and partial velocities of M in I that are used in (29) and (30) to derive the generalized forces are defined as:

$$\underline{\omega}_r^m \triangleq \frac{\partial \underline{\omega}^m}{\partial u_r} \qquad (r = 1,\ldots,6) \tag{89}$$

and

$$\underline{v}_r^* \triangleq \frac{\partial \underline{v}^*}{\partial u_r} \qquad (r = 1,\ldots,6) \tag{90}$$

respectively. Since $\underline{\omega}_r^m$ and $\underline{v}_r^*$ appear in (81) and (82) in the forms

$$\underline{\omega}^m = \sum_{r=1}^{6} \underline{\omega}_r^m u_r \tag{91}$$

and

$$\underline{v}^* = \sum_{r=1}^{6} \underline{v}_r^* u_r \tag{92}$$

they can be determined by inspection. Table 1 gives their values.

TABLE 1. Partial Velocities

r	$\underline{\omega}_r^m$	$\underline{v}_r^*$
1	$\underline{X}$	0
2	$\underline{Y}$	0
3	$\underline{Z}$	0
4	0	$\underline{X}$
5	0	$\underline{Y}$
6	0	$\underline{Z}$

(93)

C. Generalized Active Forces

As shown earlier, the generalized active forces for a rigid body are given by

$$F_r \underset{29}{=} \underline{v}_r \cdot \underline{F} + \underline{\omega}_r \cdot \underline{T} \tag{94}$$

$\underline{F}$ and $\underline{T}$ are, respectively, the vector sums of the applied contact and body forces, and the moments of those forces about a chosen point on the body. The point chosen for computing the sum of the torques will be O, the mass center of the missile, because it simplifies the computation of the generalized inertia forces later.

The forces that contribute to $\underline{F}$ and $\underline{T}$ are the gravitational and aerodynamic body forces, and the four nozzle thrusts which are shown schematically in Figure 4.

1. Gravitational

Gravitational forces will normally be determined by an analytical gravitational potential model of the type described by Britting[1]. Such a model will not be detailed here since the model chosen depends on the accuracy required in the solution of the problem. For the purpose of this analysis, the general case is assumed where three components of the gravity field vector $\mathbf{g}$ are provided by an appropriate gravity model, such that

$$\mathbf{g} = g_n\underline{N} + g_e\underline{E} + g_d\underline{D} \tag{95}$$

Thus, if m is the mass of the missile,

$$\underline{F}_g = m\mathbf{g} \underset{95}{=} mg_n\underline{N} + mg_e\underline{E} + mg_d\underline{D} \tag{96}$$

or

$$\underline{F}_g \underset{7}{=} m(g_n C_{11} + g_e C_{21} + g_d C_{31})\underline{X}$$
$$+ m(g_n C_{12} + g_e C_{22} + g_d C_{32})\underline{Y}$$
$$+ m(g_n C_{13} + g_e C_{23} + g_d C_{33})\underline{Z} \qquad (97)$$

Since $\underline{F}_g$ acts at the mass center,

$$\underline{T}_g \underset{28}{=} 0 \qquad (98)$$

2. <u>Aerodynamic</u>

For reasons analogous to the gravitational model discussed above, it is assumed for this analysis that an aerodynamic model will be used to provide the lift, drag and sideforce components of the aerodynamic force $\underline{F}_a$, such that

$$\underline{F}_a = - D_x\underline{X} + L_y\underline{Y} - L_z\underline{Z} \qquad (99)$$

This force is further assumed to act at the center of pressure as shown in Figure 4. The moment about point O is given by

$$\underline{T}_a \underset{28}{=} (h\underline{X}) \times \underline{F}_a \qquad (100)$$

Thus,

$$\underline{T}_a \underset{99}{=} hL_z\underline{Y} + hL_y\underline{Z} \qquad (101)$$

3. <u>Nozzle Thrust</u>

It is assumed that the thrust from each nozzle is equal to one-fourth of the total motor thrust, T, and that the thrust vectors are parallel to the nozzle centerlines and act at points a, b, c and d shown in Figure 2.

Nozzle 1:

$$\underline{F}_1 = (T/4)\underline{a}_1 \underset{13}{=} (T/4)[(K_1 + K_2\cos\delta_1)\underline{X} + (-K_3 + K_4\cos\delta_1)\underline{Y} + K_5\sin\delta_1\underline{Z}] \tag{102}$$

$$\underline{T}_1 = (-k\underline{X} + l\underline{Y}) \times \underline{F}_1 \tag{103}$$

$$\underline{T}_1 = (T/4)\{lK_5\sin\delta_1\underline{X} + kK_5\sin\delta_1\underline{Y} + [k(K_3 - K_4\cos\delta_1) - l(K_1 + K_2\cos\delta_1)]\underline{Z}\} \tag{104}$$

Nozzle 2:

$$\underline{F}_2 = (T/4)\underline{b}_1 \underset{14}{=} (T/4)[(K_1 + K_2\cos\delta_2)\underline{X} - K_5\sin\delta_2\underline{Y} + (-K_3 + K_4\cos\delta_2)\underline{Z}] \tag{105}$$

$$\underline{T}_2 = (-k\underline{X} + l\underline{Z}) \times \underline{F}_2 \tag{106}$$

$$\underline{T}_2 = (T/4)\{lK_5\sin\delta_2\underline{X} + [l(K_1 + K_2\cos\delta_2) + k(-K_3 + K_4\cos\delta_2)]\underline{Y} + kK_5\sin\delta_2\underline{Z}\} \tag{107}$$

Nozzle 3:

$$\underline{F}_3 = (T/4)\underline{c}_1 \underset{15}{=} (T/4)[(K_1 + K_2\cos\delta_3)\underline{X} + (K_3 - K_4\cos\delta_3)\underline{Y} + K_5\sin\delta_3\underline{Z}] \tag{108}$$

$$\underline{T}_3 = (-k\underline{X} - l\underline{Y}) \times \underline{F}_3 \tag{109}$$

$$\underline{T}_3 = (T/4)\{-lK_5\sin\delta_3\underline{X} + kK_5\sin\delta_3\underline{Y} + [-k(K_3 - K_4\cos\delta_3) + l(K_1 + K_2\cos\delta_3)]\underline{Z}\} \tag{110}$$

Nozzle 4:

$$\underline{F}_4 = (T/4)\underline{d}_{1\,16} = (T/4)[(K_1 + K_2\cos\delta_4)\underline{X} - K_5\sin\delta_4\underline{Y} + (K_3 - K_4\cos\delta_4)\underline{Z}] \quad (111)$$

$$\underline{T}_4 = (-k\underline{X} - l\underline{Z}) \times \underline{F}_4 \quad (112)$$

$$\underline{T}_4 = (T/4)\{-lK_5\sin\delta_4\underline{X} + [-l(K_1 + K_2\cos\delta_4) + k(K_3 - K_4\cos\delta_4)]\underline{Y} + kK_5\sin\delta_4\underline{Z}\} \quad (113)$$

The nozzle thrusts and torques are now summed:

$$\underline{F}_N = \sum_{i=1}^{4} \underline{F}_i \quad (114)$$

$$\begin{aligned}\underline{F}_N &= (T/4)[4K_1 + K_2(\cos\delta_1 + \cos\delta_2 + \cos\delta_3 + \cos\delta_4)]\underline{X} \\ &+ (T/4)[K_4(\cos\delta_1 - \cos\delta_3) - K_5(\sin\delta_2 + \sin\delta_4)]\underline{Y} \\ &+ (T/4)[K_4(\cos\delta_2 - \cos\delta_4) + K_5(\sin\delta_1 + \sin\delta_3)]\underline{Z}\end{aligned} \quad (115)$$

$$\underline{T}_N = \sum_{i=1}^{4} \underline{T}_i \quad (116)$$

$$\begin{aligned}\underline{T}_N &= (T/4)lK_5(\sin\delta_1 + \sin\delta_2 - \sin\delta_3 - \sin\delta_4)\underline{X} \\ &+ (T/4)[(lK_2 + kK_4)(\cos\delta_2 - \cos\delta_4) + kK_5(\sin\delta_1 + \sin\delta_3)]\underline{Y} \\ &+ (T/4)[(lK_2 + kK_4)(\cos\delta_3 - \cos\delta_1) + kK_5(\sin\delta_2 + \sin\delta_4)]\underline{Z}\end{aligned} \quad (117)$$

The net force and torque acting on the missile is determined using,

$$\underline{F} = \underline{F}_g + \underline{F}_a + \underline{F}_N \tag{118}$$

and

$$\underline{T} = \underline{T}_g + \underline{T}_a + \underline{T}_N \tag{119}$$

Thus,

$$\begin{aligned}\underline{F} = \{&(T/4)[4K_1 + K_2(\cos\delta_1 + \cos\delta_2 + \cos\delta_3 + \cos\delta_4)] \\ &+ m(g_n C_{11} + g_e C_{21} + g_d C_{31}) - D_x\}\underline{X} \\ + \{&(T/4)[K_4(\cos\delta_1 - \cos\delta_3) - K_5(\sin\delta_2 + \sin\delta_4)] \\ &+ m(g_n C_{12} + g_e C_{22} + g_d C_{32}) + L_y\}\underline{Y} \\ + \{&(T/4)[K_4(\cos\delta_2 - \cos\delta_4) + K_5(\sin\delta_1 + \sin\delta_3)] \\ &+ m(g_n C_{13} + g_e C_{23} + g_d C_{33}) - L_z\}\underline{Z}\end{aligned} \tag{120}$$

$$\begin{aligned}\underline{T} = &(T/4)lK_5(\sin\delta_1 + \sin\delta_2 - \sin\delta_3 - \sin\delta_4)\underline{X} \\ + \{&hL_z + (T/4)[(lK_2 + kK_4)(\cos\delta_2 - \cos\delta_4) \\ &+ kK_5(\sin\delta_1 + \sin\delta_3)]\}\underline{Y} \\ + \{&hL_y + (T/4)[(lK_2 + kK_4)(\cos\delta_3 - \cos\delta_1) \\ &+ kK_5(\sin\delta_2 + \sin\delta_4)]\}\underline{Z}\end{aligned} \tag{121}$$

4. Generalized Active Forces, F_r

Form $F_r \underset{29}{=} \underline{v}_r^* \cdot \underline{F} + \underline{\omega}_r \cdot \underline{T}$ using (93), (120) and (121):

$$F_1 = (T/4)lK_5(\sin\delta_1 + \sin\delta_2 - \sin\delta_3 - \sin\delta_4) \tag{122}$$

$$F_2 = hL_z + (T/4)[(lK_2 + kK_4)(\cos\delta_2 - \cos\delta_4) + kK_5(\sin\delta_1 + \sin\delta_3)] \tag{123}$$

$$F_3 = hL_y + (T/4)[(lK_2 + kK_4)(\cos\delta_3 - \cos\delta_1) + kK_5(\sin\delta_2 + \sin\delta_4)] \tag{124}$$

$$F_4 = (T/4)[4K_1 + K_2(\cos\delta_1 + \cos\delta_2 + \cos\delta_3 + \cos\delta_4)] + m(g_n C_{11} + g_e C_{21} + g_d C_{31}) - D_x \tag{125}$$

$$F_5 = (T/4)[K_4(\cos\delta_1 - \cos\delta_3) - K_5(\sin\delta_2 + \sin\delta_4)] + m(g_n C_{12} + g_e C_{22} + g_d C_{32}) + L_y \tag{126}$$

$$F_6 = (T/4)[K_4(\cos\delta_2 - \cos\delta_4) + K_5(\sin\delta_1 + \sin\delta_3)] + m(g_n C_{13} + g_e C_{23} + g_d C_{33}) - L_z \tag{127}$$

D. Generalized Inertia Forces, F_r^*

The generalized inertia forces for a rigid body are given by:

$$F_r^* = \underline{v}_r \cdot \underline{F}^* + \underline{\omega}_r \cdot \underline{T}^* \tag{128}$$

$\underline{F}^*$ and $\underline{T}^*$ are called the Inertia Force and the Inertia Torque, respectively.

1. Inertia Force, $\underline{F}^*$

$$\underline{F}^* = -m\underline{a}^* \tag{129}$$

$$\underline{F}^* = -m(\dot{u}_4 + u_2u_6 - u_3u_5)\underline{X} - m(\dot{u}_5 + u_3u_4 - u_1u_6)\underline{Y} - m(\dot{u}_6 + u_1u_5 - u_2u_4)\underline{Z} \tag{130}$$

2. Inertia Torque, $\underline{T}^*$

$$\underline{T}^* = [\omega_2\omega_3(I_2 - I_3) - \alpha_1 I_1]\underline{X} + [\omega_3\omega_1(I_3 - I_1) - \alpha_2 I_2]\underline{Y} + [\omega_1\omega_2(I_1 - I_2) - \alpha_3 I_3]\underline{Z} \tag{131}$$

From (81) and (84):

$$\underline{T}^* = [u_2u_3(I_2 - I_3) - \dot{u}_1 I_1]\underline{X} + [u_3u_1(I_3 - I_1) - \dot{u}_2 I_2]\underline{Y} + [u_1u_2(I_1 - I_2) - \dot{u}_3 I_3]\underline{Z} \tag{132}$$

3. Generalized Inertia Forces, F_r^*

Form $F_r^* \underset{30}{=} \underline{v}_r^* \cdot \underline{F}^* + \underline{\omega}_r \cdot \underline{T}^*$ using (93), (130) and (132):

$$F_1^* = u_2 u_3 (I_2 - I_3) - \dot{u}_1 I_1 \tag{133}$$

$$F_2^* = u_3 u_1 (I_3 - I_1) - \dot{u}_2 I_2 \tag{134}$$

$$F_3^* = u_1 u_2 (I_1 - I_2) - \dot{u}_3 I_3 \tag{135}$$

$$F_4^* = - m(u_2 u_6 - u_3 u_5 + \dot{u}_4) \tag{136}$$

$$F_5^* = - m(u_3 u_4 - u_1 u_6 + \dot{u}_5) \tag{137}$$

$$F_6^* = - m(u_1 u_5 - u_2 u_4 + \dot{u}_6) \tag{138}$$

E. Kane's Equations of Motion

The equations of motion of the missile M are given by:

$$F_r + F_r^* = 0 \qquad (r = 1,\ldots,6) \tag{139}$$

Combine Eqs. (122)-(127) and (133)-(138), for r = 1,...,6:

$$(T/4)lK_5(\sin\delta_1 + \sin\delta_2 - \sin\delta_3 - \sin\delta_4) + u_2u_3(I_2 - I_3) - \dot{u}_1I_1 = 0 \tag{140}$$

$$hL_z + (T/4)[(lK_2 + kK_4)(\cos\delta_2 - \cos\delta_4) + kK_5(\sin\delta_1 + \sin\delta_3)] + u_3u_1(I_3 - I_1) - \dot{u}_2I_2 = 0 \tag{141}$$

$$hL_y + (T/4)[(lK_2 + kK_4)(\cos\delta_3 - \cos\delta_1) + kK_5(\sin\delta_2 + \sin\delta_4)] + u_1u_2(I_1 - I_2) - \dot{u}_3I_3 = 0 \tag{142}$$

$$(T/4)[4K_1 + K_2(\cos\delta_1 + \cos\delta_2 + \cos\delta_3 + \cos\delta_4)] - D_x + m(g_nC_{11} + g_eC_{21} + g_dC_{31} - u_2u_6 + u_3u_5 - \dot{u}_4) = 0 \tag{143}$$

$$(T/4)[K_4(\cos\delta_1 - \cos\delta_3) - K_5(\sin\delta_2 + \sin\delta_4)] + L_y + m(g_nC_{12} + g_eC_{22} + g_dC_{32} - u_3u_4 + u_1u_6 - \dot{u}_5) = 0 \tag{144}$$

$$(T/4)[K_4(\cos\delta_2 - \cos\delta_4) + K_5(\sin\delta_1 + \sin\delta_3)] - L_z + m(g_nC_{13} + g_eC_{23} + g_dC_{33} - u_1u_5 + u_2u_4 - \dot{u}_6) = 0 \tag{145}$$

V. Initial Value Problem Formulation

A. Solution for u_i

Equations (140)-(145) represent a system of first-order, nonlinear differential equations of the form:

$$\dot{u}_i = f_i(u_1,\ldots,u_6) \qquad (i = 1,\ldots,6) \tag{146}$$

where

$$f_1 = \frac{(I_2 - I_3)}{I_1} u_2 u_3 + \frac{TlK_5}{4I_1}(\sin\delta_1 + \sin\delta_2 - \sin\delta_3 - \sin\delta_4) \tag{147}$$

$$f_2 = \frac{(I_3 - I_1)}{I_2} u_3 u_1 + \frac{hL_z}{I_2} + \frac{T}{4I_2}[(lK_2 + kK_4)(\cos\delta_2 - \cos\delta_4) + kK_5(\sin\delta_1 + \sin\delta_3)] \tag{148}$$

$$f_3 = \frac{(I_1 - I_2)}{I_3} u_1 u_2 + \frac{hL_y}{I_3} + \frac{T}{4I_3}[(lK_2 + kK_4)(\cos\delta_3 - \cos\delta_1) + kK_5(\sin\delta_2 + \sin\delta_4)] \tag{149}$$

$$f_4 = u_3u_5 - u_2u_6 + g_nC_{11} + g_eC_{21} + g_dC_{31} - \frac{D_x}{m} + \frac{T}{4m}[4K_1 + K_2(\cos\delta_1 + \cos\delta_2 + \cos\delta_3 + \cos\delta_4)] \tag{150}$$

$$f_5 = u_1u_6 - u_3u_4 + g_nC_{12} + g_eC_{22} + g_dC_{32} + \frac{L_y}{m} + \frac{T}{4m}[K_4(\cos\delta_1 - \cos\delta_3) - K_5(\sin\delta_2 + \sin\delta_4)] \tag{151}$$

$$f_6 = u_2u_4 - u_1u_5 + g_nC_{13} + g_eC_{23} + g_dC_{33} - \frac{L_z}{m} + \frac{T}{4m}[K_4(\cos\delta_2 - \cos\delta_4) + K_5(\sin\delta_1 + \sin\delta_3)] \tag{152}$$

When the values of u_i and δ_i, $(i = 1,\ldots,6)$, and the elements of the direction cosine matrix C_{ij}, $(i = 1, 2, 3;\ j = 1, 2, 3)$, are known for one instant of time, the problem formulation given by (146) is particularly convenient for a numerical integration, and thus, a time simulation of the missile.

Time updates of the direction cosine matrix can be done by integrating[1]:

$$\dot{\underline{C}} = \underline{C}\ {}^{n}\underline{\Omega}^{m} \tag{153}$$

where ${}^{n}\underline{\Omega}^{m}$ is defined to be the skew-symmetric form of ${}^{n}\underline{\omega}^{m}$. From (35), ${}^{n}\underline{\omega}^{m}$ can be computed using:

$$ {}^{n}\underline{\omega}^{m} \underset{35}{=} \underline{\omega}^{m} - \underline{\omega}^{n} \tag{154}$$

Now, using (39) and (81), (154) can be written as

$$ {}^{n}\underline{\omega}^{m} = \omega_x \underline{X} + \omega_y \underline{Y} + \omega_z \underline{Z} \tag{155}$$

where

$$\begin{aligned}
\omega_x &= u_1 - \dot{\lambda} \cos L_g C_{11} + \dot{L}_g C_{21} + \dot{\lambda} \sin L_g C_{31} \\
\omega_y &= u_2 - \dot{\lambda} \cos L_g C_{12} + \dot{L}_g C_{22} + \dot{\lambda} \sin L_g C_{32} \\
\omega_z &= u_3 - \dot{\lambda} \cos L_g C_{13} + \dot{L}_g C_{23} + \dot{\lambda} \sin L_g C_{33}
\end{aligned} \tag{156}$$

The skew-symmetric form of ${}^{n}\underline{\omega}^{m}$ is now given by

$$ {}^{n}\underline{\Omega}^{m} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{157}$$

Thus, from (7), (153), and (157),

$$\dot{C}_{11} = C_{12}\omega_z - C_{13}\omega_y \tag{158}$$

$$\dot{C}_{12} = C_{13}\omega_x - C_{11}\omega_z \tag{159}$$

$$\dot{C}_{13} = C_{11}\omega_y - C_{12}\omega_x \tag{160}$$

$$\dot{C}_{21} = C_{22}\omega_z - C_{23}\omega_y \tag{161}$$

$$\dot{C}_{22} = C_{23}\omega_x - C_{21}\omega_z \tag{162}$$

$$\dot{C}_{23} = C_{21}\omega_y - C_{22}\omega_x \tag{163}$$

$$\dot{C}_{31} = C_{32}\omega_z - C_{33}\omega_y \tag{164}$$

$$\dot{C}_{32} = C_{33}\omega_x - C_{31}\omega_z \tag{165}$$

$$\dot{C}_{33} = C_{31}\omega_y - C_{32}\omega_x \tag{166}$$

B. Solution for L_g, λ, h_g

As mentioned in Section IV.A.3, there may be a requirement to solve for the geographic parameters of latitude L_g, longitude λ, and altitude h_g. Also, as seen in the previous section, L_g, $\dot{L}_g$, and $\dot{\lambda}$ are needed to update the direction cosine matrix. By substituting (82) into (79) and equating the coefficients of $\underline{X}$, $\underline{Y}$, and $\underline{Z}$, three simultaneous equations are obtained which can be expressed in the matrix form:

$$\underline{A}\ \underline{x} = \underline{b} \tag{167}$$

where

$$\underline{x}^T = [\ \dot{L}_g,\ \ \dot{\lambda},\ \ \dot{h}_g\ ] \tag{168}$$

and

$$\underline{b}^T = [\ u_4,\ \ u_5,\ \ u_6\ ] \tag{169}$$

The matrix $\underline{A}$ is determined by first defining:

$$a_1 \triangleq h_g + R_e(1 - \varepsilon^2)(1 - \varepsilon^2 \sin^2 L_g)^{-3/2} \tag{170}$$

$$a_2 \triangleq \cos L_g[h_g + R_e(1 - \varepsilon^2 \sin^2 L_g)^{-1/2} \tag{171}$$

Then,

$$\underline{A} = \begin{bmatrix} a_1 C_{11} & a_2 C_{21} & -C_{31} \\ a_1 C_{21} & a_2 C_{22} & -C_{32} \\ a_1 C_{13} & a_2 C_{23} & -C_{33} \end{bmatrix} \tag{172}$$

The solution for $\dot{\lambda}$, $\dot{L}_g$, and $\dot{h}_g$ is obtained by inverting (172):

$$\underline{x} = \underline{A}^{-1}\ \underline{b} \tag{173}$$

Eq. (173) is a first-order nonlinear matrix differential equation which can be integrated in conjunction with (146) and (153) to solve for L_g, λ, and h_g as a function of time.

VI. Conclusion

The equations of motion and the initial-value problem which were derived are for a specific application; however, the approach used was general in form and can be applied to other near-earth flight vehicle analyses.

Specifically, the only changes that are likely to be needed to model another missile would be the expressions for the thrust vector control. In the case of a winged flight vehicle, expressions for the roll, pitch and yaw moments generated by the control surfaces would also be required. Once the correct system of forces and torques has been determined, they can be combined with exactly the same partial velocities, in precisely the same manner as before, to formulate the equations of motion and the associated initial-value problem.

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