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Conf-900906--2

LBL-30718

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

EARTH SCIENCES DIVISION

Presented at the Society of Petroleum Engineers Annual Technical Conference and Exhibition, New Orleans, LA, September 23–26, 1990, and to be published in the Proceedings

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September 1990

Prepared for the U.S. Department of Energy under Contract Number DE-AC03-76SF00098

LBL--30718

DE91 014231

An Integral Equation Formulation for Two-Phase Flow and Other Nonlinear Flow Problems Through Porous Media

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September 1990

This work was carried out under U.S. Department of Energy Contract No. DE-AC03-76SF00098, administered by the DOE Nevada Office, in cooperation with the United States Geological Survey, Denver.

ABSTRACT

Many flow problems encountered in petroleum reservoir engineering are characterized by nonlinearities and are difficult to solve analytically. The concept of a relative mass flow rate function is used to arrive at an integral equation formulation for some of these nonlinear flow problems. This formulation has some distinct advantages over existing methods of handling such nonlinear flow problems because of its generality and simplicity.

The problems considered include two-phase fluid displacement including the effect of capillary pressure and isothermal transient flow of gas. These problems can be described by nonlinear parabolic partial differential equations that have self-similar solutions.

Exact semi-analytical solutions are obtained which can be easily evaluated using a rapidly-converging iteration process. A new understanding of the mechanism of the displacement of a non-wetting phase by a wetting phase has been developed that is dependent on a critical value of the dimensionless injection rate constant.

INTRODUCTION

Many flow problems encountered in petroleum reservoir engineering are characterized by their nonlinearity and are difficult to treat analytically. These problems include the classical problem of transient flow of gas (equivalent to the unconfined Dupuit flow of groundwater), the two-phase displacement problem including capillary pressure, flow of non-Newtonian fluids, flow through pressure-sensitive media, etc. These problems are described by parabolic partial differential equations in which the coefficient of hydraulic diffusivity is either pressure- or saturation-dependent. Only a few exact solutions to these problems have been obtained for the one-dimensional case. Exact solutions for transient flow of gas (or unconfined flow of groundwater) were obtained by Boussinesq,[1] Polubarinova-Kochina[2] and Barenblatt.[3,4] Exact solutions for two-phase flow including capillary pressure were obtained by Rizhik,[5] Rizhik *et al.*,[6] Rakhimkulov and Shvidler,[7] Chen[8–10] and Yortsos and Fokas.[11] Recently, McWhorter and Sunada,[12] McWhorter[13] as well as Chen *et al.*,[14–17] used an integral equation approach, which was first proposed by McWhorter[18] and discussed by McWhorter[19] and Chen,[20] to solve nonlinear hydrology problems of this type. In this paper, the integral equation formulation is summarized and applied to two nonlinear flow problems encountered in petroleum reservoir engineering.

References and Illustrations at end of paper.

A GENERAL DESCRIPTION OF THE INTEGRAL EQUATION FORMULATION

In petroleum reservoir engineering there exists a group of nonlinear flow problems that can be described by the following second order parabolic partial differential equation:

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}\left[D(p)\frac{\partial p}{\partial x}\right] = 0 \qquad (1)$$

where the coefficient of hydraulic diffusivity, $D(p)$, is a function of p. The initial and boundary conditions are

$$p(x, 0) = p_i \qquad (2)$$

$$p(0,t) = p_0 \qquad (3)$$

$$p(\infty,t) = p_i \qquad (4)$$

Using the Boltzmann transformation

$$\xi = \frac{x}{a\sqrt{t}} \qquad (5)$$

where a is a parameter having the dimension of $L/\sqrt{T}$. Equation (1) can be transformed to an ordinary differential equation of second order

$$\frac{d}{d\xi}\left[D(p)\frac{dp}{d\xi}\right] - \frac{\xi}{2}\frac{dp}{d\xi} = 0 \qquad (6)$$

The boundary conditions are:

$$p(\xi = 0) = p_0 \qquad (7)$$

$$p(\xi = \infty) = p_i \qquad (8)$$

Equations (6)–(8) describe a self-similar problem that can be solved by means of the method proposed by Barenblatt[3,4] for the case of $p_i = 0$. We will demonstrate that it can be solved more easily, and for any p_i, through an integral equation formulation that was first used by McWhorter.[18]

Coming back to the basic equations, Darcy's law and equation of conservation of matter, we have:

$$u = -\frac{k}{\mu}\frac{\partial p}{\partial x} \qquad (9)$$

$$\frac{\partial(\phi\rho)}{\partial t} = \frac{\partial(\rho u)}{\partial x} \qquad (10)$$

Let us introduce a relative mass flow rate function $f(p)$, defined by

$$f(p) = \frac{\rho u}{(\rho u)\Big|_{x=0}} \qquad (11)$$

where $u\Big|_{x=0}$ can be expected to be inversely proportional to $\sqrt{t}$:

$$u\Big|_{x=0} = \frac{B}{\sqrt{t}} \qquad (12)$$

where B is a constant to be determined.

Using Eqs. (9), (10) and (11), a second order nonlinear ordinary differential equation for $f(p)$ can be formulated in the ξ space. It then is transformed to an integral equation which can be solved easily by a rapidly-converging iteration procedure. During the transformation process, the unknown constant B is determined from the fundamental condition that $dp/d\xi \neq 0$ until an equilibrium state in the porous medium has been reached. With the constant B and the relative mass flow rate function $f(p)$ known, the solution for p can be immediately written in a form similar to the well known Buckley-Leverett solution for frontal displacement.[21]

SOLUTION FOR TWO-PHASE DISPLACEMENT INCLUDING CAPILLARY PRESSURE

One-dimensional flow of two immiscible and incompressible fluids through a linear horizontal porous medium is governed by the equation:[10]

$$\phi\frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[u_t(t)f_1(S_w) + \frac{k}{\mu_{nw}}k_{rnw}(S_w)f_1(S_w)p_c'(S_w)\frac{\partial S_w}{\partial x}\right] = 0 \qquad (13)$$

where

$$f_1(S_w) = \frac{1}{1 + \dfrac{\mu_w k_{rnw}(S_w)}{\mu_{nw}k_{rw}(S_w)}} \qquad (14)$$

Equation (13) is a nonlinear, parabolic differential equation of second order, and no exact solution can be obtained for the general case. It is possible, however, to develop self-similar solutions under certain conditions, namely, for a semi-infinite length, a uniform initial saturation, and an injection rate inversely proportional to $\sqrt{t}$:

$$u_t(t) = A/\sqrt{t} \qquad (15)$$

where A is a given constant. At first glance condition (15) may seem unrealistic, but in fact this provides a realistic model that is equivalent to a constant pressure boundary condition at the inlet.[16] Under these conditions, the following self-similar problem can be obtained:

$$\frac{d}{d\xi}\left[k_{rnw}(S_w)f_1(S_w)J'(S_w)\frac{dS_w}{d\xi}\right] + \left[\frac{\xi}{2} - A_D f_1'(S_w)\right]\frac{dS_w}{d\xi} = 0 \qquad (16)$$

$$S_w(\xi = 0) = S_w^* \qquad (17)$$

$$S_w(\xi \to \infty) = S_{wi} \qquad (18)$$

where

$$\xi = \frac{x}{a\sqrt{t}} \qquad (5)$$

$$a = \left[\frac{\sigma\cos\theta}{\mu_{nw}}\sqrt{\frac{k}{\phi}}\right]^{1/2} \qquad (19)$$

$$A_D = \frac{A}{a\phi} \qquad (20)$$

$J(S_w)$ is the Leverett J-function which is related to the capillary pressure, $p_c(S_w)$, by:[22]

$$p_c(S_w) = \sigma\cos\theta\sqrt{\frac{\phi}{k}}\,J(S_w) \qquad (21)$$

S_w^* is the maximum obtainable saturation of the wetting phase:

$$S_w^* = 1 - S_{rnw} \qquad (22)$$

and S_{wi} is the initial saturation of the wetting phase. When $A = 0$, the problem reduces to the problem of capillary imbibition.

Now we introduce a relative infiltration function, defined by

$$f(S_w) = \frac{u_w}{u_{w0}} \qquad (23)$$

where u_{w0} is the infiltration rate of the wetting phase at the inlet that is a priori unknown and is expected to be inversely proportional to $\sqrt{t}$:

$$u_{w0} = \frac{B}{\sqrt{t}} \qquad (24)$$

where B is a constant to be determined. Substituting Eqs. (23) and (24) into the equation of conservation of matter

$$\frac{\partial u_w}{\partial x} + \phi\frac{\partial S_w}{\partial t} = 0 \qquad (25)$$

we have

$$\phi\frac{\partial S_w}{\partial t} + \frac{B}{\sqrt{t}}\frac{\partial f(S_w)}{\partial x} = 0 \qquad (26)$$

or in dimensionless form

$$\frac{df(S_w)}{d\xi} - \frac{R\xi}{2A_D}\frac{dS_w}{d\xi} = 0 \qquad (27)$$

where R is the ratio between the injection rate and the infiltration rate at the inlet

$$R = \frac{A}{B} \tag{28}$$

Equation (27) yields a solution in the Buckley-Leverett form

$$\xi = \frac{2A_D}{R}\frac{df(S_w)}{dS_w} \tag{29}$$

provided $f(S_w)$ and R can be found.

To determine $f(S_w)$ and R, a boundary value problem for $f(S_w)$ can be formulated as follows. Differentiating both sides of Eq. (29) with respect to S_w gives

$$\frac{d\xi}{dS_w} = \frac{2A_D}{R}\frac{d^2f(S_w)}{dS_w^2} \tag{30}$$

Combining Eqs. (23), (24) and the following expression for the infiltration rate of the wetting phase[10]

$$u_w = u_t(t)f_1(S_w) + \frac{k}{\mu_{nw}}k_{rnw}(S_w)f_1(S_w)p_c{'}(S_w)\frac{\partial S_w}{\partial x} \tag{31}$$

gives

$$f(S_w) = \frac{A}{B}f_1(S_w) + \frac{k\sqrt{t}}{\mu_{nw}B}k_{rnw}(S_w)f_1(S_w)p_c{'}(S_w)\frac{\partial S_w}{\partial x} \tag{32}$$

or in dimensionless form

$$f(S_w) = R\left[f_1(S_w) + \frac{1}{A_D}k_{rnw}(S_w)f_1(S_w)J{'}(S_w)\frac{dS_w}{d\xi}\right] \tag{33}$$

Substituting $d\xi/dS_w$ from Eq. (33) into Eq. (30) results in the following differential equation for $f(S_w)$

$$\frac{d^2f(S_w)}{dS_w^2} - \frac{R^2}{2A_D^2}\frac{k_{rnw}(S_w)f_1(S_w)J{'}(S_w)}{f(S_w)-Rf_1(S_w)} = 0 \tag{34}$$

From Eqs. (17) and (23) we have a boundary condition at the inlet

$$f(S_w^*) = 1 \tag{35}$$

The second boundary condition can be determined from Eqs. (33) and (18) as

$$f(S_{wi}) = Rf_1(S_{wi}) \tag{36}$$

Equation (34) is a nonlinear second order ordinary differential equation and cannot be solved analytically. However, it can be transformed into an integral equation which can be solved iteratively. A direct integration of Eq. (34) gives

$$\frac{df(S_w)}{dS_w} = \frac{R^2}{2A_D^2}\int_{S_w^*}^{S_w}\frac{k_{rnw}(\alpha)f_1(\alpha)J{'}(\alpha)}{f(\alpha)-Rf_1(\alpha)}d\alpha + C_1 \tag{37}$$

Integrating Eq. (37) once again provides an integral equation

$$f(S_w) = \frac{R^2}{2A_D^2}\int_{S_w^*}^{S_w}\frac{(S_w-\alpha)k_{rnw}(\alpha)f_1(\alpha)J{'}(\alpha)}{f(\alpha)-Rf_1(\alpha)}d\alpha + C_1(S_w-S_w^*) + C_2 \tag{38}$$

where C_1 and C_2 are arbitrary constants of integration which can be determined from the boundary conditions as

$$C_1 = \frac{1}{S_{wi}-S_w^*}\left[Rf_1(S_{wi}) - 1 - \frac{R^2}{2A_D^2}\int_{S_w^*}^{S_{wi}}\frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J{'}(\alpha)}{f(\alpha)-Rf_1(\alpha)}d\alpha\right] \tag{39}$$

$$C_2 = 1 \tag{40}$$

Now we need to examine R, the ratio between injection and infiltration rates, because at this point, we know the value of A but not of B. For the special case when $A_D = 0$, there is no injection but there will be infiltration due to imbibition. Therefore, inasmuch as the total flow rate $u_t = 0$, the infiltration rate of the wetting phase at the inlet must be equal to the counterflow rate of the non-wetting phase leaving the system. Obviously, in this case, $R = 0$.

Rakhimkulov and Shvidler[7] have investigated this same problem and have touched on a very interesting property of the flow behavior for this system. If the injection rate is sufficiently small, then at a location that is sufficiently behind the wetting-phase front, there will be a region of counterflow. This is illustrated by the conceptual diagrams on Fig. 1.

Fig. 1A shows how the location of the front (Fig. 1B) is revealed by the saturation profile. Fig. 1C illustrates how the flow rates vary behind the front. It should be noted that at the inlet, the infiltration rate of the wetting phase u_w reaches its maximum value u_{w0}, whereas the counterflow rate of the non-wetting phase u_{nw} reaches its maximum negative value u_{nw0}. The algebraic sum $u_{w0} + u_{nw0} = u_t$ gives the rate of injection. This is illustrated on Fig. 1B at the inlet to the system. Note that the magnitude of the counterflow diminishes away from the inlet and vanishes at a neutral point marked N on Fig. 1C. Beyond this neutral point, both wetting and non-wetting phases move in tha same direction, but u_w decreases and, for the particular condition $S_{wi} \le S_{iw}$, u_w must vanish at the front. Ahead of the front, only the non-wetting phase is flowing so as to satisfy the constraint $u_t = u_w + u_{nw}$ (Fig. 1C).

It is important to recognize that as the total flow rate u_t decreases, the neutral point moves further and further from the inlet; and when $u_t = 0$, the neutral point is at infinity, which means that counterflow exists everywhere. On the other hand, as u_t increases above that shown on Fig. 1C, the neutral point moves toward the inlet and the magnitude of counterflow decreases. We can anticipate that when the neutral point reaches the inlet and counterflow ceases, a critical value A_{Dcr} is reached for which $R = 1$ and $A_D = B_D$, where B_D is the dimensionless infiltration rate constant defined by

$$B_D = \frac{B}{a\phi} \tag{41}$$

Figure 2 illustrates the variation of B_D with A_D and also the variation of C_D with A_D, where C_D, the dimensionless counterflow rate constant, is defined by

$$C_D = \frac{C}{a\phi} \tag{42}$$

Thus, we see from Fig. 2 that

$$A_D = B_D + C_D \tag{43}$$

Therefore, R ranges from 0 to 1 as A_D increases from 0 to A_{Dcr}, and remains unity when $A_D > A_{Dcr}$.

Thus, when $A_D \geq A_{Dcr}$, $R = 1$, and Eq. (38) reduces to

$$f(S_w) = \frac{1}{2A_D^2}\int_{S_w^*}^{S_w} \frac{(S_w-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-f_1(\alpha)}d\alpha + \frac{S_w^*-S_w}{S_w^*-S_{wi}}\left[f_1(S_{wi})-1 - \frac{1}{2A_D^2}\int_{S_w^*}^{S_{wi}} \frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-f_1(\alpha)}d\alpha\right]+1 \quad (44)$$

In the case of $A_D < A_{Dcr}$, Eq. (38) contains an unknown parameter, R, that can be determined using the condition that in this case, $\left.\frac{dS_w}{d\xi}\right|_{\xi=0}$ must be less than zero. Then from Eq. (29), $\left.\frac{df(S_w)}{dS_w}\right|_{S_w^*}$ must equal zero in order to satisfy boundary condition (17), $S_w(\xi=0)=S_w^*$. On the other hand, from Eq. (37)

$$\left.\frac{df(S_w)}{dS_w}\right|_{S_w^*} = C_1$$

Therefore, $C_1 = 0$, which results in an equation relating R to A_D when $A_D < A_{Dcr}$:

$$A_D = R\left[\frac{1}{2[Rf_1(S_{wi})-1]} \times \int_{S_w^*}^{S_{wi}} \frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-Rf_1(\alpha)}d\alpha\right]^{1/2} \quad (45)$$

The critical value A_{Dcr} can be determined from this equation when $R = 1$. Substitution of Eq. (45) into Eq. (38) yields an integral equation for $f(S_w)$ when $A_D < A_{Dcr}$:

$$f(S_w) = 1 - \frac{\left[1-Rf_1(S_{wi})\right]\int_{S_w}^{S_w^*} \frac{(S_w-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-Rf_1(\alpha)}d\alpha}{\int_{S_{wi}}^{S_w^*} \frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-Rf_1(\alpha)}d\alpha} \quad (46)$$

Both equations, Eqs. (44) and (46), can easily be solved using an iteration process, where the first guess can be taken as $f(S_w) = (S_w - S_{wi})/(S_w^* - S_{wi})$.

From Eq. (29), the solution can now be expressed as:

$$\xi = \frac{R}{A_D}\int_{S_w^*}^{S_w} \frac{k_{rnw}(S_w)f_1(S_w)J'(S_w)}{f(S_w)-Rf_1(S_w)}dS_w \quad (47)$$

when $A_D < A_{Dcr}$, and

$$\xi = \xi^* + \frac{1}{A_D}\int_{S_w^*}^{S_w} \frac{k_{rnw}(S_w)f_1(S_w)J'(S_w)}{f(S_w)-f_1(S_w)}dS_w \quad (48)$$

when $A_D \geq A_{Dcr}$, where

$$\xi^* = \frac{2A_D}{S_{wi}-S_w^*}\left[f_1(S_{wi})-1 - \frac{1}{2A_D^2}\int_{S_w^*}^{S_{wi}} \frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-f_1(\alpha)}d\alpha\right] \quad (49)$$

It can be seen that when $A_D \leq A_{Dcr}$, no saturated region will be generated in the porous medium, whereas when $A_D > A_{Dcr}$, a saturated region will develop as shown on Figs. 3 and 4 (to be discussed below).

However, there are some complications with the fractional flow function, $f_1(S_w)$, and therefore, the solution obtained above has some limitations. From Eq. (33), when $R = 1$ we have

$$f(S_w) - f_1(S_w) \geq 0 \quad (50)$$

Then $f'(S_w)$ must be less than $f_1'(S_w)$ at $S_w = S_w^*$, otherwise $f(S_w)$ would be less than $f_1(S_w)$ in the neighborhood of S_w^* and this would contradict Eq. (50). From Eq. (44)

$$\left.f'(S_w)\right|_{S_w=S_w^*} = \frac{1}{S_{wi}-S_w^*}\left[f_1(S_{wi})-1 - \frac{1}{2A_D^2}\int_{S_w^*}^{S_{wi}} \frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-f_1(\alpha)}d\alpha\right] \quad (51)$$

and we see that $f'(S_w^*)$ increases monotonically with increasing value of A_D. The function $f_1(S_w^*)$ increases from zero at $A_D \leq A_{Dcr}$ to a maximum value

$$\left.f'(S_w^*)\right|_{max} = \frac{1}{S_w^*-S_{wi}}\left[1-f_1(S_{wi})\right] \quad (52)$$

as $A_D \rightarrow \infty$. One should recall that $f_1'(S_w^*)$ may be either greater than or equal to zero depending on the relative permeability for the non-wetting phase. If $f_1'(S_w^*) < \left.f'(S_w^*)\right|_{max}$, then the solution obtained above for $A_D > A_{Dcr}$ is valid only until the dimensionless injection rate constant A_D reaches a maximum value given by

$$A_{D,max} = \left[\frac{1}{2[1-f_1(S_{wi})-(S_w^*-S_{wi})f_1'(S_w^*)]} \times \int_{S_{wi}}^{S_w^*} \frac{(S_{wi}-\alpha)k_{rnw}(\alpha)f_1(\alpha)J'(\alpha)}{f(\alpha)-f_1(\alpha)}d\alpha\right]^{1/2} \quad (53)$$

In particular, for the case where $f_1'(S_w^*) = 0$, then $A_{D,max} = A_{Dcr}$. This means that when $f_1'(S_w^*) = 0$, and $A_D > A_{Dcr}$, no solution can be obtained by the procdure given above. This limitation is quite important because many systems are characterized by $f_1'(S_w^*) = 0$.

It should be noted that the self-similar problem stated by Eqs. (16)–(18) was first formulated and studied by Rakhimkulov and Shvidler[7] in 1962, and the special case of imbibition was solved earlier by Rizhik[5] in 1960. For such nonlinear problems, Barenblatt[3,4] has made the important finding that for the zero initial condition, the front of the disturbance must propagate with a finite velocity. Based on this finding, these workers were only able to solve this particular problem semi-analytically for the particular cases where $S_{wi} \leq S_{iw}$. The fact that the front propagates with a finite velocity implies that their solutions, as well as the solution obtained here when $S_{wi} \le S_{iw}$, also apply to a porous system of finite length as long as the front has not reached the end of the porous medium. However, the work of Rizhik[5] and Rakhimkulov and Shvidler[7] required a trial and error process, and numerical integration of the resulting ordinary differential equation is still required over most of the range of saturation. Recently McWhorter and Sunada[12] have studied this problem using the integral equation approach but only for the cases of unidirectional displacement and imbibition. They envision a laboratory setup that uses a semipermeable membrane to achieve unidirectional displacement. This integral equation formulation has some distinct advantages over existing methods in its generality and simplicity. Other existing self-similar solutions of two-phase flow including capillary pressure[6, 8–10] can also be effectively solved by this approach.[12, 19, 20]

RESULTS FOR TWO-PHASE DISPLACEMENT

The self-similar solution presented above was evaluated for certain hypothetical parameters to illustrate the nature of two-phase displacement. The relative permeabilities and the J-function were taken as

$$k_{rw}(S_w) = S_w^4 \tag{54}$$

$$k_{rnw}(S_w) = 1 - S_w \tag{55}$$

$$J(S_w) = \frac{1}{S_w^{1/2}} - 1 \tag{56}$$

These parameters imply $S_w^* = 1$ and $S_{iw} = 0$, and that the fractional flow function of the system has a non-zero derivative with respect to saturation at the maximum obtainable saturation, i.e., $f_1'(S_w^*) > 0$. Calculations were carried out for various values of A_D (the dimensionless injection rate constant) and different values of S_{wi} (the initial saturation). The viscosity ratio was basically taken as unity, but the influence of viscosity ratio on displacement was also examined.

Figure 5 shows how the relative infiltration rate function $f(S_w)$ varies with saturation. It can be seen that for the case of no injection where $A_D = 0$ and $R = 0$, the variation of $f(S_w)$ with S_w is furthest to the left. As A_D increases, the curve shifts to the right, and when $A_D \to \infty$, for this particular problem, the curve is simply the diagonal as shown on Fig. 5. The Buckley-Leverett fractional flow curve, $f_1(S_w)$, is included on this figure because the tangent to this curve, which is the well known Welge[23] technique for determining the frontal saturation, is exactly the same as the diagonal shown for $A_D \to \infty$.

The influence of the initial saturation S_{wi} on the relative infiltration function is shown on Fig. 6. Curves have been plotted for $A_D = 0$ and $A_D = A_{Dcr}$ for $S_{wi} = 0.0$, 0.2 and 0.4. The Buckley-Leverett fractional flow curve is also included because it is very useful in locating the starting point for any of these relative infiltration rate function curves. Examination of Eq. (37) reveals that for $S_{wi} > 0$, the slope of the curves at $S_w = S_{wi}$ must be vertical, whereas when $S_{wi} = 0$, the slope at $S_w = S_{wi} = 0$ is finite.

Saturation profiles for several values of A_D ranging from 0 to 0.3 for $S_{wi} = S_{iw} = 0$ are shown on Fig. 3. It can be seen that a family of curves emanates from the point $S_w = 1.0$, $\xi = 0$, which means that $S_w = S_w^*$ only at the entrance ($x = 0$), as long as A_D does not exceed A_{Dcr}. It is important to realize, however, that the increase in area beneath each curve is not proportional to the increase in the magnitude of A_D until the critical value has been exceeded. As soon as A_D exceeds the critical value, a saturated region will develop that migrates into the porous medium as A_D continues to increase. For example, A_{Dcr} on Fig. 3 is 0.2541, and the curve for $A_D = 0.3$ starts at $\xi = 0.09997$ and ends at $\xi = 0.7772$. Fig. 4 shows saturation profiles for a larger range of A_D up to 1.5 at $S_{wi} = 0$. In the case of $A_D = 0.5$, the curve starts at $\xi = 0.5710$ and ends at $\xi = 1.1313$. It can also be seen from Fig. 4 that the profile becomes steeper and steeper as A_D exceeds A_{Dcr}. The Buckley-Leverett profile, which does not consider capillary pressure, is also shown by the dashed vertical lines. At $A_D \to \infty$, the saturation profile will also be a vertical line that coincides with the Buckley-Leverett solution.

The influence of initial saturation on the saturation profiles is shown on Fig. 7. It can be seen that the saturation profiles spread out over a greater range of ξ as the initial saturation increases. Theory predicts that in the case where $S_{wi} > 0$, the initial saturation can only be reached at $\xi = \infty$, whereas when $S_{wi} = 0$, the saturation profile terminates at a finite location. This is in full agreement with the theoretical results of Barenblatt.[3,4] However, as can be seen on Fig. 7, in practice distinct fronts can also be observed at finite locations for the case of $S_{wi} > S_{iw}$.

The dependence of both the dimensionless imbibition rate constant B_D and the critical dimensionless injection rate constant A_{Dcr} on the initial saturation is illustrated on Fig. 8. It is seen that both curves are monotonic functions decreasing from their maximum values at $S_{wi} = 0$ to zero at $S_{wi} = 1$ and that the curves are roughly parallel over most of the range. Figure 9 shows the important influence of the viscosity ratio, $\mu_D = \mu_{nw}/\mu_w$, on the imbibition rate parameter, $B|_{A=0}/[\sigma\cos\theta\phi\sqrt{k\phi}]^{1/2}$, and the critical injection rate parameter, $A_{cr}/[\sigma\cos\theta\phi\sqrt{k\phi}]^{1/2}$. These two curves converge to one point at $\mu_D = 0$ (see Chen *et al.*,)[16] and deviate from each other more and more as the viscosity ratio increases. At $\mu_D = 0$, the critical injection rate parameter $A_{Dcr} = 0$, and two regions, one saturated and one unsaturated, develop immediately. This is in agreement with Philip's results.[24]

ISOTHERMAL TRANSIENT FLOW OF GAS

Let us now consider a semi-infinite horizontal porous medium which is initially saturated with gas at a uniform pressure p_i. At time $t = 0$, gas is injected at the inlet, $x = 0$, at a constant pressure $p = p_0$. This problem can be formulated as follows:

$$\frac{\partial p}{\partial t} = \frac{k}{2\phi\mu}\frac{\partial^2 p^2}{\partial x^2} \tag{57}$$

The initial and boundary conditions are

$$p(x, 0) = p_i \tag{58}$$

$$p(0, t) = p_0 \tag{59}$$

$$p(\infty, t) = p_i \tag{60}$$

The problem given by Eqs. (57)–(60) is a self-similar one. If we employ the Boltzmann transformation

$$\xi = \frac{x}{a\sqrt{t}} \tag{61}$$

where

$$a = \sqrt{\frac{kp_0}{2\phi\mu}} \tag{62}$$

Equations (57)–(60) reduce to

$$\frac{d^2 p_D^2}{d\xi^2} + \frac{\xi}{2}\frac{dp_D}{d\xi} = 0 \tag{63}$$

$$p_D(\xi = 0) = 1 \tag{64}$$

$$p_D(\xi \to \infty) = p_{Di} \tag{65}$$

where p_D is a dimensionless pressure defined by

$$p_D = \frac{p}{p_0} \tag{66}$$

This problem has been semi-analytically solved by Barenblatt[3,4] for a zero initial condition. We shall now solve this problem using the integral equation formulation for both zero and non-zero initial conditions.

If we substitute the Darcy's law from Eq. (9) into the relative mass flow rate function as defined by Eq. (10), we have

$$f(p) = -\frac{k}{\mu}\frac{\rho}{\rho_0}\frac{\sqrt{t}}{u_0}\frac{\partial p}{\partial x} \tag{67}$$

or in dimensionless form:

$$f(p_D) = -\frac{2p_D}{u_{0D}}\frac{dp_D}{d\xi} \tag{68}$$

where

$$u_{0D} = \frac{\rho_1 u_0}{a\phi\rho_0} \tag{69}$$

Substituting Eq. (11) into Eq. (10) and using an equation of state for gas given by

$$\rho = \frac{\rho_0}{p_0}p \tag{70}$$

we have

$$\frac{\phi\rho_0}{p_0}\frac{\partial p}{\partial t} + \frac{\rho_0 u_0}{\sqrt{t}}\frac{\partial f(p)}{\partial x} = 0 \tag{71}$$

or in dimensionless form

$$u_{0D}\frac{df(p_D)}{d\xi} - \frac{\xi}{2}\frac{dp_D}{d\xi} = 0 \tag{72}$$

Equation (72) immediately provides the solution to the self-similar problem in the form

$$\xi = 2u_{0D}\frac{df(p_D)}{dp_D} \tag{73}$$

if $f(p_D)$ and the unknown parameter u_{0D} can be found.

Differentiating both sides of Eq. (73) with respect to p_D and then using Eq. (68) to eliminate the variable ξ, a second order, nonlinear ordinary differential equation for $f(p_D)$ is obtained

$$\frac{d^2 f(p_D)}{dp_D^2} + \frac{p_D}{u_{0D}^2 f(p_D)} = 0 \tag{74}$$

Two boundary conditions can be specified as

$$f(p_D = 1) = 1 \tag{75}$$

$$f(p_D = p_{Di}) = 0 \tag{76}$$

Directly integrating Eq. (74) with respect to p_D gives

$$\frac{df(p_D)}{dp_D} = -\frac{1}{u_{0D}^2}\int_1^{p_D}\frac{p_D}{f(p_D)}dp_D + C_1 \tag{77}$$

Integrating once again, we have

$$f(p_D) = -\frac{1}{u_{0D}^2}\int_1^{p_D}\frac{(p_D-\alpha)\alpha}{f(\alpha)}d\alpha + C_1(p_D - 1) + C_2 \tag{78}$$

The constants of integration, C_1 and C_2, can be determined using boundary conditions (75) and (76) as

$$C_1 = \frac{1}{1-p_{Di}}\left[1 - \frac{1}{u_{0D}^2}\int_1^{p_{Di}}\frac{(p_{Di}-\alpha)\alpha}{f(\alpha)}d\alpha\right] \tag{79}$$

$$C_2 = 1 \tag{80}$$

The unknown constant u_{0D} can be determined using the fact that the pressure gradient at the inlet must not be zero until the equilibrium state within the porous medium has been reached. This leads to a condition that

$$\left.\frac{df(p_D)}{dp_D}\right|_{p_D = 1} = 0 \tag{81}$$

Then, $C_1 = 0$, and from Eq. (79), u_{0D} can be determined as

$$u_{0D} = \left[\int_1^{p_{Di}}\frac{(p_{Di}-\alpha)\alpha}{f(\alpha)}d\alpha\right]^{1/2} \tag{82}$$

Now, the integral equation for $f(p_D)$ can be expressed as

$$f(p_D) = 1 - \frac{\int_1^{p_D}\frac{(p_D-\alpha)\alpha}{f(\alpha)}d\alpha}{\int_1^{p_{Di}}\frac{(p_{Di}-\alpha)\alpha}{f(\alpha)}d\alpha} \tag{83}$$

and this can be solved iteratively using $f(p_D) = p_D$ as the first guess.

This solution has been evaluated for various values of dimensionless initial pressure, p_{Di}. The curves for the relative mass flow rate function $f(p_D)$ for different dimensionless initial pressures are given on Fig. 10. We see that these curves have the similar form and physical meaning as the well known Buckley-Leverett fractional flow curves. Thus, the generalized Welge graphical technique proposed by Chen and Song[25] can be used to determine the mass of gas between any two cross-sections with p_{Da} and p_{Db}, respectively, on this figure. Namely, two tangents of the $f(p_D)$ curve may be drawn at two points, $[f(p_{Da}), p_{Db})]$ and $[f(p_{Db}), p_{Db}]$, and the intercept of the two tangents, $\bar{p}_{Dab}$, indicates the average dimensionless pressure between these two cross-sections (see Fig. 10).

The dependence of dimensionless mass flow rate constant, u_{0D}, on dimensionless initial pressure, p_{Di}, is shown on Fig. 11. This constant decreases monotonically from its maximum value of $u_{0D} = 0.6229$ at $p_{Di} = 0$ to zero at $p_{Di} = 1$. Dimensionless pressure profiles for different dimensionless initial pressures are illustrated on Fig. 12. For comparison, the results of Barenblatt's exact solution[4] for the case of $p_{Di} = 0$ are also shown by solid circles. As expected the solution developed through the integral formulation is in excellent agreement with the results of Barenblatt.

CONCLUSIONS

Nonlinear flow problems are frequently encountered in petroleum reservoir engineering and are not easily solved by traditional mathematical methods. The nonlinearities are a result of the fact that the hydraulic diffusivity is not constant and is dependent on variables that are unknown and for which solutions are being sought. An integral equation formulation that was first developed in the field of hydrology has been applied to these problems. This new approach is dependent on casting the governing equation in terms of a relative mass flow rate function. The solution can be expressed in a form similar to that of Buckley-Leverett, with the relative flow rate function as the unknown to be determined. An integral equation for this function can be formulated and solved by iterative methods. This new approach has been applied to several types of one-dimensional problems involving two-phase displacement including the effects of capillary pressure and isothermal transient flow of gas.

A new understanding of the mechanism of the displacement of a non-wetting phase by a wetting phase has been developed that is dependent on a critical value of the dimensionless injection rate constant. When this constant is less than the critical value, a counterflow of the non-wetting phase exists; the maximum obtainable saturation of the wetting phase can only develop at the entrance to the system. On the other hand, when the injection rate constant is greater than this critical value, no counterflow exists, and the maximum obtainable saturation of the wetting phase will propagate into the system. An equation to evaluate this critical injection rate constant has been developed.

The integral equation formulation has some distinct advantages over existing methods of handling such nonlinear flow problems because of its generality and simplicity. The problems of flow in pressure-sensitive media, flow of non-Newtonian fluids, and the problem of heat transfer with temperature-dependent heat conductivity as well as the nonlinear problem of hydrodynamic dispersion in porous media can also be solved by means of this mathematical formulation.

NOMENCLATURE

a = parameter having the dimension of $L\,T^{-1/2}$ defined by Eq. (19), or (62).

A = injection rate constant $[L\,T^{-1/2}]$

B = infiltration rate constant $[L\,T^{-1/2}]$

c_t = total compresibility of the system $[L^2/F]$

C = counterflow rate constant $[L\,T^{-1/2}]$; constant of integration

D = hydraulic diffusivity $[L^2/T]$

f = relative mass flow rate function defined by Eq. (10)

f_1 = fractional flow in Buckley-Leverett problem where the capillary pressure is neglected, defined by Eq. (14)

J = dimensionless capillary pressure function

k = absolute permeability $[L^2]$

k_r = relative permeability

p = pressure $[F/L^2]$

p_c = capillary pressure $[F/L^2]$

R = ratio between the injection rate and the infiltration rate

S = saturation

S_{iw} = irreducible saturation of wetting phase

S_{rnw} = residual saturation of non-wetting phase

S_{wi} = initial saturation of wetting phase

S_w^* = maximum obtainable saturation of wetting phase

t = time $[T]$

u = flow rate $[L/T]$

x = distance $[L]$

α = dummy variable of integration

θ = contact angle

μ = viscosity $[F\text{-}T/L^2]$

ξ = similarity variable

ρ = density $[M/L^3]$

σ = interfacial tension $[F/L]$

ϕ = porosity

Subscripts

cr = critical value

D = dimensionless

i = initial condition

nw = non-wetting phase

t = total

w = wetting phase

0 = conditions at the inlet, $x = 0$

ACKNOWLEDGMENTS

The authors thank C. Doughty and C. H. Lai for critical review of this work. This work was carried out under U. S. Department of Energy Contract No. DE-AC03-76SF00098, administered by the DOE Nevada Office, in coorperation with the United States Geological Survey, Denver.

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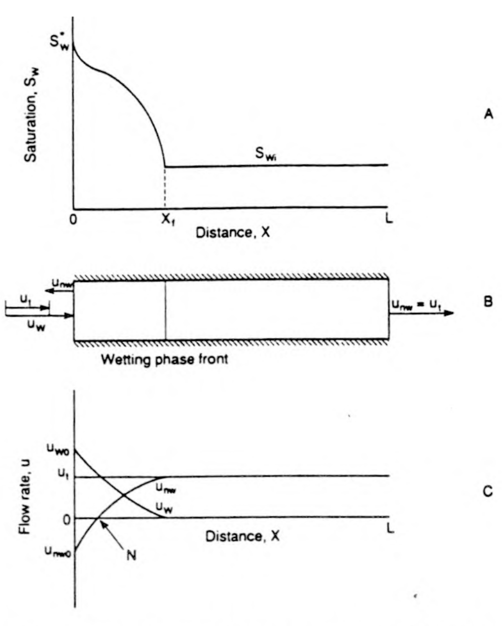

Fig. 1—A schematic of infiltration of the wetting phase at sufficiently low injection rates.

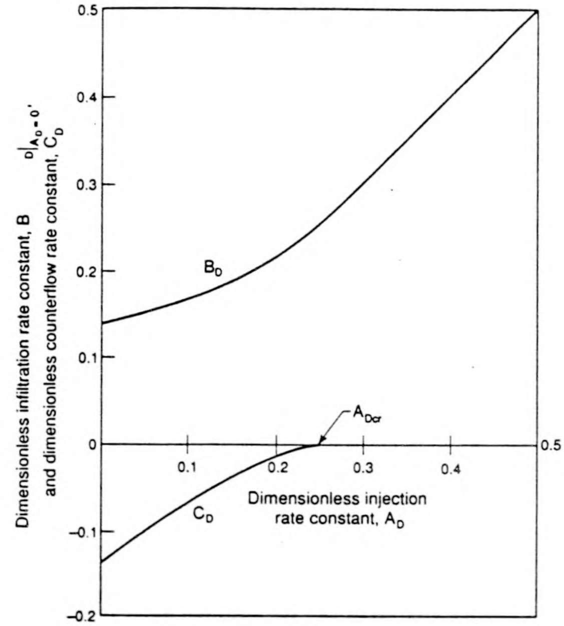

Fig. 2—Dependence of the dimensionless infiltration rate constant B_D and the dimensionless counterflow rate constant C_D on the dimensionless injection rate constant A_D.

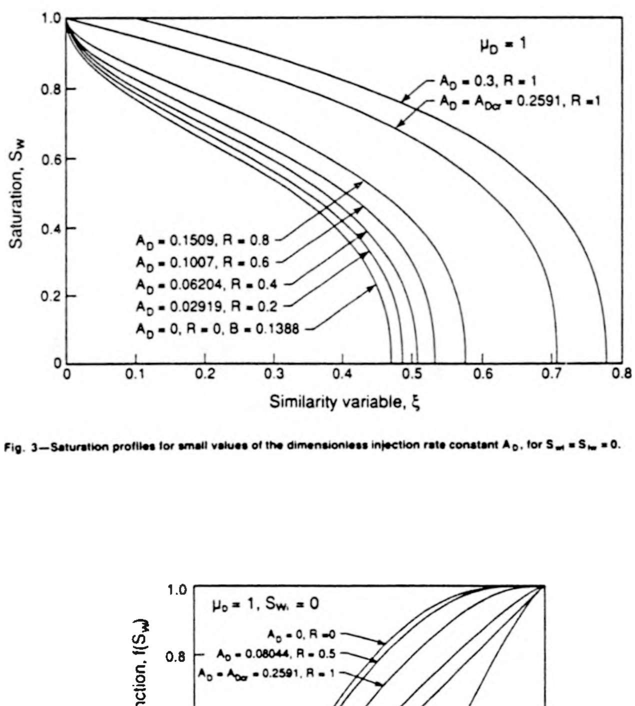

Fig. 3—Saturation profiles for small values of the dimensionless injection rate constant A_D, for $S_{wi} = S_{nw} = 0$.

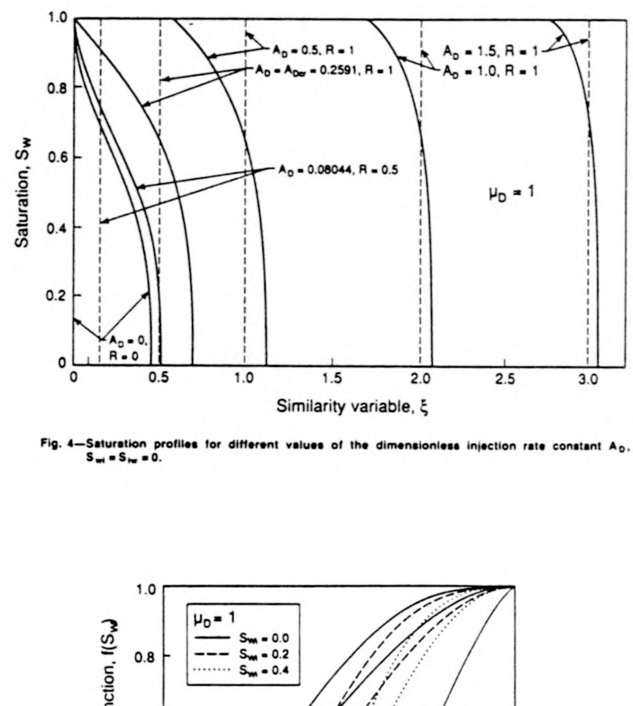

Fig. 4—Saturation profiles for different values of the dimensionless injection rate constant A_D, for $S_{wi} = S_{nw} = 0$.

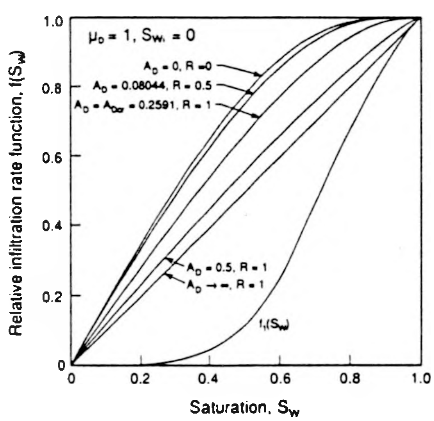

Fig. 5—The relative infiltration rate function $f(S_w)$ for $S_{wi} = S_{nw} = 0$.

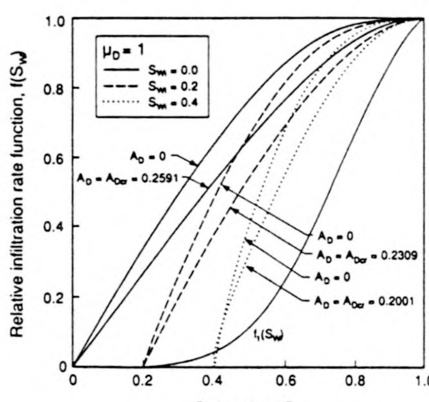

Fig. 6—The relative infiltration rate function $f(S_w)$ for different initial saturations. Values of the different dimensionless parameters are given for each curve.

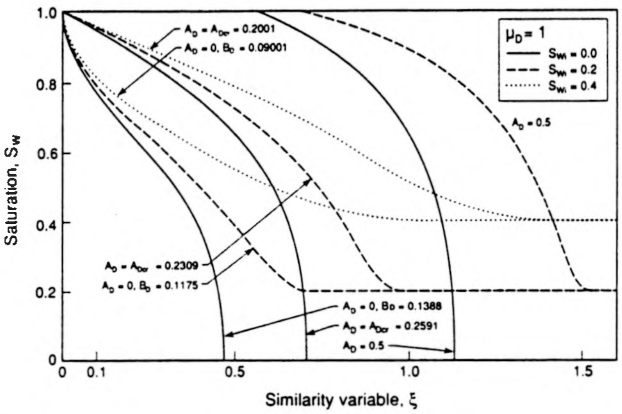

Fig. 7—Saturation profiles for different initial saturations. Values for the different dimensionless parameters are given for each curve.

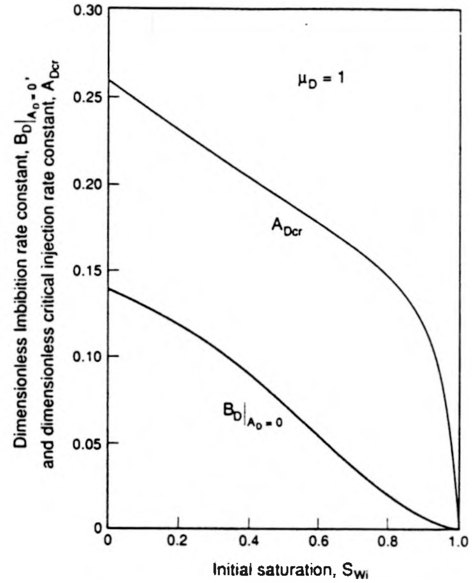

Fig. 8—Dependence of the dimensionless imbibition rate constant $B_D|_{A_D=0}$ and the dimensionless critical injection rate constant A_{Dcr} on the initial saturation.

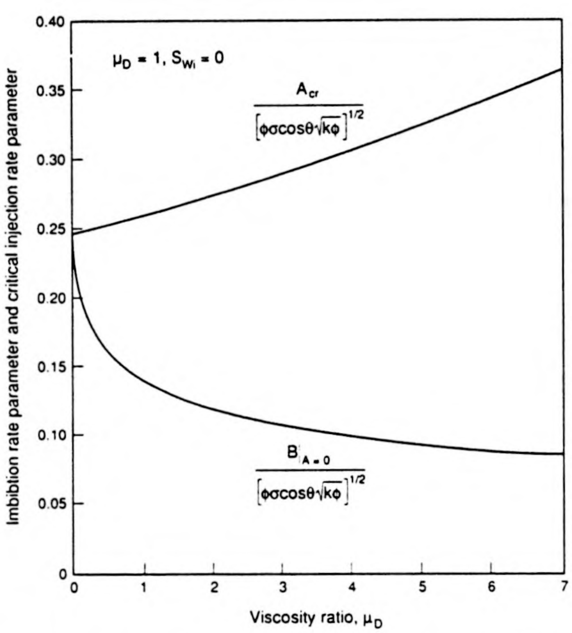

Fig. 9—Influence of the viscosity ratio on the imbibition rate parameter and the critical injection rate parameter.

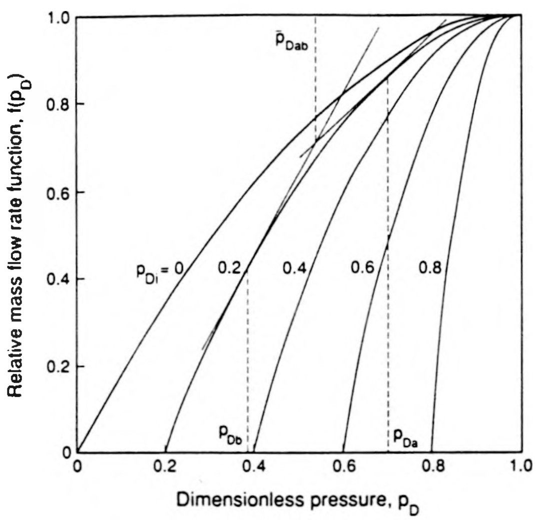

Fig. 10—The relative mass flow rate function $f(p_D)$ for various values of dimensionless initial pressure.

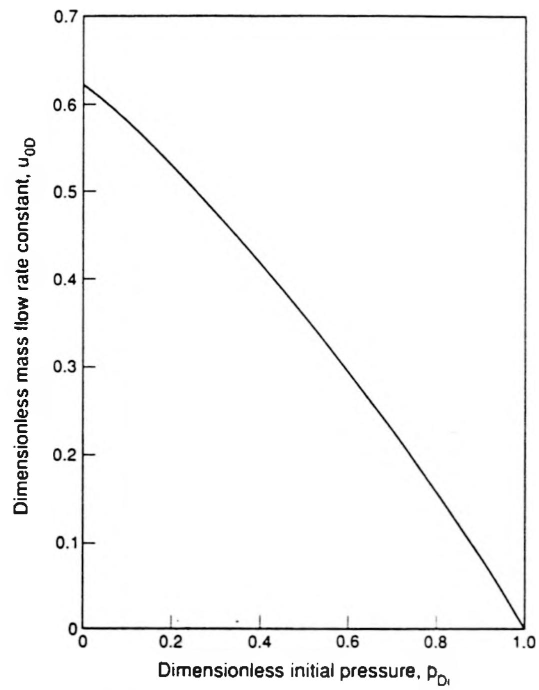

Fig. 11—Dimensionless mass flow rate constant u_{0D} vs. dimensionless initial pressure p_{Di}.

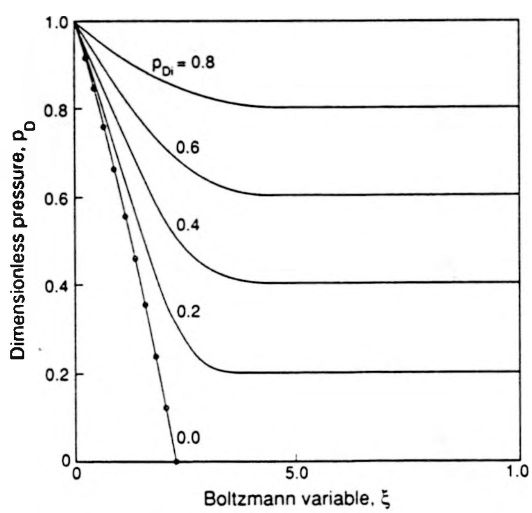

Fig. 12—Dimensionless pressure profiles for various values of dimensionless initial pressure.

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