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TITLE: QUANTIZED VORTEX FILAMENTS IN INCOMPRESSIBLE FLUIDS

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QUANTIZED VORTEX FILAMENTS

IN INCOMPRESSIBLE FLUIDS*

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Among the solutions $\vec{A}_t(\vec{x})$ to the Euler equations for a classical incompressible fluid are those describing vortex filaments. Here we discuss quantum analogues of such solutions in 2 and 3 dimensions.

Within the framework of our accompanying paper[1], we expect vortex filaments to generate coadjoint orbits in the classical phase space $T^*(\mathrm{sDiff}(R^n))$, $n = 2$ or 3. These orbits can be used as reduced phase spaces describing simple classes of fluid flow.

Consider such an orbit for $n = 2$. Let $\vec{C}(\alpha)$, for $0 \le \alpha \le 2\pi$, be a parametrized curve in R^2 (an arc or a loop). For $\vec{C}(\alpha)$ smooth, we can introduce an unparametrized curve $\Gamma = \{\vec{C}(\alpha)\}$ and a function $\gamma = d\alpha/ds$. We define $\vec{A}_{\vec{C}} \in T^*$ by its value on $\vec{v} \in T$; that is

$$\langle \vec{A}_{\vec{C}} \, , \vec{v} \rangle = \int_\Gamma ds \; \gamma(s) \; \chi_{\vec{v}}(\vec{C}(\alpha)) \; , \text{ where } \vec{\nabla} \times (\chi_{\vec{v}} \, \hat{z}) = \vec{v} \; . \qquad (1)$$

It follows from Eq. (1) that the vorticity of the generalized velocity field $\vec{A}_{\vec{C}}$ is $\vec{\nabla} \times \vec{A}_{\vec{C}} = \gamma \, \delta_\Gamma \hat{z}$. Thus Γ represents the vortex filament and γ the vortex density on this filament.

The action of $\phi \in \mathrm{sDiff}(R^2)$ on $\vec{A}_{\Gamma,\gamma}$ is given by $\phi^* \vec{A}_{\Gamma,\gamma} = \vec{A}_{\Gamma',\gamma'}$, where $\Gamma' = \phi\Gamma$ and $\gamma' = (||\hat{s} \cdot \nabla\phi||)^{-1} \gamma\circ\phi^{-1}$. Here $\hat{s}$ is the unit vector tangent to Γ. Therefore a coadjoint orbit corresponds to a collection of vortex filaments satisfying a number of additional constraints. For example: (a) the total vorticity $\int_\Gamma ds \; \gamma(s)$ is an invariant of the orbit; (b) the smoothness class of Γ (e.g. the number of kinks or jumps in derivatives of Γ) is preserved;

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(c) all topological properties of Γ are invariants including whether Γ is bounded or unbounded, whether it is an arc or a closed loop, etc.; (d) if Γ is a loop, then the area it encloses is an invariant.

For specificity consider the coadjoint orbit Δ containing the closed loop filament $\vec{A}_{\vec{c}_o}$, where $\vec{c}_o(\alpha) = (\cos\alpha, \sin\alpha)$, or equivalently $\vec{A}_{\Gamma_o,\gamma_o}$, where Γ_o is the unit circle and $\gamma_o = 1$. The stability group (or little group) K_{Γ_o,γ_o} of $\vec{A}_{\Gamma_o,\gamma_o}$ is the group of all volume-preserving diffeomorphisms ϕ which merely rotate $\vec{c}$; i.e., such that $\phi(\vec{x}) = R\vec{x}$ for all $\vec{x} \in \Gamma_o$, where R is a rigid rotation in R^2. The orbit, regarded as the reduced phase space for the classical system, is thus identified with the quotient space $sDiff(R^2))/K_{\Gamma_o,\gamma_o}$. An element of this space corresponds to a pair (Γ,γ), where Γ is a C^∞ closed loop of area π and γ is a positive C^∞ vorticity density function on Γ with total vorticity 2π.

Next we look more closely at the physical interpretation of this coadjoint orbit as describing a vortex filament. The general solution to the Euler equations for an incompressible fluid reduces in the vortex filament case to

$$\vec{A}(\vec{x},t) = \nabla \times \int_0^{2\pi} d\alpha \; \ell n|\vec{x} - \vec{c}(\alpha,t)|\hat{z} \quad , \tag{2}$$

where

$$\frac{d\vec{c}(\alpha,t)}{dt} = \vec{A}_{av}(\vec{c}(\vec{\alpha},t)), \tag{3}$$

and $\vec{A}_{av}$ is defined as follows. The generalized velocity field $\vec{A}$ has a shear, i.e. a discontinuity in its tangential component along Γ, given by $\gamma = \hat{s} \cdot (\vec{A}^+ - \vec{A}^-)$. Then $\vec{A}_{av} = (\vec{A}^+ + \vec{A}^-)/2$. Thus the coadjoint orbit Δ describes the fluid flow of a vortex ring. It can also be shown that Eqs. (2) and (3) can be written in Hamiltonian form.

Now we turn to the quantum theory. The first step in geometric quantization is to define the polarization group, K_p:

$$K_p = \{\phi \in sDiff(R^2) \; | \phi\{\Gamma\} = \{\Gamma\}\} \quad . \tag{4}$$

The notation $\phi\{\Gamma\} = \{\Gamma\}$ means simply $\phi(\vec{r}) \in \Gamma$ for $\vec{r} \in \Gamma$. Clearly K_p is a subgroup of $sDiff(R^2)$ and $K_p \supset K_{\Gamma_o,\gamma_o}$. The generators of K_p are elements of the algebra of velocity fields $\mathcal{A}_p = \{\vec{v} | \hat{n} \cdot \vec{v} = 0 \text{ on } \Gamma\}$. The essential property required of K_p is that $(\vec{A}_{\Gamma_o,\gamma_o}, [\xi,\eta]) = 0$ for $\xi,\eta \in \mathcal{A}_p$, which is easily shown.

This result allows us to decompose phase space into coordinates and momenta. In particular, coordinate space is isomorphic to $sDiff(R^2)/K_p \approx \{\Gamma\}$, i.e. to curves $\vec{C}(\alpha)$ modulo parametrization. Thus Γ is the coordinate and γ is the associated momentum variable.

The next step is to construct a character $\chi_p(\vec{\phi})$ on the polarization group. It can be shown[2] that $\chi_p(\vec{\phi}) = \exp i\Omega(\omega,\vec{\phi})$, where ω is any curve from infinity to a point on the curve Γ_o, and $\Omega(\omega,\vec{\phi})$ is the area enclosed by the curves ω, $\vec{\phi} \circ \omega$ and Γ_o.

Thus the algebraic part of the geometric quantization program has been accomplished. To obtain a unitary representation of $sDiff(R^2)$ describing vortex filaments, i.e. to fully construct a quantum theory of vortex filaments, one still needs a measure on the set of curves Γ, quasi-invariant for $sDiff(R^2)$. Unlike the finite dimensional case, the required measure does not follow simply from the existence of the canonical 2-form on the coadjoint orbit. An approach to obtaining such measures will be outlined in a future paper.

In conclusion, we note that these methods can be applied to vortex filaments in 3 dimensions. However, in this case the analysis shows that the little group is maximal and no polarization exists. Thus a 3 dimensional vortex filament can not be quantized. If the vortex filament is smeared out so as to form a vortex tube, one again expects a polarization to exist so that it will be possible to carry out the quantization program.

References

1. G. A. Goldin, R. Menikoff and D. H. Sharp, "Diffeomorphism Groups, Coadjoint Orbits and the Quantization of Classical Fluids" (These proceedings.)

2. G. A. Goldin, R. Menikoff and D. H. Sharp, (in preparation).