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MONTE CARLO RENORMALIZATION GROUP: A REVIEW‡

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ABSTRACT

The logic and the methods of Monte Carlo Renormalization Group (*MCRG*) are reviewed. A status report of results for 4-dimensional lattice gauge theories derived using *MCRG* is presented. Existing methods for calculating the improved action are reviewed and evaluated. The Gupta-Cordery improved *MCRG* method is described and compared with the standard one.

The development of Monte Carlo Renormalization group method (*MCRG*) was essentially complete in 1979 with the work of Wilson[1], Swendsen[2] and Shenker and Tobochnik[3]. Prior to this Ma[4] and Kadanoff[5] had provided key ingredients. The method is therefore relatively new, furthermore its application to field theories has been carried out only since 1982. In this short period there has been considerable activity and I shall review the methodology and summarize the status with emphasis on 4-dimensional gauge theories. There already exists extensive literature on *MCRG* and I direct the reader to it[1,3,6,7] for details and for a wider exposure. Similarly, the reviews[8,9] are a good starting point for background on Lattice Gauge Theories and on spin systems. The topics I shall cover are

1) Introduction to *MCRG* and its methodology.
2) Renormalization Group Transformations for $d = 4$ lattice gauge theories.
3) U(1) Lattice Gauge theory.
4) β-function and Scaling for SU(3) Lattice Gauge Theory.
5) Improved Actions and Methods to calculate them.
6) Improved Monte Carlo Renormalization Group.
7) Effective Field Theories.

The main results in QCD from *MCRG* are the determination of the β-function and the consequent prediction for the value of the coupling at which asymptotic scaling sets in and second an estimate of the improved gauge action[10]. These results are not spectacular in the sense of confirming that QCD is the correct theory of strong interactions, however they have led to a deeper understanding of the lattice theory and provided a quantitative estimate of the approach to the continuum limit. I shall attempt to show that this method is as yet in its infancy and should be used to tackle a number of problems.

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1) INTRODUCTION TO *MCRG*

Renormalization Group[11,12,13] (RG) is a general framework for studying systems near their critical point where singularities in thermodynamic functions arise from coherence at all length scales. This phenomenon occurs in Statistical Mechanics near and on the critical surface (defined by a divergent correlation length) and in the strong interactions of quarks and gluons. The *MCRG* method was developed to handle this problem of infinitely many coupled degrees of freedom so that sensible results can be obtained from finite computers. There are two central ideas behind *MCRG*: One is to average over these infinitely many degrees of freedom in discreet steps preserving only those which are relevant for the description of the physical quantities of interest. The interaction between these averaged (block) fields is described by an infinite set of couplings that get renormalized at each step. In QCD this discrete reduction is carried out until the correlation length is small enough so that the system can be simulated on a lattice with control over finite size effects. The second is that there are no singularities in the coupling constant space even though the correlation length diverges on the critical surface and that the fixed point is short ranged. Thus even though there are an infinite number of couplings generated under renormalization, only a few short range ones are necessary to simulate the system at a given scale and preserve the long distance physics. Present results suggest that the fixed point for QCD is short ranged.

Standard Monte Carlo: Consider a magnetic system consisting of spins $\{s\}$ on the sites of a $d-dimensional$ lattice L described by a Hamiltonian H with all possible couplings $\{K_\alpha\}$. All thermodynamic quantities can be found from a detailed knowledge of the partition function

$$Z = \sum e^{-H} = \sum e^{K_\alpha S_\alpha} \tag{1.1}$$

where S_α are the interactions. In Monte Carlo, configurations of spins on the original lattice are generated by the Metropolis[14], heat bath[15], molecular dynamics a'ias Microcanonical[16] or the Langevin[17,18] algorithm with a Boltzmann distribution $e^{-H} \equiv e^{K_\alpha S_\alpha}$. All thermodynamic quantities are given as simple averages of correlation functions over these weighted configurations. The accuracy of the calculations depend on the size of the statistical sample and the lattice size L used. Both these quantities depend on the largest correlation length ξ in the system. Near the critical temperature, T_c, associated with second order phase transitions, the correlation length and consequently thermodynamic quantities like the specific heat *etc* diverge as functions of $(T-T_c)$ with universal critical exponents that have been calculated for many systems either analytically or by the Monte-Carlo / *MCRG* method. Because of a diverging ξ, long runs are needed to counter the critical slowing down and the lattice size has to maintained at a few times ξ. The problem of critical slowing down is addressed by analyzing update algorithms (Metropolis vs. heat bath vs. Microcanonical vs. Langevin with acceleration techniques like multi-grid[19], fourier acceleration[18,20] etc). The optimum method is, of course, model dependent and has to take care of metastability (local versus global minima) and global excitations like vortices, instantons etc that are not efficiently handled by local changes. This last feature has not received adequate attention. To control the second problem in standard Monte Carlo, effects of a finite lattice especially as $\xi \to \infty$, finite size scaling

has been used with success. In this review I shall concentrate on *MCRG*. First I shall describe how universality and scaling are explained by the renormalization group.

The renormalization group transformation (*RGT*) $H^1 = R(H)$ is an operator defined on the space of coupling constants, $\{K_\alpha\}$. In practice the *RGT* is a prescription to average spins over a region of size b the scale factor of the *RGT*, to produce the block spin which interacts with an effective theory H^1. The two theories H and H^1 describe the same long distance physics but the correlation length in lattice units $\xi \rightarrow \frac{\xi}{b}$. If this *RGT* has a fixed point H^* such that $H^* = R(H^*)$, then clearly the theory is scale invariant there and ξ is either 0 or ∞. An example of a fixed point with $\xi = 0$ is $T = \infty$ and these are trivial. The interesting case is $\xi = \infty$ about which the theory is governed by a single scale ξ. If this fixed point is unstable in 1 direction only (this direction is called the Renormalized Trajectory (*RT*)), then non-critical H will flow away from H^* along trajectories that asymptotically converge to the *RT*. Thus the long distance physics of all the trajectories that converge is identical and is controlled by the *RT*. Similarly, points ϵ away from H^* on the $\infty - 1$ dimension hypersurface at which $\xi = \infty$ (the critical surface) will converge to H^*. The fact that the fixed point with its associated *RT* control the behavior of all H in the neighborhood of H^* is universality. Next, consider a non-critical H that approaches H^* along the *RT*. Thermodynamic quantities depend on a single variable i.e. distance along the *RT*. This is scaling. Corrections to scaling occur when H does not lie on the *RT*. These are governed by the irrelevant eigenvalues of the *RGT* which give the rate of flow along the critical surface towards H^* and for H not on the *RT*, the rate of convergence towards it. The relevant eigenvalue gives the rate of flow away from the fixed point (along the unstable direction *RT*) and is related to the critical exponent ν. This terse exposé ends with a word of caution; all these statements have validity close to H^*.

In the *MCRG* method, configurations are generated with the Boltzmann factor $e^{K_\alpha S_\alpha}$ as in standard Monte Carlo. The *RGT*, $P(s^1, s)$, is a prescription for averaging variables over a cell of dimension b. The blocked variables $\{s^1\}$ are defined on the sites of a sublattice L^1 with lattice spacing b times that of L. They interact with undetermined couplings K^1_α, however the configurations are distributed according to the correct Boltzmann factor e^{-H^1} i.e.

$$e^{-H^1(s^1)} = \sum P(s^1, s)\, e^{-H(s)} \tag{1.2}$$

so expectation values can be calculated as simple averages. The *RGT* should satisfy the Kadanoff constraint

$$\sum^{1} P(s^1, s) = 1 \tag{1.3}$$

independent of the state $\{s\}$. This guarantees that the two theories H and H^1 have the same partition function. The blocking is done n times to produce configurations with hamiltonians H^n describing the same long distance physics but on increasingly coarser lattices. The fixed point H^*, the *RT* and the sequence of theories, H^n, generated from a given starting H depend on the *RGT*. Many different *RGT* can be used to analyze a given model (determine the universal exponents) and I defer discussion on how to evaluate their efficiency to sections 2 and 5.

1.1) Methods to calculate the critical exponents.

There are two methods to calculate the critical exponents from expectation values calculated as simple averages over configurations. In both there is an implicit assumption that the sequence H^n stays close to H^*. The more popular method is due to Swendsen[2,7] in which the critical exponents are calculated from the eigenvalues of the linearized transformation matrix $T^n_{\alpha\beta}$ which is defined as

$$T^n_{\alpha\beta} = \frac{\partial K^n_\alpha}{\partial K^{n-1}_\beta} = \frac{\partial K^n_\alpha}{\partial \langle S^n_\sigma \rangle} \frac{\partial \langle S^n_\sigma \rangle}{\partial K^{n-1}_\beta} \quad . \tag{1.4}$$

Each of the two terms on the right is a connected 2-point correlation function

$$\frac{\partial \langle S^n_\sigma \rangle}{\partial K^{n-1}_\beta} = \langle S^n_\sigma S^{n-1}_\beta \rangle - \langle S^n_\sigma \rangle \langle S^{n-1}_\beta \rangle \, . \tag{1.5}$$

and

$$\frac{\partial \langle S^n_\sigma \rangle}{\partial K^n_\beta} = \langle S^n_\sigma S^n_\beta \rangle - \langle S^n_\sigma \rangle \langle S^n_\beta \rangle \, . \tag{1.6}$$

Here $\langle S^n_\sigma \rangle$ are the expectation values on the n^{th} renormalized lattice and K^n_σ are the corresponding couplings. The exponent ν is found from the leading eigenvalue λ_t of $T^n_{\alpha\beta}$ as

$$\nu = \frac{\ln b}{\ln \lambda_t} \tag{1.7}$$

where b is the scale factor of the RGT. The eigenvalues less than one give exponents that control corrections to scaling. The accuracy of the calculated exponents improves if they are evaluated close to the fixed point. This can be achieved by starting from a critical point and blocking the lattice a sufficient number of times i.e. for large n. Thus the convergence is limited by the starting lattice size and how close the starting H^c is to H^*. If H^* can be approximated by a small number of short range couplings (a necessary assumption in the RG), then this method can be improved if the renormalized couplings $\{K^n\}$ are determined starting from a known critical Hamiltonian. These should then be used in the update. A second possibility is to tune the RGT so that the convergence to H^* from a starting H^c is improved. In section 5, I will describe a number of methods to calculate the renormalized couplings. Tuning of the RGT is discussed in section 2.5 and a careful analysis of the accuracy of this method is deferred until section 6.

The second method to calculate the leading relevant exponent is due to Wilson[1] Consider once again the 2-point connected correlation function (the derivative of an expectation value) $\langle S^i_\alpha S^j_\beta \rangle_c$ with $j > i$. Expand S^i_α in term of the eigenoperators O^i_α of the RGT. Close to H^* the level dependence in O^i_α (equivalently in the expansion coefficients $c^i_{\alpha\beta}$) can be neglected. Then to the leading order

$$\langle S^i_\alpha S^j_\beta \rangle \sim \lambda_t^{j-i} \, c_{\alpha,t} \langle O_t S^j_\beta \rangle \tag{1.8}$$

where λ_t is the leading relevant eigenvalue and corrections are suppressed by $(\frac{\lambda}{\lambda_t})^{j-i}$. Thus for each α and β, the ratio $\frac{\langle S^i_\alpha S^j_\beta \rangle}{\langle S^{i+1}_\alpha S^j_\beta \rangle}$ gives an estimate for the leading eigenvalue

λ_t. The accuracy of the method improves if $j - i$ is large since non-leading terms are suppressed geometrically. So far this method has not been used extensively so its practical accuracy cannot be evaluated.

QCD: At the tree level, the coupling g in QCD does not renormalize and the fixed point is at $g_{bare} = 0$. At 1-loop the leading operator has eigenvalue equal to one, is relevant and the fixed point changes from simple gaussian to being asymptotically free and non-trivial. A special feature of asymptotic freedom is that even when the leading eigenvalue is one there is a flow away from the fixed point at a constant rate. At 2-loop, this operator becomes truely relevant i.e. with eigenvalue > 1. Perturbation theory also tells us that leading scaling violations are $\sim \frac{1}{b^2}$, so the second eigenvalue should be $\sim \frac{1}{b^3}$ for a RGT with scale factor b. Present studies[22b] show that the leading eigenvalue is close to 1 and the second near $\frac{1}{b^2}$. However, the statistics are poor and the calculation was done at large g_{bare}. Thus reliable quantitative results are lacking.

1.2: Wilson's method to find a critical point

Consider $MCRG$ simulations L and S with the same starting couplings K^0_α but on lattice sizes $L = b^n$ and $S = b^{n-1}$. If K^0_α is critical and after a few blockings the 2 theories are close to H^*, then all correlation functions attain their fixed point values. For non-critical starting H, expand about H^* in the linear approximation

$$\langle L^m_\alpha \rangle - \langle S^{m-1}_\alpha \rangle = \frac{\partial}{\partial K^0_\beta} \{\langle L^m_\alpha \rangle - \langle S^{m-1}_\alpha \rangle\} \Delta K^0_\beta$$

$$= \{\langle L^m_\alpha L^0_\beta \rangle_c - \langle S^{m-1}_\alpha S^0_\beta \rangle_c\} \Delta K^0_\beta \tag{1.9}$$

to determine ΔK^0_α. To reduce finite size effects the compared expectation values are calculated on the same size lattices. The critical coupling is given by

$$K^c_\alpha = K^0_\alpha - \Delta K^0_\alpha \tag{1.10}$$

and this estimate should be improved iteratively.

On the critical surface the 2-point correlation functions (like in Eq. (1.5) and (1.6)) diverge in the thermodynamic limit. However, their ratio is the rate of change of couplings and these are well behaved. The reason $MCRG$ has better control over finite size effects is that if H^* is short ranged then only short ranged correlation functions need to be evaluated in determining $T^n_{\alpha\beta}$ or in Eq. (1.9). The finite size contributions to the ratios fall off like the couplings i.e. exponentially. Thus reliable estimates are obtained from small lattices.

2: RENORMALIZATION GROUP TRANSFORMATIONS IN $d = 4$

It has been mentioned before that there is no unique RGT for a given model. There are at present four different transformations that have been proposed for 4-dimensional lattice gauge theories. In each of them the block link variable is constructed from a sum of paths $\Sigma \equiv \sum paths$. This sum of SU(N) matrices

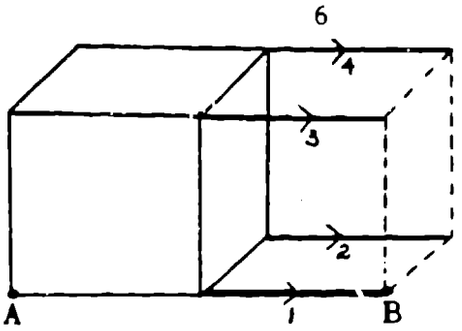

Figure 1: Wilson's b = 2 RGT. Four of the eight paths in a given direction are shown.

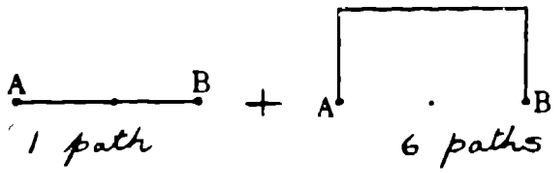

Figure 2a: Swendsen's b = 2 RGT.

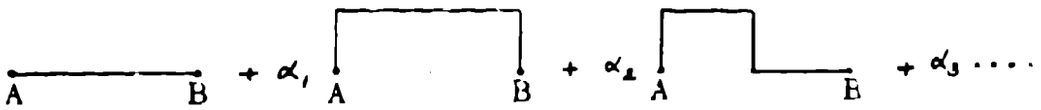

Figure 2b: Generalized Swendsen RGT with parameters α_i that have to be optimized.

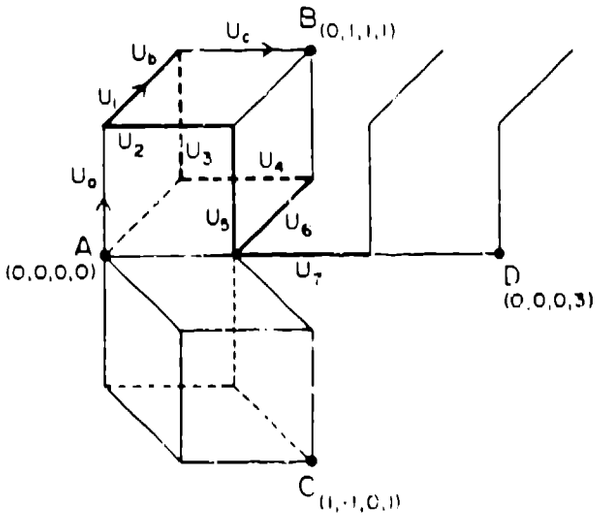

Figure 3: The geometry of the $\sqrt{3}$ block transformation.

is not an element of SU(N), and the new block link matrix is selected with the distribution

$$P(U_b) \;=\; e^{\,p\;Tr\;U_b\Sigma}\;. \qquad (2.1)$$

where p is a free parameter to be optimized. The advantage of taking the sum is that such a *RGT* preserves gauge invariance. The 4 *RGT* are (in cronological order)

2.1) $b = 2$ **by Wilson**[1]: The geometry of the transformation is shown in Fig. 1. There are 8 links in a given direction of which 4 are shown in the 3-dimensional projection. In this method the gauge has to be fixed on 15 sites other than the block site. This fixing has to take into account the fact that the ends of the 8 links are at different sites. The ansatz Wilson used was to transform the hypercube locally into the Landau gauge. The process of fixing the gauge is slow and a disadvantage of the method. The need for gauge fixing can be avoided by defining 8 paths that run between the block sites and include the same links. This modified construction violates cubic rotational invariance because of the particular choice of the ordering of the paths within the cell. In both forms only $\frac{82}{49}$ degrees of freedom are used in this approximate averaging at each level. This method has not been used since Wilson's preliminary investigation because the next two methods are simpler.

2.2) $b = 2$ **by Swendsen**[21]: The transformation in its initial form is shown in Fig. 2a. The more general version is shown in Fig. 2b where the parameters α_i have to be determined. In this construction all paths start and end at the block sites. Thus no gauge fixing is necessary and arbitrarily complex paths can be included. However calculations show that an optimization of the parameters has to be done to improve the convergence. I shall discuss this tuning later.

2.3) $b = \sqrt{3}$ **by Cordery, Gupta and Novotny**[24]: This transformation is specific to gauge theories in 4-dimensions and is based on the fact that the body diagonals of the 4 positive 3-cubes out of a site are orthogonal and of length $\sqrt{3}$. The geometry is shown in Fig. 3 and under one *RGT* the new lattice is still hypercubic but rotated with respect to the old basis. Also, the box boundary becomes jagged. This can be undone by a second application of the *RGT* with different basis vectors. So the original box geometry is recovered after every scale change by a factor of 3. The construction of the paths requires no gauge fixing, all paths are of equal length (no free parameters to be tuned) and $\frac{24}{28}$ degrees of freedom are used at each step. Further, the block cell consists of the block site and its 8 nearest neighbors. This provides an easy and natural way to include complex matter fields and block them simultaneously. It is also better suited to the fermion block diagonalization process of Mutter and Schilling[64] as is explained in section 5.10. In practice, for both SU(2) and SU(3), this *RGT* has consistently shown good convergence at strong and at weak coupling. It is therefore recommended.

2.4) $b = \sqrt{2}$ **by Callaway and Petronzio**[23]: The construction of paths shown in Fig 4a is based on a planer structure i.e. $x-y$ and $z-t$ planes are treated separately at all blocking steps. No gauge fixing is required but only 2 paths are used in the averaging i.e. in Eq. (2.1). This drawback of using only 2 planar paths can be improved by including nonplanar paths as shown in Fig. 4b. Because this *RGT* has the advantage that $b = \sqrt{2}$ is the smallest scale factor possible, a serious test should be made.

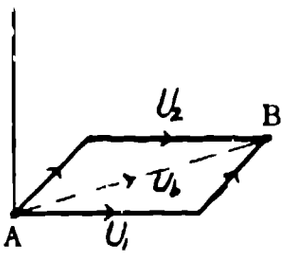

Figure 4a: The two paths in the $b = \sqrt{2}$ RGT.

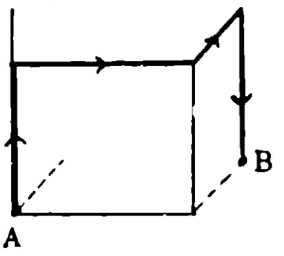

Figure 4b: Additional 4 link paths.

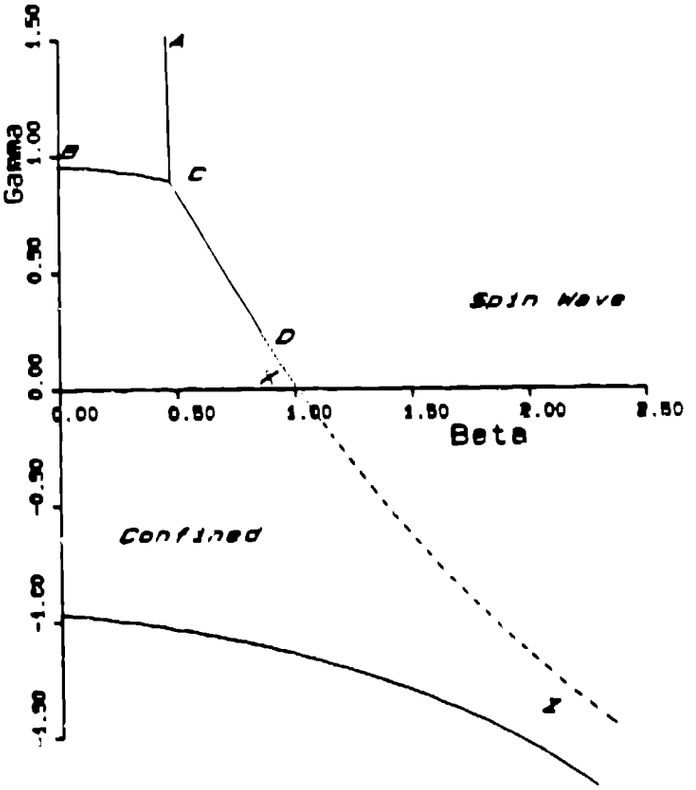

Figure 5: The phase diagram of U(1) gauge theory. The location of the TCP is somewhere near the dotted line DX.

2.5) Optimization of the *RGT*: In addition to the freedom of the choice of the *RGT*, there are the free parameters p and α_i. Hasenfratz *et al.*[24] have shown that the convergence of the original $b = 2$ Swendsen transformation is improved if p is tuned. I will give a qualitative description of how this works. Consider a set of *RGT* that are a function of the continuous parameter p i.e. R_p. Starting from a given point H, the blocked theories generated are described by $H^1(p)$. They all have the same long distance behavior as can be checked by measuring expectation values of large Wilson loops. In fact there is an effective Wilson action H_{eff} which will have the same long distance behavior. The short distance behavior of $H(p)$ will be different and for some values of p, the $\langle plaq \rangle_p$ will be larger than the $\langle plaq \rangle_w$ corresponding to H_{eff}. I have checked that this is the case for the original Swendsen transformation when $p = \infty$ and $g^2 < 1$. Lowering p reduces the blocked $\langle plaq \rangle_p$, making it agree better with H_{eff}. Thus, the tuning makes the short and long distance behavior correspond better to the same approximate H_{eff}. This improves the matching (using small loops) in the 2-lattice method to calculate the β-function. Hasenfratz *et al.*[24] estimate p using perturbation theory and by Monte Carlo using the criterion of early matching of block expectation values in Wilson's two lattice method. They found that the best value at $\frac{6}{g^2} = 6.$ given by Monte Carlo (~ 35) does not agree with the value found using perturbation theory (~ 15). So as of now this optimization is still by trial. Also, p_{opt} depends on the coupling g. This implies that the *RT* cannot be pulled close to the Wilson axis globally by this optimization. So the usefulness of such optimization is limited to the β-function calculation. The parameters α_i can similarly be optimized using the same improvement criterion.

Gupta and Patel[22] used $p = \infty$ in the $\sqrt{3}$ *RGT*. This is equivalent to choosing the matrix U such that $Tr\ U \sum$ is maximized (the δ-function construction). They find that even with this choice the small block Wilson loops are more disordered than for an H_{eff} determined using large loops. Thus lowering p would not help. The $\sqrt{3}$ *RGT* has shown good convergence properties and provided reliable results with $p = \infty$.

The freedom to choose the *RGT* and further tune the parameters α_i and p leads to the question: What are the criteria by which to decide what is the best *RGT*? I will first address the question -- what is the effect of changing the *RGT* on the fixed point and on the RT? Conjecture[25]: Changing the *RGT* moves the fixed point on the critical surface but only along redundant directions. A simple argument is as follows: Consider two different *RGT*, R_1 and R_2, and their associated fixed points H_1^* and H_2^*. There are no non-analytic corrections to scaling at either fixed points and the associated *RT*. If these two points are distinct, then under R_1 H_2^* flows to H_1^*. Consequently there are no scaling violations along the flow. This is by definition a redundant direction. This implies that the associated *RT* differ by redundant operators.

The presence of redundant operators does not effect the physics, however it can obscure results. The redundant eigenvalues are not physical, depend on the *RGT*, and can be relevant or irrelevant. If a relevant redundant operator is present then the flows will not converge to the H^* or to the *RT*. Thus it is desirable to pick a *RGT* for which the redundant eigenvalues are small. Similarly, the coefficients of the leading irrelevant operators should be reduced. To some extent the irrelevant basis vectors are a function of the position of H^*, so it is possible to simultaneously reduce the two coefficients. In *QCD* there is an additional freedom -- all possible

Wilson loops form an overcomplete set. Therefore, in order to tune the RGT and to find an efficient improved action, it is necessary to determine the operators that can be eliminated because of the overcompleteness and the redundant combinations.

Swendsen[26] has conjectured that the fixed point can be moved anywhere on the critical surface by tuning the RGT. In particular, if the simulation point is made H^*, then that RGT is optimal. There is some support for this in spin systems, where by adding terms to the RGT, one can successively kill terms in the renormalized hamiltonian. There are two things to check here: first whether the coefficients of the RGT terms fall off like the couplings, *i.e.* exponentially, and second whether the long range untuned couplings continue to fall off at least as fast as before. The quantity to optimize is the update complexity (embodied in the RGT or the hamiltonian) versus the decrease in the coefficient of the leading irrelevant operator. Swendsen[26] found that the eigenvalues for the $d = 3$ Ising model are significantly improved with a tuned 10 term RGT. However, he did not compare it with a simulation that used a 10 term truncated renormalized hamiltonian close to the H^* for a simple RGT. There is one additional anomaly in this approach: Tuning the RGT improved the thermal exponent but the results for the magnetic exponent deteriorated in quality. This is surprising because the fixed point is at zero odd couplings and these remain unchanged in tuning the RGT. The previous conjectures are in conflict and the results are ambiguous. Consequently, this subject is being explored[69] further.

The criterion for an optimum RGT is to make the H^* and the RT as short ranged as possible. In critical phenomena, the improvement can be quantified by measuring the convergence of the exponents as a function of the blocking level. In QCD we are interested in continuum mass-ratios *etc.* These have so far been hard to measure so the improvement cannot be judged. The behavior of the RT for QCD is discussed at the end of section 5. For the moment let me conclude this section by: The the question of how best to optimize $MCRG$ has not been adequately answered and is under investigation.

3: U(1) LATTICE GAUGE THEORY:

The phase diagram of the theory defined by the action

$$S = \beta \sum cos\Theta_{\mu\nu} + \gamma \sum cos2\Theta_{\mu\nu} \tag{3.1}$$

where β (γ) is the charge 1 (charge 2) coupling is known to have a phase boundary separating the confining (strong-coupling) phase from the spin-wave (QED) phase[27,28,29]. The order of the transition along the boundary DXZ in Fig. 5 is not known. In particular it is not known if the gradually weakening first order transition along CD ends in a tricritical point, and if so what is its location. Evertz *et al.*[28] claim that the location of the TCP is at $\beta = 1.05 \pm 0.04$ and $\gamma = -0.11 \pm 0.05$ on basis of a scaling analysis of the discontinuity in the energy ΔE. The mechanism driving the transition are topological excitations[30,31], *i.e.* closed loops of monopoles, whose density is observed to change at the transition[32,33]. This change in density is caused by a growth in the size of the largest monopole loop which begins to span the finite lattices used in the calculations[32,34]. Thus, the usual difficulty of finite size effects near a TCP in determining the location of the TCP by an extrapolation

of the latent heat ΔE along the phase boundary is here compounded by the presence of monopole current loops that are closed due to the lattice periodicity[32,34]. These contribute a fake piece to the ΔE which makes the extrapolation unreliable. One solution is to calculate and then subtract the contribution of these loops from ΔE before making the extrapolation. The more reliable method is $MCRG$ and in particular the 2-lattice method discussed in section 1.2 should be used to locate the TCP. A word of caution for the U(1) model when using this method: There is a large shift in the critical coupling as a function of the lattice size[32] and consequently in the contribution of the fake monopole loops. One should therefore use a starting coupling for which both lattice simulations are on the same side of the transition.

The present status is that in a $MCRG$ calculation done along the Wilson axis[32] only one relevant exponent was found using the $\sqrt{3}$ RGT. Furthermore, the value of the exponent showed a variation with β. At $\beta = 1.0075$, $\nu \approx 0.32$ and this value changes to $\nu \approx 0.43$ (or even the classical value 0.5) at $\beta = 1.01$. One explanation is that the TCP lies above the Wilson axis and in simulations along the Wilson axis one first measures the tricritical exponent and then the critical one. The same conclusion is also reached in a recent $b = 2$ $MCRG$ study[35]. Therefore the location of the TCP is still an open question.

The interest in this model (goal of MC calculations) is to know if there exists a fixed point at which a non-trivial field theory can be defined. To settle this important question requires considerably more work.

4: β-FUNCTION AND SCALING FOR SU(3) LATTICE GAUGE THEORY

The non-perturbative β-function tells us how the lattice spacing goes to zero as $g_{bare} \to 0$. Since on the lattice all dimensionful quantities, like masses, are measured in units of the lattice spacing a, we need to know how a scales in order to take the continuum limit. One option is to use the 2-loop perturbative result provided it is demonstrated that this is valid at values of g_{bare} where the calculations are done. The other is to measure the non-perturbative β-function. In case there is only asymptotic scaling, this calculation is still necessary since it provides the value of g_{bare} at which such scaling sets in.

There are two methods for calculating the non-perturbative β-function directly.

4.1) $MCRG$ using Wilson's 2 lattice method[1,3]: There are 2 groups who have used this method for SU(3); one with $b = \sqrt{3}$ RGT [37] and the second[24] with $b = 2$ proposed by Swendsen[21]. The outline of the method is: First a system of size $L = (b^n)^d$ is simulated with couplings K_α^A and the expectation values of Wilson loops are calculated on the original lattice and the n block lattices. A second system of size $S = (b^{n-1})^d$ is then simulated with couplings K_α^B (chosen judiciously) and again the expectation values are calculated on the n lattices. The expectation values from the two simulations are compared with the ones from the larger lattice L blocked one more time i.e. L^m with S^{m-1}. Finite size effects are minimized since the comparison is on approximately the same physical size lattices. The couplings K_α^B are adjusted (which requires a new simulation) until there is matching at the last, n^{th}, level. In practice it is sufficient to do two simulations S_1 and S_2 which bracket L and

then use interpolation. The test for convergence of the two theories L^m and S^{m-1} is that the expectations values should match simultaneously at the last few levels. This situation is shown in the coupling constant space in Fig. 6. At matching, the correlation length at K_α^A is larger than at K_α^B by the scale factor b. Thus if the starting trajectory is taken to be the Wilson axis (or any 1 parameter line) then the value of the β-function, $\Delta\beta$, for a scale change b is $K^A - K^B$.

Under the assumption that the fixed point action is local, and that at any scale a few short range couplings are sufficient to characterize the action, matching the expectation values of a few small Wilson loops is sufficient to guarantee that the two actions are equal. Recall that there is a one to one correspondence between the value of the couplings and the expectation values. Also note that finite size effects in expectation values are irrelevant once there is matching because then the two theories flow along a common trajectory under a RGT and continue to match. Thus it is sufficient to require that matching first take place on lattices which are large enough to accommodate the important couplings. Thereafter, the check can be on a 1^4 lattice too! It is the range of the couplings that controls finite size effects in $MCRG$ and not the correlation length and this range falls off exponentially even on the critical surface. This is why $MCRG$ has better control over finite size effects and is a powerful method.

For the simple plaquette SU(3) action with $K_F \equiv \frac{6}{g^2}$, asymptotic scaling is defined by the 2-loop perturbative β-function,

$$\frac{\partial(g^{-2})}{\partial(lna)} = -\frac{11}{8\pi^2} - \frac{51}{64\pi^4}g^2 + \cdots \quad . \tag{4.1}$$

The quantity calculated using $MCRG$ is,

$$\Delta\beta = -\frac{\partial(6g^{-2})}{\partial(lna)} \cdot lnb \quad , \tag{4.2}$$

i.e., the discrete β-function at K_F evaluated for a scale change b.

The results for the $b = \sqrt{3}$ calculation[37] are shown in Table 1, while those for $b = 2$ are shown[24] in Table 2. There is clear evidence of a dip at $\frac{6}{g^2} \sim 6.0$ which is caused by the end point of the phase transition line in the fundamental-adjoint coupling space. The conclusion of these calculations is that there is no asymptotic scaling below $\frac{6}{g^2} = 6.1$.

4.2) Loop ratio method[38,24]: This method is simpler as it uses expectation values of Wilson loops calculated in standard Monte Carlo. Thus it can be used for gauge theory with dynamical fermions while method 4.1 cannot until one learns how to block fermions. The ratios of Wilson loops that cancel the perimeter and corner terms

$$R(i,j,k,l) = \frac{W(k,l)}{W(i,j)} \qquad where \ i+j = k+l \ . \tag{4.3}$$

satisfy an approximate homogeneous renormalisation group equation

$$R(2i,2j,2k,2l,g_a,2L) = R(i,j,k,l,g_b,L) \ . \tag{4.4}$$

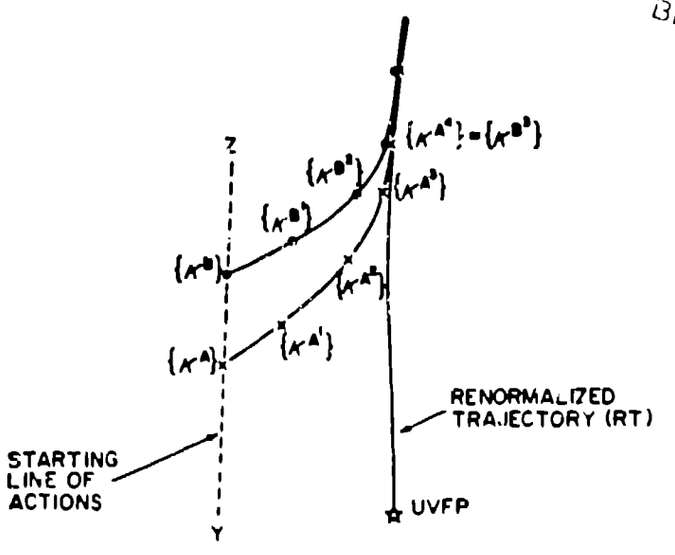

Figure 6: The evolution of actions under the renormalization group transformation. The two actions $\{K^A\}$ and $\{K^B\}$ have the same long-distance behavior and their lattice correlation lengths are related by the scale transformation factor b.

SU(3)				
9^4	$\Delta\beta$ for $b=\sqrt{3}$ from matching on			2-loop
K_F	3^4	$(\sqrt{3})^4$	1^4	$\Delta\beta$
6.0	.337(5)	.323(5)	.308(6)	.489
6.125	.387(5)	.376(5)	.351(6)	.488
6.25	.421(4)	.424(5)	.401(5)	.488
6.35	.431(4)	.452(5)	.445(9)	.487
6.45	.432(4)	.464(6)	.423(12)	.487
6.5	.435(4)	.464(6)	.449(15)	.487
6.75	.430(4)	.485(5)	.443(9)	.485
7.0	.422(7)	.503(11)	.488(20)	.484

Table 1 : The values of $\Delta\beta$ for $b=\sqrt{3}$ at different levels of matching for different values of the couplings [37]. The matching K_F on $(3\sqrt{3})^4$ were determined by linear interpolation and the errors are based on a 1σ fit. Also shown are the values of $\Delta\beta$ corresponding to asymptotic scaling.

Thus using Monte Carlo data for ratios calculated on 2 lattices of size $2L$ and L, with couplings g_a and g_b respectively, the desired function $\Delta\beta$ for $b = 2$ can be calculated. Caveats: Eq. (4.4) becomes correct only as $i \to \infty$, otherwise there are corrections due to lattice artifacts. The quality of results for large $i, j, \ldots$ are limited by statistics. The reliability of the results therefore depends on obtaining the same $\Delta\beta$ for $i = 1, 2, 3, 4, \ldots$..

The contribution of lattice artifacts can be reduced in perturbation theory. For this consider Eq. (4.4) for a linear combination of loop ratios with coefficients α_i. To determine the α_i, use the expectations values of loops calculated in perturbation theory and require that $\Delta\beta = 0$ (tree-level), 0.579 (1-loop) .... Then go back and use the monte carlo data for Wilson loops to calculate $\Delta\beta$. The drawback of this approach is that if two (or more) ratios of different scale, $i = 1$ and 4 say, are used then the difference in statistical errors is a problem. Otherwise, at weak coupling each ratio roughly satisfies Eq. (4.4) and there is a loss of sensitivity in determining α_i. At strong coupling, perturbation theory calculation/improvement of α_i breaks down. So one can, at best, expect a window where reliable results are obtained Hasenfratz *et al.*[24] claim this is true for $\frac{6}{g^2}$ in the range [6,6.6]. Their results are in agreement with their $b = 2$ *MCRG* results as shown in Table 2. It has been observed by Gutbrod[39] in SU(2) that stability with respect to loop size is reached slowly. Therefore, one has to be cautious of apparent convergence

4.3) Results: It is hard to compare the results of the $b = \sqrt{3}$ study directly with the $b = 2$ ones because of the different scale factor of the *RGT*. Petcher[40] has carried out the following analysis: he fits the $b = \sqrt{3}$ data to a smooth function which had the correct asymptotic value built in. This function can then be used to determine the discrete change $\Delta\beta$ in the couplings for any other scale factor b. In Fig 7 the smooth function found from the $\sqrt{3}$ data rescaled to $b = 2$ is compared with the $b = 2$ *MCRG* data.

Next we would like to check if the $\Delta\beta$ calculated from MC determinations of different physical observables are identical and agree with the *MCRG* calculations. This comparison tests two things, first whether there exists scaling (constant mass ratios) before (larger g) asymptotic scaling and second whether the MC measurements are reliable. The lattice value of a mass ma calculated at two values of the coupling, $\frac{6}{g_1^2}$ and $\frac{6}{g_2^2}$, gives the $\Delta\beta$ for a scale change $\frac{a_1}{a_2}$. Unfortunately the values of couplings cannot be selected to give the $\Delta\beta$ for a given constant scale change. This again introduces the problem of rescaling data thereby preventing a definite statement on scaling. In Fig. 8 we have only used pairs of data points with a scale factor close to $\sqrt{3}$. At $\frac{6}{g^2}$=6.0, the 0^{++} glueball mass[41], string tension σ[42] and the deconfinement temperature T_c [43] represent scales of 2, 5 and 8 lattice units respectively. Thus identical $\Delta\beta$ would be a reasonable test of scaling. Bearing in mind the problem of rescaling data, the only significant statement is that the behavior of the glueball mass is different.

The onset of asymptotic scaling has also been checked by plotting $\frac{ma}{\Lambda a}$ where m is the deconfinement temperature T_c and Λ is the 2-loop perturbative scale. Kuti *et al.*[43] found that for $N_\tau = 10, 12, 14$ this ratio is constant and different from the value at $N_\tau \leq 8$. From this they deduce that there is asymptotic scaling for $\frac{6}{g^2} > 6.15$. The only drawback of this method is the reliance on Λ calculated in 2-loops to define asymptotic scaling. There could be corrections i.e. $(1 + O(g^2))$ terms,

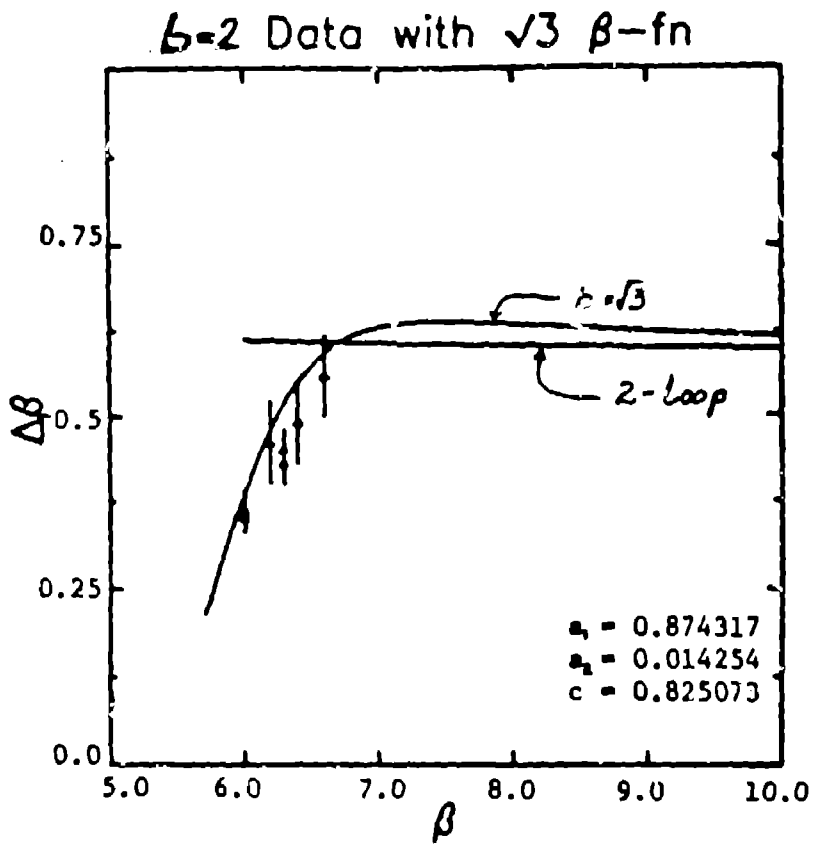

Figure 7: The smooth curve is a fit to the b = √3 data. The data points are for b = 2. (Courtesy of D. Petcher)

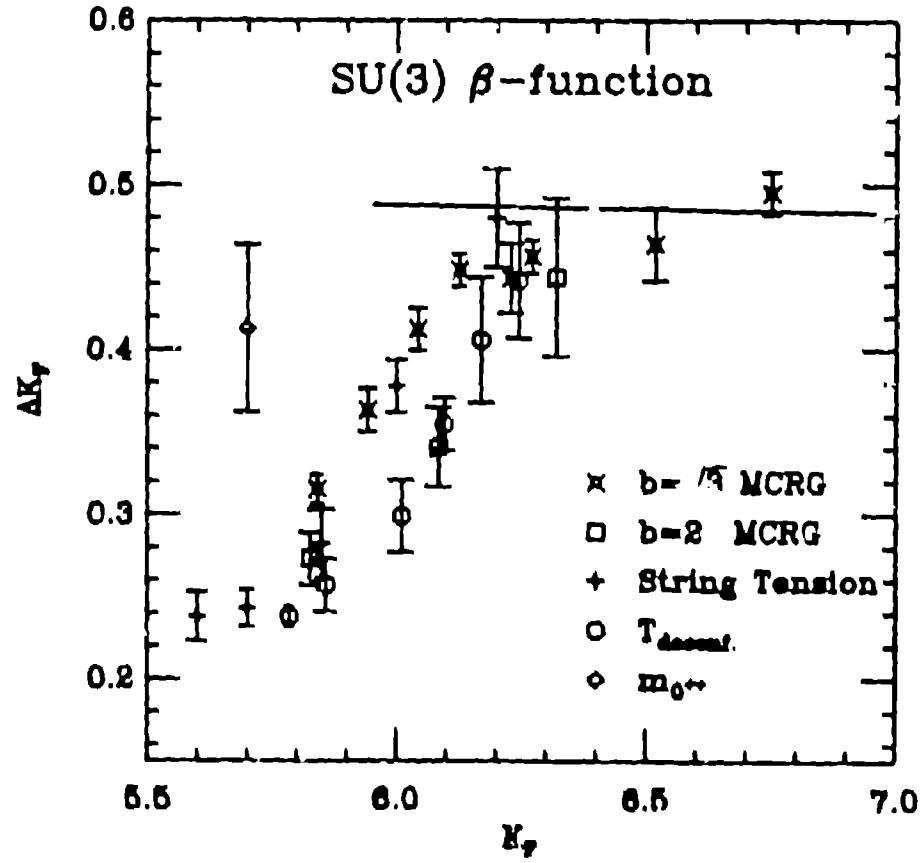

Figure 8: The discrete β-function for a scale change of $\sqrt{3}$.

that are large for $g \sim 1$. Thus these calculations should be used as a guide and the goal should always be to attain constant mass-ratios.

To conclude this section; *MCRG* calculations have not yielded any physical results so far, but they have provided us with a definitive statement on the approach to the continuum limit. This is non-trivial. The present MC determination of σ and the glueball masses need improvement before a definite statement of scaling can be made. The largest lattice calculation of σ by de Forcrand[44] show deviations from asymptotic scaling i.e. $\sqrt{\sigma} = 92\ (79)\ \Lambda_L$ at $\frac{6}{g^2} = 6.0\ (6.3)$. Since these calculations have already taxed the power of a Cray XMP-48, it leads us to the question whether improved actions can help. This is discussed next.

5: DETERMINATION OF THE IMPROVED ACTION.

The advantage of using an improved action in MC simulations is to reduce the effect of operators that lead to scaling violations. In QCD this means that corrections to mass-ratios determined from small lattices can be reduced. Second, we want to avoid regions near singularities where continuum mass-ratios are violated. A known example is the end point of the phase structure in the fundamental-adjoint plane. There are, to the best of my knowledge, 11 methods in existence to calculate the renormalized couplings. All, except for those using perturbation theory (and therefore only valid near $g = 0$ where scheme dependence is negligible), are based on *MCRG*. In fact, since the fixed point and the Renormalized trajectory is a function of the *RGT*, an improved action is content-free unless the *RGT* is specified.

I shall briefly describe the methods, state their advantages and disadvantages and mention results obtained with them. The generic problem of systematic errors in the estimate of the couplings due to a truncation in the number of couplings kept in the analysis will be referred to as "truncation errors". This is a serious drawback because the errors can be very large and there is no way of estimating them without a second long simulation. In order to consider this truncated ansatz to be the best 'fit', a criterion to judge the improvement has to be established. This is discussed after a brief description of the methods. To fix the notation, the pure gauge SU(2) action is written as

$$S = K_F \sum TrU_p + K_{6p} \sum TrU_{6p} + K_A \sum \{\frac{4}{3}(TrU_p)^2 - \frac{1}{3}\}$$
$$+ K_I \sum \{2(TrU_p)^3 - TrU_p\} \qquad (5.1)$$

while the SU(3) action is

$$S = Re\{K_F \sum TrU_p + K_{6p} \sum TrU_{6p} + K_6 \sum \{\frac{3}{2}(TrU_p)^2 - \frac{1}{2}TrU_p\}$$
$$+ K_A \sum \{\frac{9}{8} \mid TrU_p \mid^2 - \frac{1}{8}\}\}\ , \qquad 5.2)$$

Here the higher representations have been constructed from U_p, all the traces are normalised to unity and the sums are over all sites and positive orientations of the loops.

$\Delta\beta$ for SU(3) (16^4 matched with 8^4)		
K_F	$b=2$ *MCRG* method	$b=2$ 1-loop Ratio method
6.0	0.35(2)	0.36(3)
6.3	0.43(3)	0.45(3)
6.6	0.55(9)	

Table 2 : The values of $\Delta\beta$ for a scale change of $b=2$. The results are from Hasenfratz *et al.*[24].

Initial Action K_F	K_F	$\frac{K_A}{K_F}$	$\frac{K_{3/2}}{K_F}$	$\frac{K_{6p}}{K_F}$
2.50 (W)	2.571(05) 2.058(06)	-0.195(01) -0.186(06)	0.043(01) 0.038(03)	-0.0036(003) -0.010(02)
2.75 (W)	3.16(1) 2.815(35)	-0.199(03) -0.214(11)	0.042(02) 0.044(06)	-0.0208(015) -0.019(04)
3.00 (W)	3.69(1) 3.469(48)	-0.190(04) -0.211(12)	0.040(02) 0.039(04)	-0.0314(007) -0.032(03)
3.25 (W)	4.12(2) 4.003(37)	-0.160(05) -0.182(10)	0.025(03) 0.032(06)	-0.0374(004) -0.040(03)
3.50 (W)	4.71(2) 4.396(67)	-0.168(05) -0.150(15)	0.028(03) 0.007(06)	-0.0402(004) -0.049(02)
4.35 (MK)	3.42(1) 3.098(33)	-0.211(02) -0.235(12)	0.044(01) 0.055(04)	-0.0268(011) -0.029(03)

Table 3. Projection of the renormalised SU(2) action onto the $[K_F, K_A, K_{3/2}, K_{6p}]$ space for several starting actions. For each starting action, the first row shows the couplings after one $b=\sqrt{3}$ *RGT* on starting lattices of size 9^4 calculated by the 2-lattice method [23b]. The second row shows the couplings after two *RGT* on starting lattices of size 18^4 calculated using the microcanonical demon method [63]. The last set, $K_F=4.35$, is on the MK trajectory Eq. (5.6).

5.1) Symanzik Program[45]: This is a perturbation theory method to remove all $O(a^2)$ corrections in physical observables. At the tree level, at 1-loop[46] and in the leading log[47] analysis, the $O(a^2)$ corrections are removed by including the 6-link planar loop with strength

$$\frac{K_{6p}}{K_F} = -0.05 \quad . \tag{5.3}$$

There have been some SU(3) calculations[36] done with this action, but they are inconclusive and no statement for an improvement in mass-ratios can be made.

5.2) Block Spin Renormalization Group (perturbation theory): The first work in this direction is by Wilson[1] who wrote down the ansatz

$$\frac{K_{6p}}{K_F} = -0.0576 \quad , \quad \frac{K_{6t}}{K_F} = -0.0388 \quad , \tag{5.4}$$

where K_{6t} is the twisted 6-link coupling. No calculation of physical observables has been done with this action. The group of Iwasaki *et al.*[48] have made a large independent effort in this direction of improvement. They find that near $g = 0$ the action after 3 *RGT* can be approximated by including the 6-link planar loop with strength

$$\frac{K_{6p}}{K_F} \approx -\frac{0.331}{3.648} \quad . \tag{5.5}$$

They show that for both the Wilson ansatz, Eq. (5.4), and for this action instantons are stable on the lattice. Since this is not true of the simple plaquette action, they regard it as another criterion for improvement. They have recently calculated the string tension and the hadron masses in the quenched approximation using the improved gauge action of Eq. (5.5) and the standard Wilson action for the quark propagator on a $12^3 \times 24$ lattice at an effective $\frac{6}{g^2} \sim 5.9$. Their results for mass ratios are impressive. A comparison with an equivalent calculation on the Wilson axis is limited because a number of parameters are different. The accuracy of their results warrants more attention.

5.3) Migdal-Kadanoff Recursion Technique: This calculation[49] is limited to the plaquette in the fundamental and higher representations. The integration over links is done by expanding the action in terms of the characters and then using the recursion formula. In the improved action, the effect of the singularity in the fundamental adjoint plane is reduced but the leading irrelevant coupling K_{6p} is not included. For SU(2)[49], the convergence in the character expansion was good, the recursion was stable on keeping 20 characters. The improved action is dominated by the spin 1 and 3/2 representations, and the K-M improved trajectory was approximated by

$$\frac{K_A}{K_F} = -0.24 \quad . \tag{5.6}$$

It was later shown by Bitar *et al.*[50] that the heat Kernal action works very well in the recursion scheme and in fact is the solution in the perturbative limit. For a SU(2) calculation of the β-function along the K-M improved trajectory $K_A = -0.24 K_F$, and for an analysis of the improved action see Ref. 22b.

5.3b) Phenomenological (Lines Of Constant String Tension): The continuum limit is taken along directions perpendicular to the lines of constant string

tension in the negative fundamental-adjoint plane. Rebbi[52] *et al.* have measured the $q\bar{q}$ potential, while Samuel[53] has spearheaded a calculation with scalar quarks. The effective coupling for comparison on the Wilson axis is defined by using the large N resummation technique[53a]. Since no direct comparison has been made it is hard to state if better mass ratios are obtained.

5.4) Swendsen's method[54] using the Callen representation: The block expectations values of Wilson loops are calculated in two ways. First as simple averages over block configurations, and second using the Callen representation[55] with a guess for the block couplings. From these two estimates, the block couplings are determined iteratively. The method is fast and easy to implement. It does have undetermined truncation errors. Lang[56] has used this method to show that the quartic coupling $\lambda\phi^4$ in the self-interacting scalar field theory renormalizes to zero. Recently Burkitt[35] has used it to map the flow of the action under the $b = 2$ *RGT* (section 2.2) for the U(1) model. From a difference in the flows he can estimate the transition point on the Wilson axis. It would be instructive to extend the U(1) analysis to $\pm\gamma$ coupling values along the phase transition line and check if there is a qualitative change at the TCP.

5.5) Callaway-Petronzio-Wilson[57,58] method of fixed block spins: This method is useful for discrete spin systems like the Ising model and models in the same universality class. A *MCRG* calculation is modified by fixing all the block spins except one such that only a controllable few block interactions are non-zero. The system is simulated with the *RGT* used as an additional weight in the Metropolis algorithm. The ratio of probability of this unfixed spin being up to it being down is equal to a determined function of a certain number (depending on how many block interactions are non-zero) of block couplings. By using different configurations of fixed block spins a system of linear equations is set up from which the block couplings are determined. The drawback of this method, even for the Ising model, is that it is hard to set up the block spins so that only a few (≈ 10) block interactions are nonzero. Wilson showed that this can be done if one uses the lattice gas representation *i.e.* 0 or 1 for spin values. The couplings in the ± 1 representation are then given by an expansion in the lattice gas couplings. This expansion has been shown to converge rapidly for the $d = 2$ Ising model. The second improvement due to Wilson is that instead of a MC determination of the ratio of probabilities, the exact result can be obtained in the transfer matrix formalism. The drawback here is that a Vax 11/780 type machine can handle only up to 12 spins. However its non-statistical nature makes it useful as a test.

5.6) Character Expansion method of Bitar[59]: I will describe this method with a restriction to simple plaquette actions. The character expansion for the action is $S = \sum_p \sum_r K_r \chi_r(U_p)$ where χ_r is the character in the r^{th} representation and K_r is the corresponding coupling. Similarly the Boltzmann factor F_p for each plaquette p can be expanded in a character expansion $F_p = \sum_r d_r f_r \chi_r(U_p)$ where d_r is the dimension and f_r the coefficient for r^{th} representation. The couplings K_r are given by

$$K_r = \int d(U_p)\, ln F_p(U_p)\, \chi_r(U_p)\,. \qquad (5.7)$$

The crucial step is that the ratio $\frac{d_r f_r}{f_1}$ can be calculated as a ratio of expectation val-

ues over block configurations. From this the Boltzmann factor F_p and consequently K_r can be determined. The method is sensitive to the convergence of the character expansion i.e. the number of terms in r needed to determine F_p accurately. After this there are no truncation errors in determining K_r. The method grows in complexity if larger loops are to be included in the analysis. The first results[59] for the simple plaquette action in SU(2) are encouraging.

5.7) The Schwinger-Dyson Equation method of Falconi *et al.*[60]: In this method the lattice Schwinger-Dyson equations (equations of motion for expectation values of n-point functions) are used to write down a set of inhomogeneous linear equations for the couplings. The coefficients and the inhomogeneous term are given in terms of expectation values of n-point functions. In deriving these equations the action has to be truncated to the subspace of couplings to be determined. Thus the method has truncation errors. Preliminary results for the O(3) non-linear σ-model in $d = 2$ are encouraging.

5.8) 2-Lattice *MCRG* method[61,6]: The calculation steps are the same as Wilson's 2-Lattice method to determine the β-function. The method consists of expanding the block expectation values (with unknown couplings) about those from a simulation with known couplings. Keeping just the linear term in the expansion gives the difference between the two sets of couplings. The main advantage is that this comes free with the calculation of the β-function. The method has a statistical drawback that it requires two different simulations so there is no possibility of cancellation of statistical errors. Also, far from the RT, only the first renormalized couplings can be determined accurately. There exist extensive calculations for both the SU(2) and the SU(3) models using the $\sqrt{3}$ RGT. The estimate for the improved action in a 4-parameter space for SU(2) is[22]

$$\frac{K_{6p}}{K_F} = -0.06 \quad , \quad \frac{K_A}{K_F} = -0.19 \quad , \quad \frac{K_{\frac{1}{2}}}{K_F} = 0.03 \tag{5.8}$$

and for SU(3) is[37]

$$\frac{K_{6p}}{K_F} = -0.04 \quad , \quad \frac{K_8}{K_F} = -0.12 \quad , \quad \frac{K_6}{K_F} = -0.12 \, . \tag{5.9}$$

The truncation errors are known to be large and the reliability of the results is being tested by using the estimated improved action in the update and repeating the calculation of the β-function and the improved action. Also the hadron spectrum is being calculated to test if better mass-ratios are obtained. A detailed comparison of the results for the renormalized action is made with the microcanonical method discussed next.

5.9) Microcanonical (Creutz's Demon) Method[62]: This method is very efficient if from a previous *MCRG* calculation expectation values of n block Wilson loops at each of the l block levels are determined. To determine the corresponding couplings at the l^{th} level, a microcanonical simulation is then done (on a same size lattice as on which the block expectation values were calculated) with the corresponding n energies fixed and with one demon per interaction. The desired n couplings are then determined from the distribution of demon energies. P. Stolorz[63]

at Caltech used the block expectations values obtained after two applications of the $\sqrt{3}$ *RGT* for SU(2). From these he obtained the second, ($l = 2$), renormalized action in a truncated coupling constant space (four couplings of Eq(5.1)). The results are shown in Table 3 and compared with the first renormalized couplings obtained from the 2-Lattice *MCRG* method described above. The results show a rapid convergence of the action to the *RT* consistent with the estimates given in Eqs (5.8). This is evidence that the $\sqrt{3}$ *RGT* transformation has good convergence properties after two steps. In this calculation it was easy to thermalize the four energies. The simulation is faster than the 2-Lattice method and has better statistical properties. Also the block couplings at all levels can be determined once the block expectation values are known. The truncation errors are the same as in the 2-Lattice method.

5.10) Block Diagonalization method of Mutter and Schilling[64]: This is at present the only method that attempts to improve both the gauge and the fermion action. The main idea is that quark propagators are calculated on blocked gauge configurations using a blocked fermion action. The blocked fermion action is calculated as follows: Let the starting action be the Wilson action

$$\Psi\, M\, \Psi \quad , \tag{5.10}$$

where M is the interaction matrix. The lattice is now divided into blocks which for the $\sqrt{3}$ *RGT* contain 9 sites each. The site action is then cast into a block action

$$\Xi\, \Gamma\, \Xi \tag{5.11}$$

where Ξ is a 9 component Dirac fermion field and Γ is the interaction matrix set up to reproduce Eq. (5.10). The mass term part of Γ, Γ_m, is now diagonalized to provide the non-interaction fermion basis vectors. For the $\sqrt{3}$ *RGT*, the 9 eigenvalues of Γ_m are 0 and 8 degenerate ones with value $\frac{9}{a}$. The light mode alone is kept on the blocked lattice. The interaction between the light and heavy modes is calculated in perturbation theory and these terms are added to the Wilson action to give the improved fermion coupling matrix. It is to be noted that this fermion diagonalization is only approximate. Thus lattice masses will not a priori change by the scale factor b between the original and the blocked lattice. It is therefore necessary to first check how good the transformation is in preserving mass-ratios of the unblocked system. The results on a twice blocked set of configurations using $b = 2$ are encouraging[65]. Results of a test of preservation of mass ratios under blocking should be available soon for both the $b = 2$ and $b = \sqrt{3}$ *RGT*. At this point it is worth mentioning that the following advantages were observed in the diagonalization process for the $\sqrt{3}$ *RGT* in comparison to $b = 2$.

(a) The separation between the light modes $m \sim 0$ and the heavy modes is better i.e. $\frac{9}{a}$ versus $\frac{2}{a}$, so the perturbative corrections are more reliable.
(b) Rotational invariance is not broken as is in the $b = 2$ transformation.
(c) No closed gauge loops which manifest themselves as additional contact terms in the fermion operators arise. This implies that the value of the Wilson parameter r does not get modified and κ_c would remain the same on the blocked lattice for the Wilson fermions if the exact fermion coupling matrix was used in the calculations.
(d) The blocking of gauge links is the same as defined in section 2.3.

Discussion: There are some features of the improved action that seem common to the various analysis done. The details will certainly depend on the specific *RGT*.

(a) The leading irrelevant operator is dominated by K_{6p}, the $6-link$ planar Wilson loop. Thus a *RGT* that kills it is an improvement.

(b) From the $\sqrt{3}$ *RGT* analysis, one gets an estimate of $\frac{K_A}{K_F} \sim \frac{K_a}{K_F} \sim -0.12$. Thus near $\frac{6}{g^2} = 6.$, the phase structure in the $\{K_F,\ K_A\}$ plane is avoided. This is necessary because in the vicinity of the end point of the phase structure universality is violated.

(c) The *RT* for the $b = \sqrt{3}$ *RGT* shows significant deviations from linearity in the region accessible to Monte Carlo. The ratios given in Eqs. (5.8) and (5.9) are an estimate of the asymptotic behavior.

(d) The *RT* out of the fixed point is local *i.e.* dominated by small loops. The Wilson axis is tangent to the strong coupling *RT* at the trivial fixed point at $K_\alpha = 0$. The change from the weak coupling *RT* to flow close to the Wilson axis takes place in the region where current Monte Carlo calculations have been done *i.e.* between 5.7 and 6.5. This feature needs to be investigated since current mass-ratios show a behavior that is in between strong coupling and the expected continuum one.

It is still necessary to evaluate whether constant mass-ratios in the quenched approximation are obtained earlier with an improved action. The results have to justify the factor of ~ 5 by which the gauge update slows down when the above four couplings are used. The key lies in improving the fermion sector. For dynamical quarks, the gauge update is a small fraction of the update time. So, an investment in improving the action is justified.

6: IMPROVED MONTE CARLO RENORMALIZATION GROUP[66]

I shall review the Gupta-Cordery Monte Carlo Renormalization Group method (*IMCRG*) in some detail. In this method the Renormalized Hamiltonian and the Linearized Transformation Matrix, *LTM*, are determined without any truncation errors. There are no long time correlations even on the critical surface and the block n-point correlation functions like $\langle S^1_\alpha S^1_\beta\rangle - \langle S^1_\alpha\rangle\langle S^1_\beta\rangle$ are calculable numbers. Also, the method allows a careful error analysis in the determination of the renormalized couplings and in the *LTM*.

In the *IMCRG* method the configurations $\{s\}$ are generated with the weight

$$P(s^1,s)e^{-H(s)+H^g(s^1)} \tag{6.1}$$

where H^g is a guess for H^1. Note that both the site and block spins are used in the update of the site spins. In analogue to Eq. (1.2), the distribution of the block spins is given by

$$e^{-H^1(s^1)+H^g(s^1)} = \sum P(s^1,s)e^{-H(s)+H^g(s^1)} \ . \tag{6.2}$$

If $H^g = H^1$, then the block spins are completely uncorrelated and the calculation of the n-point functions on the block lattice is trivial.

$$\langle S^1_\alpha\rangle = 0 \qquad \langle S^1_\alpha S^1_\beta\rangle = n_\alpha\delta_{\alpha\beta} \qquad \ldots \tag{6.3}$$

where for the Ising model (and most other models) the integer n_α is simply a product of the number of sites times the multiplicity of interaction type S_α. When $H^\theta \neq H^1$, then to first order

$$\langle S_\alpha^1 \rangle \;=\; \langle S_\alpha^1 S_\beta^1 \rangle_{H^\theta = H^1} \; (K^1 - K^\theta)_\beta \tag{6.4}$$

and using Eq. (6.3), the renormalized couplings $\{K_\alpha^1\}$ are determined with no truncation errors as

$$K_\alpha^1 \;=\; K_\alpha^\theta \;+\; \frac{\langle S_\alpha^1 \rangle}{n_\alpha} \quad . \tag{6.5}$$

This procedure can be iterated -- use H^{n-1} as the spin H in Eq. (6.1) to find H^n. If the irrelevant eigenvalues are small, then after two or three repetitions of the RGT, the sequence H^n converges to the fixed point Hamiltonian H^* which is assumed to be short ranged. For the $d = 2$ Ising model, the method has been shown to be extremely stable[67]. The only limitations of this method are the linearity approximation, Eq. (6.4), (this is trivially handled by iterating H^θ) and the use of a truncated H^{n-1} for the spin Hamiltonian in the update to find H^n. The second limitation can be overcome and the solution is straightforward: In Eq. (6.1) use H^θ as the guess for H^n. The update now involves the original spins and all block spins up to the n^{th} *level* in the Boltzmann weight

$$P(s^n, s^{n-1}) \;\ldots\ldots\; P(s^1, s) e^{-H(s) + H^\theta(s^n)} \quad . \tag{6.6}$$

The four Eqs. (6.2-6.5) are unchanged except that the *level* superscipt is replaced by n, i.e. the n^{th} *level* block-block correlation matrix is diagonal and given by Eq. (6.3). With this modification, the H^n is calculated directly. The limitation on n is the size of the starting lattice. Such a check is necessary because errors in long range couplings due to finite statistics and the effects of a truncation in the spin H^{n-1} get magnified and the system rapidly flows away from the fixed point.

The calculation of the LTM proceeds exactly as in the standard $MCRG$ i.e. Eqs. (1.4) to (1.6). However, in the limit $H^\theta = H^1$, the block-block correlation matrix is diagonal and given by Eq. (6.3). Thus it has no truncation errors, can be inverted with impunity and the final LTM elements are also free of all truncation errors. The only error is in finding the eigenvalues from a truncated matrix. These errors can be estimated and the results improved as explained below.

$IMCRG$ is therefore more complicated than $MCRG$ and requires a simultaneous calculation of a many term $H(s)$ and $H^\theta(s^1)$ at update. However, the system does not have critical slowing down. Secondly, the correlation length ξ can always be made of $O(1)$, so finite size effects are dominated by the range of interactions, which by assumption of a short range H^* fall off exponentially. Thus, critical phenomenon can be studied on small lattices and with no hidden sweep to sweep correlations that invalidate the statistical accuracy of the results.

6.1: Truncation Errors In The LTM

Consider the matrix equation for T in block form

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \;=\; \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \tag{6.7}$$

where D_{11} and U_{11} are the 2 derivative matrices calculated in some truncated space of operators that are considered dominant. The elements of the sub-matrix T_{11} will have no truncation errors provided we can calculate

$$T_{11} = D_{11}^{-1} \{U_{11} - D_{12}T_{21}\} . \tag{6.8}$$

In the *IMCRG* method the matrix D is diagonal and known, so D_{12} is 0. Thus elements of T_{11} determined from U_{11} have no truncation errors. The errors in the eigenvalues and eigenvectors arise solely from diagonalizing T_{11} rather than the full matrix T. Calculations in the $d = 2$ Ising model have shown that these errors are large, i.e. of order 10%, if all operators of a given range are not included. An open problem right now is a robust criterion for classifying operators into sets such that including successive sets decreases the truncation error geometrically by a large factor.

The errors arising from using a sub-matrix T_{11} can be reduced significantly by diagonalizing

$$T_{11} + T_{11}^{-1}T_{12}T_{21} = D_{11}^{-1} U_{11} + \{-D_{11}^{-1}D_{12} + T_{11}^{-1}T_{12}\} T_{21} \tag{6.9}$$

as shown by Shankar, Gupta and Murthy[68]. The correction term $T_{11}^{-1}T_{12}T_{21}$ is the 2^{nd} order perturbation result valid for all eigenvalues that are large compared to those of T_{22}. This correction matrix can be calculated in *IMCRG* from $(T^2)_{11} - (T_{11})^2$. I am here overlooking the errors due to the *RG* flow, because of which T^2 is evaluated at a different point than T. Another aspect of these errors is their behavior as a function of how close to H^* the calculation is done. For the $d = 2$ Ising model we[66,69] find that the truncation errors in the relevant eigenvalues are large. Adding more operators does not monotonically decrease the error. The fluctuations can be as large as 2% even after the 20 largest operators are included in T_{11}.

In standard *MCRG*, the calculations with $T_{11} = D_{11}^{-1}U_{11}$ have shown good convergence once few operators, $O(5-10)$, are included in T_{11}. The reason for this is an approximate cancellation of a term ignored and the correction term. Using Eq. (6.7), ignoring terms with T_{22} and approximating $T_{11} = D_{11}^{-1}U_{11}$ we get

$$-D_{11}^{-1}D_{12} + T_{11}^{-1}T_{12} \sim -D_{11}^{-1}D_{12} + U_{11}^{-1}U_{12} .$$

Further, these derivative matrices are roughly proportional, i.e. $U \sim \lambda_t D$ and the corrections fall off as the ratio of non-leading eigenvalues to the leading one λ_t. This follows from the arguments of section 1.1 and can be checked by expanding operators in term of eigenoperators. Thus Swendsen[7] by calculating just $D_{11}^{-1}U_{11}$ and ignoring all truncation problems was effectively incorporating a large part of the perturbative correction piece. This explains his success. Shankar[70] has found a correction term to further decrease the truncation effects in *MCRG*. However, given the assumptions, the flow under a *RG* and the success of the procedure as it exists, an improvement may be hard to evaluate.

Thus, at present the best way to get accurate results is to use *IMCRG* to calculate the Renormalized couplings and Swendsen's *MCRG* method to calculate the eigenvalues. The topics that need more work are the accuracy of perturbative improvement in *IMCRG*, the classification of interactions into complete sets and a quantitative understanding of the tuning of the *RGT*.

Let me also summarize some of the other results obtained from the study of the $d = 2$ Ising model.

[1] The LTM has elements that grow along rows and fall along columns[68], therefore it can be arranged to look like

$$\begin{pmatrix} A & B \\ \epsilon & D \end{pmatrix} \tag{6.10}$$

with A the minimal truncated $n \times n$ block matrix that should be calculated. The case $\epsilon = 0$ is simple; there are no truncation errors in either method and diagonalizing A gives the n largest eigenvalues. Otherwise for $IMCRG$ the truncation error depends on the dot product of terms in ϵ and B. The requirement of absolute convergence in the dot product only guarantees that this product is finite but it may be arbitrarily large *i.e.* $O(1)$. Therefore for each model, a careful study of the signs and magnitude of the elements in ϵ as a function of the RGT becomes necessary. This is being done at Cornell[67].

[2] The leading left eigenvector is normal to the critical surface[68]. Its elements give an estimate of the growth in the elements along the rows of the LTM.

[3] Using H^0 as the known nearest-neighbor critical point $K^c_{nn} = 0.44068$, the $IMCRG$ results[67] for H^1 are independent (within statistical accuracy) of finite size effects for lattice sizes 16, 32, 64 and 128.

[4] The results for H^n converged provided the couplings in H^0 were correct to $O(10^{-3})$. This initial accuracy can be achieved[67] with a few thousand sweeps on a 128^2 lattice.

[5] The statistical errors in $IMCRG$ can be evaluated very reliably[67]. Detailed binning analysis showed that each sweep is approximately independent and an accuracy of 10^{-5} is obtained in all couplings with $\sim 2 \cdot 10^6$ sweeps on a 64^2 lattice. This could be achieved with 3000 Vax 11/780 hours.

To conclude, I believe that $IMCRG$ provides a complete framework to analyze the critical behavior of spin and gauge models. With the increased availability of supercomputer time we shall have very accurate and reliable results.

7: EFFECTIVE FIELD THEORIES

The point of effective field theories is that physical phenomena at some given length scale can be described by some effective/composite degrees of freedom. The couplings between these variables are determined by the underlying microscopic theory. Thus we would like to know these effective degrees of freedom and the couplings. So far the discussion of $MCRG$ has focused on the change of scale without a change of variables. To make full use of its power, a transformation of variables at the appropriate scale should be added *i.e.* in addition to a RGT that just averages over degrees of freedom, consider a change from the microscopic theory to an effective theory with new variables at some give length scale. These variables can be composite (as is the case in going from QCD to a theory where the degrees of freedom are hadrons) or represent a freezing as in SU(2) at high temperatures where the interaction between the Wilson lines is described by an effective $d = 3$ Ising model. Here one transforms from link variables to Wilson lines to Ising spins.

Once the effective theory has been constructed, it is important to know the universality class to which it belongs. This would provide a detailed knowledge of the critical/long distance behavior. Little work has been done in actually exploring universality classes by mapping flows that incorporate a change of variables.

The way to do this in standard MC is to define the composite degrees of freedom and their n-point functions in terms of the microscopic variables. From the expectation values of these n-point correlation functions calculated as simple averages, the corresponding couplings can then be determined by a Microcannonical simulation as described in section 5.9. One such calculation is by Ogilvie and Gocksch[70] in which they determine the nearest neighbor couplings between the Wilson lines in SU(2).

In $MCRG$, the transformation from the microscopic degrees of freedom to the composite variables is made on the original lattice (same as in MC). The RGT is defined on the composite variables and the critical exponents of the effective theory are calculated from the LTM. The couplings can be determined by one or more of the methods of section 5. This process also maps the universality class. Similarly, $IMCRG$ can be used provided H^0 is a guessed hamiltonian for the effective theory. This subject is being actively pursued in collaboration with A. Patel, C. Umrigar and K. G. Wilson and we hope it will blossom.

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REFERENCES

[1] K. G. Wilson, in *Recent Developments in Gauge Theories*, Cargese (1979), eds. G. 't' Hooft, *et al.* (Plenum, New York, 1980).

[2] R. H. Swendsen, Phys. Rev. Lett. **42**, (1979) 859

[3] S. H. Shenker and J. Tobochnik, Phys. Rev. **B22** (1980) 4462.

[4] S. K. Ma, Phys. Rev. Lett. **37**, (1976) 461.

[5] L. P. Kadanoff, Rev. Mod. Phys. **49**, (1977) 267.

[6] K. G. Wilson, in *Progress in Gauge Field Theories*, edited by G. 't Hooft *et al.*, (Plenum, New York 1984).

[7] R. H. Swendsen, in *Real Space Renormalization, Topics in Current Physics*, Vol 30, edited by Th. W. Burkhardt and J. M. J. van Leeuwen (Springer, Berlin, 1982) pg. 57.

[8] J. M. Drouffe and C. Itzykson, Phys. Reports **38** (1978) 133.
J. B. Kogut, Rev. Mod. Phys. **51** (1979) 659 and **55** (1983) 775.
M. Creutz, L. Jacobs and C. Rebbi, Phys. Rep. **95**, (1983) 201.
M. Creutz, *Quarks, Gluons and Lattices* , Cambridge Univ. Press (1984).
J. M. Drouffe and J. B. Zuber, Phys. Reports **102** (1983) 1.

[9] K. Binder, in *Monte Carlo Methods in Statistical Physics*, edited by K. Binder (Springer, Berlin,1979) Vol 7 , and in *Applications of Monte Carlo Methods in Statistical Physics*, (Springer Verlag, Heidelberg, 1983).

[10] I have taken the liberty to use as synonymous the terms action and Hamiltonian since the meaning is clear from the context.

[11] K. G. Wilson and J. Kogut, Phys. Rep. **12C**, (1974) 76.

[12] P. Pfeuty and G. Toulouse, *Introduction to the Renormalization Group and Critical Phenomenon*, (John Wiley & Sons, New York 1978).

[13] D. Amit, *Field Theory, the Renormalization Group and Critical Phenomenon*, (World Scientific, 1984).

[14] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller and E. Teller, J. Chem. Phys. **21** (1953) 1087.

[15] M. Creutz, Phys. Rev. D **21** (1980) 2308.

[16] D. Callaway and A. Rehman, Phys. Rev. Lett. **49** (1982) 613.
M. Creutz, Phy. Rev. Lett. **50** (1983) 1411.
J. Polonyi and H. W. Wyld, Phys. Rev. Lett. **51** (1983) 2257.

[17] G. Parisi and Wu Yongshi, Sci. Sin. **24** (1981) 483.

[18] G. G. Batrouni, G. R. Katz, A. S. Kornfeld, G. P. Lapage, B. Svetitsky, and K. G. Wilson, Cornell Preprint CLNS-85(65), May 1985.

[19] A. Brandt, in *Multigrid Methods*, Lecture Notes in Math 960, (Springer Verlag 1982) and references therein.

[20] The idea was first discussed by G. Parisi in *Progress in Gauge Field Theories*, edited by G. 't Hooft, *et al.*, (Plenum, New York, 1984).

[21] R. Swendsen, Phys. Rev. Lett. **47** (1981) 1775.

[22] R. Cordery, R. Gupta and M. A. Novotny, Phys. Lett. B **128** (1983) 425.
R. Gupta and A. Patel, Nucl. Phys. B **251** (1985) 789.

[23] D. Callaway and R. Petronzio, Phys. Lett. B (1985) .
This transformation was also known to R. Gupta, B. Svetitsky and K. G. Wilson but not pursued in favor of the first three.

[24] K. C. Bowler, A. Hasenfratz, P. Hasenfratz, U. Heller, F. Karsch, R. D. Kenway, I. Montvay, G. S. Pawley, and D. J. Wallace, Nucl. Phys. **B257** (1985) [FS14] 155, and
A. Hasenfratz, P. Hasenfratz, U. Heller, and F. Karsch, Phys. Lett. **140B** (1984) 76.
[25] A more careful statement is made by M. E. Fischer and M. Randeria, Cornell Note (1985).
[26] R. H. Swendsen, Phys. Rev. Lett. **52** (1984) 2321.
[27] G. Bhanot, Nucl. Phys. **B205** (1982) 168.
[28] H. G. Evertz, J. Jersák, T. Neuhaus and P. M. Zerwas, Nucl. Phys. **B251** (1985) 279.
[29] The presence this phase transition was shown analytically by A. Guth, Phys. Rev. **D21** (1980) 2291; and by J. Frolich and T. Spencer, Commun. Math. Phys. **83** (1982) 411.
[30] T. Banks, R. Mayerson and J. Kogut, Nucl. Phys. **B129** (1977) 493.
[31] T. A. DeGrand and D. Toussaint, Phys. Rev. **D22** (1980) 2478.
[32] R. Gupta, M. A. Novotny and R. Cordery, Northeastern Preprint 2654 (1984). A condensed version to appear in Phys. Lett. B
[33] J. Barber, Phys. Lett. B **147** (1984) 330.
[34] V. Grosch, K. Jansen, J. Jérsak, C. B. Lang, T. Neuhaus and C. Rebbi, CERN preprint 4237/85.
[35] A. N. Burkitt, Liverpool preprint LTH 138 October (1985).
[36] P. de Forcrand and C. Roiesnel, Phys. Lett. **137B** (1984) 213, and **143B** (1984) 453.
P. de Forcrand, Ecole Polytechnique preprint A615.0784 (July 1984).
[37] R. Gupta, G. Guralnik, A. Patel, T. Warnock and C. Zemach, Phys. Rev. Lett. **53** (1984) 1721.
R. Gupta, G. Guralnik, A. Patel, T. Warnock and C. Zemach, Phys. Lett. **161B** (1985) 352.
[38] M. Creutz, Phys. Rev. **D23** (1981) 1815.
[39] F. Gutbrod, DESY preprint 85-092 (1985).
[40] D. Petcher, Private Communication.
[41] P. de Forcrand, G. Schierholz, H. Schneider and M. Taper, Phys. Lett. **143B** (1985) 107.
[42] S. W. Otto and J. Stack, Phys. Rev. Lett. **52** (1984) 2328.
D. Barkai, K. J. M. Moriarty and C. Rebbi, Phys. Rev. **D30** (1984) 1293.
[43] S. Gottlieb, A. D. Kennedy, J. Kuti, S. Meyer, B. J. Pendleton, R. Sugar and D. Toussaint, Phys. Rev. Lett. **55** (1985) 1958.
[44] P. de Forcrand, in the proceedings of this conference.
[45] K. Symanzik, in *Proceedings of the Trieste workshop on non perturbative field theory and QCD (dec. 1982)*, World Scientific (1983) 61.
[46] P. Weisz and R. Wohlert, Nucl. Phys. **B236** (1984) 397.
[47] G. Curci, F. Menotti and G. P. Paffuti, Phys. Lett. **130B** (1983) 205.
[48] Y. Iwasaki, preprint UTHEP-118 (1983).
S. Itoh, Y. Iwasaki and T. Yoshie, preprint UTHEP-134 and UTHEP-146.
[49] K. M. Bitar, S. Gottlieb and C. Zachos, Phys. Rev **D26** (1982) 2853.
[50] K. Bitar, D. Duke and M. Jadid, Phy. Rev. **D31** (1985) 1470.
[51] D. Barkai, K. J. M. Moriarty and C. Rebbi, Phys. Rev. **D30** (1984) 2201.

[52] O. Martin, K. Moriarty and S. Samuel, Nucl. Phys. **B261** (1985) 433.
[53] B. Grossmann and S. Samuel, Phys. Lett. **120B** (1983) 383.
A. Gonzales Arroyo and C. P. Korthals Altes, Nucl. Phys. **B205** [FS5] (1982) 46.
R. K. Ellis and G. Martinelli, Nucl. Phys. **B135** [FS11] (1984) 93.
[54] R. H. Swendsen, Phys. Rev. Lett. **52** (1984) 1165.
[55] H. B. Callen, Phys. Lett. **4B** (1961) 161.
[56] C. Lang in *The Proceedings of the Tallahassee Conference on Advances in Lattice Gauge Theory,* (World Scientific, 1985).
[57] D. Callaway and R. Petronzio, Phys. Lett. **139B** (1984) 189.
[58] K. G. Wilson and C. Umrigar, unpublished.
[59] K. Bitar, FSU preprint SCRI-85-7 (1985).
[60] M. Falcioni, G. Martinelli, M. L. Paciello, G. Parisi, B. Taglienti, Nucl. Phys. **B265** (1986) [FS15] 187.
[61] R. Gupta and A. Patel, Phys. Rev. Lett. **53** (1984) 531 and R. Gupta and A. Patel in *Proceedings of the Argonne Conference on Gauge Theory On a Lattice,* (1984).
[62] M. Creutz, A. Gocksch, M. Ogilvie and M. Okawa, Phys. Rev. Lett. **53** (1984) 875.
[63] P. Stolorz, Caltech Preprint CALT-68-1323 (1986).
[64] K. H. Mutter and K. Schilling, Nucl. Phys. **B230** [FS10] (1984) 275.
[65] A. Konig, K. H. Mutter and K. Schilling, Nucl. Phys. **B259** (1985)33, and in this Conference's Proceedings.
[66] R. Gupta and R. Cordery, Phys. Lett. **A105** (1984) 415.
[67] R. Gupta in *Proceedings of the Tallahassee Conference on Advances in Lattice Gauge Theory,* World Scientific (1985).
[68] R. Shankar, R. Gupta and G. Murthy, Phys. Rev. Lett. **55** (1985) 1812.
[69] R. Gupta, C. Umrigar and K. G. Wilson, under progress.
[70] R. Shankar, Yale preprint YTP 85-25 Nov. (1985).
[71] A. Gocksch and M. Ogilvie, Phys. Rev. Lett. **54** (1985) 1985.