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WEAK- AND STRONG-TURBULENCE REGIMES
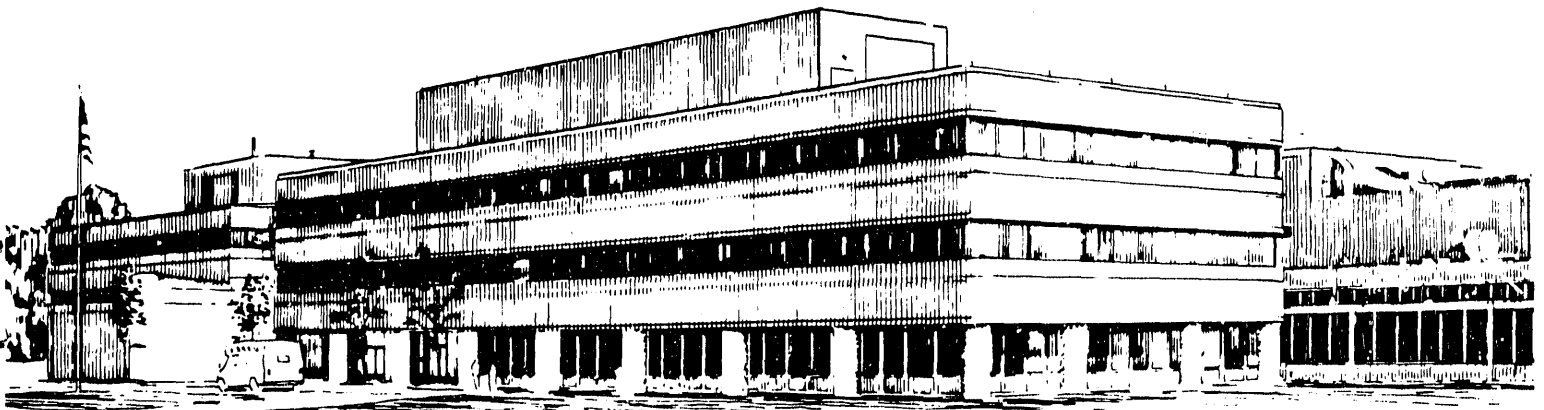# OF THE HASEGAWA-MIMA EQUATION

BY

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February 1992

PPPL

PRINCETON
PLASMA PHYSICS
LABORATORY

PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY

Weak- and strong-turbulence regimes of the Hasegawa–Mima equation

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A Kolmogorov-type analysis of the energy- and enstrophy-cascading ranges of the forced Hasegawa–Mima equation allows one to derive a criterion for the threshold of the transition between the weak turbulence and the strong turbulence regimes. It is found that, due to the inverse energy cascade, the large-scale portion of the inertial range is in the strong turbulence regime in the limit of infinite Reynolds-like numbers.

PACS numbers: 52.35.Ra, 47.25.-c

Among the tools employed to analyze the various reduced models of plasma turbulence, the weak turbulence approximation [1] (WTA) is perhaps the most popular. Formally, the WTA for a given nonlinear model is introduced as an expansion in terms of the magnitude of the coupling coefficients [2] or as a multiple-time-scale expansion [3]. The physical WTA expansion parameter, however, turns out to be a dimensionless quantity representing the ratio of the rate of energy injection to some measure of wave dispersion. For example, if the turbulence model involves just a single scalar field (as in the case of the

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two-dimensional Navier-Stokes equation or the Hasegawa–Mima equation [4], the proper WTA expansion parameter ϵ_{WT} is

$$\epsilon_{WT} \doteq \gamma_k/\Delta\omega_k,$$

where γ_k is the growth rate of a given instability as a function of the wavenumber k and $\Delta\omega_k \doteq \omega_k - k \cdot v_{g,k}$, with ω_k being the wave frequency and $v_{g,k}$ being the group velocity, is a measure of wave dispersion.

It is generally assumed that the condition that ϵ_{WT} is somewhat smaller than one is sufficient for the applicability of the WTA. Moreover, in many applications the spectrum of unstable modes, driven by γ_k, is localized to scales substantially smaller than the size of the system. One typically considers a region in wavenumber space around $k_\perp \rho_i \approx 1$, where ρ_i is the ion gyroradius. Since the long-wavelength modes are stable or only weakly unstable, one generally concludes that large-scale turbulent dynamics are appropriately described by weak turbulence equations.

Implicit to this viewpoint is the assumption that the shape of the turbulent spectrum somehow resembles the profile of the instability growth rate. Therefore, in the wavenumber region where ϵ_{WT} is significantly smaller than unity nonlinear interactions are assumed to be small compared to wave dispersion and wavelike phenomena are considered to be dominant.

That this line of thought is at least questionable can be understood by noting that the primary effect of nonlinear interactions, which are generally conservative, is to transfer

energy to scales different from those into which it is injected. Therefore, even in regions of negligible growth rate the fluctuation level can be so high that the dynamics are dominated by nonlinearity. The conclusion is that in order to assess the validity of the weak turbulence approach one must perform a more detailed analysis of the nonlinear dynamics.

In this letter we carry out such analysis for a specific model, the forced Hasegawa–Mima (HM) equation [4], the paradigm for a large class of plasma turbulence models. The main conclusion will be that large scales are always in the regime of strong turbulence in the limit of infinite Reynolds-like numbers.

Upon employing the usual normalizations (lengths normalized to ρ_s, where $\rho_s \doteq c_s/\omega_{ci}$, $c_s \doteq (T_e/m_i)^{1/2}$, and $\omega_{ci} \doteq eB/m_i c$; times normalized to L_n/c_s, where L_n is the density scale length), the forced Hasegawa–Mima equation can be written as

$$\partial_t(1 - \nabla^2)\Phi + \partial_y\Phi + \widehat{\gamma}\Phi + V_E\cdot\nabla(-\nabla^2\Phi) = -\nu_L\nabla^{p_L}\Phi - \nu_S\nabla^{p_S}\Phi,$$

where $\widehat{\gamma}$ is a linear growth-rate operator associated with the energy injection and the right-hand side represents phenomenological dissipation terms associated with large-scale and small-scale damping coefficients [(hyper-)viscosities] ν_L and ν_S respectively. Also, V_E is the $E \times B$ velocity: for any scalar field ψ, $V_E\cdot\nabla\psi = \partial_x\Phi\,\partial_y\psi - \partial_y\Phi\,\partial_x\psi$.

In the following it is assumed that γ_k vanishes outside a small band of width Δk_f centered around the forcing wavenumber k_f. Then the model possesses three dimensionless control parameters: two Reynolds-like numbers, inversely proportional to ν_S and ν_L; and the weak turbulence parameter ϵ_{WT}. In the limit $\nu_S \to 0$, $\nu_L \to 0$, the inertial ranges

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virtually extend from zero wavenumber to infinite wavenumber. Then, varying ϵ_{WT} allows one to pass from regimes of fully developed weak turbulence to regimes of fully developed strong turbulence. Note that, depending on the value of ϵ_{WT}, different regimes may occur in different wavenumber ranges.

The analysis of the turbulent cascade is made difficult by the fact that the present model is not scale-invariant due to the operator $1 - \nabla^2$. In addition, the presence of waves makes the spectrum anisotropic. These difficulties can be circumvented in the strong turbulence (ST) regime. Indeed, in the ST regime wave effects are negligible and one can drop the wave operator altogether. In addition, one can break the wavenumber space into two regions: $k \ll 1$ and $k \gg 1$. In each of these regions the dynamics are described by scale-invariant equations. A Kolmogorov-type analysis can then be carried out separately in the two regions and the spectra connected afterwards.

Let us first consider the case $k \ll 1$. In the absence of forcing and damping, the Hasegawa–Mima equation reduces to

$$\partial_t + \boldsymbol{V}_E \cdot \boldsymbol{\nabla}(-\nabla^2 \Phi) = 0. \tag{1}$$

The scale transformation $(x, y) \rightarrow \lambda(x, y)$, $t \rightarrow \tau t$ leaves Eq. (1) unchanged provided that $\Phi \rightarrow (\lambda^4/\tau)\Phi$. Therefore the intrinsic dimensions of the field Φ associated with scale invariance in the $k \ll 1$ regime are

$$[\Phi] \sim [\text{length}]^4 [\text{time}]^{-1} \quad (k \ll 1). \tag{2}$$

The two invariants of the full Hasegawa–Mima equation reduce to $E = \sum_k |\Phi_k|^2$ (energy) and $Z = \sum_k k^2 |\Phi_k|^2$ (enstrophy). As usual, energy cascades to low k while enstrophy cascades to high k in the limit of infinite Reynolds numbers. In the energy-cascading range the rate of energy transfer ϵ across wavenumber space is constant. Dimensionally, upon using Eq. (2) one finds $[\epsilon] \sim [\text{length}]^8 [\text{time}]^{-3}$. Then the energy-transfer timescale (turnover time) at wavenumber k is given by

$$\tau_k^{(E)} \sim \epsilon^{-1/3} k^{-8/3}.$$

Similarly one obtains $\tau_k^{(Z)} \sim \eta^{-1/3} k^{-2}$ in the enstrophy-cascading range, where η is the rate of enstrophy transfer. Upon defining the potential spectrum $E_\phi(k)$ such that $\int_0^\infty dk\, E_\phi(k) = \sum_k |\Phi_k|^2$ and using again Eq. (2), one obtains

$$E_\phi(k) \sim \begin{cases} C_K \epsilon^{2/3} k^{-11/3} & (k < k_f), \\ C_K \eta^{2/3} k^{-5} & (k > k_f). \end{cases} \tag{3}$$

In the opposite case $k \gg 1$ the Hasegawa–Mima equation reduces to the two-dimensional Navier-Stokes equation. Then invariants are the usual energy $E = \sum_k k^2 |\Phi_k|^2$ and enstrophy $Z = \sum_k k^4 |\Phi_k|^2$. The potential has the dimensions of a stream function and one recovers the well known expressions for the turnover times [5]: $\tau_k^{(E)} \sim \epsilon^{-1/3} k^{-2/3}$ and $\tau_k^{(Z)} \sim \eta^{-1/3}$. Still, the same dependence (3) on k is obtained. Indeed, Eqs. (3) yield the usual Kolmogorov expressions when written in terms of the energy spectral density.

One can recognize that ϵ and η are the rate of transfer of the invariants of the full Hasegawa–Mima equation $E = \sum_k (1 + k^2) |\Phi_k|^2$ and $Z = \sum_k k^2 (1 + k^2) |\Phi_k|^2$. Then it is natural to assume that the spectral functions join smoothly at $k_f \approx 1$, thus implying the same value of the Kolmogorov constant C_K throughout the whole k space.

The validity of the ST approximation requires that energy transfer due to nonlinearity dominates over wave dispersion. Since the appropriate measure of the rate of energy transfer in wavenumber space is the local turnover time τ_k, the ST approximation is valid when

$$\frac{1}{\tau_k \Delta \omega_k} > 1. \tag{4}$$

Upon using the limiting expressions $\Delta \omega_k \sim k_y k^2 \sim k^3$ and $\Delta \omega_k \sim k_y / k^2 \sim k^{-1}$ for $k \ll 1$ and $k \gg 1$ respectively, one can evaluate the conditions given in Eq. (4). In the long-wavelength limit one finds that $1/(\tau_k \Delta \omega_k) \sim \epsilon^{1/3} k^{-1/3}$. Therefore long wavelengths are always found in the ST regime in the limit of zero large-scale dissipation (infinite large-scale Reynolds number).

The behavior of $1/(\tau_k \Delta \omega_k)$ as a function of k is depicted in Figs. 1 and 2 for $k_f < 1$ and $k_f > 1$ respectively. For each k, the criterion for the transition from WT to ST depends on the forcing. In any case, one can see from Figs. 1 and 2 that the wavenumber region that first enters the WT regime as the forcing is reduced is $k \approx 1$. The transition criterion can be recast in terms of the integral of the growth rate over the unstable domain: $\Gamma \doteq \int_{\gamma > 0} dk \, \gamma_k \approx 2\pi \gamma_{k_f} k_f \Delta k_f$. For $k_f < 1$ one can relate Γ to η using $\eta = \sum_k \gamma_k k^2 |\Phi_k|^2$ for $k \approx k_f$. Omitting constants of order unity (such as the Kolmogorov constant) one obtains $\eta^{1/3} \approx \Gamma k_f^{-4}$. Upon imposing the condition given by Eq. (4) around $k \approx 1$ (the most restrictive case) one finally obtains

$$\Gamma \gtrsim k_f^4 \quad (k_f < 1) \tag{5}$$

for the uniform validity of the ST approximation in the whole wavenumber space. Also, one can rewrite Eq. (5) in terms of the weak turbulence parameter ϵ_{WT} evaluated at $k \approx k_f$. Assuming that $\Delta k_f \approx k_f$ one finds $\gamma_{k_f}/\omega_{*,k_f} \gtrsim k_f \rho_s$ or $\epsilon_{WT} \gtrsim (k_f \rho_s)^{-1}$, where the original normalization length ρ_s^{-1} has been restored and $\omega_{*,k} \doteq k_y \rho_s c_s / L_n$.

Finally we would like to comment on the relation between the Hasegawa–Mima equation and the very similar Rossby wave (RW) equation that is employed in the modeling of atmospheric turbulence. The RW equation can be seen as the large gyroradius limit of the HM equation. Therefore only $k_f > 1$ and the $k > 1$ portion of Fig. 1 must be considered. At low wavenumbers, one has $1/(\tau_k \Delta \omega_k) \sim \epsilon^{1/3} k^{5/3}$, and large scales are always found in the weak turbulence regime. This implies the formation of zonal flows [6]. No such phenomena are expected in the Hasegawa–Mima equation.

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FIGURE CAPTIONS

Figure 1. Strong-turbulence parameter $(\tau_k \Delta \omega_k)^{-1}$ as a function of wavenumber in the case of large-scale forcing ($k_f \ll 1$, indicated by the dotted line).

Figure 2. Behavior of the strong-turbulence parameter in the case of small-scale forcing ($k_f \gg 1$).
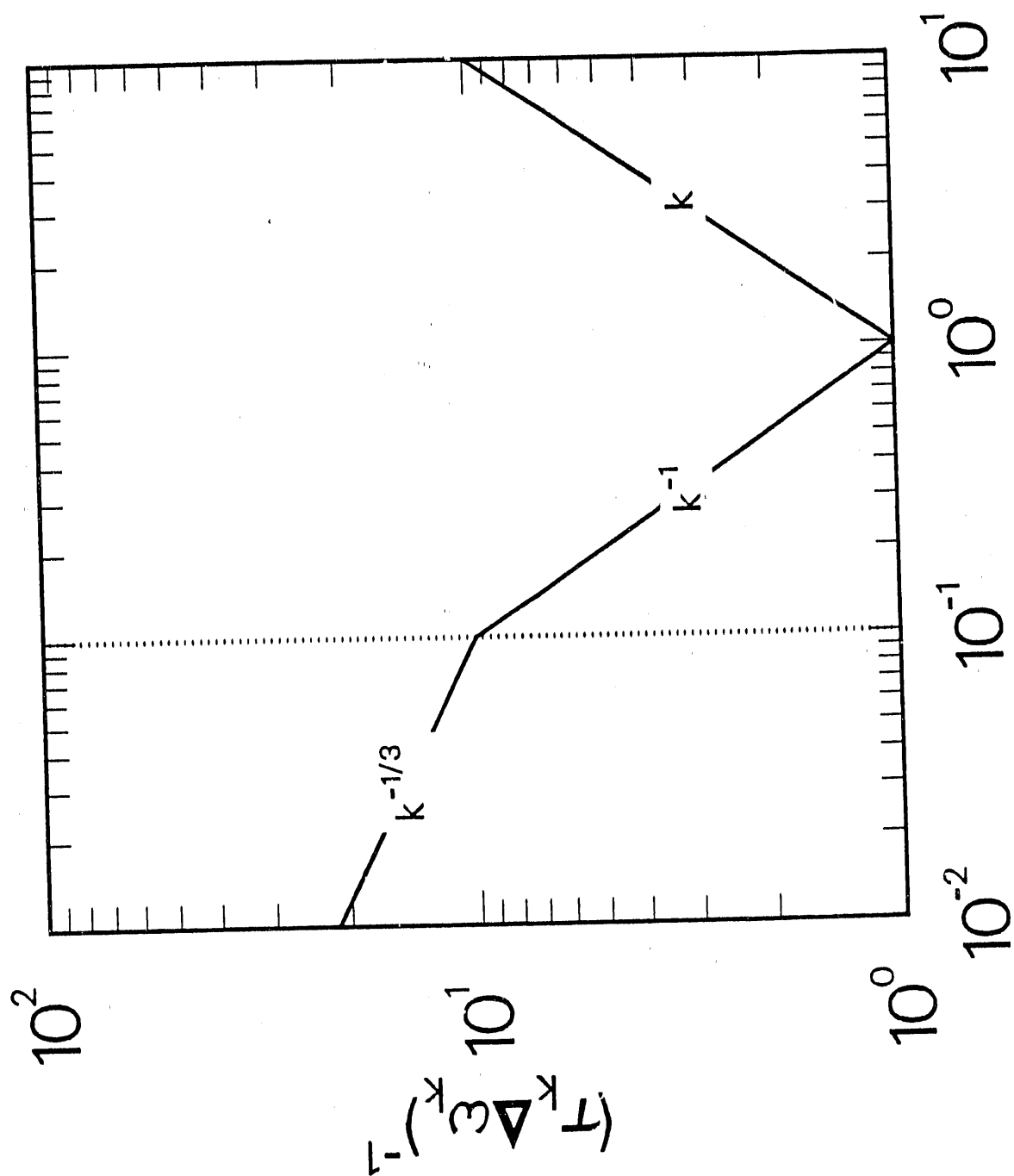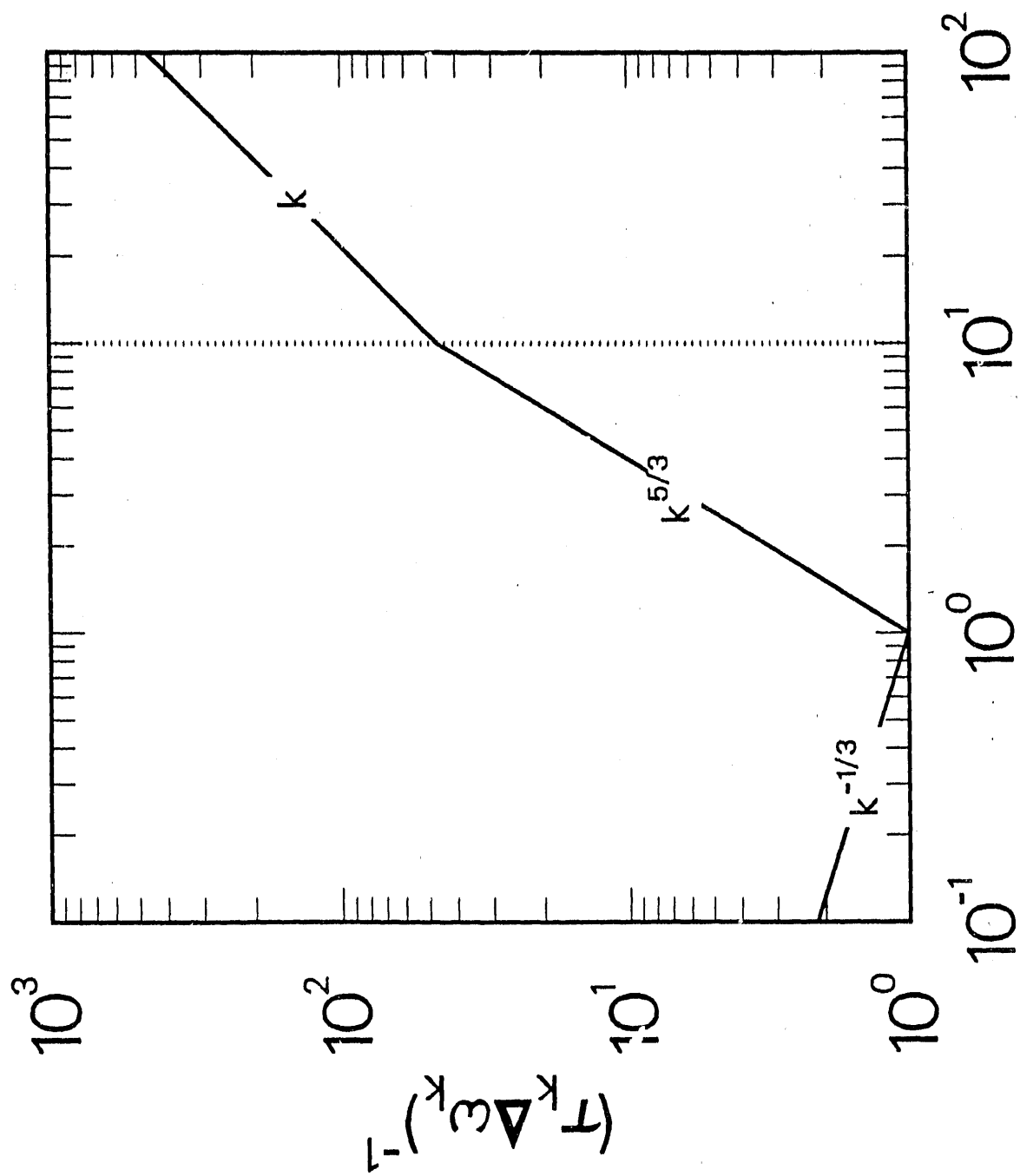
Fig. 1

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Fig. 2

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