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THE CHAOTIC DYNAMICAL APERTURE*

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Introduction

Nonlinear magnetic forces become more important for particles in the modern large accelerators.[1] These nonlinear elements are introduced either intentionally to control beam dynamics or by uncontrollable random errors.[2] Equations of motion in the nonlinear Hamiltonian are usually non-integrable. Because of the nonlinear part of the Hamiltonian, the tune diagram of accelerators is a jungle. Nonlinear magnet multipoles are important in keeping the accelerator operation point in the safe quarter of the hostile jungle of resonant tunes. Indeed, all the modern accelerator design have taken advantages of nonlinear mechanics. On the other hand, the effect of the uncontrollable random multipoles should be evaluated carefully.[3] A powerful method of studying the effect of these nonlinear multipoles is using a particle tracking calculation, where a group of test particles are tracing through these magnetic multipoles in the accelerator hundreds to millions of turns in order to test the dynamical aperture of the machine. These methods are extremely useful in the design of a large accelerator such as SSC, LEP, HERA and RHIC. These calculations unfortunately takes tremendous amount of computing time. In this paper, we are trying to apply the existing method in the nonlinear dynamics to study the possible alternative solution. When the Hamiltonian motion becomes chaotic, the tune of the machine becomes undefined. The aperture related to the chaotic orbit can be identified as chaotic dynamical aperture. In the following review the method of determining chaotic orbit and apply the method to nonlinear problems in accelerator physics. We then discuss the scaling properties and effect of random sextupoles.

Chaotic Transition

The equations of motion of a particle in the Hamiltonian system is given by

$$\mathring{q}_i = \frac{\partial H}{\partial p_i}, \qquad \mathring{p}_i = -\frac{\partial H}{\partial q_i}, \qquad (i = 1, N) \tag{1}$$

with certain given initial values $(q_i(0), p_i(0))$. Nonlinear Hamiltonian are in general non-integrable. Henon and Heiles[4] found that a nonintegrable motion does not lead to chaos, which is in accord with the KAM theorem.[5] As the amplitude of the Hamiltonian motion grows, the motion may become chaotic because of larger nonlinear perturbation.

A distinct character of transition to chaos is the positive Liapunov exponent,[6] which was observed in the numerical experiment of Henon-Heiles potential.[4] For a chaotic orbits, the distance between two trajectories, with infinitesimal separation initially, will grow exponentially with "time". The growth rate is called the Liapunov exponent. For a regular orbit, the Liapunov exponent is zero. A simple test of the Liapunov exponent criteria is BDT test,[7] where one study the Liapunov exponents around the transition boundary surfaces. To be specific, let us consider two trajectories $(q^o(t), p^o(t))$ and $(q(t), p(t))$, with infinitesimal separation. Let η and ξ be the separation of these two trajectories, i.e.

$$\eta \equiv p - p^o, \qquad \xi \equiv q - q^o \tag{2}$$

The equation of motion for η and ξ are given by

$$-\mathring{\eta} = \frac{\partial H}{\partial q} - \frac{\partial H}{\partial q^o} = \frac{\partial^2 H}{\partial q^o \partial p^o}\eta + \frac{\partial^2 H}{\partial q^o \partial q^o}\xi \quad \cdots$$
$$\mathring{\xi} = \frac{\partial H}{\partial p} - \frac{\partial H}{\partial p^o} = \frac{\partial^2 H}{\partial p^o \partial p^o}\eta + \frac{\partial^2 H}{\partial p^o \partial q^o}\xi + \cdots \tag{3}$$

where we have assumed Taylor series expansion for the Hamiltonian around these two infinitesimal near trajectories.

A truncation of eq. (2), up to the first order, gives us a linear system of equations.

$$\begin{pmatrix} \mathring{\eta} \\ \mathring{\xi} \end{pmatrix} = D \begin{pmatrix} \eta \\ \xi \end{pmatrix} \tag{4}$$

where

$$D = \begin{pmatrix} -\dfrac{\partial^2 H}{\partial q^o \partial p^o} & -\dfrac{\partial^2 H}{\partial q^o \partial q^o} \\ \dfrac{\partial^2 H}{\partial p^o \partial p^o} & \dfrac{\partial^2 H}{\partial p^o \partial q^o} \end{pmatrix} \tag{5}$$

Linear equation can be solved by assuming

$$\begin{pmatrix} \eta \\ \xi \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \hat{\eta} \\ \hat{\xi} \end{pmatrix}. \tag{6}$$

Equation (4) becomes the secular equation for solving the eigenvalues.

$$||\lambda - D|| = 0 \tag{7}$$

The stability or the regulatory of the Hamiltonian motion requires that the real part of the eigenvalues are negative on the invariants of the Hamiltonian. This method has been successfully applied to analyze Hamiltonian systems. As an example, the eigen-exponents are found to be negative for the Henon-Heiles potential,

$$H = \frac{1}{2}(\mathring{x}^2 + \mathring{y}^2 + x^2 + y^2) + (x^2 y - \frac{1}{3}y^3), \tag{8}$$

provided that the energy $E < 1/12$. The escape energy is $E_{es} = 1/6$ for the Henon-Heiles potential.

Application to Accelerator Physics

Strong focusing large accelerator has alternate focusing and de-focusing elements. The average effect for a particle moving in the accelerator will experience a transverse focusing and moves approximately in a harmonic oscillator potential well. Because of strong focusing, the particle in the beam will suffer chromatic effect. Sextupole elements are used to correct the chromaticity of the accelerators. The Hamiltonian for a particle in the accelerator can be expressed as

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$$H(1) = \frac{1}{2}(\mathring{x}^2 + \mathring{y}^2 + b_1(s)x^2 - b_1(s)y^2) + b_2(s)(x^2y - \frac{1}{3}y^3) \tag{9}$$

Here s is the position along the accelerator and $\mathring{x} = dx/ds$ $\mathring{y} = dy/ds$ and $b_1(s)$, $b_2(s)$ are respectively strengths of quadrupoles and sextupoles. The Hamiltonian is piecewise conserved. The Hamiltonian is however not conserved along the accelerator. Without the sextupoles, the conserved quantity is the Courant-Snyder invariant[8] or the emittance of the beam. Because of the alternating gradient principle and small betatron phase advance across each elements, we approximate the Hamiltonian by the average focusing and sextupole strength, i.e.

$$\bar{H} = \frac{1}{2}(\mathring{x}^2 + \mathring{y}^2 + \bar{b}_1x^2 + \bar{b}_1y^2) + \bar{b}_2(x^2y - \frac{1}{3}y^3) \tag{10}$$

Let us change the coordinate into ξ and η, with

$$\xi = \frac{\bar{b}_2}{\bar{b}_1}x \qquad \eta = \frac{\bar{b}_2}{\bar{b}_1}y , \tag{11}$$

The transformed Hamiltonian becomes the Henon-Heiles potential, i.e.

$$h = \frac{\bar{b}_2^2}{\bar{b}_1^3} H = \frac{1}{2}(\mathring{\xi}^2 + \mathring{\eta}^2 + \xi^2 + \eta^2) + (\xi^2\eta - \frac{1}{3}\eta^3) \tag{12}$$

It is well-known that the critical energy for the chaotic orbit transition is $E_c = 1/12$.

Let us now make the following assumption: The energy of the Henon-Heiles potential in eq. (12) is related to the amplitude of the transverse motion in the accelerator, i.e.

$$h = A_o^2 + \frac{2}{3}A_o^3 = \frac{1}{12} \tag{13}$$

or

$$A_o = .266$$

The amplitude can be translated into the critical emittance and β function as

$$X_c = \sqrt{\frac{\beta\epsilon_c}{\pi}} = \frac{\bar{b}_1}{\bar{b}_2}A_o = .266\frac{\bar{b}_1}{\bar{b}_2} \tag{14}$$

Thus the critical amplitude of the transverse motion are related to the average focusing strength and sextupole nonlinear strength. A larger $\bar{b}_1$ and smaller $\bar{b}_2$ can give larger dynamical aperture of the beam. But we shall see that a larger $\bar{b}_1$ will naturally lead to a larger $\bar{b}_2$.

Scaling Properties of Accelerators

Let us consider a simple large accelerator, which consists of only regular FODO cells. The following lattice properties can be deduced

$$\delta = b_1\ell_Q , \quad L\delta = 4\sin\frac{\mu}{2}$$

$$\beta_F = \frac{L}{\sin\mu}(1 + \sin\frac{\mu}{2}) , \quad \beta_D = \frac{L}{\sin\mu}(1 - \sin\frac{\mu}{2}) \tag{15}$$

$$x_p(F) = \frac{L\phi}{4}\frac{1 + \frac{1}{2}\sin\frac{\mu}{2}}{\left(\sin\frac{\mu}{2}\right)^2} , \quad x_p(D) = \frac{L\phi}{4}\frac{1 - \frac{1}{2}\sin\frac{\mu}{2}}{\left(\sin\frac{\mu}{2}\right)^2}$$

where L, μ and ϕ are respectively length, phase advance and beam bending angle of the FODO cell. δ is the quadrupole focusing strength, $\int B'd\ell/B\rho$, and $b_1 = B'/B\rho$ is the renormalized quadrupole gradient. The natural chromaticity can be obtained from

$$\xi_N = \frac{\frac{\Delta Q}{Q}}{\frac{\Delta p}{p}} = -\frac{1}{4\pi Q}\int\beta b_1(s)ds = -\frac{\tan\frac{\mu}{2}}{\frac{\mu}{2}} \tag{16}$$

The chromatic sextupoles are used to correct the natural chromaticity, i.e.

$$\begin{aligned}\xi_s &= \frac{1}{4\pi Q}\int\beta(s)x_p(s)b_2(s)ds = \frac{1}{2\mu}(\beta_{max}x_p(F)\ell_s \\ &\quad - \beta_{min}x_p(D)b_2(D)\ell_s) \\ &= \frac{(1 + \frac{1}{2}\sin\frac{\mu}{2})L^2\phi}{8\mu\cos\frac{\mu}{2}(\sin\frac{\mu}{2})^2}\left(b_2(F)\ell_s\right) ,\end{aligned} \tag{17}$$

where the sextupole strength is obtained by making the total chromaticity of the machine zero, $\xi_N + \xi_S = 0$.

We found that

$$s_2 = b_2\ell_s = \frac{16(\sin\frac{\mu}{2})^3}{L^2\phi(1 + \frac{1}{2}\sin\frac{\mu}{2})} \tag{18}$$

The average focusing and sextupole strength are

$$\bar{b}_1 = \frac{\delta}{L} = \frac{4\sin\frac{\mu}{2}}{L^2} \tag{19}$$

$$\bar{b}_2 = \frac{s_2}{L} = \frac{16(\sin\frac{\mu}{2})^3}{L^3\phi(1 + \frac{1}{2}\sin\frac{\mu}{2})} \tag{20}$$

Thus the ratio $\bar{b}_1/\bar{b}_2$ depends on the length of the cell only through Lϕ, which is a more or less constant number for all accelerators. The dynamical aperture is thus a function of only phase advance μ, i.e.

$$x_c = .266\frac{L\phi(1 + \frac{1}{2}\sin\frac{\mu}{2})}{4(\sin\frac{\mu}{2})^2} \tag{21}$$

We note that the dynamical aperture due to the chromatic sextupoles are larger for smaller phase advance per cell. This is understandable because we have assumed only the chromatic sextupoles. Therefore the strength of sextupole will be much smaller for a weaker quadrupole field. As an example, in the SSC reference design A, $L\phi = 3.25$m, $\mu = 80°$, we obtain $x_c = 690^{mm}$. Tracking calculation gives 600^{mm} dynamical aperture[9] with only FODO cells. For RHIC, $L\phi = 1.15^m$, $\mu = 90°$

we obtain x_c = 87mm. The tracking calculation gives x_c = 92mm, which include the insertion.

Effect of random sextupoles. The estimated random sextupoles in SSC design A due to errors in coil placement are $\Delta b_2 = 1.6 * 10^{-4}$ cm^{-2} and $\Delta a_2 = 1.4 * 10^{-4}$ cm^{-2}. Therefore, we have

$$\overline{b_2^r} = \frac{1}{\sqrt{2}} [(1.4)^2 + (1.6)^2]^{1/2} \cdot \frac{10^{-4} * 10^4}{\rho} m^{-3} , \qquad (22)$$

where ρ is the magnetic radius of curvature. For 6.5T field strength and 20 TeV energy,

$$\rho = 1.03 * 10^4 \text{ m}$$

and

$$\overline{b_2^r} = 1.5 * 10^{-4} \text{ m}^{-3} .$$

Since this is much larger than the chromatic sextupoles, we can neglect the effect of chromatic sextupoles. The dynamical aperture becomes

$$x_c = .266 \frac{4 \sin\frac{\mu}{2}}{L^2 * 1.5 * 10^{-4}} \qquad (23)$$

At L = 200m $\mu = 80^\circ$, we obtain x_c = 114mm. The dynamical aperture is reduced by a factor of six due to random sextupoles.

Conclusion

In conclusion, we have made an effort in applying the nonlinear dynamics to define a chaotic dynamical aperture for accelerator. We examine, as a simplest example, the problems related to the chromatic correction sextupole and discuss the scaling properties of the accelerator. Random sextupoles are found to be much more important than the chromatic sextupoles.

We did not discuss, at all, the effect of higher random multipoles. The analysis becomes somewhat more complicated. More work is needed.

In the present analysis, we do not discuss the resonance condition. Here we have assumed no resonance condition for the tune of the machine. The effect of resonance may further decrease the dynamical aperture.

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