LA-UR -86-2301

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LA-UR--86-2301

DE86 013857

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SUBMITTED TO 1986 Applied Superconductivity Conference
Baltimore, MD, Sept. 28-Oct. 3, 1986

TRANSIENT HEAT TRANSPORT IN SUPERFLUID HELIUM IN CYLINDRICAL GEOMETRY

by

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ABSTRACT

Heat transport in cylindrical space from a round superconductor immersed in a finite bath of superfluid has been analyzed both numerically and analytically. The computer and closed analytical results are essentially the same. Analytical equations are given for the temperature of the helium bath as a function of radius, time, and heat flux from the conductor and for the time to reach the superfluid to normal helium transition temperature at the conductor as a function of heat flux.

INTRODUCTION

Interest in the use of superfluid helium for superconductor coolant exists because of the increased temperature margin of operation at the lower temperature of the superfluid for a given current density in the superconductor, the attendant capability to use less superconductor in a given magnet design to attain a given field, and the unique high thermal conductivity and heat capacity characteristics of the superfluid helium. The last feature has been the subject of extensive experimental research and analytical analyses.[1-8] Dresner has obtained equations, by means of self similarity solutions, to three conditions -- a half-space with clamped heat flux at the surface,[6] a half-space with clamped temperature at the surface,[8] and an infinite superfluid helium bath with a pulsed plane heat source,[8] all in one dimensional rectangular coordinates. Dresner[9] has also solved the steady state heat transport problem in cylindrical coordinates for an infinitely extended superfluid helium bath with the heat source temperature at radius r_o fixed at $T=T_\lambda$, the temperature of transition to the normal fluid state. For these conditions, the heat transport is given by

$$q = K\ (2(T_\lambda - T_b)/r_o)^{1/3},$$

where subscript b denotes the superfluid bath temperature at $r=\infty$. For the steady state solution, the bath temperature profile is given by

$$T = (T_\lambda - T_b)\ (r_o/r)^2 + T_b.$$

This paper develops equations for the time dependent temperature profiles for a finite cylindrical region in which a specified heat flux is imposed at one boundary.

ANALYTIC SOLUTION

Heat transport in superfluid helium is characterized by the Gorter-Mellink relation[10]

$$\tilde{q} = -K(\nabla \tilde{T})^{1/3}, \tag{1}$$

where $\tilde{q}$ is the heat flux, W/cm^2;

$\nabla \tilde{T}$ is the temperature gradient, K/cm; and

K is a thermal conductance parameter, $W/(cm^{5/3}\ K^{1/3})$

Equation (1) can be combined with the energy conservation equation,

$$\nabla \cdot \tilde{q} + \rho c(\partial \tilde{T}/\partial \tilde{t}) = 0, \tag{2}$$

where ρ is the fluid density, g/cm^3, and c is the fluid heat capacity, J/(gK),

to obtain

$$\frac{\partial}{\partial \tilde r}\left(\tilde r\left(\frac{\partial \tilde T}{\partial \tilde r}\right)^{1/3}\right) = \tilde r\,\frac{\rho c}{K}\,\frac{\partial \tilde T}{\partial \tilde t}. \qquad (3)$$

The tilde notation designates dimensioned variables. Initial and boundary conditions are chosen as follows:

$$\tilde T(\tilde r, 0) = 1.8\ \mathrm{K} = \text{initial bath temperature}, \qquad (4a)$$

$$K^3\,\frac{\partial \tilde T}{\partial \tilde r}(\tilde r_o, \tilde t) = -\tilde q^3 = \text{cube of the heat flux in the fluid at the conductor surface, and} \qquad (4b)$$

$$\frac{\partial \tilde T}{\partial \tilde r}(\tilde r_{max}, \tilde t) = 0 = \text{temperature gradient at the insulated bath wall}, \qquad (4c)$$

where $\tilde r_o$ is the radius of the heat source surface and $\tilde r_{max}$ is the maximum cylindrical bath radius. Dimensionless variables are introduced for convenience with arbitrary values introduced for scaling as follows:

$$r = \tilde r/\tilde r_{max}, \qquad (5a)$$

$$t = \tilde t/\tau = \tilde t\,(\tilde r_{max}^{4/3}\,\theta^{2/3}\,\rho c/K)^{-1} = \tilde t/(0.020/\tilde r_{max}), \qquad (5b)$$

$$T = (\tilde T - \theta_o)/\theta, \qquad (5c)$$

and

$$q = \tilde q/((\theta/\tilde r_{max})^{1/3}\,K) = \tilde q/(7.892/\tilde r_{max}^{1/3}) \qquad (5d)$$

where Θ=0.24 K and Θ_o=1.8 K. Thus, at T=1, $\tilde{T}$=2.04 K. The value 2.04 was determined by Dresner[4] by using the constant property values K=12.7 W/(cm$^{5/3}$K$^{1/3}$) and ρc=0.681 J/(cm^{3}K) to obtain an excellent curve approximation to the experimental points of Van Sciver[2] for a bath temperature of 1.8 K and a conductor heat flux of 2.2 W/cm^2. The value $\tilde{T}$ = 2.04 K is, in a sense, a pseudo transition temperature between the superfluid and normal fluid helium states. Clearly, the problem can be restated with $\tilde{T} = \tilde{T}_\lambda$ = 2.17 K at T=1 with somewhat different values for K and ρc.

The dimensionless forms of equations (3) and (4a,b,c) become

$$\frac{\partial}{\partial r}\left(r\left(\frac{\partial T}{\partial r}\right)^{1/3}\right) = r\,\frac{\partial T}{\partial t}, \tag{6}$$

$$T\,(r,0) = 0, \tag{7a}$$

$$\frac{\partial T}{\partial r}\,(r_o,t) = -q^3 \quad , \tag{7b}$$

$$\frac{\partial T}{\partial r}\,(1,t) = 0 \; . \tag{7c}$$

There is an analytical solution to equation (6), which is of the form

$$T\,(r,t) = at + u(r), \tag{8}$$

that increases linearly with time. Because of the surface boundary condition, the integral of the solution for constant heat transport q must be linear with time. Equation (8) is the simplest function with these properties. Substitution of (8) into (6) yields

$$ar = \frac{\partial}{\partial r}\,\left(r\left(\frac{\partial u}{\partial r}\right)^{1/3}\right) \tag{9}$$

which, after one integration, becomes

$$r\left(\frac{\partial u}{\partial r}\right)^{1/3} = \frac{\alpha r^2}{2} + A \qquad (10)$$

Equation (10), after it is cubed and integrated, becomes

$$u(r) = \frac{\alpha^3 r^4}{32} + \frac{3\alpha^2 A r^2}{8} + \frac{3\alpha A^2 \ln r}{2} - \frac{A^3}{2r^2} + B. \qquad (11)$$

The term α and the integration constant A are determined from the boundary conditions, equations (7b,c), together with the cube of equation (10), to be

$$A = qr_o/(r_o^2 - 1) \qquad (12a)$$

and

$$\alpha = -2A = 2qr_o/(1-r_o^2). \qquad (12b)$$

The solution is very close to that computed numerically, Fig. 1. An approximate solution results by taking

$$T(r,0) = u(r) \qquad (13)$$

with B of equation (11) chosen such that

$$u(1) = 0. \qquad (14)$$

Thus,

$$B = \frac{3}{32}\alpha^3, \qquad (15)$$

and the transient temperature profile in the superfluid bath, if no convection (other than that from counter flow of the superfluid and normal fluid components) occurs, is

$$T(r,t) = \frac{2qr_o t}{1-r_o^2} + \frac{q^3 r_o^3}{(1-r_o^2)^3}\left(\frac{1}{4} r^4 - \frac{3}{2} r^2 + 3 \ln r + \frac{1}{2r^2} + \frac{3}{4}\right). \qquad (16)$$

Because equation (16) is a monotonically decreasing function of r, the maximum fluid temperature always occurs at r_o. Thus, the time the fluid reaches the transition temperature occurs when $T(r_o,t) = 1$. For this condition

$$t = \frac{1-r_o^2}{2qr_o} - \frac{q^2 r_o^2}{2(1-r_o^2)^2}\left(\frac{1}{4} r_o^4 - \frac{3}{2} r_o^2 + 3 \ln r_o + \frac{1}{2r_o^2} + \frac{3}{4}\right). \qquad (17)$$

Dresner[9] suggests that equation (6) can be integrated with respect to r from r_o to 1 to get

$$r_o qt = \int_{r_o}^{1} rT dr \qquad (18)$$

for which there is no constant of integration because $T(r,0) = 0$. Substitution of equation (8) into (18), combined with equations (12a,b), gives

$$\int_{r_o}^{1} ur dr = 0 \qquad (19)$$

Equation (19) can then be solved directly for a more rigorous determination of the integration constant B consistent with the initial condition of equation (7a). B is then given by

$$B = \frac{\alpha^3}{8(1-r_o^2)}\left(13/6 + r_o^6/12 - \frac{3}{4} r_o^4 - \frac{3}{2} r_o^2 + (3r_o^2 + 1) \ln r_o\right) \qquad (20)$$

The values of B calculated from equations (15) and (19) are in very close aagreement, better than 1%, when r_o is large (the vessel wall is close to the heat source) and the value of B contributes significantly to u(r) of equation (11). As r_o becomes smaller the values of B calculated from equations (15) and (19) agree somewhat less well, but the contribution of B to u(r) is substantially diminished and the lack of agreement becomes unimportant.

NUMERICAL SOLUTION

If equation (6) is divided through by r and then differentiated with respect to r, the following differential equation for the nondimensional heat flux q(r,t) can be derived by using relation (1):

$$\frac{\partial}{\partial t}q^3 = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rq)\right). \tag{21}$$

Saul'ev's finite difference method for linear diffusion equations[11] was modified to be appropriate for equation (21) and was used to solve that equation numerically with initial and boundary conditions

$$q(r,0) = 0, \tag{22a}$$

$$q(1,t) = 0, \tag{22b}$$

$$q(r_o,t) = \begin{cases} q_o t/t_{ramp}, & \text{for } t<t_{ramp}; \\ q_o, & \text{for } t>t_{ramp}. \end{cases} \tag{22c}$$

The temperature T(r,t) was then computed from the heat flux by solving

$$\frac{\partial T}{\partial r} = -q^3 \quad \text{for } r_o < r < 1 \tag{23}$$

at each time step with the boundary condition

$$\frac{\partial T}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(rq), \qquad (24)$$

evaluated at $r = r_o$.

DISCUSSION

Figure 1, in dimensionless coordinates, shows the results of one such numerical calculation, plotted with temperature versus radius with time as a parameter, superimposed on the analytical results for equation (16). The case shown is for $\tilde{r}_o = 3.5$ cm, $\tilde{r}_{max} = 13.5$ cm, $\tilde{T}(\tilde{r},0) = 1.8$ K, and $\tilde{q} = 3$ W/cm^2 for the analytical case with the value of $\tilde{q}$ ramped from 0 to 3 W/cm^2 in 6 ms and held constant thereafter at 3 W/cm^2 for the numerical solution. The initial heat source ramp is used to avoid the singularity in the numerical computation that would occur at the surface if the heat flux is initiated at $t = 0$. The 6 ms ramp period was chosen as a reasonable rise time for the onset of Joule heating. The numerical results lie above those from the analytical solution for this case, but not for all geometries and conditions, and generally differ by such a small amount that the use of equation (16) is an adequate solution. The time to reach $T = 1.0$ or $\tilde{T}_\lambda$ is indicated to be only slightly less for the numerical computation than that calculated from equation (16).

The effect of the linear heat ramp for the solution is explored in Figs. 2 and 3 and compared to the analytical results. Figure 2, also in dimensionless coordinates, shows the very early time dependence of the numerical and analytical calculations. For the times up to $T = 0.1$ ($\tilde{T} = 28$ ms) the analytical results lie above those of the numerical computation, opposite the trend observed in Fig. 1 for later times. Because of the inital condition of the analytical solution with the choice of equation (8) as the form of the solution, there is a non zero contribution to the temperature at $t = 0$. This non physical solution at very small values of t can be misleading; however, convergence to near agreement to the physically more realistic numerical solution is approached at $t = 0.1$.

A plot of the time derivative of the temperature versus time in Fig. 3 shows a spike in the numerical solution at the end of the heat ramp at $T = 0.02$ ($\tilde{T} = 5.6$ ms). That feature is not exhibited in the analytical solution. Also, from Figs. 1 and 2, the contribution of the spike to the temperature and hence the heat transport is observed to be markedly small as evidenced by the good agreement between the two methods of solution, especially for $T > 0.1$. The value of the numerically computed temperature time derivative at $T = 0.20$ is 0.50316, whereas the analytical value at $T = 0.20$ is 0.50315 in remarkable agreement.

Table I examines the times at which the numerically computed temperature reaches the critical temperature for two values of the ramp time t_{ramp} and various values for r_{max}. The analytically predicted values are also given for comparison. For these computations, the applied flux at $r = r_0$ was 3 W/cm^2. The agreement between the two methods of solution is quite good and relatively insensitive to the ramp time.

Thus, despite the neglect of (7a) to obtain B as given in equation (15), the form of the solution in equation (8) is found to be a good representation. Further, the choice of equations (13) and (14) to determine the integration constant B preserved the similarity between the numerical and analytical solutions except at very early times.

TABLE I

Nondimensional Times to Reach T_λ
for $\tilde{q} = 3$ W/cm^2

Ramp Time, ms		t		
	$\tilde{r}_{max}$, cm =	3.8	5.5	13.5
2		0.132	0.675	1.809
6		0.151	0.687	1.812
analytical		0.138	0.681	1.860

The rigorous approach to determine B from equation (20), based upon an integration that takes into account the initial condition (7a), gives results that differ only minimally from those calculated from equations (16) and (17). For this reason, the simplicity of (16) and (17) dictates their use for convenience in most cases.

Table II, for a set of arbitrary configurations, represented by several sizes of superfluid baths with a single conductor size, gives the times to reach the pseudo transition temperature at the conductor surface for several heat flux values. The dimensioned times and heat flux values in seconds and W/cm^2, respectively, were calculated by using the multipliers of equations (5a,b,c). The linear time dependent term of (8) and (16) dominates, thus,

$$\alpha t \gg u(r) \tag{25}$$

for the dimensions and heat flux values considered. Simply stated, the dominant term of (16) is merely the temperature rise from the surface heat flux into the annular superfluid helium bath surrounding the conductor. This condition is readily shown by neglecting the low order correction term u(r), rearranging the equation, and reverting to the dimensioned variables to obtain the energy balance as

$$2\tilde{r}_o \, \tilde{t} \, \tilde{q} = (\tilde{T}-\Theta_o) \, (\tilde{r}^2_{max}-\tilde{r}^2_o) \, \rho c. \tag{26}$$

Equations (16) and (17), solved by using only the dominant term, αt, give values for $T(r_o,t)$ and $t\ (r_o,T=1)$, for most configurations, within a few percent or better of the values from the full equations. Thus, the complex analysis leads to a simple physical representation.

Not included in the analysis presented here is the effect of the boundary layer Kapitza resistance at the surface of the conductor. Heat transport across this resistance for the heat flux values considered here could create a large temperature difference between the surface and the superfluid bath.[12] This difference could, in fact, be so great that the heat transport mechanism would rapidly shift to one of thermal conduction through a normal helium boundary layer with the

TABLE II

Time to Reach the Transition Temperature

T_b (t=0) = 1.8 K and r_o = 3.5 cm

q, W/cm^2		$\tilde{t}$, s			
	$\tilde{r}_{max}$, cm =	3.8	5.5	13.5	53.5
0.5		0.016	0.48	3.3	35
1.0		0.008	0.24	1.7	18
2.0		0.004	0.12	0.82	8.6
3.0		0.002(6)	0.08	0.52	5.4

conductor surface temperature above the superfluid to normal helium transition temperature.

ACKNOWLEDGEMENT

The authors express their appreciation to E. Ann Stanley for many helpful discussions on this problem and to L. Dresner for his insights into the solution of the equations.

REFERENCES

1. G. Claudet, C. Meuris, J. Parain, B. Turck, IEEE Trans. Mag. MAG-15, 340 (1979).

2. S. W. Van Sciver, Cryogenics 19, 385 (1979).

3. P. Seyfert, J. Lafferranderie, G. Claudet, Cryogenics 22, 401 (1982).

4. L. Dresner, Adv. Cryog. Eng. 27, 411 (1982).

5. C. Meuris, IEEE Trans. Mag. MAG-19, 272 (1983).

6. "Similarity Solutions of Nonlinear Partial Differential Equations," L. Dresner, Pitman Publishing Inc., Boston, 1983.

7. S. W. Van Sciver, Adv. Cryog. Eng. 29, 315 (1984).

8. L. Dresner, Adv. Cryog. Eng. 29, 323 (1984).

9. L. Dresner, personal communication.

10. "The Properties of Liquid and Solid Helium," J. Wilks, Oxford University Press, Oxford, 1967.

11. "Difference Methods for Initial-Value Problems," R.D. Richtmyer, K.W. Morton, New York, Interscience Publishers, 1967, p. 191.

12. G. Bon Mardion, G. Claudet, P. Seyfert, "Steady State Heat Transport in Superfluid Helium at 1 Bar," 7th Int. Cryog. Eng. Conf., London, G. B. (1978).

APPENDIX

The solution to the superfluid helium heat transport problem in one dimensional linear coordinates for the condition of a heat source of $\tilde{q}$ from an infinite extent slab at $\tilde{x}_o$ with an insulated bath wall at $\tilde{x}_{max}$ yields the following equivalent dimensionless equations, respectively, for equations (16,17,15,20) of the main text:

$$T(x,t) = \frac{qt}{(1-x_o)} + \frac{q^3}{(1-x_o)^3}\left(x^4/4 - x^3 - 3x^2/2 - x\right) + B, \qquad (1a)$$

$$t = (1-x_o)/q - \frac{q^2}{(1-x_o)^2}\left(x_o^4/4 - x_o^3 + 3x_o^2/2 - x_o\right) - B(1-x_o)/q, \qquad (2a)$$

and

$$B = \frac{q^3}{4(1-x_o)^3}, \qquad (3a)$$

or

$$B = \frac{q^3}{(1-x_o)^4}\left(1/5 + x_o^5/20 - x_o^4/4 + x_o^3/2 - x_o^2/2\right). \qquad (4a)$$

The dimensionless variables are identical to equations (5a,b,c,d) with the variable r replaced by x.

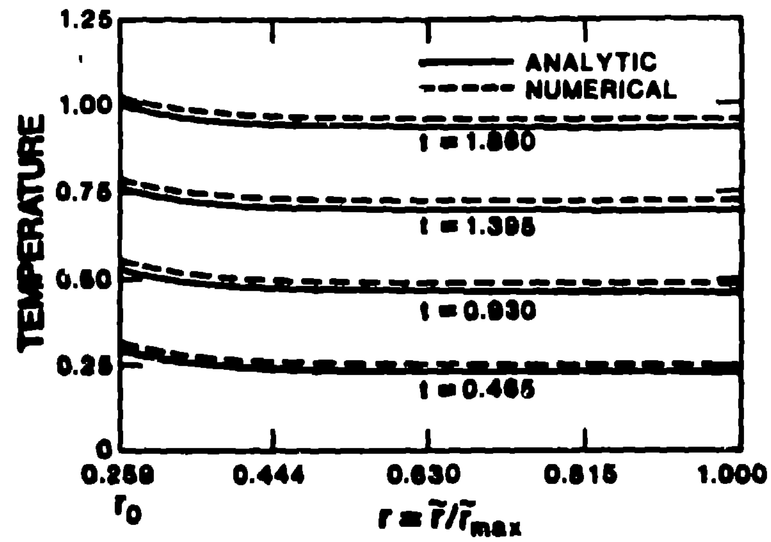

Fig. 1. Combined plot of numerical and analytical solutions at four different times.

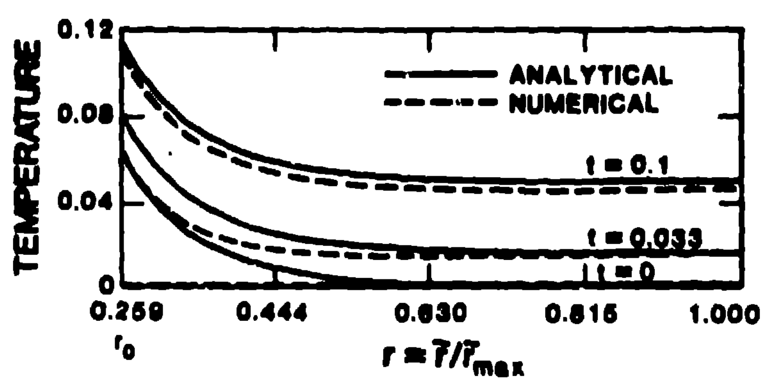

Fig. 2. Combined plot of numerical and analytical solutions at early times.

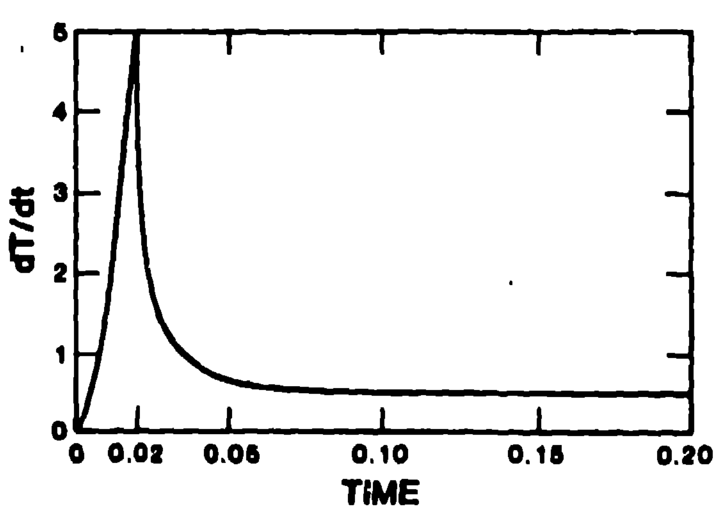

Fig. 3. Time derivative of temperature at the heat source surface for the numerical solution.