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TITLE: EXACT INVARIANTS FOR TIME-DEPENDENT NONLINEAR HAMILTONIAN SYSTEMS

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EXACT INVARIANTS FOR TIME-DEPENDENT NONLINEAR HAMILTONIAN SYSTEMS

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Two methods, one based on canonical transformations and one based on an assumed structure, are used to determine exact invariants for certain Hamiltonians of the type $H = \frac{1}{2}p^2 + V(q,t)$. Invariants are found explicitly for a class of nonlinear, time-dependent potentials $V(q,t)$. The former method is then developed to find exact invariants for Hamiltonians of the form $H = H(q,p,\rho(t),\delta(t))$.

I. INTRODUCTION

The search for exact invariants for nonautonomous Hamiltonian systems has been prompted by theoretical studies in plasma physics and quantum mechanics where such systems play an important role. Apart from applications in these and other fields, our results are also of interest in the theory of canonical transformations in analytical dynamics.

Exact invariants for nonautonomous linear systems have been found during recent years by using a variety of methods [1], and invariants have also been found for some nonlinear systems [2]. The work that is reported briefly here has as its aim the determination of exact invariants for much broader classes of nonautonomous systems. In Section II, we outline a method that uses canonical transformations for Hamiltonians of the form

$$H = \frac{1}{2}p^2 + V(q,t) \tag{1.1}$$

to obtain invariants for a certain class of potentials. The contents of Section II are described in detail in a forthcoming article [3]. The invariants obtained in Section II are all quadratic in p. In Section III, we begin with the ansatz that an invariant is quadratic in p,

$$I = f_0(q,t) + pf_1(q,t) + p^2f_2(q,t) \quad , \tag{1.2}$$

and solve the equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I,H] = 0 \tag{1.3}$$

directly, where H has the form (1.1). The result is invariants for a wider class of potentials than in Section II. In Section IV we return to the method of Section II, but do not restrict the dependence of the Hamiltonian on p to be as in (1.1). The details of Sections III and IV will be published elsewhere [4,5].

II. CANONICAL TRANSFORMATION APPROACH

In the conventional treatment of canonical transformations, the generating function contains a mixture of new and old variables. For this work we employ an unconventional type of generating function, which is a function of the old variables only. Consider a canonical transformation from (q,p) to (Q,P) where

$$Q = Q(q,p,t) \quad , \quad P = P(q,p,t) \quad . \tag{2.1}$$

The original Hamiltonian H(q,p,t) and the new Hamiltonian K(Q(q,p,t), P(q,p,t),t) are related by

$$p\frac{dq}{dt} - H = P\frac{dQ}{dt} - K + \frac{dF}{dt} \quad , \tag{2.2}$$

where F(q,p,t) is the generating function of the transformation. Treating all functions in (2.2) as functions of the old coordinates only, we separate the coefficients of $\frac{dq}{dt}$ and $\frac{dp}{dt}$ and the part that does not involve either to obtain

$$p - P\frac{\partial Q}{\partial q} - \frac{\partial F}{\partial q} = 0 \quad , \tag{2.3}$$

$$P\frac{\partial Q}{\partial p} + \frac{\partial F}{\partial p} = 0 \tag{2.4}$$

$$H + P\frac{\partial Q}{\partial t} - K + \frac{\partial F}{\partial t} = 0 \quad . \tag{2.5}$$

We can eliminate $F(q,p,t)$ from (2.3)-(2.5) in two ways, viz., by differentiating (2.5) with respect to q and (2.3) with respect to t or by differentiating (2.5) with respect to p and (2.4) with respect to t. We obtain, respectively,

$$\frac{\partial K}{\partial q} = \frac{\partial H}{\partial q} + \frac{\partial P}{\partial q}\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial t}\frac{\partial Q}{\partial q} \quad , \tag{2.6}$$

$$\frac{\partial K}{\partial p} = \frac{\partial H}{\partial p} + \frac{\partial P}{\partial p}\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial t}\frac{\partial Q}{\partial p} \quad . \tag{2.7}$$

So far the discussion has been quite general. We now introduce some constraints that will enable a result to be obtained. We introduce a nontrivial auxiliary function, $\rho(t)$, which is such that $\dot{\rho} \neq 0$, and we assume that the time dependence of the transformation is expressed completely by dependence of Q and P on ρ and $\dot{\rho}$.

$$Q = Q(q,p,\rho,\dot{\rho}) \quad , \quad P = P(q,p,\rho,\dot{\rho}) \quad . \tag{2.8}$$

We take

$$H = \frac{1}{2}p^2 + V(q,t) \quad , K = K(P,\rho) \quad , \tag{2.9}$$

and assume that $V(q,t)$ cannot be written with its time dependence expressed entirely through $\rho(t)$ and $\dot{\rho}(t)$. Because the new Hamiltonian, K, does not depend on Q, the new momentum, P, will be an invariant.

With these constraints, (2.6) and (2.7) become

$$\frac{\partial K}{\partial P}\frac{\partial P}{\partial q} = \frac{\partial V}{\partial q} + \dot{\rho}[P,Q]_{q\rho} + \ddot{\rho}[P,Q]_{q\dot{\rho}} \quad , \tag{2.10}$$

$$\frac{\partial K}{\partial P}\frac{\partial P}{\partial p} = p + \dot{\rho}[P,Q]_{p\rho} + \ddot{\rho}[P,Q]_{p\dot{\rho}} \quad , \tag{2.11}$$

in which the bracket $[P,Q]_{\alpha\beta}$ is defined as

$$[P,Q]_{\alpha\beta} = \frac{\partial P}{\partial \alpha}\frac{\partial Q}{\partial \beta} - \frac{\partial P}{\partial \beta}\frac{\partial Q}{\partial \alpha} \quad . \tag{2.12}$$

In addition, we use the Poisson bracket relation for Q and P and the requirement of consistency of the time evolution of Q in the two coordinate systems, i.e.,

$$[Q,P]_{qp} = 1 \quad , \tag{2.13}$$

$$\frac{\partial K}{\partial P} = p\frac{\partial Q}{\partial q} - \frac{\partial V}{\partial q}\frac{\partial Q}{\partial p} + \dot{\rho}\frac{\partial Q}{\partial \rho} + \ddot{\rho}\frac{\partial Q}{\partial \dot{\rho}} \quad . \tag{2.14}$$

The latter equation is also a consequence of (2.10), (2.11), and (2.13). The analysis of (2.10), (2.11), (2.13), and (2.14) is lengthy and will be given in detail elsewhere [3]. It is based on examination of the equations to uncover restrictions of functional dependences of the unknown functions on the variables q, p, ρ, and $\dot{\rho}$. We can illustrate the type of argument by describing how the functional dependence of P on its arguments is restricted as an immediate consequence of (2.10) and (2.11). In (2.11), each quantity except $\ddot{\rho}$ manifestly depends on t only through explicit dependence on $\rho(t)$ and $\dot{\rho}(t)$. Therefore, either $[P,Q]_{p\dot{\rho}}$ vanishes, or $\ddot{\rho}$ is expressible completely in terms of ρ and $\dot{\rho}$, or both. If $\ddot{\rho}$ were expressible completely in terms of ρ and $\dot{\rho}$, then (2.10) would require that the t dependence of $\frac{\partial V}{\partial q}$ be expressible solely in terms of ρ and $\dot{\rho}$; but we have assumed this not to be the case. Therefore, we must have

$$[P,Q]_{p\dot{\rho}} = 0 \quad . \tag{2.15}$$

Considering this as a first-order partial differential equation for P, we then find

$$P(q,p,\rho,\dot{\rho}) = \Gamma(Q,q,\rho) \quad , \tag{2.16}$$

Where Γ is an arbitrary function.

The result of carrying the analysis to a conclusion can be summarized as follows [3]. We have derived an explicit invariant for any potential of the form

$$V(q,t) = \frac{f_0(a,t)}{c_2 - c_1 a} \left(\frac{1}{2} c_1 q^2 - c_0 q\right) + \frac{W(u)}{(c_2 - c_1 a)^2} \quad , \tag{2.17}$$

where a(t) is any particular solution of the differential equation

$$\ddot{a} = f_0(a,t) \quad , \tag{2.18}$$

f_0 is an arbitrary function of its arguments, c_0, c_1 and c_2 are arbitrary constants such that $c_1^2 + c_2^2 = 1$, W(u) is an arbitrary function of u, and u is defined by

$$u = \frac{q - c_0(c_1 + c_2 a)}{c_2 - c_1 a} \quad . \tag{2.19}$$

We have found a canonical transformation for which the new Hamiltonian is

$$K(P,\rho) = \frac{P}{(c_2 - c_1 a)^2} \quad . \tag{2.20}$$

The invariant is the new momentum, which is given by

$$P = \frac{1}{2} v^2 + W(u) \quad , \tag{2.21}$$

where

$$v = (c_2 - c_1 a)p + \dot{a}(c_1 q - c_0) \quad . \tag{2.22}$$

The new coordinate is

$$Q = \frac{v}{|v|} \int^u \frac{du'}{\{2(P-W(u'))\}^{1/2}} + T(P) \quad , \tag{2.23}$$

where T is an arbitrary function.

These results may be interpreted in terms of a transformation to action-angle variables under a generalized canonical transformation

$$(q,p,t) \rightarrow (Q,P,T) \quad ,$$

where Q and P are given by (2.23) and (2.21), respectively, and

$$T = \int^t (c_2 - c_1 a(t'))^{-2} \, dt' \quad . \tag{2.24}$$

The new Hamiltonian is then simply the invariant P.

III. INVARIANTS QUADRATIC IN THE MOMENTUM

Because the invariants found in Section II are all quadratic in the momentum p, it is natural to inquire whether the admissible class of potentials associated with those invariants contains all potentials for which there exists an invariant quadratic in p. Thus, we assume a Hamiltonian

$$H = \frac{1}{2} p^2 + V(q,t) \tag{3.1}$$

and make the ansatz that there exists an invariant of the form

$$I = f_0(q,t) + p f_1(q,t) + p^2 f_2(q,t) \quad . \tag{3.2}$$

That is, I is to satisfy

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I,H]_{qp} = 0 \quad . \tag{3.3}$$

Substitution of (3.1) and (3.2) into (3.3) gives the system of partial differential equations

$$\frac{\partial f_2}{\partial q} = 0 \quad , \tag{3.4}$$

$$\frac{\partial f_1}{\partial q} + \frac{\partial f_2}{\partial t} = 0 \quad , \tag{3.5}$$

$$\frac{\partial f_0}{\partial q} - 2f_2 \frac{\partial V}{\partial q} + \frac{\partial f_1}{\partial t} = 0 \quad , \tag{3.6}$$

$$- f_1 \frac{\partial V}{\partial q} + \frac{\partial f_0}{\partial t} = 0 \quad . \tag{3.7}$$

The solution of (3.4) - (3.7) is relatively straightforward and will be described elsewhere [4]. The result is the following. There exists an invariant quadratic in p if, and only if, the potential is of the form

$$V(q,t) = - F(t)q + \frac{1}{2} \Omega^2(t)q^2 + \frac{1}{\rho_1^2} W\left(\frac{q-\rho_2}{\rho_1}\right) \quad , \tag{3.8}$$

where F, Ω, and W are arbitrary functions, ρ_1 and ρ_2 are any particular solutions of

$$\ddot{\rho}_1 + \Omega^2(t)\rho_1 - \frac{k}{\rho_1^3} = 0 \quad , \tag{3.9}$$

$$\ddot{\rho}_2 + \Omega^2(t)\rho_2 = F(t) \quad . \tag{3.10}$$

and k is an arbitrary constant. The invariant quadratic in p is

$$I(q,p,t) = \frac{1}{2}[\rho_1(p - \dot{\rho}_2) - \dot{\rho}_1(q - \rho_2)]^2$$

$$+ \frac{1}{2} k(\frac{q-\rho_2}{\rho_1})^2 + W(\frac{q-\rho_2}{\rho_1}) \quad . \tag{3.11}$$

This result is a generalization of the result obtained in Section II because the potential may now contain two independent functions of time.

IV. FURTHER DEVELOPMENT OF THE CANONICAL TRANSFORMATION APPROACH

In Section II, our canonical transformation approach was applied to a Hamiltonian equal to $\frac{1}{2} p^2$ plus a potential whose time dependence was assumed <u>not</u> to be expressible entirely through dependence on $\rho(t)$ and $\dot{\rho}(t)$. We now assume that the time dependence of the Hamiltonian <u>is</u> expressible solely in terms of $\rho(t)$ and $\dot{\rho}(t)$, and we allow a general dependence of the Hamiltonian on p. That is, we take

$$H = H(q,p,\rho,\dot{\rho}) \quad . \tag{4.1}$$

The new canonical variables Q and P again are taken to be functions of q, p, ρ and $\dot{\rho}$. Equations (2.10) and (2.11) now read

$$\frac{\partial K}{\partial P}\frac{\partial P}{\partial q} = [H,p]_{qp} + \dot{\rho}[Q,P]_{\rho q} + \ddot{\rho}[Q,P]_{\dot{\rho}q} \quad , \tag{4.2}$$

$$\frac{\partial K}{\partial P}\frac{\partial P}{\partial p} = -[H,q]_{qp} - \dot{\rho}[Q,P]_{p\rho} - \ddot{\rho}[Q,P]_{p\dot{\rho}} \quad . \tag{4.3}$$

In (4.2) and (4.3), all quantities except $\ddot{\rho}$ depend on t only through dependence on ρ and $\dot{\rho}$. Therefore, we either take

$$[Q,P]_{\dot{\rho}q} = 0 \quad , \quad [Q,P]_{p\dot{\rho}} = 0 \quad , \tag{4.4}$$

or require ρ to satisfy a differential equation of the form

$$\ddot{\rho} = g(\rho,\dot{\rho}) \quad . \tag{4.5}$$

(As before, we require $\ddot{\rho} \neq 0$.) The former choice is too restrictive. Adopting the latter, we write

$$\ddot{\rho}[Q,P]_{\dot{\rho}q} = f_1(q,p,\rho,\dot{\rho}) \quad , \quad \ddot{\rho}[Q,P]_{p\dot{\rho}} = - f_2(q,p,\rho,\dot{\rho}) \quad . \tag{4.6}$$

These two equations may be combined to give a single homogeneous partial differential equation for P,

$$[Q,P]_{\dot{\rho}q}\, f_2(q,p,\rho,\dot{\rho}) + [Q,P]_{p\dot{\rho}}\, f_1(q,p,\rho,\dot{\rho}) = 0 \quad . \tag{4.7}$$

The solution is

$$P(q,p,\rho,\dot{\rho}) = \Gamma\{Q(q,p,\rho,\dot{\rho}),\ u(q,p,\rho),\ \rho\} \tag{4.8}$$

where $u(q,p,\rho)$ is related to f_1 and f_2 by

$$f_1(q,p,\rho,\dot{\rho}) = f(q,p,\rho,\dot{\rho})\,\frac{\partial u}{\partial q} \quad , \quad f_2(q,p,\rho,\dot{\rho}) = f(q,p,\rho,\dot{\rho})\,\frac{\partial u}{\partial p} \quad , \tag{4.9}$$

and f is an arbitrary function.

Substituting Γ for P in (4.2) and (4.3), taking two combinations of the resulting equations, and rewriting the Poisson bracket requirement on Q and P, we obtain the fundamental equations

$$\frac{\partial K}{\partial P} = [Q,H] + \dot{\rho}\,\frac{\partial Q}{\partial \rho} + \ddot{\rho}\,\frac{\partial Q}{\partial \dot{\rho}} \quad , \tag{4.10}$$

$$\frac{\partial K}{\partial P}\frac{\partial \Gamma}{\partial Q} = -\frac{\partial \Gamma}{\partial u}\dot{u} - \frac{\partial \Gamma}{\partial \rho}\dot{\rho} \quad . \tag{4.11}$$

$$\frac{\partial \Gamma}{\partial u} = \frac{i}{[Q,u]} \quad . \tag{4.12}$$

where

$$\dot{u} \equiv \dot{\rho}\frac{\partial u}{\partial \rho} + [u,H] \quad . \tag{4.13}$$

Detailed analysis of restrictions on functional dependences required by these equations finally leads to a result. The entire derivation will be published elsewhere [5]. Here we summarize the results.

We find that Q may be written as a function of two canonical variables, u (as defined above) and v:

$$Q(q,p,\rho,\dot{\rho}) = R(u,v) \quad , \tag{4.14}$$

where

$$v = I(q,p,\rho) - J(u,\rho,\dot{\rho}) \tag{4.15}$$

and

$$[u,v] = [u,I] = 1 \quad . \tag{4.16}$$

The functions u and v also satisfy

$$\frac{\partial \dot{u}}{\partial \dot{\rho}} + \frac{\partial J}{\partial \dot{\rho}}[u,\dot{u}] = 0 \quad , \tag{4.17}$$

$$\frac{\partial \dot{v}}{\partial \dot{\rho}} + \frac{\partial J}{\partial \dot{\rho}}[u,\dot{v}] = 0 \quad . \tag{4.18}$$

The dot denotes total time differentiation, as in (4.13), and a bracket without subscripts denotes the usual Poisson bracket with respect to q and p.

As in the problem considered in Section II, there is no loss of generality in taking $K(P,\rho)$ to be linear in P and, in fact, we take

$$K(P,\rho) = \beta(\rho)P$$

$$= \beta(\rho)\, N(u,v) \quad , \tag{4.19}$$

where $N(u,v)$ is the invariant. The permissible form of the Hamiltonian is given by

$$H(q,p,\rho,\dot{\rho}) = \beta(\rho)\, N(u,v) + a_0(q,p,\rho) + \dot{\rho}\, a_1(q,p,\rho) + a_2(u,\rho,\dot{\rho}) \quad , \tag{4.20}$$

where $a_2(u,\rho,\dot{\rho})$ is not linear in $\dot{\rho}$. The functions a_0, a_1, and a_2 are related to u, v and $g(\rho,\dot{\rho})$ $(=\ddot{\rho})$ by

$$[u,a_0] = 0 \quad , \quad \frac{\partial u}{\partial \rho} + [u,a_1] = 0 \quad , \tag{4.21}$$

$$[I,a_0] = G_0(u,\rho) \quad , \quad \frac{\partial I}{\partial \rho} + [I,a_1] = G_1(u,\rho) \quad , \tag{4.22}$$

$$g(\rho,\dot{\rho})\, \frac{\partial J}{\partial \dot{\rho}} + \dot{\rho}\, \frac{\partial J}{\partial \rho} + \frac{\partial a_2}{\partial u} = G_0(u,\rho) + \dot{\rho}\, G_1(u,\rho) \quad . \tag{4.23}$$

Equations (4.21) - (4.23) together with the Poisson bracket relation for u and v constitute a consistent set of equations in themselves and are consistent with (4.17) and (4.18).

V. DISCUSSION

The results of the various analyses that have been described briefly here indicate that it is possible to find invariants for a wide class of time-dependent one-dimensional Hamiltonian systems. Such results will have generalizations to higher dimensions as has been found by Lewis [6] for the three-dimensional counterpart of the problem described in Section III. The results reported in Section IV mark a fundamental change from those of Section II and III and earlier results. In the earlier work, canonical coordinates, here called (u,v), have been obtained for which the transformation from (q,p) to (u,v) consisted in a coordinate transformation q to u with the momentum transformation from p to v being induced by the coordinate transformation. In Section IV, the possibility for more general transformations is admitted. Indeed, one might list the types of problem that could be treated in terms of the p dependence of u.

In Sections II and IV, the time dependence of the canonical transformations was through the single function $\rho(t)$ whereas in Section III we saw that two independent functions may arise. Part of our program of investigation of invariants for Hamiltonian systems is to consider, using the method of Sections II and IV, the effect of introducing several independent functions. The other part is to apply the method of Section IV to a Hamiltonian whose time dependence is not through $\rho(t)$ and $\delta(t)$ alone. We hope to report on these matters soon.

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