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LIMITS ON LINEARITY OF MISSILE ALLOCATION OPTIMIZATION

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The limits of linearization in optimizations of missile allocations are explored and found to result from the interaction of more weapons with the shift of allocation from missiles to value that they induce. For current target sets, linearization appears accurate up to about 1,000 weapons.

Optimizations of missile allocation based on linearized exchange equations produce accurate allocations, but the limits of validity of the linearization are not known. These limits are explored in the context of the upload of weapons by one side to initially small, equal forces of vulnerable and survivable weapons. The analysis compares analytic and numerical optimizations and stability indices based on aggregated interactions of the two missile forces, the first and second strikes they could deliver, and they resulting costs.

This note discusses the costs and stability indices induced by unilateral uploading of weapons to an initially symmetrical low force configuration. These limits are quantified for forces with a few hundred missiles by comparing analytic and numerical optimizations of first strike costs. For forces of 100 vulnerable and 100 survivable missiles on each side, the analytic optimization agrees closely with the numerical solution. For 200 vulnerable and 200 survivable missiles on each side, the analytic optimization agrees with the indices to within about 10%, but disagrees with the allocation of the side with more weapons by about 50%. The disagreement comes from the interaction of the possession of more weapons with the shift of allocation from missiles to value that they induce.

Exchange Model used is an aggregated, probabilistic treatment of the interaction of two unequal missile forces described in Appendix A,[1] which models the first and second strikes each side could deliver. The two sides are denoted by "unprime" and "prime," in accord with the symbols used for their forces and parameters. The unprime side has M vulnerable missiles with m weapons each and N survivable missiles with n weapons each. The prime side has M' vulnerable missiles with m' weapons each and N' survivable missiles with n' weapons each. This example studies the effect of additional land missiles on the prime side, so $M = N = M' = N'$. . $M = 100$ and 200 are studied to examine the effect of changing the number of missiles. m' is increased from 1 to 5 while holding $m = n = n' = 1$ to study the effect of additional weapons.

If unprime strikes first, allocating a fraction f of his total of $W = mM + nN$ weapons at the M' prime vulnerable missiles, his counter force strike is fW and his first strike on value is $F = (1 - f)W$. The counter force portion of the strike delivers an average of $r = fW/M'$ weapons per vulnerable missile, which gives an average survival probability of $Q' \approx q^r$, where $q = 1 - p$, and p is the single shot probability of kill, taken to be 0.6 for all weapons and vulnerable missiles for both sides. The prime second strike is S' = Q'm'M' + n'N'. The prime first strike F' and unprime second strike S can be derived by "conjugating" these expressions, i.e., interchanging prime and unprime symbols in them.

The resulting first and second strikes are converted into first and second strike costs through exponential approximations to the fractions of military value targets destroyed, as described in Appendix A. The calculations assume that the unprime and prime sides each have 1,000 value targets, which are denoted below by the symbols k = k' = 1/1000 for ease in generalization. The cost of damage to self and incomplete damage to other are joined in a weighted average using a weighting parameter L for unprime and L' for prime, which measure the attacker's relative preference for damage to the other and prevention of damage to self.[2] The calculations below use L = L' = 0.5 for both sides. Sensitivity to L, which is significant, is studied in earlier notes. The first strikes of each side are optimized by minimizing the cost of executing them through an approach used in the Russian literature[3] described in Appendix C,[4] which has been shown to be sufficiently accurate for the modest force levels treated here.[5]

Forces. The forces treated here are small because it is desired to both explore the limits of the linearization of the exchange optimization and to study the scaling of these exchanges and the stability indices they produce at post START III levels. For the example calculations below, each side initially has 100 or 200 vulnerable missiles (depending on the scenario) with 1 warhead each and 100 or 200 survivable missiles with 1 weapon each for a total of W = W' = 200 or 400 weapons. Prime then increases the number of weapons on each vulnerable missile in increments of 0.5 weapons, so that after 8 increments, in the 100 + 100 missile case there are 5 weapons on each prime vulnerable missile for a total of W' = 5 x 100 + 1 x 100 = 600 weapons, of which 83% are vulnerable, while unprime remains at 1 x 100 + 1 x 100 = 200 weapons.

Attack allocations are treated both numerically and analytically. The numerical solution is found iteratively by minimizing the cost to the first striker, as given in Eq. (B3). The linearization of the exchange model and the optimization of the weapon allocation are given in Eq. (C2) of Appendix C, which produces the optimal fraction of the unprime attack allocated to vulnerable missiles

$$f_o = (M'/W \ln q) \ln(-Lk'/km' \ln q), \qquad (1)$$

which depends directly on M'/W and logarithmically on m', L, and the ratio of unprime and prime targets k'/k. Note that the form Eq. (1) means unprime's counter-force first strike, $f_o W$, is independent of the number of unprime weapons. A similar relationship holds by conjugation for prime's first strike.

Figure 1 shows the optimal allocations for the case where both sides have 100 vulnerable and survivable missiles, i.e., M = N = M' = N' = 100. The solid curves are the numerical f and f';

the lighter curves are the numerical values. The agreement between the numerical and analytic unprime allocation f is quite good for all m'. Even at m' = 3-3.5, where the discrepancy is largest, it is only about 5%.

The lower two curves for f' have similar shapes, but the discrepancies are larger. The numerical curve falls more slowly than the analytic result, and both are decreasing. Thus, while the discrepancy between them is only 10% at m' = 2-2.5, it is 30-50% by m' = 5.

Figure 2 shows the optimal allocations for the case where both sides have 200 vulnerable and survivable missiles. The solid curves are again the numerical f and f', and the lighter curves the numerical values. The agreement between the numerical and analytic unprime allocations f is still reasonably good for most m', although it degrades from 10% at m' = 2 to 15% at m' = 3.5.

The agreement between the numerical and analytic prime allocations is not as good. It degrades from 10% at m' = 2 to 50% at m' = 5. The reason for the degradation is the further flattening of the numerical f' curve. The analytic f', which by Eq. (1) is insensitive to missile levels, has not changed from Fig. 1.

Stability indices. Figure 3 shows the analytic and numerical stability indices for M = N = 100 missiles. The top two curves are the analytic and numerical prime indices, $I' = C_1'/C_2'$. The bottom two curves are the analytic and numerical unprime indices $I = C_1/C_2$. The two middle curves are the analytic and numerical composite indices, I x I'. In each pair, the agreement is quite good, even at large m'. After an initial drop, the prime index stays relatively constant, the unprime index drops because of his increasingly disadvantaged position, and the composite index falls with an intermediate slope. However, it is clear that al of these indices are essentially the same, so to the extent that stability indices are the primary quantities of concern, even the large discrepancies at large m' in Fig. 1 do not degrade that essential agreement for up to 600 weapons.

Figure 4 shows the analytic and numerical indices for M = N = 200 missiles. Again, the top two curves are the analytic and numerical I', the bottom two the analytic and numerical I, and the two middle curves the composite indices, I x I'. Here, the agreement is mixed, even at moderate m'. The unprime index drops much as for the 100 missile case. But I' increases sharply through m' = 3-4 before dropping to the previous value. That pulls the composite I x I' up so that it is about 10% greater than the analytic value by m' ~ 3. A 10% change in a stability index seems significant, and a change in slope from flat to positive seems even more significant. Thus, it would appear that at around 1,000-1,200 weapons per side, the validity of the analytic optimization is compromised.

Limits of linearization. The above discussion makes it clear that the validity of analytic optimization is compromised at around 1,000-1,200 weapons per side. Not unsurprisingly, that is about the number of value targets (~ 1/k ~ 1/k') attributed to each side. It is expected that the augmented attacks would saturate these targets at about these levels. However, it is possible to trace the reason for the compromise more carefully. Unprime is seeking to minimize Eq. (B3)

$$C_1 = (1 - e^{-kS'} + Le^{-k'F})/(1 + L), \qquad (2)$$

which contains the products kS' and k'F. In general, second strikes are much smaller than first strikes, so in forces with only 100 survivable weapons, $kS' << 1$, and the approximation $1 - e^{-kS'} \approx kS'$ used in deriving Eq. (1) is valid. While first strikes can be large, $F \sim (1 - f)W$, and when m' increases, $f \sim 1$, so $F \sim 0$. Thus, the variation of the unprime forces does not impact the validity of linearization. The cost to prime of striking first is

$$C_1' = (1 - e^{-k'S} + Le^{-kF'})/(1 + L'). \qquad (3)$$

S is on the order of S', so the approximation $1 - e^{-k'S} \approx k'S$ is valid. However, the last term becomes $F' \sim (1 - f_o')W'$, and

$$f_o' = (M/W'\ln q) \ln(-L'k/k'm\ln q) \sim M/W'\ln q \sim 1/(1 + m') \sim 0, \qquad (4)$$

so that $F' \sim W'$, which saturates $e^{-kF'}$ by $W' \sim 1{,}000$. Thus, it is the combination of prime's possession of more weapons, and the shift of his allocation that they produce from missiles to value targets that ultimately invalidates the linearization used in the optimal allocation.

Equations (2) and (3) could be solved without linearization, but they are transcendental, which means that the effort involved in doing so is comparable to that in the direct iterative solution of the exchange equations, as used above. It is interesting that the optimization gives relatively accurate predictions of allocations and indices even for $W' \sim 1/k$. It is thus likely that the optimization could give adequate results to bridge the gap from the ~ 2,000 weapons of START III to the ~ 1,000 weapons of START III+ where the approximation is valid. The main cautions are that the approximation underestimates f', which it does because the large W' reduces the effectiveness of f' in the third term of Eq. (3).

Summary and conclusions. This note discusses the sensitivity of allocations and indices to the linearization used in deriving optimal analytic allocations. It discusses the costs and stability indices induced by unilateral uploading of weapons to an initially symmetrical low force configuration. Linearized exchange equations produce accurate allocations up to the limit of their validity. These limits are quantified for forces with a few hundred missiles by comparing analytic and numerical optimizations of first strike costs. For forces of 100 vulnerable and 100 survivable missiles on each side, the analytic optimization agrees with the numerical solution almost exactly. For 200 vulnerable and 200 survivable missiles on each side, the analytic optimization agrees with the indices to within about 10%, but disagrees with the allocation of the side with more weapons by about 50%. It is shown that the disagreement comes from the allocation of the side with more weapons—specifically from the interaction of his possession of more weapons with the shift of his allocation they induce.

The index equations could be solved without linearization, but that would involve about as much effort as the in direct iterative solution used above. Since the optimization gives relatively accurate predictions of allocations and indices even for large forces, it is likely that the optimization gives adequate results to bridge the gap from START III to START III+ where it is rigorously valid. However, the limitations for such accuracy cannot be stated simply, other than the smallness of computed first and strike forces relative to the value target sets held at risk.

APPENDIX A. EXCHANGE MODEL

It is possible to model exchanges between equal missile missiles forces in terms of the first, F, and second, S, strikes one side could deliver. That analysis can be extended to unequal forces by treating the strikes F' and S' that the second side forces (denoted by primes for simplicity) could deliver. In it, the symbol convention is that unprime has M vulnerable missiles with m weapons each and N survivable missiles with n weapons each, and prime has M' vulnerable missiles with m' weapons each and N' survivable missiles with n' weapons each. If unprime is the first striker and a fraction f of his weapons is directed at prime's vulnerable missiles, unprime's first strike on value targets is

$$F = (1 - f)(mM + nN). \tag{A1}$$

The average number of weapons delivered on each of prime's vulnerable missile is

$$r = f(mM + nN)/M'. \tag{A2}$$

For r large, their average probability of survival is approximately[6]

$$Q' \approx q^r \approx e^{fW \ln q/M'}, \tag{A3}$$

where q = 1 - p, and p is the attacking missile's single shot probability of kill, which is taken to be p = 0.6 for all missiles. Prime's second strike is

$$S' = m'M'Q' + n'N' \approx m'M'q^r + n'N', \tag{A4}$$

which is delivered on value, as missiles remaining at the end of the exchange are taken to have no value in this two strike engagement. The corresponding equations for prime's strike can be derived either by repeating the logic from his perspective or simply by conjugating the equations above, i.e., interchanging primed and unprime symbols in Eqs. (A1) - (A4).

APPENDIX B. STRIKE COSTS AND STABILITY INDICES.

These first and second strike magnitudes can be converted into the costs of striking first and second through exponential approximations to the fractions of value targets destroyed. The cost of damage to unprime when he strikes first is approximated by

$$C_{1s} = (1 - e^{-kS'})/(1 + L), \tag{B1}$$

where the constant $k \approx 1/1000$ is roughly equal to the inverse of the size of unprime military value target set that prime wishes to hold at risk,[7] and L is a parameter. For a small prime second strike, $C_1 \sim kS'/(1 + L)$, which is small; for a large prime second strike, $C_1 \sim 1/(1 + L)$. The cost to unprime of incomplete damage to prime is approximated as

$$C_{1o} = Le^{-k'F}/(1 + L), \tag{B2}$$

where $k' \approx 1/1000$ is roughly the inverse of unprime value that unprime wishes to hold at risk. C_{1o} is small for F large and large for F small. C_{1s} and C_{1o} are formally incommensurate, as they represent damage to different parties, but a conventional approximation to a total cost for striking first is their weighted sum[8]

$$C_1 = C_{1s} + C_{1o} = (1 - e^{-kS'} + Le^{-k'F})/(1 + L), \quad \text{(B3)}$$

where L is a constant that represents the attacker's relative preference for inflicting damage on the other and preventing damage to self. L small means that as a first striker, unprime is primarily concerned about denying damage; L large means he is more concerned about inflicting damage on the other. The conventional assumption that $L \leq 1$ and construction of C_1 as a weighted average is plausible but not unique.[9] Second strike costs are also composed of damage to self and other. The former is approximated for unprime by

$$C_{2s} = (1 - e^{-kF'})/(1 + L); \quad \text{(B4)}$$

the latter by

$$C_{2o} = Le^{-k'S}/(1 + L); \quad \text{(B5)}$$

and the total cost for unprime striking second by

$$C_2 = C_{2s} + C_{2o} = (1 - e^{-kF'} + Le^{-k'S})/(1 + L), \quad \text{(B6)}$$

The first and second strike costs for prime can be obtained either by re-deriving these results from prime's viewpoint or by conjugating Eqs. (B3) and (B6), which introduces a second constant L', which reflects prime's attack preference[10]

There is some arbitrariness in converting C_1 and C_2 into stability indices.[11] It is conventional to use the ratio of first and second strike costs, $I = C_1/C_2$, as a stability index for unprime, and $I' = C_1'/C_2'$, as a stability index for prime. When this index is large, there is no advantage to striking first, and when it is small, there is an advantage, which may be perceived as an incentive to first attack in a crisis. For unequal forces, the product of the stability indices of the two sides is used as a compound index

$$\text{Index} = I \times I' = (C_1/C_2)(C_1'/C_2'). \quad \text{(B7)}$$

APPENDIX C. OPTIMAL ATTACK ALLOCATION.

For unprime, optimal attack allocation amounts to choosing f that minimizes his first strike cost C_1, which is accomplished by differentiating Eq. (B3) with respect to f, setting the result to zero, and solving for f. For large forces the resulting equation is transcendental, but the optimal f for small forces ($F, S \ll 1/k$) holds sufficiently accurately for moderate forces ($F, S < 1/k$), for which Eq. (B3) reduces to

$$(1 + L)\, C_1 \approx k(m'M'e^{fW\ln q/M'} + n'N') + L[1 - k'(1 - f)W], \quad \text{(C1)}$$

whose derivative with respect to f has a minimum at

$$f_o = (M'/W\ln q)\,\ln(-Lk'/km'\ln q). \quad \text{(C2)}$$

The equation for prime's first strike allocation is the conjugate of Eq. (C2). f_o scales directly on the opponent's vulnerable missiles M' and inversely on one's own total weapons W = mM + nN. In a first strike, the distinction between vulnerable and survivable missiles is not significant, so the degree of fractionation of each is unimportant, only W matters. It is plausible that the number of weapons allocated to missiles should be proportional to the number of missiles, i.e., $Wf_{opt} \sim M'$, so that the vulnerable missiles are roughly covered. If in addition, the number of vulnerable weapons mM is proportional to the number of survivable weapons, mM ~ nN, then $f_o \alpha M'/mM$., i.e., f_o scales in proportion to the relative number of the opponent's vulnerable missiles and inversely with one's own weapon inventory.

If the number of vulnerable missiles on each sides change proportionally, M ~ M', then $f_o \sim 1/m$. If the number of weapons per missile does not change, f_o is constant. For m = n, i.e., equal fractionation of vulnerable and survivable weapons, W = m(M + N), so that the allocation only depends on the total number of missiles, not on whether they are vulnerable or survivable. To first order it is unprime's weapons per missile that determines his allocation; prime's weapons per missile m' enters only logarithmically, as do L, k, and k'. The allocation decreases with L; it is insensitive to prime's attack parameter L', which does not enter C_1.

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Fig. 1 Allocations for 100 missiles

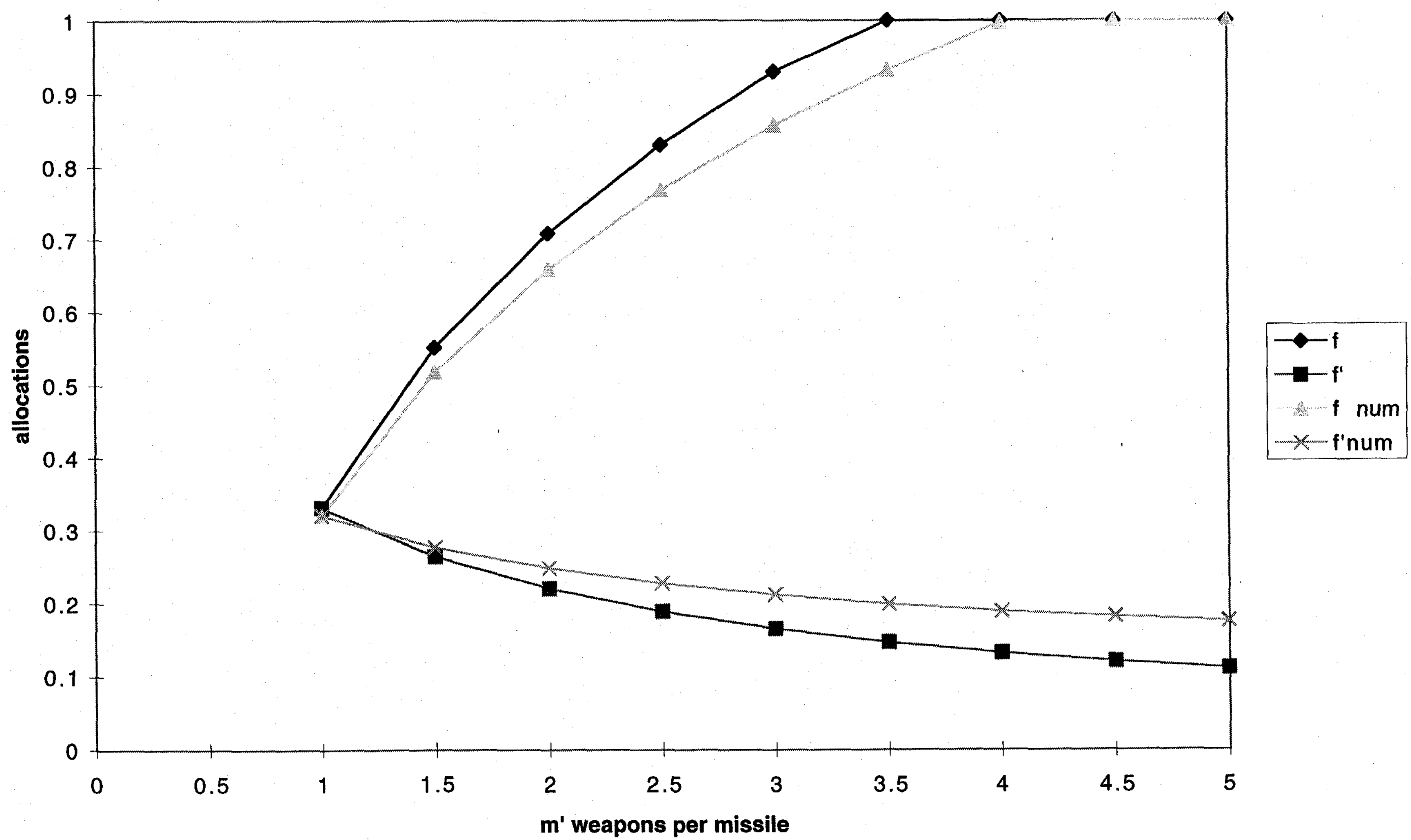

Fig. 2. f & f' vs m' for M = N = 200

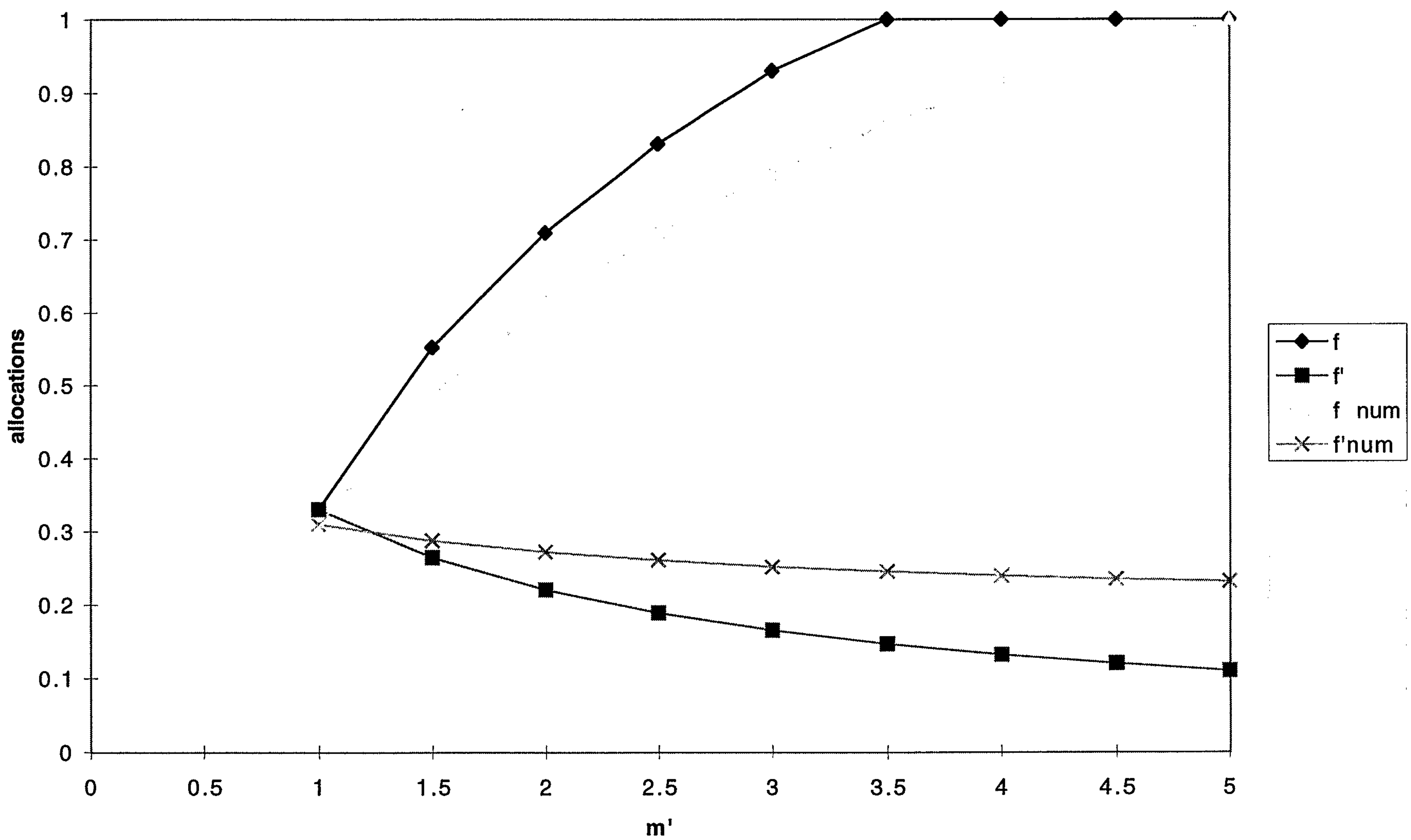

Fig. 3. Indices vs m' M = 100

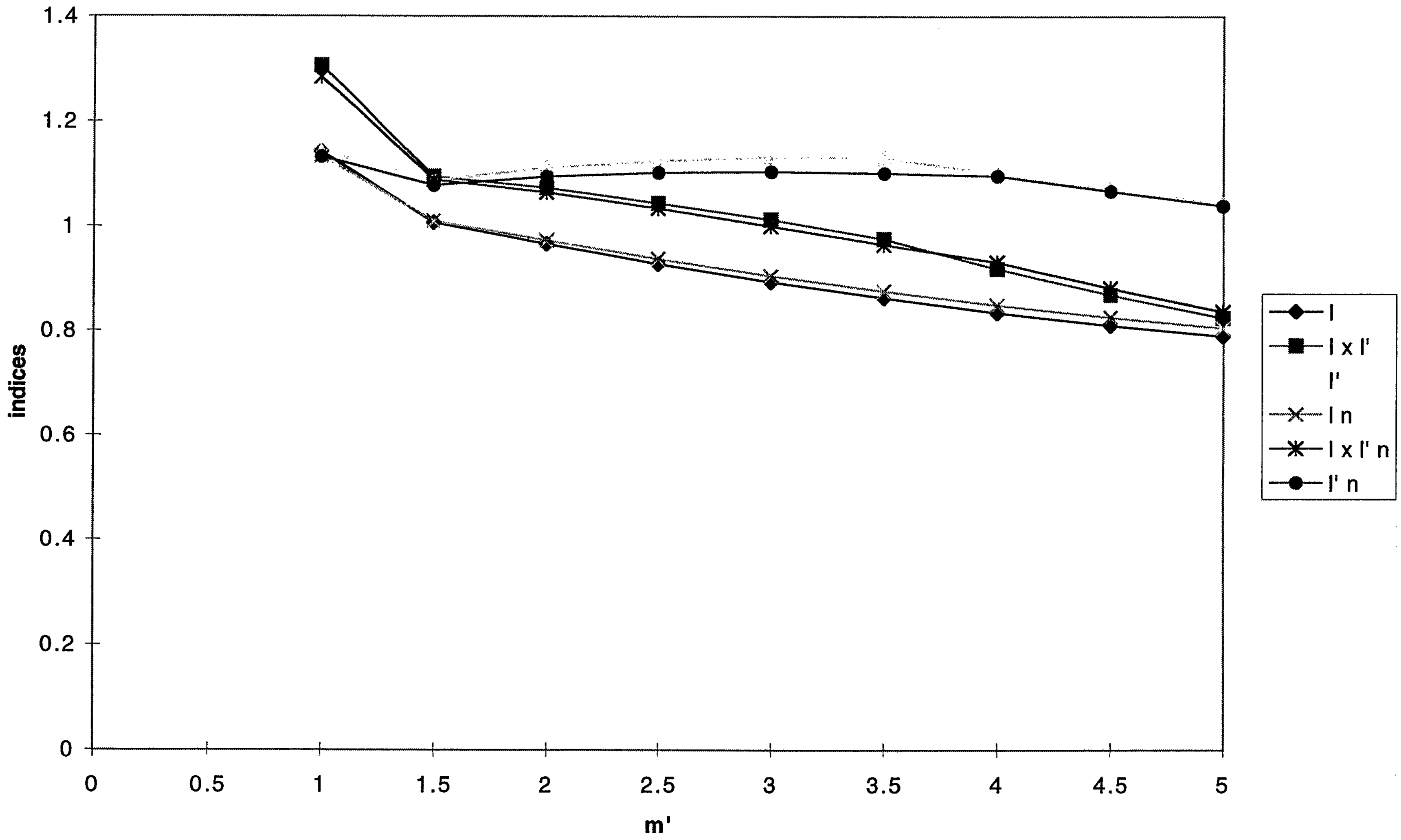

Fig. 4. Indices vs m'