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DYNAMICAL SUPERSYMMETRIC DIRAC HAMILTONIANS

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INTRODUCTION

Most recently a relativistic quantum mechanical approach to nuclear physics has proven promising.[1] However we know from the interacting boson model of nuclei[2] and the fermion dynamical symmetry model of nuclei[3-5] that nuclear spectra exhibit dynamical symmetries.* Perhaps these symmetries have their basis in a realtivistic theory, particularly since the spin-orbit potential is a relativistic effect. For this reason I would like to explore in this Symposium certain Dirac Hamiltonians, although I don't pretend to come anywhere near the above-stated goal in this talk. The Dirac Hamiltonian I would like to consider is that of a neutral fermion interacting with a tensor field, say for example the electromagnetic field tensor. In this paper I shall use the language of QED since it is our best-known realtivistic quantum theory. However, the Hamiltonians can have a more general applicability for example to QHD[1] or QCD[6].

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*See also talks by R. Casten, D. H. Feng, and C. L. Wu in this Symposium.

First we shall discuss a supersymmetry found for a general Dirac Hamiltonian of this type. Then we shall discuss a special case of this type of Dirac Hamiltonian.

THE DIRAC HAMILTONIAN

A neutral fermion can interact with an external electormagnetic field through its "anomalous" magnetic moment μ', i.e., the difference between its measured magnetic moment and the Dirac magnetic moment. For example the anomalous magnetic moment of the neutron could be due to its meson cloud. In any case the Dirac equation is

$$H\Psi_s = \beta\left[\vec{\gamma}\cdot\vec{p} + \frac{i\mu'}{c}\,\gamma_\mu\gamma_\nu F^{\mu\nu} + mc\right]\psi_s = \epsilon_s\psi_s \tag{1}$$

We use the usual notation for the 4 × 4 Dirac matrices,

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \beta \tag{2a}$$

$$\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & \end{pmatrix} = \beta\,\alpha_k, \quad k = 1, 2, 3 \tag{2b}$$

The bound states are denoted by the quantum number s and σ_k are the 2 × 2 Pauli matrices.

The electromagnetic field tensor is

$$F^{\mu\nu} = \frac{\partial}{\partial x_\nu} A^\mu - \frac{\partial}{\partial x_\mu} A^\nu \tag{3a}$$

where the four-vector potential, electric field and magnetic field are

$$A_\mu = (\phi, \vec{A}) \tag{3b}$$

$$E_i = F^{oi}\ , \quad i = 1, 2, 3 \tag{3c}$$

$$B_i = \epsilon_{ijk}F^{jk} \tag{3d}$$

Furthermore we assume that B = 0, and the electric field is centrally symmetric $\vec{E} = \hat{r}\,\frac{\partial\phi(r)}{\partial r}$. In this case the Dirac Hamiltonian is spherically symmetric and conserves the total anagular momentum j and projection μ, helicity ν,

$$\nu = \left[\frac{\vec{\sigma}\cdot\vec{j}+1}{j+\frac{1}{2}}\right]\beta \tag{4a}$$

which has eigenvalues ± 1, and parity π,

$$\pi = \nu(-1)^{j-\frac{1}{2}} \tag{4b}$$

Thus we shall have

$$s = (n\nu j\mu) \tag{4c}$$

where n is the radial quantum number.

We write the Dirac wavefunction as $(\psi_s^{(+)}, \psi_s^{(-)})$ where $\psi_s^{(+)}$ is the upper component, and $\psi_s^{(-)}$ the lower component. Then the Dirac Hamiltonian (1) reduces to

$$H = \begin{pmatrix} mc & h^\dagger \\ h & -mc \end{pmatrix} \tag{5a}$$

where

$$h^\dagger = \sigma\cdot(\vec{p} + i\,\mu'\vec{E}) \tag{5b}$$

If we square H, we get a diagonal Hamiltonian H

$$\bar{H} = H^2 = \begin{bmatrix} \bar{H}_+ & 0 \\ 0 & \bar{H}_- \end{bmatrix} \tag{6a}$$

where

$$\bar{H}_+ = (mc)^2 + h^\dagger h = p^2 + V - W \tag{6b}$$

$$\bar{H}_- = (mc)^2 + hh^\dagger = p^2 + V + W \tag{6c}$$

and the potentials V and W are

$$V = (mc)^2 + \left[\frac{\mu'}{c}\frac{d\phi}{dr}\right]^2 \tag{6d}$$

$$W = \frac{\hbar\mu'}{c}\left[\frac{1}{r^2}\left(\frac{d}{dr}r^2\frac{d\phi}{dr}\right) + \frac{2}{r}\frac{d\phi}{dr}\vec{\sigma}\cdot\vec{L}\right] \tag{6e}$$

Clearly the eigenvalues of $\bar{H}$ are ϵ_s^2. Furthermore $\bar{H}_+$ and $\bar{H}_-$ are each Schrodinger equations which have different potentials, but the same eigenvalues:

$$\bar{H}_+\psi_s^{(+)} = \epsilon_s^2\psi^{(+)} \tag{7a}$$

$$\bar{H}_-\psi_s^{(-)} = \epsilon_s^2\psi^{(-)} \tag{7b}$$

This means that the square of the Dirac Hamiltonian is a supersymmetric Schrodinger Hamiltonian[7,8] in three-dimensional space[9]. The supersymmetric generators are

$$Q^\dagger = \begin{bmatrix} 0 & h^\dagger \\ 0 & 0 \end{bmatrix} \tag{8a}$$

$$Q = \begin{bmatrix} 0 & 0 \\ h & 0 \end{bmatrix} \tag{8b}$$

and the supersymmetric Hamiltonian is

$$\bar{H} = (mc)^2 + \{Q^{+},Q\} \qquad (9)$$

and clearly

$$[Q^{\dagger},\bar{H}] = [Q,\bar{H}] = 0 \qquad (10)$$

Thus $Q^{\dagger}, Q, \bar{H}$ generate an sl(1/1) supersymmetry.

For a singlet representation of this algebra we would have

$$Q\psi_s = Q^{\dagger}\psi_s = 0 \qquad (11)$$

which implies that

$$h^{\dagger}\psi_s^{(-)} = 0 \qquad (12a)$$

$$h\psi_s^{(+)} = 0 \qquad (12b)$$

From (5) this leads to

$$\psi_s^{(\pm)} = A_{\nu j}^{(\pm)} e^{\frac{\mp\mu'}{\hbar c}\phi(r)} r^{t} \qquad (13a)$$

where

$$t = \pm\nu(j + \frac{1}{2}) - 1 \qquad (13b)$$

and $A_{\nu j}^{(\pm)}$ is the normalization.

From (9) we see that these singlet eigenfunctions will be eigenfunctions of the supersymmetric Hamiltonian with $\epsilon_s^2 = (mc)^2$. Of course whether or not they are well-behaved eigenfunctions, i.e., regular at the origin and normalizable depends on the details of $\phi(r)$. However it is clear from (13) that there may be more than one of these eigenfunctions with the same eigenenergy.

Furthermore the Dirac eigenfunctions will be $\psi_s = \left(\psi_s^{(+)}, 0\right)$ $\psi_s' = \left(0, \psi_s^{(-)}\right)$ with $\epsilon_s = + mc$ and $\epsilon_s = - mc$ respectively.

In particular if $\phi(r) \overset{r\to\infty}{\to} r^P$, where P>0, then clearly $\psi_s^{(+)}$ will be normalizable and well-behaved for positive helicity states, $\nu = 1$. But $\psi_s^{(-)}$ will not be normalizable. Hence there will exist an infinite multiplet of states ($j = \frac{1}{2}, \frac{3}{2}, \ldots$) with the same energy $\epsilon_s = mc$, and which do not have a lower component.

For a doublet representation, the algebra gives for each eigenfunction with $\epsilon_s^2 \neq mc^2$,

$$Q^+\Psi_s^{(-)} = (\epsilon_s - mc)\Psi_s^{(+)} \tag{14a}$$

$$Q\Psi_s^{(+)} = (\epsilon_s + mc)\Psi_s^{(-)} \tag{14b}$$

$$Q^+\Psi_s^{(+)} = Q\Psi_s^{(-)} = 0 \tag{14c}$$

The covariant Dirac Hamiltonian will then have a dynamic supersymmetry,

$$H_c = \beta H - m + Q^{\dagger} - Q$$

since it is a linear combination of the generators. The eigenfunctions will be either supersymmetric doublets or singlets. In fact we could have used H_c and $H_c^{\dagger}$ as the supersymmetry generators since

$$\bar{H} = \frac{1}{2}\{H_c, H_c^{\dagger}\} \tag{15}$$

In this Symposium, a similar supersymmetry has been found for the Dirac equation with minimal coupling to the external four-vector potential*. However this supersymmetry differs from the one we have discussed in that γ_5 is diagonal rather than γ_0. Hence chiral symmetry is preserved rather than parity. We can call the supersymmetry discussed here γ_0-supersymmetry, and the other γ_5-supersymmetry.

A SPECIAL CASE

Let us consider the special case of a harmonic electric potential $\phi = \phi_0 r^2$. Then the potential V is also harmonic $V = (mc)^2 + 4\left[\frac{\mu'}{c}\phi_0\right]^2 r^2$, and the potential W has no radial dependence, $W = W_0(\vec{\sigma}\cdot\vec{L} + \frac{3}{2})$, where $W_0 = (2\hbar\omega m)$ and $\omega = 2\mu'\phi_0/mc$. Then the supersymmetric Hamiltonian can be written in terms of the harmonic oscillator quantum number operator,

$$\bar{H}_+ = (mc)^2 + 2\ m\hbar\omega[\hat{N} - \vec{\sigma}\cdot\vec{L}] \tag{16a}$$

$$\bar{H}_- = (mc)^2 + 2\ m\hbar\omega[\hat{N} + 3 + \vec{\sigma}\cdot\vec{L}] \tag{16b}$$

where $\hat{N} = a^{\dagger}\cdot\tilde{a}$ and counts the number of oscillator quanta

$$a_q^{\dagger} = \frac{1}{\sqrt{2m\hbar\omega}}\left[p_q + im\omega r_q\right] \tag{17a}$$

and

$$h^{\dagger} = \sigma\cdot a^{\dagger} \tag{17b}$$

$$h = \sigma\cdot a \tag{17c}$$

*See talk by L. O'Raifeartaigh in this Symposium.

This special Dirac Hamiltonian[10] has been considered as a model for spinor quarks[11], before supersymmetric quantum mechanics. Later the supersymmetric Hamiltonian (16) was considered without (apparently) recognizing its relationship to the Dirac Hamiltonian.[9,12,13]

The radial eigenfunctions are clearly given in terms of the harmonic oscillator eigenfunctions with radial quantum number n = 0, 1, ...,. From (13), and as noted in the last section, there are normalizable and well-behaved eigenfunctions with ϵ_s = mc which correspond to n = 0 and ν = 1,

$$\Psi^{(+)}_{n=0,\ \nu=1,j} = A^{(+)}_j r^{j-\frac{1}{2}} e^{-\frac{\mu'}{\hbar c}\phi_0 r^2} \tag{18}$$

However the lower component $\Psi^{(-)}_{n=0,\nu=1,j}$ in (13) is not normalizable. This corresponds to other results[7,8,9,12,13,14] that the ground state of a supersymmetric Hamiltonian has only a "boson" normalizable eigenfunction.

We also note that the ground state exists for an infinite number of j, $j = \frac{1}{2}, \frac{3}{2}, \ldots$, as noted in the last section. In fact this infinite degeneracy is even more widespread. If we look at the exact eigenergies for all states[11], which we can find easily from (16),

$\nu = 1$

$$\epsilon_{n\nu=1j} = \left[(mc)^2 + 4\, m\hbar\omega n \right]^{\frac{1}{2}} \tag{19a}$$

$\nu = -1$

$$\epsilon_{n\nu=-1j} = \left[(mc)^2 + 4\, m\hbar\omega(n + j + 1) \right]^{\frac{1}{2}} \tag{19b}$$

Hence we see that, for the states with positive helicity, the energy depends only on the radial quantum number n and not on j, and hence these states are infinitely degenerate for each radial quantum number, not only for n=0. The states with negative helicity have finite degeneracies. For example for $\nu=-1$, n=0, $j=\frac{3}{2}$ and n=1, $j=\frac{1}{2}$ are degenerate. Because of these degenerate bands, it is convenient to define a new quantum number in place of the radial quantum number

$$k = 2n + (1 - \nu)(j+1) \tag{20}$$

Then all the states in a degenerate band have the same value of j. Also the bands with $\nu=1$ have k even, k=0, 2, ... and those with $\nu=-1$ have k odd, k = 3, 5, ...

The eigenenergies will then depend only on the integer k,

$$\epsilon_k = [(mc)^2 + 2mh\omega k]^{\frac{1}{2}} \tag{21a}$$

and the allowed values of k are

$$k = 0, 2, 3, 4, 5, \ldots \tag{21b}$$

and the allowed values of j are

$$j = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{k}{1-\nu} - 1 \tag{21c}$$

Because of these additional degeneracies we may ask whether or not there is a higher symmetry in this special case similar to the higher (SU_3) symmetry that occurs in the non-relativistic harmonic oscillator. This does not seem to be the case[14,15]. However we have found the ladder operator which steps up from one Dirac state in the degenerate band to the next Dirac state. This operator is

$$\vec{A}^{\dagger} = \begin{pmatrix} (\vec{\sigma}\times\vec{j})h^{\dagger} & 0 \\ 0 & h^{\dagger}(\vec{\sigma}\times\vec{j}) \end{pmatrix} \quad (22a)$$

and it has the ladder property

$$\vec{A}^{\dagger}\psi_{kj} = c_{kj}\psi_{k,j+\nu} , \quad (22b)$$

where c_{kj} is some constant. That is, it leaves k constant but increases (decreases) the angular momentum for positive (negative) helicity states thereby generating all the states in a band. The $\vec{A}^{\dagger}$ and $\vec{A}$ commute with the Hamiltonians,

$$[\vec{A}^{\dagger},\bar{H}] = [\vec{A},\bar{H}] = [\vec{A}^{\dagger},H] = [\vec{A}, H] = 0, \quad (23)$$

but $\vec{A},\vec{A}^{\dagger}$ do not seem to form a closed algebra.

CONCLUSIONS

We have shown that the square of the Dirac Hamiltonian of a neutral fermion interacting via an anomolous magnetic moment in an electric potential is equivalent to a three-dimensional supersymmetric Schrodinger equation. If the potential grows as a power of r as r increases, $\phi \xrightarrow{r\to\infty} \phi_C r^P$, P>0, then the lowest energy of the Hamiltonian equals the rest mass of the fermion, and the Dirac eigenfunction has only an upper component which is normalizable. Furthermore there will be an infinite number of states with the same energy but different angular momenta, $j = \frac{1}{2}, \frac{3}{2}, \ldots$. Also, the higher energy states have upper and lower components which form a supersymmetric doublet, and each separately are eigenfunctions of the supersymmetric Hamiltonian with the same eigenvalue.

REFERENCES

1. B. D. Serot and J. D. Walecka, The Relativistic Nuclear Many-Body Problem, Adv. in Nuclear Physics, 16:1 (1986).
2. A. Arima and F. Iachello, The Interacting Boson Model, Adv. in Nuclear Physics, 13:139 (1984).

3. J. N. Ginocchio, A Schematic Model for Monopole and Quadrupole Pairing in Nuclei, Ann. Phys., 126:234 (1980).
4. C. L. Wu, D. H. Feng, X. G. Chen, J. Q. Chen, and M. W. Guidry, A Fermion Dynamical Symmetry Model and the Nuclear Shell Model, Phys. Lett. 168B:313 (1986).
5. M. W. Guidry, C. L. Wu, D. H. Feng, and J. N. Ginocchio, A Fermion Dynamical Symmetry Model for High Spin Physics, Los Alamos preprint LA-UR-86-407, to be published in Phys. Lett. (1986).
6. I. J. R. Aitchison, An Informal Introduction to Gauge Field Theories, Cambridge University Press, New York (1982).
7. E. Witten, Constraints on Supersymmetry Breaking, Nucl.Phys. B202:253 (1982).
8. F. Cooper and B. Freedman, Aspects of Supersymmetric Quantum Mechanics, Ann. Phys., 146:262 (1983).
9. H. Ui, Supersymmetric Quantum Mechanics in Three-Dimensional Space. I, Prog. Theor. Phys. 72:813 (1984).
10. N. V. V. J. Swamy, Exact Solution of the Dirac Equation with an Equivalent Oscillator Potential, Phys. Rev. 180:1225 (1969).
11. Y. M. Cho, A Potential Approach to Spinor Quarks, Il Nuovo Cimento 23A:550 (1974).
12. A. B. Balantekin, Accidental Degeneracies and Supersymmetric Quantum Mechanics, Ann. Phys. 164:277 (1986).
13. V. A. Kostelecky, M. M. Nieto, and D. R. Truax, Supersymmetry and the Relationship Between Coulomb and Oscillator Problems in Arbitrary Dimensions, to be published in Phys. Rev. D.
14. H. Ui and G. Takeda, Does Accidential Degeneracy Imply a Symmetry Group?, Prog. Theor. Phys. 72:266 (1984).
15. M. Moshinsky and C. Quesne, Does Accidental Degeneracy Imply a Symmetric Group?, Ann. Phys. 148:462 (1983).