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Theory of the Tune Shift Due to Linear Coupling

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1. Introduction

The presence of skew quadrupole fields will linearly couple the x and y motions. The x and y motion can then be written as the sum of two normal modes[1] which have the tunes ν_1 and ν_2 which are different from the tune, ν_x, ν_y, in the absence of the skew quadrupole fields. New beta functions, β_1 and β_2, can be defined[2] which are the beta functions of the normal modes and which are different from β_x and β_y, the beta functions of the unperturbed accelerator.

This paper presents analytical perturbation theory results for ν_1, ν_2. The results for ν_1, ν_2 are first found correct to lowest order in the skew quadrupole fields. The results for ν_1, ν_2 are then carried one step further to include the next higher order terms in the skew quadrupole fields. Results for β_1, β_2 will be given in a future paper.

These analytical results show that for the higher order shift in tune the important harmonics of the skew quadrupole field are the harmonics near $\nu_x + \nu_y$. However the harmonics closest to $\nu_x + \nu_y$ do not contribute to the higher order tune splitting, $|\nu_1 - \nu_2|$, as they shift ν_1 and ν_2 about equally. This results in a lack of a dominant harmonic for the higher order contribution of $|\nu_1 - \nu_2|$, which complicates the understanding and correction[3] of the higher order contribution to $|\nu_1 - \nu_1|$.

Analytical results are found for the residual tune splitting which is the $|\nu_1 - \nu_2|$ that remains after the driving term of the nearby difference resonance has been corrected.

2. Lowest Order Solution for the Motion and the Tune

The equations of motion can be written as

$$\begin{aligned}
\left(\frac{d^2}{d\theta_x^2}+\nu_x^2\right)\eta_x &= b_x(s)\ \eta_y \\
\left(\frac{d^2}{d\theta_y^2}+\nu_y^2\right)\eta_y &= b_y(s)\ \eta_x \\
x = \beta_x^{\frac{1}{2}}\eta_x,\ & y = \beta_y^{\frac{1}{2}}\eta_y \\
\theta_x &= \int ds\,(1/\nu_x\beta_x) = \psi_x/\nu_x \\
\theta_y &= \int ds\,(1/\nu_y\beta_y) = \psi_y/\nu_y \\
b_x(s) &= \nu_x^2\beta_x\,(\beta_x\beta_y)^{1/2}\,(a_1/\rho) \\
b_y(s) &= \nu_y^2\beta_y\,(\beta_x\beta_y)^{1/2}\,(a_1/\rho)\ .
\end{aligned} \tag{2.1}$$

The skew quadrupole field is described by $a_1(s)$. On the median plane, the field B_x is given by

$$B_x = -B_0\ a_1\ x\ ,$$

where B_0 is the main dipole field. ρ is the radius of curvature in the main dipole.

To simplify the solutions of Eq. (2.1), we introduce ζ_x and ζ_y such that

$$\begin{aligned}
\eta_x &= \zeta_x + \text{c.c.} \\
\eta_y &= \zeta_y + \text{c.c.}
\end{aligned} \tag{2.2}$$

ζ_x and ζ_y also satisfy Eq. (2.1). In addition, when $a_1 = 0$, the solution for ζ_x, ζ_y is

$$\begin{aligned}
\zeta_x &= A\exp(i\nu_x\theta_x) \\
\zeta_y &= B\exp(i\nu_y\theta_y)
\end{aligned} \tag{2.3}$$

We are looking for a solution of Eq. (2.1) which is valid when ν_x, ν_y are close to the coupling resonance $\nu_x - \nu_y = p$, p being some integer. The solution for ζ_x, ζ_y will be assumed to have the form

$$\begin{aligned}
\zeta_x &= A_s\exp(i\nu_{x,s}\theta_x) + \sum_{r\neq s} A_r\exp(i\nu_{x,r}\theta_x)\,, \\
\zeta_y &= B_s\exp(i\nu_{y,s}\theta_x) + \sum_{r\neq s} B_r\exp(i\nu_{y,r}\theta_y)\ . \\
\nu_{x,s} &- \nu_{y,s} = p\ .
\end{aligned} \tag{2.4}$$

The A_r are assumed to be small compared to A_s, and the B_r small compared to B_s. $\nu_{x,s}$, $\nu_{y,s}$ will give the ν-values of the normal modes. The normal mode ν-values are ν_1, ν_2 and we assume $\nu_1 \to \nu_x$ and $\nu_2 \to \nu_y$ when $a_1 \to 0$, then $\nu_{x,s} \to \nu_x$ for the ν_1 mode, and $\nu_{y,s} \to \nu_y$ for the ν_2 mode, when $a_1 \to 0$. The justification for choosing this form for the solutions, and the choice of the $\nu_{x,r}$ and the $\nu_{y,r}$ present will come out of the solution one finds using this form.

The $\nu_{x,r}$ and $\nu_{y,r}$ for $r \neq s$ will be seen to have the form

$$\begin{aligned} \nu_{x,r} &= \nu_{x,s} + n \\ \nu_{y,r} &= \nu_{y,s} + m \end{aligned} \tag{2.5}$$

where n, m are integers. This could be assumed from the beginning. An alternative procedure is not to restrict $\nu_{x,r}$ and $\nu_{y,r}$, and to make the $\exp(i\nu_{x,r}\theta_x)$ an orthogonal set by choosing $\nu_{x,r} = (2\pi/T)\,q$, q is some integer and T is some very large angle, and treating $\nu_{y,r}$ similarly. Putting Eq. (2.4) into Eq. (2.1) and using the orthogonal property, one finds

$$\begin{aligned} \left(\nu_{x,r}^2 - \nu_x^2\right) A_r &= -2\nu_x \sum_{r'} b_x\left(\nu_{x,r}, \nu_{y,r}\right) B_r \\ \left(\nu_{y,r}^2 - \nu_y^2\right) B_r &= -2\nu_y \sum_{r'} b_y\left(\nu_{y,r}, \nu_{x,r'}\right) A_{r'} \\ b_x\left(\nu_{x,r}, \nu_{y,r'}\right) &= \frac{1}{2T}\int_0^T d\theta_x \beta_x \left(\beta_x\beta_y\right)^{\frac{1}{2}} (a_1/\rho) \exp\left[i\left(-\nu_{x,r}\theta_x + \nu_{y,r'}\theta_y\right)\right], \\ b_y\left(\nu_{y,r}, \nu_{x,r'}\right) &= \frac{1}{2T}\int_0^T d\theta_y \beta_y \left(\beta_x\beta_y\right)^{\frac{1}{2}} (a_1/\rho) \exp\left[i\left(-\nu_{y,r}\theta_y + \nu_{x,r'}\theta_x\right)\right]. \end{aligned} \tag{2.6}$$

In Eq. (2.6) we assume $B_r << B_s$, $A_r << A_s$ for $r \neq s$ and find the first order results

$$\begin{aligned} \left(\nu_{x,s}^2 - \nu_x^2\right) A_s &= -2\nu_x b_x\left(\nu_{x,s}, \nu_{y,s}\right) B_s \\ \left(\nu_{y,s}^2 - \nu_y^2\right) B_s &= -2\nu_y b_y\left(\nu_{y,s}, \nu_{x,s}\right) A_s \\ \left(\nu_{x,r}^2 - \nu_x^2\right) A_r &= -2\nu_x b_x\left(\nu_{x,r}, \nu_{y,s}\right) B_s \\ \left(\nu_{y,r}^2 - \nu_y^2\right) B_r &= -2\nu_y b_y\left(\nu_{y,r}, \nu_{x,s}\right) A_s \end{aligned} \tag{2.7}$$

The first two equations in Eq. (2.7) are homogeneous equations for A_s and B_s, and the ν-values $\nu_{x,s}, \nu_{y,s}$ are determined by requiring the matrix of the coefficients of A_s, B_s

to vanish. This gives

$$\begin{aligned}
&\left(\nu_{x,s}^2-\nu_x^2\right)\left(\nu_{y,s}^2-\nu_y^2\right)=4\nu_x\nu_y\left|\Delta\nu\left(\nu_{x,s},\nu_{y,s}\right)\right|^2\\
&\Delta\nu\left(\nu_{x,s},\nu_{y,s}\right)=\frac{1}{4\pi}\int_0^{2\pi}ds\left(\beta_x,\beta_y\right)^{\frac{1}{2}}\left(a_1/\rho\right)\exp\left[i\left(-\nu_{x,s}\theta_x+\nu_{y,s}\theta_y\right)\right]\\
&\nu_{x,s}-\nu_{y,s}=p
\end{aligned}\tag{2.8}$$

Eq. (2.8) can be simplified by assuming that ν_x, ν_y are close to the resonance line $\nu_{x,s} - \nu_{y,s} = p$ and $\nu_{x,s} \simeq \nu_x$ and $\nu_{y,s} \simeq \nu_y$. Keeping terms of lowest order only, one gets

$$\begin{aligned}
&\left(\nu_{x,s}-\nu_x\right)\left(\nu_{y,s}-\nu_y\right)=\left|\Delta\nu\left(\nu_{x,s},\nu_{y,s}\right)\right|^2\\
&\nu_{x,s}-\nu_{y,s}=p
\end{aligned}\tag{2.9}$$

Eq. (2.9) has two solutions for $\nu_{x,s}$, $\nu_{y,s}$. We denote by ν_1 the value of $\nu_{x,s}$ that goes to ν_x when $a_1 \to 0$, and ν_2 the value of $\nu_{y,s}$ that goes to ν_y when $a_1 \to 0$. The solutions can be written as

$$\begin{aligned}
&\nu_1=\overline{\nu}_x\pm\left\{\left(\frac{\nu_x-\nu_y-p}{2}\right)^2+\left|\Delta\nu\left(\overline{\nu}_x,\overline{\nu}_y\right)\right|^2\right\}^{\frac{1}{2}},\\
&\nu_2=\overline{\nu}_y\mp\left\{\left(\frac{\nu_x-\nu_y-p}{2}\right)^2+\left|\Delta\nu\left(\overline{\nu}_x,\overline{\nu}_y\right)\right|^2\right\}^{\frac{1}{2}}\\
&\overline{\nu}_x=\left(\nu_x+\nu_y+p\right)/2,\overline{\nu}_y=\left(\nu_y+\nu_x-p\right)/2
\end{aligned}\tag{2.10}$$

For the $\pm$, the $+$ sign is used when $\nu_x > \nu_y + p$ for ν_1 and the opposite sign for ν_2. In $\Delta\nu\left(\nu_{x,s}, \nu_{y,s}\right)$, $\nu_{x,s}$ has been replaced by $\overline{\nu}_x$, and $\nu_{y,s}$ by $\overline{\nu}_y$, which introduces a higher order error that can be neglected.

From Eq. (2.10) one finds

$$\begin{aligned}
&\left|\nu_1-\nu_2-p\right|=2\left\{\left(\frac{\nu_x-\nu_y-p}{2}\right)^2+\left|\Delta\nu\left(\overline{\nu}_x,\overline{\nu}_y\right)\right|^2\right\}^{\frac{1}{2}}\\
&\nu_1+\nu_2=\nu_x+\nu_y
\end{aligned}\tag{2.11}$$

3. Higher Order Shifts in ν_1 and ν_2

To find a higher order result for ν_1 and ν_2, one has to find higher order equations for A_s, B_s by putting the lower order solution for $A_r, B_r, r \neq s$, given by Eq. (2.7) into Eq. (2.6).

Eq. (2.7) for A_r, B_r can be somewhat simplified by assuming that ν_x, ν_y are close to the resonance line $\nu_{x,s} = \nu_{y,s} + p$ so that one can assume that $\nu_{x,s} \simeq \nu_x$ and $\nu_{y,s} \simeq \nu_y$ and then

$$\begin{aligned} A_r &= \frac{-2\nu_x b_x\left(\nu_{x,r}, \nu_{y,s}\right)}{\left(n+\nu_x+\nu_y\right)(n-p)} B_s, n \neq p \\ B_r &= \frac{-2\nu_y b_y\left(\nu_{y,r}, \nu_{x,s}\right)}{\left(n+\nu_x+\nu_y\right)(n+p)} A_s, n \neq -p \end{aligned} \tag{3.1}$$

where $\nu_{x,r} = \nu_{y,s} + n$ and $\nu_{y,r} = \nu_{x,s} + n$.

Putting these results for A_r, B_r in Eq. (2.6) one finds the improved equations for A_s, B_s

$$\begin{aligned} \left(\nu_{x,s}^2 - \nu_x^2 - \Delta_x\right) A_s &= -2\nu_x b_x\left(\nu_{x,s}, \nu_{y,s}\right) B_s, \\ \left(\nu_{y,s}^2 - \nu_y^2 - \Delta_y\right) B_s &= -2\nu_y b_y\left(\nu_{y,s}, \nu_{x,s}\right) A_s . \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Delta_x &= 4\nu_x\nu_y \sum_{n\neq p} \frac{|c_n|^2}{\left(n-\nu_x-\nu_y\right)(n-p)}, \\ \Delta_y &= 4\nu_x\nu_y \sum_{n\neq -p} \frac{|b_n|^2}{\left(n-\nu_x-\nu_y\right)(n+p)} \\ b_n &= \frac{1}{4\pi\rho}\int ds\ a_1\left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[i\left(\left(n-\nu_y\right)\theta_x + \nu_y\theta_y\right)\right] \\ c_n &= \frac{1}{4\pi\rho}\int ds\ a_1\left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[i\left(\left(n-\nu_x\right)\theta_y + \nu_x\theta_x\right)\right] \end{aligned}$$

Eq. (3.2) gives the equation for $\nu_{x,s}$ and $\nu_{y,s}$

$$\begin{aligned} &\left(\nu_{x,s}^2 - \nu_x^2 - \Delta_x\right)\left(\nu_{y,s}^2 - \nu_y^2 - \Delta_y\right) = 4\nu_x\nu_y\left|\Delta\nu\left(\nu_{x,s}, \nu_{y,s}\right)\right|^2 \\ &\nu_{x,s} = \nu_{y,s} + p \end{aligned} \tag{3.3}$$

Eq. (3.3) was obtained by using the result for A_r, B_r which is first order in a_1. By iterating Eq. (2.6) one can find a result for A_r, B_r to second order in a_1 which will change Eq. (3.3) by replacing $\Delta\nu$ by

$$\Delta\nu \rightarrow \Delta\nu + \Delta\nu^{(3)} \tag{3.4}$$

where $\Delta\nu^{(3)}$ is third order in a_1. By going one step further and iterating Eq. (2.6) to find results for A_r, B_r to third order in a_1 will change Eq. (3.3) by replacing Δ_x, Δ_y by

$$\begin{aligned}\Delta_x &\to \Delta_x + \Delta_x^{(4)} \\ \Delta_y &\to \Delta_y + \Delta_y^{(4)}\end{aligned} \tag{3.5}$$

where $\Delta_x^{(4)}$, $\Delta_y^{(4)}$ are fourth order in a_1. One can write down all these higher order terms. However, the expression Eq. (3.3) keeping terms up to second order in a_1 is probably sufficient here.

One should also note that in Eq. (3.3) $\nu_{x,s}$ and $\nu_{y,s}$ also occur implicitly in $\Delta\nu\left(\nu_{x,s}, \nu_{y,s}\right)$ which complicates the solution of Eq. (3.3) for $\nu_{x,s}$ and $\nu_{y,s}$. Solutions can be found depending on the size of $\Delta\nu$ and the distance from the resonance line $\nu_x = \nu_y + p$.

One interesting case is when a 2 family a_1 correction system is used to make $\Delta\nu = 0$, and when ν_x, ν_y are very close to the resonance line $\nu_x - \nu_y = p$, so that $\nu_1 = \nu_x$ and $\nu_2 = \nu_y$ with an error that is second order in a_1. Very close to the resonance line, so that in Eq. (2.10) $\left(\nu_x - \nu_y - p\right)^2/4$ can be neglected compared to $|\Delta\nu|^2$, then the above can be achieved by making $\Delta\nu\left(\overline{\nu}_x, \overline{\nu}_y\right) = 0$ as shown in Eq. (2.10).

This corresponds roughly to the situation when a 2 family a_1 correction is used to cancel the driving term of the nearby difference resonance, $\nu_x - \nu_y = p$. In this situation, one can find the shift in $\nu_{x,s}$ and $\nu_{y,s}$ due to the second order Δ_x, Δ_y. Then in Eq. (3.3) $\Delta\nu\left(\nu_{x,s}, \nu_{y,s}\right)$ is not zero but differs from zero by terms of order a_1^3, and thus $|\Delta\nu|^2$ is of order a_1^6. For this result, the previous observation, that higher order terms can only change the $\Delta\nu$ term by $\Delta\nu^{(3)}$, a term of third order, is significant. As $|\Delta\nu|^2$ is of order a_1^6, one can treat it as being zero, and Eq. (3.3) becomes

$$\left(\nu_{x,s}^2 - \nu_x^2 - \Delta_x\right)\left(\nu_{y,s}^2 - \nu_y^2 - \Delta_y\right) = 0\ , \tag{3.6}$$

which gives the normal modes

$$\begin{aligned}\nu_1 &= \nu_x + \frac{1}{2\nu_x}\Delta_x \\ \nu_2 &= \nu_y + \frac{1}{2\nu_y}\Delta_y\ .\end{aligned} \tag{3.7}$$

Thus for the case when $\Delta\nu = 0$ and close to the resonance line, there is a second order in a_1 shift in the ν-values given by $\Delta_x/2\nu_x$ and $\Delta_y/2\nu_y$. Eq. (3.2) for Δ_x and Δ_y show that the largest second order ν–shifts will come from harmonics in a_1 close to $\nu_x + \nu_y$. The driving terms b_n and c_n for n closest to $\nu_x + \nu_y$ contribute most to the second order ν–shifts.

One may also notice that b_n, c_n, as given by Eq. (3.2), are just the usual stop–band results for the $\nu_x + \nu_y = n$ resonance but evaluated at particular points on the resonance line. b_n corresponds to the point $n - \nu_y, \nu_y$ and c_n to the point $\nu_x, n - \nu_x$. For the n–values corresponding to resonance lines closest to the unperturbed ν_x, ν_y, these points on the resonance are not far apart and the b_n and c_n are about equal. Thus for the $\nu_x + \nu_y = n$ lines closest to the unperturbed ν_x, ν_y, ν_1 and ν_2 are shifted about equally and these b_n, c_n do not contribute much to the residual $|\nu_1 - \nu_2|$. This lack of a dominant harmonic for the residual $|\nu_1 - \nu_2|$ makes the correction of the residual $|\nu_1 - \nu_2|$ more difficult.

Eq. (3.7) has been checked[4] by comparing these results with numerical computations of ν_1, ν_2. For the case of $\nu_x = \nu_y$ resonance line, $p = 0$, Eq. (3.3) may be solved for $\nu_{x,s}$, $\nu_{y,s}$ and written as

$$\begin{aligned}
\nu_1 &= \frac{1}{2}\left(\tilde{\nu}_x^2 + \tilde{\nu}_y^2\right) \pm \left\{ \left(\frac{\tilde{\nu}_x^2 - \tilde{\nu}_y^2}{2} \right)^2 + 4\nu_x\nu_y |\Delta\nu\,(\nu_1, \nu_1)\,|^2 \right\}^{\frac{1}{2}} \\
\nu_2 &= \frac{1}{2}\left(\tilde{\nu}_x^2 + \tilde{\nu}_y^2\right) \mp \left\{ \left(\frac{\tilde{\nu}_x - \nu_y^2}{2} \right)^2 + 4\nu_x\nu_y |\Delta\nu\,(\nu_2, \nu_2)\,|^2 \right\}^{\frac{1}{2}} \qquad (3.8) \\
\tilde{\nu}_x^2 &= \nu_x^2 + \Delta_x, \quad \tilde{\nu}_y^2 = \tilde{\nu}_y^2 + \Delta_y
\end{aligned}$$

ν_1 is the mode that goes to ν_x when $a_1 \to 0$, and ν_2 goes to ν_y. For the $\pm$ sign, the $+$ sign is used when $\nu_x > \nu_y$ for ν_1 and the opposite sign for ν_2. One can derive Eq. (3.7) from Eq. (3.8) when $\Delta\nu\,(\overline{\nu}, \overline{\nu}) = 0$, $\overline{\nu} = \frac{1}{2}\left(\nu_x + \nu_y\right)$, and close to the resonance line $\nu_x = \nu_y$.

4. ν–Shifts when ν_x, ν_y are far from the $\nu_x - \nu_y = p$ Resonance

In the derivation of the previous results, ν_x, ν_y were assumed to be close to the $\nu_x - \nu_y = p$ resonance line. When ν_x, ν_y are far from the resonance line the results are less interesting as the ν-shifts are of higher order and smaller. However, it is interesting to see how the results for the ν shifts in these two cases will fit together.

Up to Eq. (2.6), the previous derivation will hold when ν_x, ν_y are far from the $\nu_x - \nu_y = p$ resonance line. Let us first consider the ν_1 mode where $\nu_1 \to \nu_x$ when $a_1 \to 0$. In this case, it is assumed that not only the A_r are small compared to A_s, but also B_s is small.

To lowest order, Eq. (2.7) become

$$\begin{aligned}
&\left(\nu_{x,s}^2 - \nu_x^2\right) A_s = 0 \\
&\left(\nu_{x,r}^2 - \nu_x^2\right) A_r = 0 \\
&\left(\nu_{x,r}^2 - \nu_y^2\right) B_r = -2\nu_y b_y \left(\nu_{y,r}, \nu_{x,s}\right) A_s \\
&\qquad \nu_{y,r} = \nu_{x,s} + n \ .
\end{aligned} \tag{4.1}$$

Thus to lowest order, $\nu_1 = \nu_x$, and the tune shift is a higher order effect in a_1. To find the second order shift in ν_1, the result for B_r in Eq. (4.1) is put into Eq. (2.6) and the A_s equation becomes

$$\begin{aligned}
&\left(\nu_{x,s}^2 - \nu_x^2\right) A_s = \overline{\Delta}_x A_s \\
&\overline{\Delta}_x = 4\nu_x\nu_y \sum_n \frac{|c_n|^2}{(n - \nu_x)^2 - \nu_y^2} \\
&c_n = \frac{1}{4\pi\rho} \int ds\ a_1 \left(\beta_x\beta_y\right)^{\frac{1}{2}} \exp\left[i\left((n - \nu_x)\theta_y + \nu_x\theta_x\right)\right] \ .
\end{aligned} \tag{4.2a}$$

This gives the shift in ν_x,

$$\nu_1^2 = \nu_x^2 + \overline{\Delta}_x \ . \tag{4.2b}$$

The $\overline{\Delta}_x$ is similar to the Δ_x in Eq. (3.2) except that we now do not assume that $\nu_x - \nu_y \cong p$ and the sum over n is over all n. This result, Eq. (4.2b), can be obtained from Eq. (3.3) if in Eq. (3.3) we assume that

$$\left(\nu_{y,s}^2 - \nu_y^2 - \Delta_y\right) \simeq \left((\nu_x - p)^2 - \nu_y^2\right) \ ,$$

and not replace $\nu_x - \nu_y$ by p in Eq. (3.2) for Δ_x.

In the same way one finds for the ν_2 mode,

$$\begin{aligned}
\nu_2^2 &= \nu_y^2 + \overline{\Delta}_y \\
\overline{\Delta}_y &= 4\nu_x\nu_y \sum_n \frac{|b_n|^2}{(n-\nu_y)^2 - \nu_x^2} \\
b_n &= \frac{1}{4\pi\rho}\int ds\ a_1 (\beta_x\beta_y)^{\frac{1}{2}} \exp\left[i\left((n-\nu_y)\theta_x + \nu_y\theta_y\right)\right] \ .
\end{aligned} \tag{4.3}$$

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