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The Phase Gradient Autofocus Algorithm: An Optimal Estimator of the Phase Derivative

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Abstract

The phase gradient algorithm represents a powerful new signal processing technique with applications to aperture synthesis imaging. These include, for example, synthetic aperture radar phase correction and stellar image reconstruction. The algorithm combines redundant information present in the data to arrive at an estimate of the phase derivative. In this report, we show that the estimator is in fact a linear, minimum variance estimator of the phase derivative.

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I. Introduction

We have recently presented a new non–parametric autofocus technique, the Phase Gradient Autofocus (PGA) algorithm for phase correction of Synthetic Aperture Radar (SAR) imagery [1]. Independently, several investigators in the optical astronomy community have begun using essentially the same algorithm, the Phase Gradient (PG) algorithm, for stellar image reconstruction [2]. The phase gradient algorithm as presented in those papers appears somewhat ad–hoc in that no formal treatment is made of the specific manner in which redundant information (aperture phase error for the SAR case, and object phase for the stellar imaging case) is combined to arrive at the estimate of the phase derivative. In this paper we show that the form of the estimator described in those papers is in fact the solution to a minimum variance estimation problem. Such formalism not only lends credibility to the basic algorithm, but provides additional insight into the behavior of, and several extensions to, that algorithm.

The phase gradient estimation problem for SAR imagery is very similar to that for stellar image reconstruction. Several differences between the two problems are manifest. First, for SAR imagery, the point spread function due to the phase error is redundant but the target data varies throughout the image while in speckle imaging, the underlying stellar image is constant but the speckle noise and atmospheric phase varies from exposure to exposure. Furthermore, the SAR image is complex while only intensity data are collected by the speckle imaging technique. Phase gradient estimation is fundamentally two dimensional in speckle imaging while only one dimensional processing is typically required in radar. However, several colleagues have recently demonstrated two dimensional processing for SAR images [3], and presented a more powerful method of estimating 2–D phases from 2–D phase differences using fast elliptic PDE solvers [4]. Finally, the estimator is shown to be unbiased where the signal–to–noise ratio is high, as is typical in SAR. In stellar imaging, however, additional terms must be added to make the estimator unbiased [2].

II. Signal and Noise Model

We will present the derivation of the minimum variance estimator for the phase derivative in the context of SAR processing. However, its application to the speckle imaging problem is direct. Because the speckle noise model used in this development is characterized by random, independent, complex sinusoids, the 2–D gradient estimation problem is separable. Therefore, only a 1–D derivation is presented here.

We will model the range compressed phase history data in the following way. Let each range bin be composed of a single complex exponential modulated by a phase error plus speckle noise. The targets and noise are independent from range bin to range bin, but the phase error function is common to all (a 1–D phase error). Thus, each range bin can be expressed as (the independent variable t represents aperture position in SAR):

$$g(t) \;=\; ae^{j(\omega_0 t+\phi_\epsilon(t))}+n(t) \qquad (1)$$

where $ae^{j(\omega_0 t+\phi_\epsilon(t))}$ is a complex sinusoid due to a hard target of magnitude a and doppler frequency ω_0 that has been modulated by the phase error $\phi_\epsilon(t)$. The additive speckle noise $n(t)$ is modelled in the conventional manner as the sum of random complex sinusoids with Rayleigh distributed magnitudes and uniformly distributed phases [5]:

$$n(t) \;=\; \sum_{m=-M}^{M} b_m e^{j(\omega_m' t+\theta_m)} \tag{2}$$

where the b_m are independent, Rayleigh distributed random variables, and the θ_m are independent and uniform on $[0,2\pi]$.

We have at our disposal a method of measuring the instantaeous phase derivative of $g(t)$ for each range bin k as:

$$\dot{\phi}_k(t) \;=\; \frac{Im\left\{\dot{g}_k(t)g_k^*(t)\right\}}{|g_k(t)|^2} \tag{3}$$

The question to be resolved here is how to combine these measurements from the many range bins to form the best estimate, in some sense, of the common phase error function $\phi_\epsilon(t)$. The approach will be to form the best estimator in the sense of minimum variance. We will demonstrate that the PG and PGA algorithms are in fact linear unbiased minimum variance (LUMV) estimators for $\dot{\phi}_\epsilon(t)$. To do this, we first show that, given the above model of the phase history data, the measurements given by Eq. (3) can be expressed as a linear system of signal plus noise. We then derive the first and second order statistics of the noise process. Finally, we solve for the LUMV estimator.

III. The Noise Process

We begin by examining the noise process $n(t)$ for any one range bin. Without loss of generality in what follows, we will shift the hard target to zero doppler by complex multiplication of $g(t)$ by $e^{-j\omega_0 t}$ and change variables $\omega_m = \omega_m' - \omega_0$. We denote the in-phase and quadrature–phase components of $n(t)$ as:

$$\begin{aligned} n_i(t) &= Re\left\{n(t)\right\} = \sum_m b_m \cos\left(\omega_m t+\theta_m\right) \\ n_q(t) &= Im\left\{n(t)\right\} = \sum_m b_m \sin\left(\omega_m t+\theta_m\right) \end{aligned} \tag{4}$$

Since the b_m are independent and identically distributed (iid), they have equal variances. Furthermore, the Rayleigh distribution is α–integrable for some $\alpha > 2$ ($i.e.$, $\alpha = 3$), Thus, the Central Limit Theorem is applicable to the sums $n_i(t)$ and $n_q(t)$, so $n_i(t)$ and $n_q(t)$ are assumed to be Gaussian distributed. We can immediately prove the following properties:

1. $\overline{n_i(t)} = \overline{n_q(t)} = 0$

$$
\begin{aligned}
E\{n_i(t)\} &= \sum_m E\{b_m\}E\{\cos(\omega_m t + \theta_m)\} \\
&= 0
\end{aligned} \tag{5}
$$

$n_q(t)$ is similar.

2. $R_{n_i}(t+\tau, t) = R_{n_i}(\tau) = R_{n_q}(\tau) = \eta/2 \sum_m \cos(\omega_m \tau)$

$$
\begin{aligned}
&E\left\{\sum_m b_m \cos(\omega_m(t+\tau)+\theta_m) \sum_l b_l \cos(\omega_l t + \theta_l)\right\} \\
&\quad = \eta \sum_m E\left\{\cos(\omega_m(t+\tau)+\theta_m)\cos(\omega_m t + \theta_m)\right\} \\
&\quad = \eta \sum_m E\left\{1/2\cos(\omega_m \tau) + 1/2\cos(\omega_m(2t+\tau) + 2\theta_m)\right\} \\
&\quad = \eta/2 \sum_m \cos(\omega_m \tau)
\end{aligned} \tag{6}
$$

The first step results from: $E\{\cos(\omega_m(t+\tau)+\theta_m)\cos(\omega_l t+\theta_l)\} = 0 \quad \forall\ m \neq l$. $n_q(t)$ is similar.

3. (1) + (2) imply that $n_i(t)$ and $n_q(t)$ are wide sense stationary.

4. $\overline{n_i(t)^2} = \overline{n_q(t)^2} = R_{n_i}(0) = M\eta$

5. $n_i(t)$ and $n_q(t)$ are jointly Gaussian.

$$
\begin{aligned}
&C_1 n_i(t) + C_2 n_q(t) \\
&\quad = \sum_m b_m \left(C_1 \cos(\omega_m t + \theta_m) + C_2 \sin(\omega_m t + \theta_m)\right) \quad \forall\ C_1, C_2
\end{aligned} \tag{7}
$$

The Central Limit Theorem applies here as well. Therefore, since $C_1 n_i(t) + C_2 n_q(t)$ is Gaussian $\forall\ C_1, C_2$, this implies that $n_1(t)$ and $n_q(t)$ are jointly Gaussian.

6. $n_i(t)$ and $n_q(t)$ are orthogonal.

$$
\begin{aligned}
E\{n_i(t)n_q(t)\} &= E\left\{\sum_m b_m \cos(\omega_m t + \theta_m) \sum_l b_l \sin(\omega_l t + \theta_l)\right\} \\
&= 0
\end{aligned} \tag{8}
$$

since $E\{\cos(\cdot)\sin(\cdot)\} = 0$.

7. (1) + (5) + (6) imply $n_i(t)$ and $n_q(t)$ are independent.

Next, we perform a change of variables to express the noise process in the form of magnitude and phase:

$$\begin{aligned} n(t) &= n_i(t) + jn_q(t) \triangleq \alpha(t)e^{j\phi_n(t)} \quad (9) \\ \text{where}: \quad \alpha(t) &= \left[n_i^2(t) + n_q^2(t)\right]^{1/2} \\ \phi_n(t) &= \tan^{-1}\frac{n_q(t)}{n_i(t)} \end{aligned}$$

It may be easily proved (see Papoulis [6]) that, since $n_i(t)$ and $n_q(t)$ are independent and Gaussian, the amplitude $\alpha(t)$ is Rayleigh distributed, the phase $\phi_n(t)$ is uniformly distributed, and $\alpha(t)$ and $\phi_n(t)$ are independent. Thus, the phase history data may be written:

$$\begin{aligned} g(t) &= ae^{j\phi_\epsilon(t)} + \alpha(t)e^{j\phi_n(t)} \\ &= ae^{j\phi_\epsilon(t)}\left[1 + \frac{\alpha(t)}{a}e^{j(\phi_n(t)-\phi_\epsilon(t))}\right] \quad (10) \end{aligned}$$

The instantaneous phase of $g(t)$ will be denoted as $\phi(t)$ and is given by:

$$\phi(t) = \angle g(t) = \phi_\epsilon(t) + \tan^{-1}\left\{\frac{\frac{\alpha(t)}{a}Im\left\{e^{j(\phi_n(t)-\phi_\epsilon(t))}\right\}}{1 + \frac{\alpha(t)}{a}Re\left\{e^{j(\phi_n(t)-\phi_\epsilon(t))}\right\}}\right\} \quad (11)$$

In what follows, the t dependence in $\alpha(t), \phi_\epsilon(t)$, and $\phi_n(t)$ will be implicitly understood but occasionally omitted for clarity. If we assume a high signal/noise, then: $\alpha/a \ll 1$ and we have the following approximation:

$$\phi(t) \cong \phi_\epsilon(t) + \frac{\alpha}{a}Im\left\{e^{j(\phi_n-\phi_\epsilon)}\right\} \quad (12)$$

We will return to this assumption later. Let us denote:

$$V(t) \triangleq \frac{\alpha}{a}Im\left\{e^{j(\phi_n-\phi_\epsilon)}\right\} \quad (13)$$

so,

$$\phi(t) \cong \phi_\epsilon(t) + V(t) \tag{14}$$

Thus, the instantaneous phase of the range compressed data $g(t)$ is equal to the desired phase error function plus an additive noise term. Since our processing technique actually extracts the phase derivative (Eq. (3)), we have:

$$\dot{\phi}(t) \cong \dot{\phi}_\epsilon(t) + \dot{V}(t) \tag{15}$$

We now require the first and second order statistics for $\dot{V}(t)$. The difficulty lies in finding the variance. To do this, we solve for the autocorrelation function for $V(t)$, determine its power spectrum, find the power spectrum for $\dot{V}(t)$, and finally integrate to get the variance. Rewriting Eq. (13),

$$\begin{aligned} V(t) &= \frac{\alpha}{a} Im\left\{e^{j(\phi_n - \phi_\epsilon)}\right\} \\ &= 1/a \; Im\left\{\alpha e^{j\phi_n} e^{-j\phi_\epsilon}\right\} \\ &= 1/a \; Im\left\{n(t) e^{-j\phi_\epsilon(t)}\right\} \\ &= 1/a \; Im\left\{(n_i(t) + j n_q(t))\left(\cos(\phi_\epsilon(t)) + j\sin(\phi_\epsilon(t))\right)\right\} \\ &= 1/a\left[n_q(t)\cos(\phi_\epsilon(t)) + n_i(t)\sin(\phi_\epsilon(t))\right] \end{aligned} \tag{16}$$

The autocorrelation of the phase noise $V(t)$ is then:

$$\begin{aligned} R_v(\tau) &= \frac{1}{a^2} E\left\{\left[n_q(t+\tau)\cos(\phi_\epsilon(t+\tau)) + n_i(t+\tau)\sin(\phi_\epsilon(t+\tau))\right]\cdot\right. \\ &\qquad \left.\left[n_q(t)\cos(\phi_\epsilon(t)) + n_i(t)\sin(\phi_\epsilon(t))\right]\right\} \\ &= \frac{1}{a^2}\left[R_{n_q}(\tau) E\left\{\cos(\phi_\epsilon(t+\tau))\cos(\phi_\epsilon(t))\right\} + R_{n_i}(\tau) E\left\{\sin(\phi_\epsilon(t+\tau))\sin(\phi_\epsilon(t))\right\}\right] \\ &= \frac{1}{a^2} R_{n_q}(\tau) E\left\{\cos(\phi_\epsilon(t+\tau) - \phi_\epsilon(t))\right\} \end{aligned} \tag{17}$$

where the second step results from the fact that $n_i(t)$ and $n_q(t)$ are independent, zero mean, and the third because $R_{n_q}(\tau) = R_{n_i}(\tau)$. If we denote:

$$h(\tau) \triangleq E\left\{\cos(\phi_\epsilon(t+\tau) - \phi_\epsilon(t))\right\} \tag{18}$$

then:

$$\begin{aligned} R_v(\tau) &= \frac{1}{a^2} R_{n_q}(\tau) h(\tau) \\ &= \frac{1}{a^2} \left[\frac{\eta}{2} \sum_m \cos(\omega_m \tau) \right] \cdot h(\tau) \end{aligned} \tag{19}$$

The power spectrum of the process $V(t)$ is therefore:

$$\begin{aligned} S_v(f) &= \mathcal{F}\{R_v(\tau)\} \\ &= \frac{\eta}{2a^2} \left\{ \left[\sum_m \delta(f - f_m) \right] * H(f) \right\} \\ &= \frac{\eta}{2a^2} \sum_m H(f_m) \end{aligned} \tag{20}$$

If the processing bandwidth $-\omega_M < 2\pi f < \omega_M$ is wide enough to capture nearly all the energy in $e^{j\phi_\epsilon(t)}$, then:

$$\sum_{m=-M}^{M} H(f_m) \approx \int_{-\infty}^{\infty} H(f) df = h(0) \tag{21}$$

where:

$$\begin{aligned} h(0) &= E\{\cos(\phi_\epsilon(t+0) - \phi_\epsilon(t))\} \\ &= 1 \end{aligned} \tag{22}$$

and so we have:

$$S_v(f) = \frac{\eta}{2a^2} \tag{23}$$

After differentiation, the power spectrum of the process $\dot{V}(t)$ is:

$$\begin{aligned} S_{\dot{V}}(f) &= |j2\pi f|^2 S_v(f) \\ &= \frac{2\pi^2 \eta f^2}{a^2} \end{aligned} \tag{24}$$

We thus have finally:

$$
\begin{aligned}
\overline{\dot{V}^2(t)} &= R_{\dot{V}}(0) = \int_{-W}^{W} S_{\dot{V}}(f)df \\
&= \frac{2\pi^2\eta}{a^2}\int_{-W}^{W} f^2 df \\
&= \frac{4\pi^2\eta W^3}{3a^2} \\
&= \frac{C_0}{a^2} \qquad C_0 \triangleq \frac{4\pi^2\eta W^3}{3}
\end{aligned} \tag{25}
$$

where $\pm W = \frac{\pm\omega_M}{2\pi}$ is the processing bandwidth.

Equation (25) tells us the noise variance. We can immediately calculate the expected value of the noise:

$$
\begin{aligned}
E\left\{\dot{V}(t)\right\} &= E\left\{\frac{\dot{\alpha}(t)}{a}\sin\{\phi_n(t)+\phi_\epsilon(t)\} + \frac{\alpha(t)}{a}\cos\{\phi_n(t)+\phi_\epsilon(t)\}\left[\dot{\phi}_n(t)+\dot{\phi}_\epsilon(t)\right]\right\} \\
&= 0
\end{aligned} \tag{26}
$$

and show that the signal and noise are uncorrelated:

$$
\begin{aligned}
E\left\{\dot{V}(t)\dot{\phi}_\epsilon(t)\right\} &= E\left\{\frac{\dot{\alpha}(t)}{a}\sin\{\phi_n(t)+\phi_\epsilon(t)\}\dot{\phi}_\epsilon(t)\right. \\
&\qquad \left. + \frac{\alpha(t)}{a}\cos\{\phi_n(t)+\phi_\epsilon(t)\}\left[\dot{\phi}_n(t)+\dot{\phi}_\epsilon(t)\right]\dot{\phi}_\epsilon(t)\right\} \\
&= 0
\end{aligned} \tag{27}
$$

Both of these result from the fact that $\phi_n(t)$ is uniformly distributed and independent of $\phi_\epsilon(t)$. Consequently:

$$
\begin{aligned}
E\left\{\dot{\phi}(t)\right\} &= E\left\{\dot{\phi}_\epsilon(t)\right\} + E\left\{\dot{V}(t)\right\} \\
&= E\left\{\dot{\phi}_\epsilon(t)\right\}
\end{aligned} \tag{28}
$$

so that the phase derivative processor output for each range bin is unbiased. In summary, we have shown:

$$\begin{aligned} E\left\{\dot{V}\right\} &= 0 \\ E\left\{\dot{V}^2\right\} &= \frac{C_0}{a^2} \\ E\left\{\dot{V}\dot{\phi}_\epsilon\right\} &= 0 \\ E\left\{\dot{\phi}\right\} &= E\left\{\dot{\phi}_\epsilon\right\} \end{aligned}$$

IV. The Linear Minimum Variance Estimator

Now, let us return to the estimation problem. We are given K range bins of data:

$$g_k(t) = a_k e^{j\phi_\epsilon(t)} + n_k(t) \quad k\epsilon\,[1, K] \tag{29}$$

We assume the noise is uncorrelated from range bin to range bin, but with the same statistics. $\phi_\epsilon(t)$ is common to all range bins, but the target magnitude a_k varies. For each $g_k(t)$, we have a processor that extracts the instantaneous phase derivative:

$$\dot{\phi}_k(t) = \frac{Im\left\{\dot{g}_k(t)g_k^*(t)\right\}}{|g_k(t)|^2} \quad k\epsilon\,[1, K] \tag{30}$$

But from Eq.(15), the processor output for each range bin is simply the desired phase derivative of the phase error function plus additive noise. Also, recall that $\alpha(t)/a \ll 1$, so that from Eq.(10), $|g_k(t)|^2 \cong a_k^2$. We may thus write:

$$\dot{\phi}_k(t) = \frac{Im\left\{\dot{g}_k(t)g_k^*(t)\right\}}{a_k^2} = \dot{\phi}_\epsilon(t) + \dot{V}_k(t) \quad k\epsilon\,[1, K] \tag{31}$$

Where the assumption of high signal–to–noise ratio is not valid, an additional term is required to compensate for the bias in the denominator [2]. Writing Eq.(31) in vector form:

$$\dot{\mathbf{\Phi}} = \mathbf{H}\dot{\phi}_\epsilon + \mathbf{V} \tag{32}$$

where:

$$
\begin{aligned}
\dot{\mathbf{\Phi}} &= \begin{bmatrix} Im\left\{\dot{g}_1(t)g_1^*(t)\right\}/a_1^2 \\ \vdots \\ Im\left\{\dot{g}_K(t)g_K^*(t)\right\}/a_K^2 \end{bmatrix} \quad \text{is } K \times 1 \\
\mathbf{H} &= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \quad \text{is } K \times 1 \\
\mathbf{V} &= \begin{bmatrix} \dot{V}_1(t) \\ \vdots \\ \dot{V}_K(t) \end{bmatrix} \quad \text{is } K \times 1
\end{aligned}
$$

The covariance matrix of the noise process is (from Eq.(25)):

$$
\mathbf{R}_V = C_0 \begin{bmatrix} \frac{1}{a_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{a_K^2} \end{bmatrix} \tag{33}
$$

A linear unbiased minimum variance (LUMV) estimator for $\dot{\phi}_\epsilon(t)$ is given by [7]:

$$
\hat{\dot{\phi}}_\epsilon(t) = \left[\mathbf{H}^T\mathbf{R}_V^{-1}\mathbf{H} + \sigma_{\dot{\phi}}^{-2}\right]^{-1}\left[\mathbf{H}^T\mathbf{R}_V^{-1}\dot{\mathbf{\Phi}} + \sigma_{\dot{\phi}}^{-2}\mu_{\dot{\phi}}\right] \tag{34}
$$

where $\sigma_{\dot{\phi}}^2$ is the variance of $\dot{\phi}(t)$, and $\mu_{\dot{\phi}}$ is the mean. In both the PGA and PG algorithms, it is assumed that we have no knowledge of the statistics of $\dot{\phi}(t)$. That is, we treat it as a deterministic but unknown function. This is conventionally represented as:

$$
\begin{aligned}
\sigma_{\dot{\phi}}^{-2} &= 0 \\
\mu_{\dot{\phi}} &= 0
\end{aligned} \tag{35}
$$

in which circumstance Eq. (34) becomes:

$$
\hat{\dot{\phi}}_\epsilon(t) = \left[\mathbf{H}^T\mathbf{R}_V^{-1}\mathbf{H}\right]^{-1}\mathbf{H}^T\mathbf{R}_V^{-1}\dot{\mathbf{\Phi}} \tag{36}
$$

Substituting, we find:

$$
\begin{aligned}
\hat{\dot{\phi}}_\epsilon(t) &= \left[[11\cdots1]\ \frac{1}{C_0} \begin{bmatrix} a_1^2 & & 0 \\ & \ddots & \\ 0 & & a_K^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right]^{-1} \cdot \\
&\quad [11\cdots1]\ \frac{1}{C_0} \begin{bmatrix} a_1^2 & & 0 \\ & \ddots & \\ 0 & & a_K^2 \end{bmatrix} \begin{bmatrix} Im\left\{\dot{g}_1(t)g_1^*(t)\right\}/a_1^2 \\ \vdots \\ Im\left\{\dot{g}_K(t)g_K^*(t)\right\}/a_K^2 \end{bmatrix} \\
&= \left[\frac{1}{C_0}\sum_{k=1}^{K} a_k^2\right]^{-1} \frac{1}{C_0}\left[a_1^2\cdots a_K^2\right] \begin{bmatrix} Im\left\{\dot{g}_1(t)g_1^*(t)\right\}/a_1^2 \\ \vdots \\ Im\left\{\dot{g}_K(t)g_K^*(t)\right\}/a_K^2 \end{bmatrix} \\
&= \left[\sum_{k=1}^{K} a_k^2\right]^{-1} \left[\sum_{k=1}^{K} Im\left\{\dot{g}_k(t)g_k^*(t)\right\}\right] \\
&= \frac{\sum_{k=1}^{K} Im\left\{\dot{g}_k(t)g_k^*(t)\right\}}{\sum_{k=1}^{K} |g_k(t)|^2}
\end{aligned}
\tag{37}
$$

which is precisely the form of the estimator in the PGA and PG algorithms. We have thus shown that the phase derivative estimator used in those algorithms is in fact a minimum variance estimator. That is, no other linear estimator could do as well as this in the mean square sense.

V. Discussion

There is a remarkable connection between the phase gradient estimation problem addressed here and the problem of analyzing the noise performance of a frequency-modulated communication system. In both cases, one is attempting to measure the phase derivative of a signal in the presence of additive, narrowband noise. The "carriers" in the SAR problem are the complex sinusoidal phase histories of hard targets, while in the stellar imaging problem they are associated with the sources being imaged. The "message" is the phase error function we seek which, in fact, does frequency modulate the "carriers". (Phase modulation and frequency modulation are identical save for an integration of the modulating signal.) Of course, there is no concept of redundancy in FM while it was just such redundancy that was exploited in the current problem to give rise to a minimum variance estimator. However, the analysis of the noise process given here follows closely that originally developed for the FM problem.

It is natural to ask what else may be learned from the example of FM. In particular, we return to Eq. (12) and the assumption of high signal–to–noise ratio. A like assumption is made in the analysis of FM to arrive at a tractable expression for the noise spectrum. Where this assumption does not hold, an FM receiver exhibits a so–called threshold behavior. That is, as the received SNR falls, the detector output suddenly experiences the onset of message mutilation at a threshold SNR. Experimentally, the author and his colleagues have observed a similar behavior in processing synthetic SAR imagery. Using simulated targets and noise, we have demonstrated that phase error estimation can only be accomplished above a certain threshold SNR, below which the phase error estimate is hopelessly garbled.

A second question is that of post–detection filtering. Because of the non–linearity of exponential modulation, the pre–detection bandwidth can be much larger than the message bandwidth. In fact, the pre–detection bandwidth is a function of the message *amplitude* as well as its frequency content. As a result, FM systems often employ a post–detection filter to improve the output signal–to–noise ratio. A similar strategy might be advantageous for SAR and speckle image processing. In the case of the former, the PGA algorithm uses an adaptive pre–detection filter but no post–detection filter. We are currently investigating the use of a post–detection filter for the situation where the statistics of the phase error process are known, such as power law spectra for ionospheric induced errors.

VI. Conclusions

In this paper, we have shown that the phase derivative estimator used in the Phase Gradient Autofocus algorithm for SAR image processing and the Phase Gradient algorithm for stellar image reconstruction is in fact a linear minimum variance estimator. Using a conventional speckle noise model, we have shown that the estimator is linear and unbiased, we have solved for the power spectrum of the noise process, and have derived the linear minimum variance estimator of the phase derivative given redundant observations. This was seen to be identical to the estimator used in the PGA and PG algorithms.

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