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ON A CLASS OF NONLINEAR DISPERSIVE-DISSIPATIVE INTERACTIONS

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Abstract

We study the prototypical, genuinely nonlinear, equation; $u_t + a(u^m)_x + (u^n)_{xxx} = \mu(u^k)_{xx}$, $a, \mu = \text{consts.}$, which encompasses a wide variety of dissipative – dispersive interactions. The parametric surface $k = (m + n)/2$ separates diffusion dominated from dissipation dominated phenomena. On this surface dissipative and dispersive effects are in detailed balance for all amplitudes. In particular, the $m = n + 2 = k + 1$ subclass can be transformed into a form free of convection and dissipation making it accessible to theoretical studies. Both bounded and unbounded oscillations are found and certain exact solutions are presented. When $a = (2\mu/3)^2$ the map yields a *linear equation*; rational, periodic and aperiodic solutions are constructed.

1. Introduction.

The complexity of nonlinear phenomena, and the very limited analytical means presently available for their modelling, severely limits the scope of our scientific endeavors. Though in reality one rarely encounters phenomena which are either purely dissipative or dispersive, the means available for the

study of these phenomena differ to such an extent that, unless a head-on computing is employed, with rare exceptions, these phenomena are studied separately. In this context the celebrated KdV and the Burgers equations

$$u_t + uu_x + u_{xxx} = 0 \quad \text{and} \quad u_t + uu_x = u_{xx} \quad (1a, b)$$

have became the outstanding paradigms describing convective-dispersive and convective-dissipative interaction, respectively.

However, it is not only a matter of fashion, convenience, or herd instincts, that we study separately patterns shaped by dispersion and dissipation. Pushed to extremity one might say that while dispersive systems cannot forget their past, the dissipative ones do not remember it. The old parable of Archilochus; the fox knows many things, but the hedgehog knows one big thing, acquires a new meaning when one identifies dispersion with the fox and dissipation with the hedgehog. Indeed, while the conservation laws of a dispersive system, at all times carry the memory of the initial startup, dissipative systems respond like the fabled hedgehog; care very little about their initialization, shaping the future according to their own, predetermined, blue-print.

It is thus not surprising that models which combine 'fox-like' with 'hedgehog-like' features, say; the combined *KdV – Burgers* equation

$$u_t + uu_x + u_{xxx} = u_{xx}, \quad (2)$$

are so hard to analyze. The competition between such a different entities as dispersion and dissipation, very rarely turns into a cooperative interaction, but when it does, an analytical glimpse into these phenomena becomes possible.

The typical model, derived in the weakly nonlinear limit, eliminates most of the phenomena related to large gradients and/or amplitudes like wave breaking, their collapse, fusion or saturation. The use of linear dissipation or dispersion in such a model is, as a rule, done out of convenience or necessity

(as usually nothing better is available) then from the conviction that this is the true state of affairs. The linearity, however, takes its toll; it brings in undesirable features as, say, the infinite tail of the typical soliton (or the infinite Gaussian tail), being the consequence linear dispersion (or diffusion). The compactification of thermal pulses due to the nonlinear conductivity, is perhaps the simplest and the most striking example, that reveals how nonlinearity can combat the nuisance of an infinite tail.

Similarly, it was recently found that *nonlinear dispersion* can compactify solitary waves and generate compactons-solitons with a compact support [1-3]. As a prototypical dispersive model that describes compact patterns, I have recently proposed [1-3] to extend the K-dV type equations into a genuinely *nonlinear dispersion regime* and to consider

$$\mathbf{K}(\mathbf{m}, \mathbf{n}); \quad u_t + a(u^m)_x + (u^n)_{xxx} = 0, \quad m, n \geq 1, \quad a = \text{const.} \quad (3a)$$

For $a > 0$, compact solitary travelling structures are possible, and for $n = m$ have a very simple form

$$u = \left\{ \frac{2\lambda n}{n+1} \cos \left[\frac{n-1}{4n} (x - \lambda t) \right] \right\}^{\frac{2}{n-1}} \quad \text{for} \quad |x - \lambda t| \leq \frac{2n\pi}{n-1} \quad (4)$$

and zero otherwise. For $a < 0$ solitary patterns having cusps, peaks or infinite slope may form, all being the manifestation of nonlinear dispersion in action [4]. Its dissipative counterpart, a fully nonlinear variant of the Burgers equation

$$\mathbf{B}(\mathbf{m}, \mathbf{k}); \quad u_t + a(u^m)_x = \mu(u^k)_{xx} \quad (3b)$$

also admits simple solutions [5]. Returning to our modest goal we *propose* herein a new model equation which goes beyond Eq.(2), and merges into one equation the interaction between convection, dispersion and dissipation, all assumed to be *genuinely nonlinear* functions of state variable. Merging Eq. (3a) and (3b) we thus propose a combined dispersive-dissipative entity

$$\text{DD}(\mathbf{k}, \mathbf{m}, \mathbf{n}); \quad u_t + a(u^m)_x + (u^n)_{xxx} = \mu(u^k)_{xx}, \quad a, \mu = \text{consts.} \quad (5a)$$

The exponents m, n and k in (5a) span a wide variety of nonlinear scenarios. In fact while in each of equations (3) the pattern is formed via balance between dispersive (or dissipative) forces and convection, having three mechanisms enables, roughly speaking, three kinds of scenarios.

- A.** Phenomena dominated by balance between dispersion and convection with dissipation playing secondary role that manifests itself mainly on long temporal scales.
- B.** Phenomena dominated by balance between dissipation and convection with dispersion playing a secondary role and
- C.** phenomena characterized by a detailed, three ways, balance between dissipation, convection and dispersion.

To unfold these classes we look first at the scaling properties of solutions to equation (5a) as a function of the exponents k, m and n . Invariance of Eq. (5a) under shifts in space and time affords steadily progressing waves and the associated scaling relations between speed, width and the amplitude of these structures.

So far the $\text{DD}(\mathbf{k}, \mathbf{m}, \mathbf{n})$ was addressed only in two special cases, the Burgers-KdV, Eq.(2),[6], and *Burgers – mKdV*, [7]. In both cases the main effort was directed to elucidate the limiting behaviour when either the dissipation or dispersion tend to zero. This usually is motivated by a mathematical quest to understand how weak solution of the purely convective problem $u_t + (u^m)_x = 0$ are modified by dispersion and/or dissipation. From physical point of view, the main interest is to understand the formation of patterns and their topology, and will be our main concern here.

One could say that since the combined structure is not integrable there is no *a priori* reason to consider a KdV-like like extension of dissipative process

rather than a BBM like extension. With an equal vigor one could make an argument in favor of

$$u_t + a(u^m)_x = u_{xxt} + \mu(u^k)_{xx}, \quad a, \mu = \text{consts.} \quad (5b)$$

This equation and its fully nonlinear dispersive extensions, have a life of their own that perhaps warrants a separate consideration. However, as a first step, Eq.(5a) should certainly suffice. We thus return to Eq.(5a) and consider

2. Scales and scalings.

Let $s = x - \lambda t$, then integrating once Eq. (5) and setting

$$u = \alpha U(s), \quad \alpha = \lambda^{1/(m-1)} \quad (6a)$$

we obtain

$$-U + aU^m + \alpha^{n-m}(U^n)_{ss} = \mu\alpha^{k-m}(U^k)_s + P_0(\text{const.}) \quad (6b)$$

Let further define $\eta = \lambda^\varpi s$, $\varpi = (m - n)/(2m - 2)$. Rescaling again and neglecting the integration constant P_0 , and setting $U = U[\lambda^\varpi(x - \lambda t)]$, we obtain

$$-U + aU^m + (U^n)_{\eta\eta} = \sigma(U^k)_\eta, \quad \sigma = \mu\lambda^\beta, \quad \beta = \frac{2k - (m + n)}{2(m - 1)}. \quad (6c, d, e)$$

β vanishes when

$$k = (m + n)/2 \quad (7)$$

In this special case the emerging patterns are universal in the sense that they are independent of speed, and thus of amplitude. Otherwise $U = U(\eta, \sigma)$ and the effective dissipation coefficient depends on the amplitude of the

wave. For $2k < m+n$, the dissipation is inhibited at high amplitudes and the pattern is dominated by dispersion, while for $2k > m+n$ the opposite occurs; the process is governed by balance between convection and dissipation, with dispersion playing secondary role.

A. Consider first a dissipative extension of the KdV equation; now $\sigma = \mu\lambda^{k-3/2}$. For $k = 1$ we have the *KdV - Burgers* equation (2) (or the DD(1,2,1) equation, in the notation (5a)), with $\sigma = \mu/\sqrt{\lambda}$. At large amplitudes σ decreases and, on a short time scale, the dissipative effects are secondary. The opposite occurs for small amplitudes, which are dominated by dissipation, and therefore will quickly merge into high amplitude patterns and disappear. Recent numerical simulations carried for equation(2),[8], fully confirm these conclusions.

Now let the assumed dissipation be quadratic in u ; then $k = 2 \implies \sigma = \mu\sqrt{\lambda}$. The effective dissipation increases with amplitude and high amplitudes patterns dissipate quickly, while the low amplitude phenomena persist on a much longer time scale.

B. Consider now a dissipative extension of the m-KdV equation; $m = 3$ and $n = 1$ imply that $\sigma = \mu\lambda^{k-2}$. For $k = 1, 2$ and 3 , $\sigma = \mu/\lambda, \mu$, and $\mu\lambda$, respectively. Here $k = 2$ is the critical value; the effective dissipation does not depend on amplitude.

The total mass of a travelling structure is another way to look at the effects of scaling. In the purely dispersive $K(m, n)$ case the mass scales as

$$M_0 = \int u dx = \lambda^\beta \int U(s) ds, \quad \beta = (n+2-m)/2(m-1), \quad (9)$$

while for the dissipative case (3b) we have; $\beta = (k+1-m)(m-1)$. Consequently the total mass of a pattern in the $DD(k, m, n)$ equation is independent of the amplitude, **iff** $m = n + 2 = k + 1$.

Let us now consider the implications of invariance under a group of stretchings and the consequent similarity solutions. For the $K(m, n)$ one has

$$u = t^{1/\Delta} F(\zeta), \quad \zeta = xt^{-\alpha}, \quad \alpha = (n-m)/2\Delta, \quad \Delta = 1 + (n-3m)/2 \quad (10a)$$

provided that $\alpha > 0$, while for the purely dissipative case (3a) ($\alpha > 0$)

$$u = t^{1/\Delta} F(\zeta) = xt^{-\alpha}, \quad \alpha = (k-m)/\Delta, \quad \Delta = 1 + k - 2m. \quad (10b)$$

When $\alpha < 0$, the self-similar solutions represent phenomena that terminate within a finite, say t_0 , time. Instead of (10a) we now have

$$u = (t_0 - t)^{1/\Delta} F(\zeta), \quad \zeta = x(t_0 - t)^{(m-n)/2\Delta} \quad (11)$$

and a similar modification applies to the dissipative variant. It thus follows that the combined case admits a stretching symmetry iff $2k = m + n$, which is nothing more than the universality condition (7). In this case the symmetries of the dissipative and dispersive processes have a non-empty overlap. In Sec. 4.2 we shall present a family of such solutions. If consistency condition (7) is not satisfied, self-similar structures may emerge only asymptotically, when either the dissipative or the dispersive mechanism becomes suppressed.

We also note two exceptional cases that occur when either α or Δ vanish. Exploiting the invariance under shifts in time or space, one finds that each of these cases reflects a spiral symmetry, and induces similarity structures of the form

$$u = t^{1-m} F(x + \lambda lnt), \quad \text{if } m = n \quad \text{and} \quad m = k, \quad (12a)$$

and

$$u = e^{-\lambda t} F[xe^{2(m-1)t}] \quad \text{if} \quad n = 3m - 2 \quad \text{and} \quad k = 2m - 1. \quad (12b)$$

The presented discussion is not a systematic study of symmetries of Eq. (5a) but rather an outline of system's response to changes in scales. For a particular choice of nonlinearity exponents, additional symmetries may emerge. For instance, the KdV-Burgers equation (2) has a solution in an accelerated frame of reference

$$u(x, t) = U(\xi) - \lambda t, \quad \xi = x + a\lambda t^2. \quad (13)$$

The reader is challenged to find $U(\xi)$.

3. Travelling Patterns

In this section we shall outline some basic features of the steadily travelling structures of Eq.(6). We shall

1. Determine the topology of these structures and in particular their limiting forms like kinks or solitons.
2. Seek their explicit forms.

We start with Eq.(6) and note that without dispersion this equation is immediately integrable. Without dissipation, energy integral is available which as a rule reduces the problem into an integrable, albeit not necessarily in terms of elementary functions. However, the presence of both dissipation and dispersion makes the problem next to impossible and save for the special case $m = n + 2 = k + 1$, a first integral of motion is not available. One has therefore resort either to numerical or to phase space methods. Let $M = \partial_\eta U$. Since $\partial_\eta = M\partial_u$ then from (6c) we have

$$-U + aU^m + nM\partial_u(U^{n-1}M) = k\sigma U^{k-1}M \quad (14a)$$

$$\text{or , if } N = U^{n-1}M \quad (14b)$$

$$-U^n + aU^{m+n-1} + nN\partial_u N = k\sigma U^{k-1}N. \quad (14c)$$

In what follows we shall assume that $a \pm 1$.

3.1 Elements of phase portrait.

It is clear from(6c) that $U = 0$ and $U = 1$ (or -1 if $a = -1$) are singular points that need special attention. When $u = 0$ is not involved, we linearize the flow near the singular manifold at $U = 1$. If $U \sim \exp(\gamma\eta)$, then

$$\tilde{\gamma} = \frac{\sigma k \pm \sqrt{\Delta}}{2n} \text{ where } \Delta = \sigma^2 k^2 - 4n(m-1). \quad (15)$$

When $\Delta > 0$ and $n > 0$, $U = 1$ is a node which is characteristic of kink type solutions. When $\Delta < 0$; $U = 1$ is a spiral and the solution in its vicinity is oscillatory, a typical setup for underdamped oscillations. In the present context this means that dispersion dominates dissipation. The critical value of the normalized dissipation

$$\sigma_c \equiv \mu_c \lambda_c^\beta = \frac{2}{k} \sqrt{n(m-1)}. \quad (16a, b)$$

separates between dissipation and dispersion dominated solutions. It associates a critical speed with every value of μ ; $\lambda_c = (\mu/\sigma_c)^{1/\beta}$. Thus the traveling wave is oscillatory if $\sigma < \sigma_c$ or, what amounts to the same, if $\lambda^\beta < \lambda_c^\beta$.

3.2 The front line.

If $n > 1$ and $k > 1$, then at $u = 0$ nonlinearity degenerates and, typically, the solution has there a weak discontinuity. It is clear that this point plays a special role in our discussion as the singular manifold may be essentially nonlinear. The assumption that both $k > 1$ and $n > 1$ is needed (and occasionally that $a > 0$) to assure the existance of a front, otherwise the front of the propagating wave(s) will run away to infinity. Consideration of the $K(m, n)$ reveals that for dispersive structures like compactons [1-3], near the front line located at, say $x = 0$, we have $u \approx x^{2/(n-1)}$ while for the dissipative Eq. (3b), $u \approx x^{1/(k-1)}$. Both effects are in balance only when $n + 1 = 2k$. Otherwise, comparing the dispersive part with the dissipative

contribution, we find that near the front dispersion (dissipation) dominates for $n + 1 < 2k$ ($n + 1 > 2k$).

Thus, for $n = k = 2$, the behavior near the front line is shaped by the dispersive part and $u \approx x^2$. Without the dispersion, the behavior of the front is determined by the nonlinear diffusion which dictates here $u \approx x$. For $n = 3$ and $k = 2$, dispersion and dissipation are in balance; each considered separately predicts that $u \approx x$. Note that convection has not entered into these considerations. Its crucial role is reserved to the overall dynamics.

3.3 Explicit solutions.

The difficulty to derive explicit solutions makes it necessary not to dismiss any approach even if of a very limited scope. In what follows we describe two attempts at our problem;

1. Factoring of Operators ;

A. Let $n = 1$, $\sigma = a$, and $k = m$, then Eq.(6c) may be rewritten as ($\partial_\eta \equiv \partial$)

$$(\partial^2 - 1)U = a(\partial - 1)U^m. \quad (17a)$$

B. Let $a = -1$, $k = \sigma = 1$, $m = n$, and $\lambda = 1 \rightarrow \lambda = -1$, then Eq.(6c) in its defocusing version may be rewritten as

$$(\partial^2 - 1)U^m = (\partial - 1)U. \quad (17b)$$

The presence of a common factor enables to simplify our search for solutions. For (17a) the reduced problem reads

$$(\partial + 1)U - aU^m = U_0 e^\eta. \quad (18)$$

When $U_0 = 0$ the solutions are easily found to be kinks ($a = 1$)

$$U = \frac{1}{[1 + A \exp(-\eta/\alpha)]^\alpha}, \quad \alpha = \frac{1}{m-1} \quad (19)$$

Let $U_0 \neq 0$. For $m = 2$ (18) is a Riccati equation. When put into a linear form it generates an unbounded solution. Though for $m \neq 2$ no explicit solutions of (18) are known, the presence of the exponential part in (18), precludes bounded solutions for $m \neq 2$ as well.

The operator factoring may be better seen if we note that for $m = k$ and $n = 1$ equation (5a) may be rewritten as

$$u_t + A^2 u_x + (-A + \partial_x) \partial_x [u_x - \nu u^m + Au] = 0 \text{ where } A = a/\nu. \quad (20)$$

Similarly, for $m = n$ and $k = 1$ we write

$$u_t + A\nu u_x + (A + \partial_x) \partial_x [(u^m)_x - Au^m - \nu u] = 0 \text{ where } A^2 = -a > 0. \quad (21)$$

We now move into a frame of reference free of the linear advection (i.e., $x \rightarrow y = x - \lambda t$ and $t \rightarrow \tau = t$ where $\lambda = A^2$ and $A\nu$, respectively). In the new travelling coordinates we seek a stationary solution. This yields exactly the previous results and clarifies their very special nature. Since the resulting subsystem is governed by a polynomial first order differential equation, for those cases that afford such a split, in general one should not expect oscillatory patterns.

A more evolved operator factoring is found via

$$[\partial + A(1-u)][\partial + B(1+u)]u = u_{xx} + (2B - A)uu_x + (A + B)u_x + ABu(1 - u^2). \quad (22)$$

There are two natural choices; either $A = 2B$ or $A = -B$. In the first case

$$u_t - 2B^2(u^3)_x + u_{xxx} = 3Bu_{xx} \quad (23)$$

is factored into two first order pieces

$$u_t - 2B^2u_x + [\partial + 2B(1-u)][\partial + B(1+u)]u = 0. \quad (24a)$$

Eliminating the linear advection via $x \rightarrow y = x + 2B^2t$, $t \rightarrow t$, we find a kink solution: $u = 1/[C_0 \exp(By) + 1]$. The second case enables to factor

$$u_t - \alpha u_x + \alpha(u^3)_x + u_{xxx} = \nu(u^2)_{xx} \quad \text{where } \alpha = 4\nu^2/9 \quad (24b)$$

into two parts. This case will be shown shortly to be exactly linearizable and thus need not to be pursued any further here.

The possibility to factor operators is a topic of a far wider applicability than can be treated here. Let us only note that if $m = k$ and $a = \mu$ then equation (5b) can be factored as

$$(1 - \partial_x)[(1 + \partial_x)u_t + \mu\partial_x u^m] = 0$$

which enables to find solutions to equation (5b) by solving the second order equation

$$u_{xt} + u_t + \mu(u^m)_x = 0.$$

2. A Direct Ansatz; the various approaches presented overlap to some extent but not completely. This gives some hope for something new. The idea explored now is simple: a change of variables introduces a degree of freedom which is then utilized to decompose the problem into a solvable sequence of simpler ones.

Consider the *KdV – Burgers* case; $k = n = 1, m = 2$. We restore the integration constant P_* in (6c) and use the ansatz

$$U = U_0 + e^{2\alpha\eta} F(e^{\alpha\eta}).$$

The choice of the powers in the exponent is dictated by the balance between nonlinear advection and dispersion. Eq.(6c) now takes the form

$$A_0 + A_2 e^{2\alpha\eta} + A_3 e^{3\alpha\eta} + A_4 e^{4\alpha\eta} = 0. \quad (25a)$$

We solve (25a) equating each A_n to zero. This leads to four equations which determine the parameters as follows

$$\alpha = \sigma/5, \quad U_0 = \frac{1}{2a} + \frac{3\sigma^2}{25a}, \quad P_* = 2U_0 - U_0 a \quad (25b)$$

and an equation for F ;

$$\sigma^2 F'' + 25aF^2 = 0. \quad (25c)$$

The solution of (25c) is given in terms of an elliptic function, but the solution to U

$$U = U_0 + s^2 F(s), \quad \text{where } s = \exp[\sigma\eta/5], \quad (26a)$$

develops unbounded oscillations as $\eta \rightarrow \infty$. A bounded solution is obtained when F degenerates into an algebraic entity;

$$U = U_0 - \frac{F_0}{(1 + e^{-\alpha\eta})^2}, \quad F_0 = 6\sigma^2/25a. \quad (26b)$$

The solution trajectory connects U_0 with $U_0 - F_0$. Since we have to connect the steady states, $U = 0$ and $U = 1$, we need $U_0 = F_0 = 1$, which implies $\nu/\lambda \equiv \sigma^2 = 25/6$. Thus the speed of the kink for a given upstream-downstream pair is, again, limited to **one definite value**.

A similar ansatz may be used in a number of other cases, say; $m = k = 2$ and $n = 1$ or $n = 1, k = 2, m = 3$. In each case a balance between the dominant parts dictates an ansatz of the form

$$U = U_0 + sF(s), \quad \text{where } s = e^{\alpha\eta}. \quad (27)$$

Repeating the analysis we find that the sought after bounded solutions are restricted to the simple kinks already presented. Of course, another ansatz could perhaps do the trick, but the challenge to find it remains unanswered.

4. The Distinguished Case: $m = n + 2 = k + 1$.

This case will emerge as the only where a meaningful progress can be made. We start with

4.1: Potential Representation.

Let $\mu = \theta_x$. Integrating once, Equation (5a) becomes

$$\theta_t + a(\theta_x)^m + [(\theta_x)^n]_{xx} = \mu[(\theta_x)^k]_x, \quad a, \mu = \text{consts.} \quad (28)$$

The next step is to look for an integrating factor which enables to cast (28) into a conserved form. Upon integration by parts, we find this to be possible only if $\mathbf{m} = \mathbf{n} + 2 = \mathbf{k} + 1$, and yields

$$f_t + \partial_x \left\{ f'(\theta)[(\theta_x)^n]_x + af(\theta)(\theta_x)^{n+1} \right\} = 0, \quad (29)$$

provided that the integrating factor $f'(\theta)$ satisfies the auxiliary linear condition

$$\frac{n}{n+1}f'''(\theta) + \mu f''(\theta) + af'(\theta) = 0. \quad (30)$$

If $\mu \geq \sqrt{4an/(n+1)}$ then f takes the form $f(\theta) = \exp(r_n \theta)$, where

$$r_n = -\mu_n \pm \sqrt{\mu_n^2 - \frac{a(n+1)}{n}} \quad \text{and} \quad \mu_n = \frac{n+1}{2n}\mu. \quad (31a)$$

These solutions represent an 'overdamped' mode of propagation. In contrast, if $\mu < \sqrt{4an/(n+1)}$

$$f(\theta) = \exp(-\mu_n \theta) \cos(\omega_n \theta) \quad \text{where} \quad \omega_n = \sqrt{\frac{a(n+1)}{n} - \mu_n^2}. \quad (31b)$$

We digress to note that the *purely dissipative* equation (3b), begets

$$\theta_t + a\theta_x^m = \mu(\theta_x^k)_x. \quad (32a)$$

Under the action of the integrating factor, in normalized time units, this equation is cast into

$$v_t = \left[v(v_x/v)^k \right]_x. \quad (32b)$$

For $k = 1$, the quest for an integrating factor in (32a) (which for $k = 1$ is a potential representation of the Burgers equation), reconstructs the Hopf-Cole map, and Eq. (32b) reduces to the linear heat equation. For $k = 1/2$ or $k = 2$ using another potential function w ($v = w_x$) we have

$$(w_t)^2 = w_x w_{xx} \text{ and } w_x w_t = [w_{xx}]^2, \quad (33)$$

respectively.

Returning to our problem one observes that in the overdamped case, the $f - \theta$ relations enable to use f as a dependent variable and cast Eq.(29) into

$$\begin{aligned} \mathbf{n} = \mathbf{1}; \quad \gamma f_t &= \left[f^{1+\gamma} (f^{-\gamma})_{xx} \right]_x \quad \text{where } \gamma = 1/2 + \mu/r_1 \\ \mathbf{n} = \mathbf{2}; \quad f_t &= \left[f_x f^\gamma (f^{-\gamma})_{xx} \right]_x \quad \text{where } \gamma = 1/3 + \mu/2r_2 \end{aligned} \quad (34a, b)$$

etc., Clearly, the trace of dissipation is carried by γ . The original variable u is recovered via

$$u = \frac{f_x}{r_n f}. \quad (35)$$

The γ as defined in (34a) has two branches

$$\gamma_{\pm} = \frac{1}{2} + \frac{1}{-1 \pm \sqrt{1-2\delta}} \implies \gamma_+ \gamma_- = \frac{1}{4}, \quad (34c)$$

where

$$\delta \equiv a/\mu^2. \quad (34d)$$

Observe that a and μ enter into the problem only via δ . While γ as a function of δ is continuous, with $-1/2 < \gamma_- < 1/2$, γ_+ is discontinuous and unbounded at $\delta = 0$ (this corresponds to $a = 0$). As is clear from (34c), both branches

coincide at $\delta = 1/2$ (the upper limit of the overdamped mode), and $\delta = -\infty$ the purely dispersive limit.

We note the following invariance property of Eq.(34a);

Let $f^ = f^{-2\gamma}$ and $\gamma^* = 1/4\gamma$, then Eq.(34a) is invariant under $f \rightarrow f^*$, $\gamma \rightarrow \gamma^*$ (and $t \rightarrow t/2\gamma$).* Proof by substitution.

The lemma assures that every solution obtained for a particular value of γ , can be used to generate another solution via its ‘conjugate’ value γ^* corresponding to the same δ . In the purely dispersive case $\gamma = 1/2$, $\gamma = \gamma^*$ and Eq.(34a) is invariant under $f \rightarrow 1/f$. This is the potential counterpart of the invariance of the mKdV, which associates with every speed λ two solutions; u and $-u$.

That γ depends only on δ , merely reflects the fact that for the distinguished class, we are dealing here, defining

$$v = \mu u, \quad \delta = a/\mu^2 \quad \text{and} \quad T = t/\mu^{n-1}, \quad (34e)$$

yields

$$v_T + \delta(v^{n+2})_x + (v^n)_{xxx} = (v^{n+1})_{xx}. \quad (34f)$$

Thus δ is the only relevant parameter in terms of which all properties of this class can be expressed.

Is the representation (34) of Eq.(5a), really a simplification? Though the convection and dissipation parts of the original problem have been eliminated, their replacement is quite cumbersome. Yet, insofar as special solutions are concerned, equations (34a,b) offer a great advantage over the original Eq.(5a). Here we consider the implications of two obvious symmetries, that of stretchings and Galilean boosts;

Let $\mathbf{n} = 1$. The invariance of Eq.(34a) under the group of stretching provides similarity solutions of the form

$$f = \tau^{-1/3}\Phi(\eta), \quad \eta = x/(3\tau^{1/3}), \quad \tau = t/\gamma, \quad (36a)$$

and Φ satisfies

$$-\eta\Phi = \Phi^{\gamma+1}(\Phi^{-\gamma})_{\eta\eta} + C_0, C_0 = \text{const.} \quad (36b)$$

If $C_0 = 0$ then in terms of $Z = \Phi^{-\gamma}$ we have a linear equation

$$Z_{\eta\eta} + \eta Z = 0, \quad (37a)$$

solved via the Airy functions $\text{Ai}(\eta)$ and $\text{Bi}(\eta)$;

$$f = \tau^{-1/3}[A_i(\eta) + C_1 B_i(\eta)]^{-\gamma}, \quad C_1 = \text{const.} \quad (37b)$$

In terms of u we have ($t \equiv d/d\eta$)

$$u = \frac{-u_0}{(3\tau)^{1/3}} \left[\frac{A'_i(\eta) + C_1 B'_i(\eta)}{A_i(\eta) + C_1 B_i(\eta)} \right], \quad u_0 \equiv \gamma/r_1. \quad (38)$$

However, due to the divergence of either $A'_i(\eta)$ or $B'_i(\eta)$ as $|\eta| \rightarrow \infty$, this solution is unbounded. If the integration constant C_0 is kept, the problem is far more difficult. Apart of two special cases; $\gamma = -1$, which renders the problem linear

$$-\eta\Phi = \Phi_{\eta\eta} + C_0, \quad (39)$$

and the purely dispersive $\gamma = 1/2$ case, which yields a second Painleve transcendent in terms of $V = \Phi^{-1/2}$, I cannot say much about its solution.

The constant C_0 in (39) induces an additional part in the solution;

$$f_p = C_0 \eta^2 \sum a_{3n} \eta^{3n} \quad \text{where } a_{3n} = (-1)^n \frac{3^n n!}{(3n+2)!}.$$

To understand the origin of the special linear subcase (37a), let us derive the similarity form directly. Since $m = k + 1 = n + 2$, we obtain, see sec.1, $u = t^{-\frac{1}{n+2}} F(\zeta = x/t^{-\frac{1}{n+2}})$. After one integration

$$\frac{-1}{n+2}(\zeta F) + a F^{n+2} + (F^n)'' = \mu(F^{n+1})' + C_1, \quad (40a)$$

where C_1 is a constant. We now observe that in the overdamped case presently considered, equation (40a) may be factored into

$$(\partial_\zeta + r_n F) \left[-\frac{1}{n+1} \zeta + a F^{n+1} + r_n (F^{n+1})' \right] = C_2, \quad (40b)$$

where $C_2 = C_1 - 1/(n+1)$ is another constant. If $C_2 = 0$, then Eq.(40b) reduces into a first order equation. For $n = 1$ it is a Riccati equation, which can be linearized via an uplift to a second order. The resulting form coincides with (37a). For $n \neq 1$ even the first order equation does seem to be solvable in an explicit form.

4.2: A Few Special Cases

For certain values of γ , Equation (34a) takes a particularly attractive form. Thus when the radical in (31a) vanishes, we have a double root $\mu/r_1 = -1$ ($\Rightarrow \gamma = -1/2$), then in terms of $\phi = \sqrt{f}$ our problem reads

$$\gamma \phi \phi_t = [\phi \phi_{xx}]_x. \quad (41a)$$

For future reference we record another interesting form for $n = 2$ and $\gamma = -1$

$$w_x w_t = w_{xx} w_{xxx} \quad \text{where } f = w_x. \quad (41b)$$

Perhaps the most remarkable particular case occurs when $\gamma = -1$ (thus $\delta = 4/9$), for in this case Eq.(34a) reduces to the **linear dispersive equation**

$$f_t + f_{xxx} = 0. \quad (42)$$

To recapitulate; Eq.(42) represents the $DD(k = 2, m = 2, n = 1)$ for $a = 4\mu^2/9$. We turn now to exploit the linearity of Eq.(42) and consider

Travelling waves; depending on the direction of propagation there are kinds of waves;

A. Waves propagating to the left. We set $s = x - \lambda t$ and f has to be of the form $f = f_0 + \sum a_n \cos(\sqrt{\lambda_n} s)$. A one mode solution is ($u_0 = 3/2\mu$)

$$\frac{u}{u_0} = -\frac{\sqrt{\lambda_1} \sin(\sqrt{\lambda_1} s)}{f_1 + \cos(\sqrt{\lambda_1} s)}, \quad f_1 = \text{const.} \quad (43)$$

Certainly, for $|f_1| > 1$ this is a nice periodic wave. However when two (or more) modes are involved and their speed ratio $\sqrt{\lambda_1/\lambda_2}$ is not commensurate, periodicity is lost. Let $\lambda_1 = 1$ and $\lambda_2 = 2$ and write a two mode solution as

$$\frac{u}{u_0} = -\frac{b_1 \sin(s) + b_2 \sqrt{2} \sin(\sqrt{2}s)}{1 + b_1 \cos(s) + b_2 \cos(\sqrt{2}s)} \quad (44)$$

If $|b_1| + |b_2| < 1$, the solution stays bounded but the periodicity is lost. However if $|b_1| + |b_2| = 1$ (or ≥ 1), things change drastically; depending on how closely the two periods overlap one, obtains bursts of intermittency. The theory of Diophante approximations assures that in an arbitrarily large interval, u will attain any value with a frequency inversely dependent on the amplitude. This means that on an open interval we should expect both bounded and unbounded burst(s).

For a periodic motion the strength of the intermittency depends on the period; there is a minimal interval of periodicity in which u , with a predetermined level of proximity, will attain at least once a prescribed value.

B. Waves propagating to the right; Now f is of exponential type and can be used to describe interaction of kinks. However, a more interesting interaction emerges if one mixes waves moving in both directions. A simple example is one in which a periodic waves (traveling to the left) are superimposed on a kink which travels to the right (α and β are arbitrary constants);

$$\frac{u}{u_0} = -\frac{\alpha \sinh[\alpha(x + \alpha^2 t)] - b_0 \beta \sin[\beta(x - \beta^2 t)]}{\cosh[\alpha(x + \alpha^2 t)] + b_0 \cos[\beta(x - \beta^2 t)]}, \quad |b_0| < 1. \quad (45)$$

The resulting solution is a bounded ($|b_0| < 1$) oscillating breather; the largest oscillations are at the center, they travel out and decay exponentially.

The overall solution is best seen in a frame moving with the kink. If $s = x + \alpha^2 t$, then the composite solution reads

$$\frac{u}{u_0} = -\frac{\alpha \sinh(\alpha s) - b_0 \beta \sin \beta[s - (\beta^2 + \alpha^2)t]}{\cosh(\alpha s) + b_0 \cos \beta[s - (\beta^2 + \alpha)t]}, \quad |b_0| < 1. \quad (46)$$

As can be seen from the example in Fig.(1), the oscillatory part decays quickly away from the center.

Rational solutions; In analogy with the polynomial solutions to the Laplace (or heat) equation, one constructs polynomial solutions $P_n(x)$ to Eq.(42). In terms of u , these are rational functions. We use, for instance

$$P_6(x, t) \equiv \frac{x^6}{6 \cdot 5 \cdot 4} - tx^3 + 3t^2 + C_0, \quad \text{and } C_0 = \text{const.}, \quad (47a)$$

to construct

$$\frac{u}{u_0} = -\frac{x^5 - 60tx^2}{P_6(x, t)}. \quad (47b)$$

This solution, which starts at $t = 0$ as a nice pulse

$$\frac{u}{u_0} = -\frac{6x^5}{x^6 + 120C_0},$$

develops a singularity within a finite time, t_* . This singularity is due to the emergence of a double, bi-cubic root (for $t_* = \sqrt{C_0/27}$ and $x_* = 60t_*^{1/3}$) in P_6 . Since C_0 is a constant, the time of blow-up and its location are adjustable. This evolution is clearly seen in Fig.(2). A similar effect is observed for other, even order, polynomials, c.f., $P_4 = x^4 - 24tx + C_1$ (In polynomials of odd order singularity is present at all times). The solution (47b) is useful to elucidate the emergence of a singularity.

Remarks:

A. The linearization 'miracle' which occurs for a specific ratio between the coefficients of dissipation and dispersion, i.e., when $\delta = 4/9$, provides us with an analytical handle which otherwise is completely beyond our reach. Since, however, this value of δ has no particular physical distinction, one

can use the explicit forms at hand as an initial input, to study numerically the formation of patterns for other values of δ .

B. The formation of a singularity in the last example, reflects the fact that for negative u 's the dissipation in the DD(3,1,2) equation turns into an anti-dissipation, which is to say that for $u < 0$ energy is deposited to the wave. This is a highly destabilizing mechanism which in the present example leads to a blow up in a finite time. However, in other cases, to be discussed shortly, this instability will cooperate with dispersion and convection to generate stable, permanent, patterns.

C. Observe (see (34b)) that the $n = 2$ case does not admit a linearizable subcase. In fact looking at the 'reverse problem'

$$f_t + f_{xxxx} = 0 \quad (48)$$

and taking $u = f_x/f$, we find that u satisfies

$$u_t + (u^4)_x + \frac{4}{3}(u^3)_{xx} + \frac{3}{2}(u^2)_{xxx} + u_{xxxx} = 0. \quad (49)$$

This equation is the DD(3,4,2), but with an additional, fourth order, term. Its presence thus precludes the *DD(3, 4, 2)* from being linearizable, at least in the sense that applies to the Burgers and DD(2,3,1) equations.

D. We remind the reader that all the special solutions presented so far this section, are based on the fact that f was used as a new dependend variable, and are thus limited to the overdamped mode. The fact that in the underdamped case such representation is impossible, restricts our analysis to travelling waves.

4.3: Travelling waves

We now consider three ways of finding traveling waves in the distinguished case. Two are presented next and one in the Appendix.

First approach; we reconsider ab initio the solutions of Eq.(6c) which now reads

$$-U + aU^{n+2} + (U^n)_{\eta\eta} = \sigma(U^{n+1})_\eta. \quad (50)$$

In terms of $v = U_\eta$, $z = U^{n-1}v$ and $y = U^{n+1}$ we have

$$-1 + ay + n(n+1)z \frac{dz}{dy} = \sigma(1+n)z.$$

Define $\zeta = (ay - 1)/(n+1)$ and $F = z/\zeta$. The resulting equation

$$\frac{dF}{d\ln \zeta} = \frac{\sigma F - 1 - \delta_* F^2}{\delta_* F} \quad \text{where} \quad \delta_* = \frac{an}{1+n} \quad (51a, b)$$

may be immediately integrated to yield

$$\ln \frac{\zeta}{\zeta_0} = -\delta_* \int \frac{FdF}{D} \quad \text{where} \quad D = \delta_* F^2 - \sigma F + 1. \quad (52)$$

Clearly, the nature of the solutions depends on the roots of D . Each of the three different cases yields a different solution manifold $\Phi(\zeta, F)$ which still has to be unfolded in terms of the original variables U and v , for the last integration $\eta = \int dU/v$ to be carried out. The program, while quite straightforward, is in practice too involved to be carried out even when D has a double root. In this case the integration of (50) yields an implicit expression for z

$$\sigma z - 2\zeta = \mu \zeta_0 \exp\left(\frac{2\zeta}{\sigma z - 2\zeta}\right). \quad (53)$$

Further unfolding yields an expression for v , which cannot be made explicit in terms of U and thus cannot be used to determine η . For other cases the affairs are even more complicated. Thus even the availability of a first integral could not render the problem solvable explicitly.

Second Approach; we now exploit directly the potential representation (29) of our problem. Each n has to be considered separately. In addition, it is also necessary to treat separately the overdamped and the underdamped cases.

The overdamped, $n=1$, case :

Let $s = x - \lambda t$ and $R = f^{-1/\gamma}$, then after two integrations Eq.(34a) yields ($\gamma \neq -1/2$)

$$R_s^2 + P_*(R) = E, \quad \text{where} \quad P_* = \lambda\gamma R^2 + AR^{2+1/\gamma}, \quad (54a)$$

and

$$u = -\frac{\gamma R_s}{r_1 R} = \pm \frac{\gamma}{r_1} \sqrt{ER^{-2} - \lambda\gamma - AR^{1/\gamma}}. \quad (54b)$$

A and E are integration constants (E plays a role of the "total energy"). In terms of R we thus arrive at an equation describing a motion of a particle in a central field. **The only trace of dissipation** is carried by γ . Recall that two γ 's correspond to the same δ . In fact, we can transform equation (54a) into an equivalent form with γ_* replacing γ .

Consider first the purely dispersive case; this is the defocusing, $a < 0$, variant of the $m - KdV$. Now $\gamma = 1/2$, $f = 1/R^2$, and potential function $P_*(R)$ has a shape of a double hump, if $\lambda > 0$, and $A < 0$, or a double well, if $\lambda < 0$ and $A > 0$. (see Figs.(3a) and (3b)).

1. The double hump case; depending on whether $\lambda^2/16AE < 1$, $= 1$, or > 1 , we have in $R(s)$, a periodic motion, a runaway or a kink, respectively. However, since u is a logarithmic derivative of R , passing through the bottom of the potential implies that at $R = 0$ u becomes unbounded unless R_s vanishes as well. Thus, for instance, the zero of the kink in R -units, is an unbounded crest of a soliton in u .

2. the double well; for $E < 0$, R undergoes a periodic motion with $R \neq 0$ consequently the motion is bounded and periodic in both R and u . When $E = 0$, R describes a soliton, with R and R_x vanishing simultaneously. In terms of u this trajectory is a kink. When $E > 0$ the periodic motion in R samples $R = 0$, (here $R_x = \sqrt{E}$), thus in u , the resulting wave is periodic, but unbounded.

Let us now restore the dissipation, thus $\gamma \neq 1/2$.

Given the wave speed λ , the freedom to choose E and A , generates a large variety of patterns. Seeking the bounded ones, we have to assure that R is

kept away from 0, which necessitates to take $E < 0$ and consider a negative potential well. Taking into account that $a < 0 \rightarrow \gamma > 0$ and $a > 0 \rightarrow \gamma < 0 \Rightarrow$, when $\gamma > 0$, $AR^{1/\gamma}$ is superquadratic, and thus controls the behavior of the potential at large R's, it is then easily seen that bounded oscillations in u will be found for

$$E < 0, \lambda < 0, \text{ and } A\gamma > 0.$$

We note the two special values of $\gamma = +1$ and -1 . In the first case the periodic waves can be expressed in terms of ratio of elliptic functions, in the other, which is the exactly linearizable case, in terms of trigonometric functions.

Actually, when $\gamma = 1, A > 0$, and $\lambda < 0$ the resulting potential (see Fig.(3c)), has exactly the same form as the potential of the travelling waves of the KdV!. For $E < 0$, R never vanishes and we obtain bounded oscillations with periodicity expressed in terms of elliptic function. When $E = 0$, motion of R describes a soliton with u being again a nice kink.

The fact that we obtain a self sustained oscillatory motion can, again, be attributed to the peculiar form of our dissipative term, which for $u < 0$ becomes anti-dissipative. Thus for $u > 0$ the wave deposits energy to the medium, but for $u < 0$ the opposite occurs; the medium transfers energy to the wave! When the resulting motion is bounded and unattenuated, dissipation and energy deposition are in a detailed balance. Indeed, since R_x changes sign at each of its crests, u takes both positive and negative values.

This process is a continuum analog of a well known phenomenon occurring in nonlinear oscillations; pumping energy into a damped motion can sustain oscillations, and in a nonlinear case, even induce new ones.

n ≥ 2; Let $s = x - \lambda t$ and $n = 2$, then after two integration Eq.(b) yields

$$F_s^3 - \gamma\lambda F^3 + AF^{3+1/\gamma} = E \text{ here } F = f^{-\gamma}. \quad (55)$$

The form of this equation is quite unusual and defies a simple physical interpretation. When either of the integration constants E or A vanishes,

one can derive an a simple solitary solution. However it is easier to derive this solution directly from (28); an integration in a traveling frame, and simple rearrangements, yield

$$-\lambda f + 2f'\theta_s\theta_{ss} + af(\theta_s)^3 = C_0. \quad (56a)$$

The sought after solution is obtained upon assuming that $C_0 = 0$. Since $f = \exp(r_2\theta)$ and $u = \theta_s$, equation (56a) simplifies

$$-\lambda + r_2(u^2)_s + au^3 = 0. \quad (56b)$$

Let us assume that $a\lambda > 0$ and introduce $u = (\lambda/a)^{1/3}V$, then

$$\int \frac{VdV}{1 - V^3} = \frac{1}{2ar_2}(\frac{\lambda}{a})^{1/3}s \equiv k_2s. \quad (56c)$$

This is an elementary integral which yields;

$$\ln[\frac{(V-1)^2}{V^2 + V + 1}] + 2\sqrt{3}\tan^{-1}(\frac{1+2V}{\sqrt{3}}) = 6k_2s + \frac{\pi}{\sqrt{3}}. \quad (56d)$$

Note that if in (56d) dissipation is removed, the resulting defocusing $m - KdV$ yields a dark soliton with a cusp at the origin. This can be seen from (56); now $r_2 = \pm\sqrt{-8a/3}$, and for $a < 0$ the solution is composed from two branches of the solution that join to form a cusped soliton (a cuspon, see ref[4]). When dissipation is restored its impact is to distort the cusped solitary wave to the effect that the resulting solitary wave is asymmetric, see Fig.(4). Again to construct this solitary wave, we use the two branches of the solution (r_2 takes both positive and negative values, see (31a)). If instead dissipation, dispersion is removed, the resulting solution is a kink with u vanishing on the front line $x_f(t)$. Near the front, $u \sim \sqrt{x_f}$ for $x \leq 0$ (and $u = 0$, for $x \geq 0$), and as required, the dissipative flux vanishes at this point. However, when both dissipation and dispersion are present then, since the dispersive flux $((u^2)_{xx})$ does not vanish at $u = 0$, this point can no longer serve as a front line.

For the solitary wave of the last example, the integration constant was discarded, which enabled us to return to the original variable u . In this case the long detour via the potential representation was needed to deduce Eq.(56b). This procedure can be reapplied to other solitary structures in the overdamped case as follows;

Let $s = x - \lambda t$, and integrate equation (29) once. Discard the integration constant to obtain

$$-\lambda f + f'(\theta)[(\theta_s)^n]_s + af(\theta_s)^{n+1} = 0$$

Using $f = \exp(r_n \theta)$ and $u = \theta_s$ we obtain

$$-\lambda + r_n(u^n)_s + au^{n+1} = 0, \quad (56e)$$

which yields ($u = (\lambda/a)^{\frac{1}{n+1}} V$)

$$\int \frac{V^{n-1} dV}{1 - V^{n+1}} = \frac{1}{nar_n} \left(\frac{\lambda}{a}\right)^{1/(n+1)} s \equiv k_n s. \quad (56f)$$

For integer n 's the last integral is known. For $n = 1$ and $n = 3$ we obtain kink solutions as follows

$$n = 1; \quad V = \tanh(k_1 s) \quad (56g)$$

$$n = 3; \quad \ln\left[\frac{1+V}{1-V}\right] - 2 \tan^{-1} V = 4k_3 s \quad (56h)$$

Note that all kinks which we have obtained so far, are monotone. Oscillatory kinks will be found in the underdamped case to be considered shortly.

Observe that the last kink has a perfectly vertical slope at the origin which is in the center of the sharp transit ($u \sim s^{1/3}$), and where dissipation collapses exactly.

Unlike the $n = 2$ case, for odd n 's, both positive and negative a 's are permitted, provided that $a\lambda > 0$. Since

$$a < 0 \implies k_n^+ > 0, k_n^- < 0 \text{ while } a > 0 \implies k_n^+ < 0, k_n^- < 0,$$

and k_n^\pm is corresponding branch of $r_n \pm$ (see (56e)). In the first case, the orientation of the resulting kinks is opposite, while in the second case, we have an unusual situation, wherein both kinks have the same orientation. Example of such pair of kinks is shown in Fig.(5). The various questions which this situation raises, must at this stage be left unanswered.

One cannot leave the present topic without pausing to understand how a first order equation in u , Eq.(56e), relates to the full travelling wave equation (50) (apart of a slightly different normalization). The answer is to be found in the possibility to factor Eq.(50) into a product of two operators. Using the present notation we have

$$(\partial_s + r_n u) [-\lambda + r_n (u^n)_s + a u^{n+1}] = 0, \quad (56i)$$

which clarifies how factorization leads to the solitary wave solution. Clearly, for other solutions one has to address the full equation. Note, that unlike the cases considered in sec.2, the present factorization does not restrict the solution to one, definite, speed of propagation. Similarly to the factorization in (40a), the factorization is limited to the overdamped mode. *In the purely dispersive case this allows factorization only in the defocusing variant of the m-KdV, with $r_1 = \pm\sqrt{-2a}$.*

The Underdamped $n = 1$ Case;

Now $\mu < \sqrt{2a}$, $\mu_1 = \mu$ and $\omega_1 = \omega$. Since in the present case e no longer can replace θ with f , We shall use equation (29) directly. Integration in a travelling frame yields

$$-\lambda f + f' \theta_{ss} + f \theta_s^2 = C_0, \quad s = x - \lambda t. \quad (57)$$

no simplification of the kind made in (56) seem to be possible now. As it stands, Eq.(57) looks even more complicated then the original in (6). What

makes the difference is the existence of an an integrating factor; multiply (57) by $v\theta_s$ - the integrating factor is then defined via

$$(f'v)' = 2afv.$$

If $R = f'v/2$, then our problem may be cast into

$$-\lambda R_s + a[R\theta_s^2]_s = C_0 v\theta_s.$$

Integrating and solving for v and R yields

$$\theta_s^2 + R_* \cos(\omega\theta) e^{\mu\theta} = \frac{\lambda}{a}, \quad R_* = \text{const.} \quad (58a)$$

Equation (58a) looks like an energy equation for a nonlinear oscillator. The trace of dissipation is kept in the exponential part of the potential shown in Fig.(6).The invariance of the problem under shifts in θ is used to assure that the potential function vanishes when $\theta = \pi/2$. This rescales the integration constant (in Fig.(6) we assumed $a = 1$, and $R_* = 1$). One is now left with one, final, integration to determine θ .

The crucial features of the solution will be now deduced directly from (58a), without solving explicitly for θ .

We consider first the $m - KdV$ equation, now $a > 0$, and set $\mu = 0$ in(58a). ($\Rightarrow \omega = \sqrt{2a}$).

$$(\theta_s)^2 + R_* \cos(\omega\theta) = \frac{\lambda}{a}, \quad R_* = \text{const.} \quad (58b)$$

In terms of θ , Eq.(58a) is now an energy equation for the travelling waves of the $m - KdV$. Here λ/a plays the role of the total energy of the system. When λ/a is above the potential well (i.e., when $R_* < \lambda/a$), then $\theta_s (\equiv u)$ never vanishes (u always resides in one of the wells in (59)). The class of large oscillations in u is obtained when λ/a resides within the potential well, i.e., $\lambda/a < R_*$. The transitory state wherein $R_* = \lambda/a$ describes one kink solution in θ which in terms of u is a one soliton solution.

Note that Eq.(58b) has exactly the same form as the equation describing the motion of a nonlinear pendula with θ describing the angle of deflection. In this description u represents the angular velocity. The soliton in u would then represent that particular scenario wherein the pendula starts at one horizontal position and, as $t \rightarrow \infty$, approaches the other horizontal position.

It is also useful to write the energy equation for the $m - KdV$ in terms of the original variable u ;

$$u_s^2 - \lambda u^2 + au^4/2 = E. \quad (59)$$

For $\lambda > 0$ its potential has a double well, with two equilibrium points $u = \pm\sqrt{\lambda/a}$. It is seen from (58a) that these equilibrium points are attained when $\theta \rightarrow -\infty$. For $E < 0$, Eq.(59) has two separate bands ($u > 0$ and $u < 0$) of travelling waves, which for $E > 0$ merge into a one, large amplitude domain. We see that the particle motion in the potential well in (59) corresponds in (58b) to an unconfined motion (so that θ_s never vanishes) and vice versa.

Returning to the dissipative problem we note that the exponential factor causes the potential function to oscillate with an ever increasing amplitude thus, irrespective of the value of the 'total energy', λ/a , it has to cut the potential well at a certain point, say; $\theta = \theta_m$.

Assume first this point is not one of the crest points of the potential. Then starting at $\theta \rightarrow -\infty$ where $\theta_s \equiv u^2 = \lambda/a$, the solution develops an ever growing oscillations until $\theta = \theta_m$ where $\theta_s = 0$ and thus u vanishes. This is a turning point of the potential well in (58a) (see Fig.(6)), which means that now u has changed its sign, on its way back to the equilibrium point at $\theta = -\infty$. In terms of u , the traveling wave is thus an oscillatory kink connecting the two equilibrium points; $u = -\sqrt{\lambda/a}$ with $u = \sqrt{\lambda/a}$. Using a mechanical analogy, the anti-dissipative nature of the diffusive part for negative u 's, drives the 'particle' out of $u = -\sqrt{\lambda/a}$ in the left well, toward the stable equilibrium point $u = +\sqrt{\lambda/a}$ in the right well in (59). Throughout

this drive the 'particle' oscillates with an ever decreasing amplitude, around the final rest point at the bottom. The traveling wave thus acts as a bi-stable system. Three such kinks are shown in Fig.(7). The crest near $\theta = 0$ is at 1.0032. In Fig.(7a), $\lambda = 0.9$. In Fig.(7b), $\lambda = 1.002$, placing it very close to the top. The resulting change in the pattern is clearly visible. Now note the kink in Fig.(9c) with $\lambda = 1.0034$, which is slightly 'over the top', and allows the particle to cross the top and roll into the next well where it executes one more oscillation and then is reflected back by the barrier of the well. This one extra oscillation is clearly noticeable in the resulting shape of the kink. In the limiting case, shown in Fig.(7c), wherein $\lambda = 1.0032$, to be discussed shortly, the uphill push is such that as $s \rightarrow \infty$ the particle approaches the top (where $u = 0$), 'without having the time' to roll to the second well. *We note that unlike the monotone kinks found in the overdamped regime, all kinks found here oscillate!*

The same 'energy level' λ/a which generates the kink, also enables an infinite number of oscillating waves, each of which resides in its own potential well. As can be anticipated from Fig.(6), as one moves in the right direction from one well to another, the well deepens and narrows. The corresponding amplitudes and the frequency of these waves, increase in discrete quantas. With each wave having its own eigen-frequency. Two such cases are shown in Fig.(8a) and (8b), respectively (using the notation in Fig.(6); one wave resides in valley (1) and the other in valley (3)). The pattern shown in Fig.(9a), share the same well with the one in (8a), but the 'energy level' λ is different. As in all other cases, the persistence of undamped oscillations is due to energy deposition whenever u assumes negative values.

The exceptional case, aluded before, wherein the energy line is tangent to one of the crests, say at $\psi_c \equiv \omega\theta_c$, necessitates that

$$\beta = -\tan(\psi_c) \text{ and } \lambda = aR_*\cos(\psi_c)e^{\beta\psi_c} \quad (60a)$$

where

$$\beta \equiv \frac{\mu}{\omega} = \frac{1}{\sqrt{2\delta - 1}}, \quad (60b)$$

to be satisfied. Given R_* as a measure of the initial energy of the wave, condition (60) is satisfied by a denumerable number of wave eigen-speeds λ_i , $i = 1, 2, \dots$ given via $\psi_c + 2i\pi$. Each eigen-speed represents energy line tangent to a specific crest. Approaching now the crest point near $\theta = 0$, corresponds to the tail of $u \rightarrow 0$ as $s \rightarrow \infty$. The resulting solution is also a kink in u and is accompanied by a train of damped oscillations connecting the upstream at $u = 0$ with a downstream at $u = +\lambda/a$ (see Fig.(9a)). A countable number kinks corresponds to a countable number of eigen speeds in (60a). In addition to the oscillatory bands of travelling solutions residing each in its own well, one well, namely; the well to the right of the critical crest (θ_c), stands out. It hosts a solitary wave in θ . In terms of u , this is a **traveling doublet** - see Fig.(9b). Using the potential double-well of the $m - KdV$, (59), we can interpret this solution as follows; a particle starting at the $u = 0$ top, rolls into the left, unstable well. Due to anti-dissipation, it gains energy and its return to the top of the hill at $u = 0$, occurs in a finite time. Now it rolls into the right well. Here, however, the process is dissipative, and the motion is such, that its return from to the top of the hill, will take now an infinite time.

The underdamped cases for $n > 1$; Similarly to the $n = 1$ case, we obtain an expression which is completely analogous to (58a), namely;

$$(\theta_s)^{n+1} + R_* \cos(\omega_n \theta) e^{\mu_n \theta} = \frac{\lambda}{a}, \quad R_* = \text{const.} \quad (61)$$

The main difference is between odd and even n 's. For odd n 's, say $n = 3$, the analysis for $n = 1$, carries through, with quartic root replacing the quadratic root, but where for $n = 1$ we had a solitonic tail, the pulse has a sharp front, compare Fig.(7c) with Fig.(11a) In fact, even the doublet seen in Fig. (10a), has a compact support! In Fig.(12) we display a typical periodic waves which have a weak discontinuity at $u = 0$. These waves are exactly

the $n = 3$ analog of Fig.(8) Similarly, the kinks in Fig.(11) are the $n = 3$ replica of Fig.(7a) and (9c), respectively. As for $n = 1$, the frequency and the amplitude of these waves change as we move to another potential well.

For $n = 2$, taking third root in (61) has a very different meaning. Because both positive and negative values are admitted, we can no longer interpret equation (61) as describing a motion of a particle in a central field. In fact all oscillations grow *indefinitely*. The only bounded solutions arise in the exceptional case when λ/a is tangent to the crest. In terms of u , this represents a semi-infinite wave train with a sharp front.

5. Summary

We have seen in this work that unless the distinguished $m = k+1 = n+2$ case, is considered, very little can be said about patterns emerging from dispersive-dissipative interactions. Explicit solutions for particular wave speeds have been obtained and this is pretty much all that can be said in terms of the ensuing patterns. *It is only in the distinguished case that a glimpse into the dispersive- dissipative interactions became possible.* While the stability and the attraction basin of the presented patterns is yet to be determined, their variety is truly remarkable. Some of the permanently oscillating patterns emerge as a result of a global balance between dissipated and deposited energy. We obtain a global bi-stable dynamical system in which for negative u 's the system deposits energy to the wave, while for positive u 's it dissipates it. Without nonlinear convection and dispersion, a system with negative dissipation, is unstable to the point of ill posedness. It is the presence of these mechanisms which mitigates the unstable process and generates stable patterns. One expects that if at $t = 0$ the negative part of u is not too large, these bi-stable patterns will be evolutionary. Otherwise, as the example in Fig.(1) clearly demonstrates, we can expect a blow-up in a finite time.

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Appendix; Traveling Waves Via A Lagrange Map

We define new variables via;

$$y = U^{n+1} \text{ and } \zeta = \int U d\eta. \quad (1a)$$

When $m = n + 2 = k + 1$, then in terms of y and ζ equation (6c) yields

$$-1 + ay + \frac{n}{n+1} \frac{d^2y}{d\zeta^2} = \sigma \frac{dy}{d\zeta}, \quad (1b)$$

which is a **linear equation**. Define

$$\frac{2n}{n+1} \gamma \equiv \sigma \pm \sqrt{\Delta} \text{ where } \Delta \equiv \sigma^2 - 4an/(n+1), \quad (1c)$$

then the solution reads

$$y = 1/a + y_0 e^{\gamma\zeta} \cosh(\sqrt{\Delta}\zeta) \text{ when } \Delta > 0, \quad (2a)$$

and

$$y = 1/a + y_0 e^{\gamma\zeta} \cos(\sqrt{|\Delta|}\zeta) \text{ when } \Delta < 0. \quad (2b)$$

When $\Delta = 0$ then $y = 1/a + y_0 \zeta e^{\gamma\zeta}$. In each case the solution has to be reexpressed in terms of the original coordinate η , via the integral

$$\eta = \int_{-\infty}^{\zeta} \frac{d\zeta}{[y(\zeta)]^{\frac{1}{n+1}}}. \quad (3)$$

The present approach provides a uniform representation for all n 's, however, inspite of its simplicity the map is limited to U 's that do not change sign, which considerably restricts its applicability.

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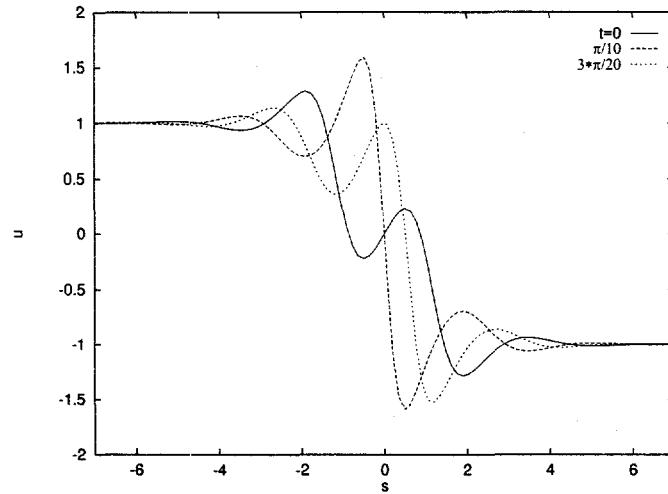


Figure 1: A breathing kink in a steadily traveling frame (see Eq.(46)).

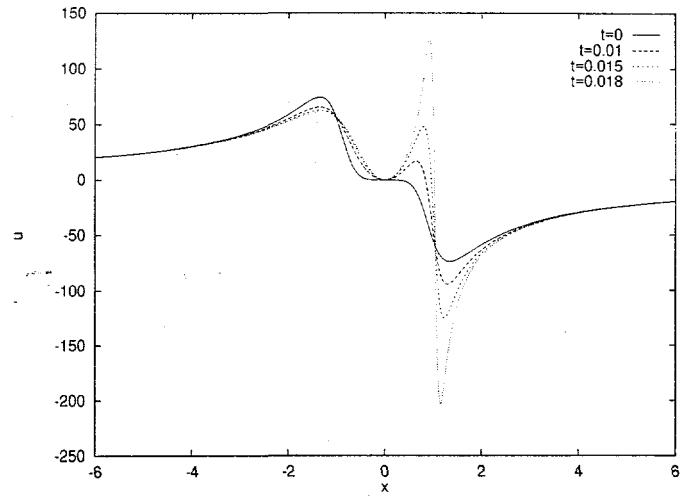


Figure 2: Formation of a singularity within a finite time ($C_0 = 0.1$, see Eq.(47)).

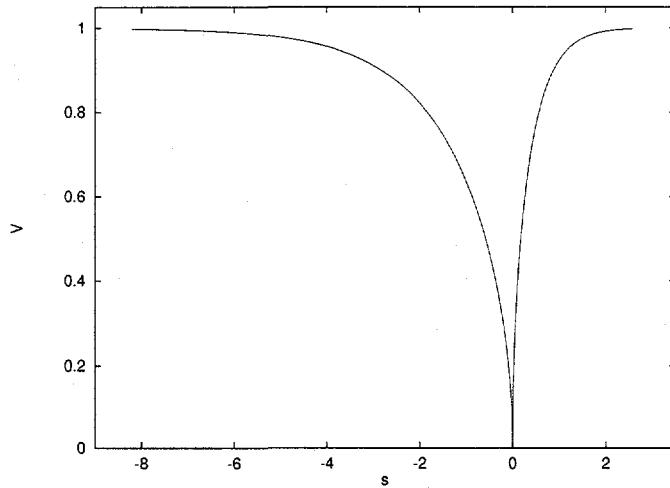


Figure 4: A dark solitary wave for the overdamped $n = 2$ mode. Note its asymmetric shape (see Eq.(56d)).

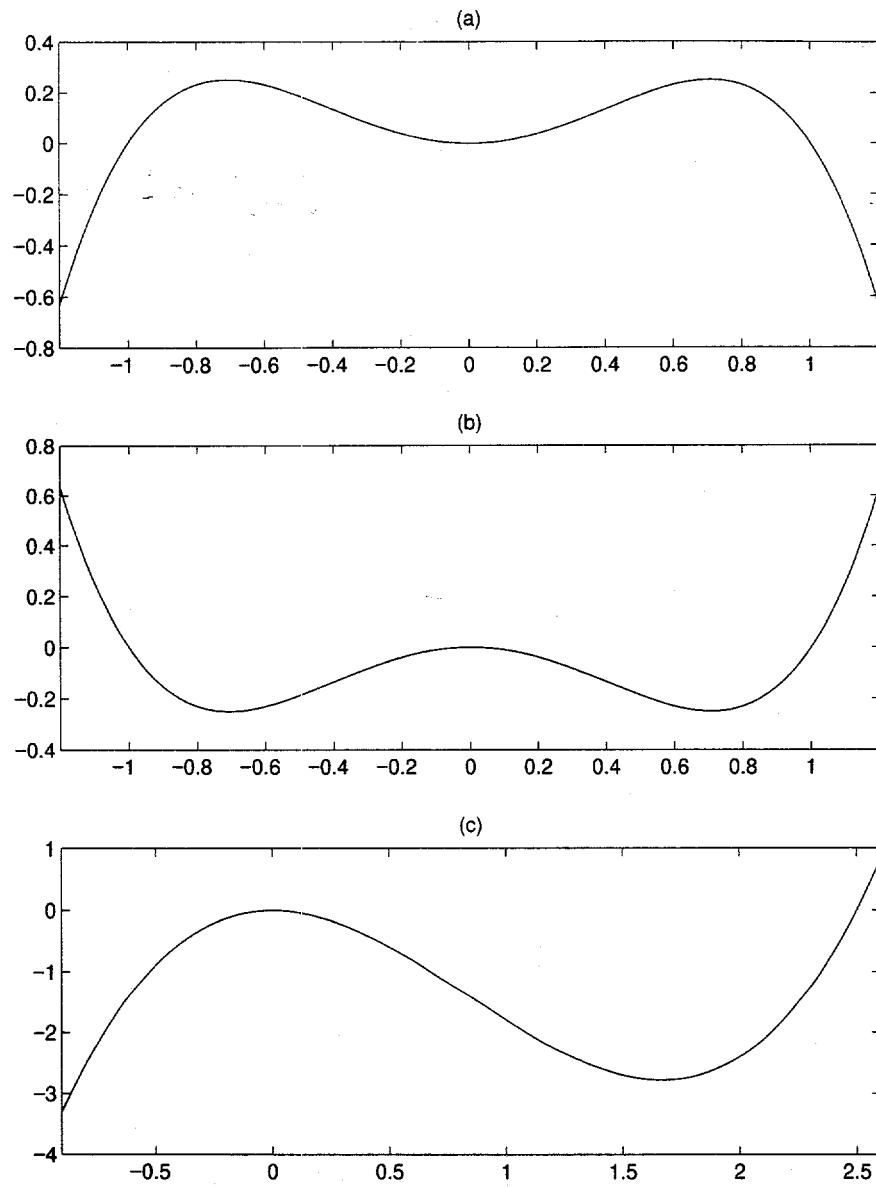


Figure 3: Types of effective potentials in the overdamped mode, that support bounded oscillations. In cases (a) and (b), $\gamma = 1/2$, $\lambda = \pm 1$, and $A = \mp 1$, respectively. In (c); $\gamma = 1$, $\lambda = -3$ and $A = 1.2$

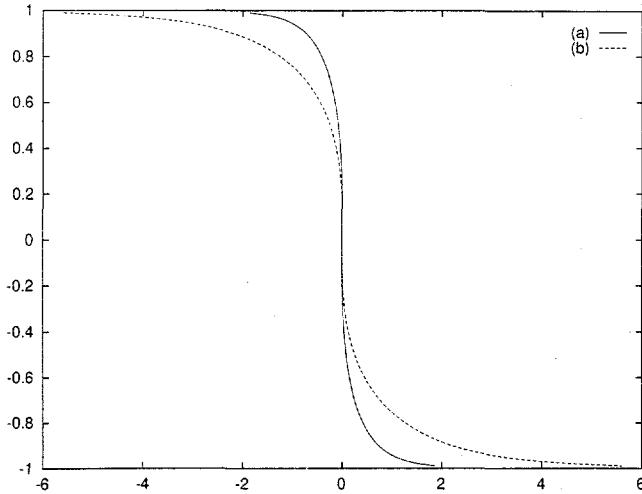


Figure 5: A kink with a sharp transient at $u = 0$, where the dissipation flux vanishes. Here $n = 3$, $a = 1$ and $\mu = 2$. Note that for $a > 0$, two kinks, having the same orientation, as shown, are possible. In (a), $k_3 = -1/2$, and in (b), $k_3 = -1/6$ (see Eq.(56)).

Captions of Figures 6-12

Figure 6.- Potential function of the underdamped traveling waves (see Eq.(58a)).

Figure 7.- Three kinks for the $n=1$ case. The point of reflection of the kink in (7b) is very close to the crest as is evident from its shape near $u = 0$. The kink in (7c) corresponds to the limiting case wherein the 'energy line' is tangent to the crest and 'the particle' is never reflected.

Figure 8.- Two travelling waves having the same speed $\lambda = 0.9$ but residing in a different wells. The wave in (a) resides in well (1) and the one in (b) resides in well (3), see Fig.(6).

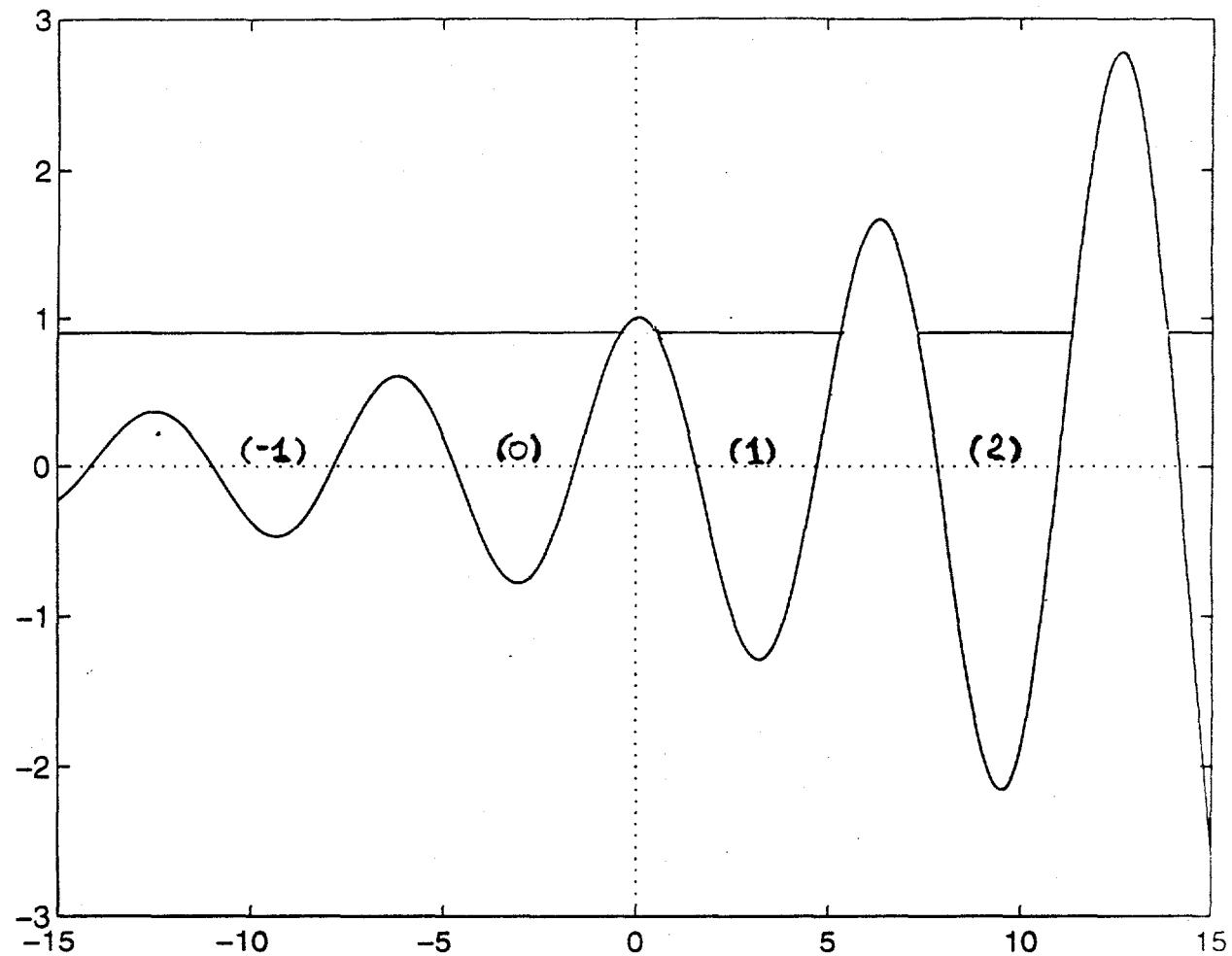
Figure 9.- Three typical patterns as they occur in the potential well (2) for $n = 1$. The periodic wave in (a) has the same speed as the kink in (7b); its counterpart on the other side of the hill. Both are very close to the crest. The doublet in (9b) is the counterpart of the special kink in (7c). Their speed is tangent to the crest.

Figure 10.- Two kinks for $n = 3$. Note the sharp angle of approach near $u = 0$. The kink in (10a) and (10b) are the $n = 3$ counterparts of the kinks in (7a) and (9c), respectively.

Figure 11.- The $n = 3$ counterparts of the $n = 1$ doublet in (9b) and the kink in (7c). Note the sharp front of these structures.

Figure 12.- The $n = 3$ counterparts of the $n = 1$ travelling waves in Fig.(8).

Fig 6



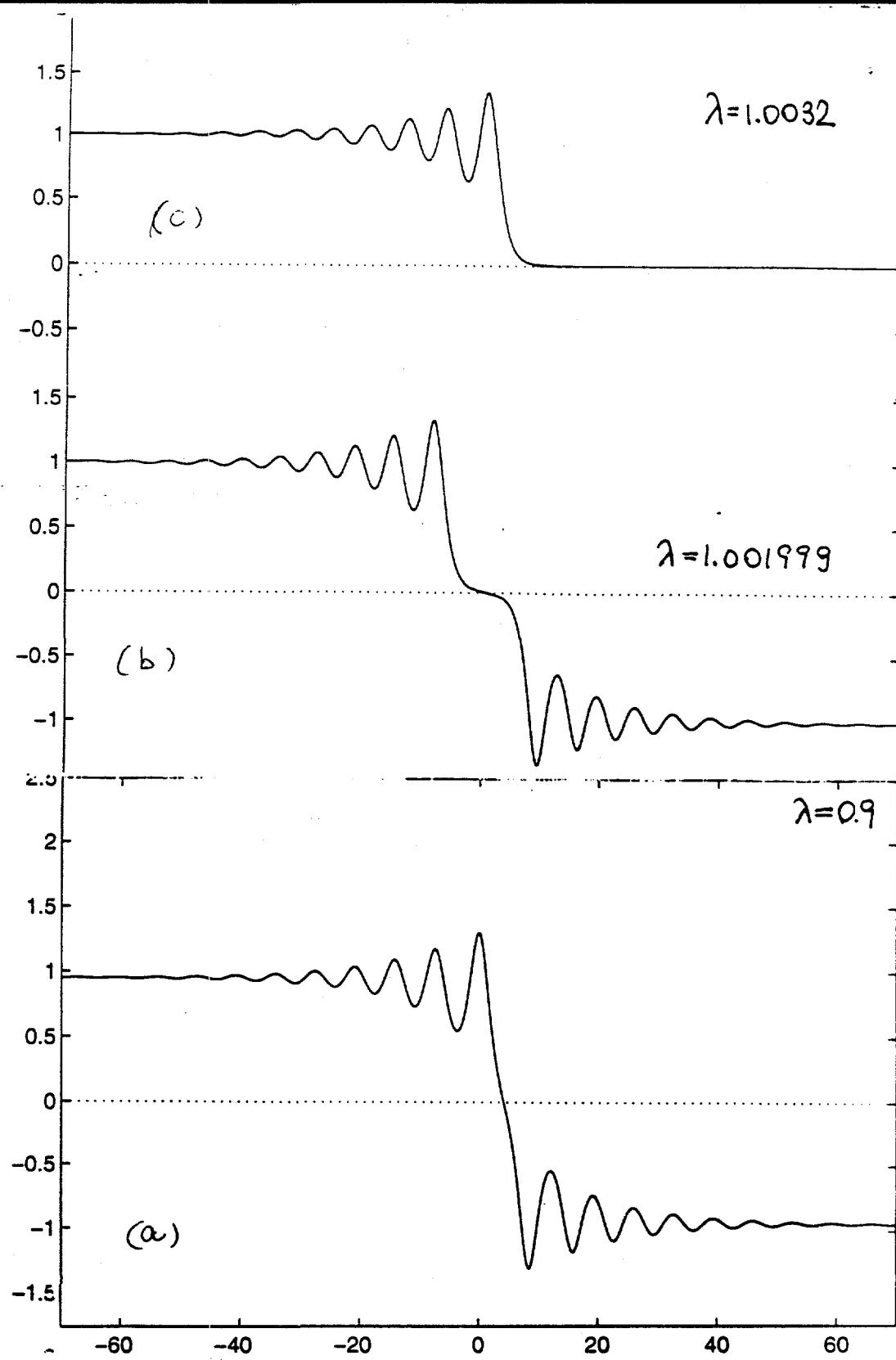
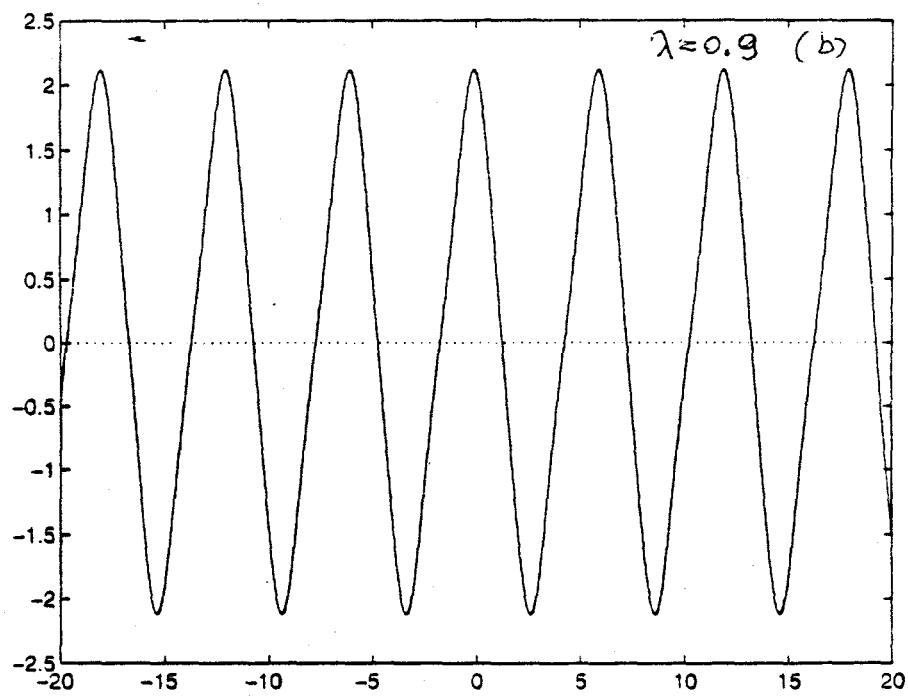
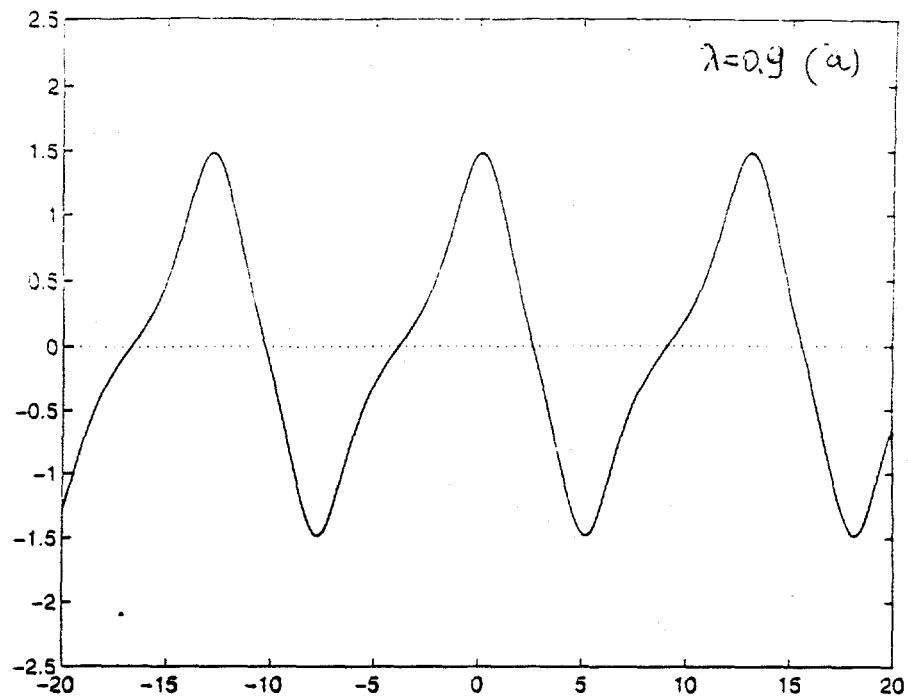


Fig 7

Fig 8



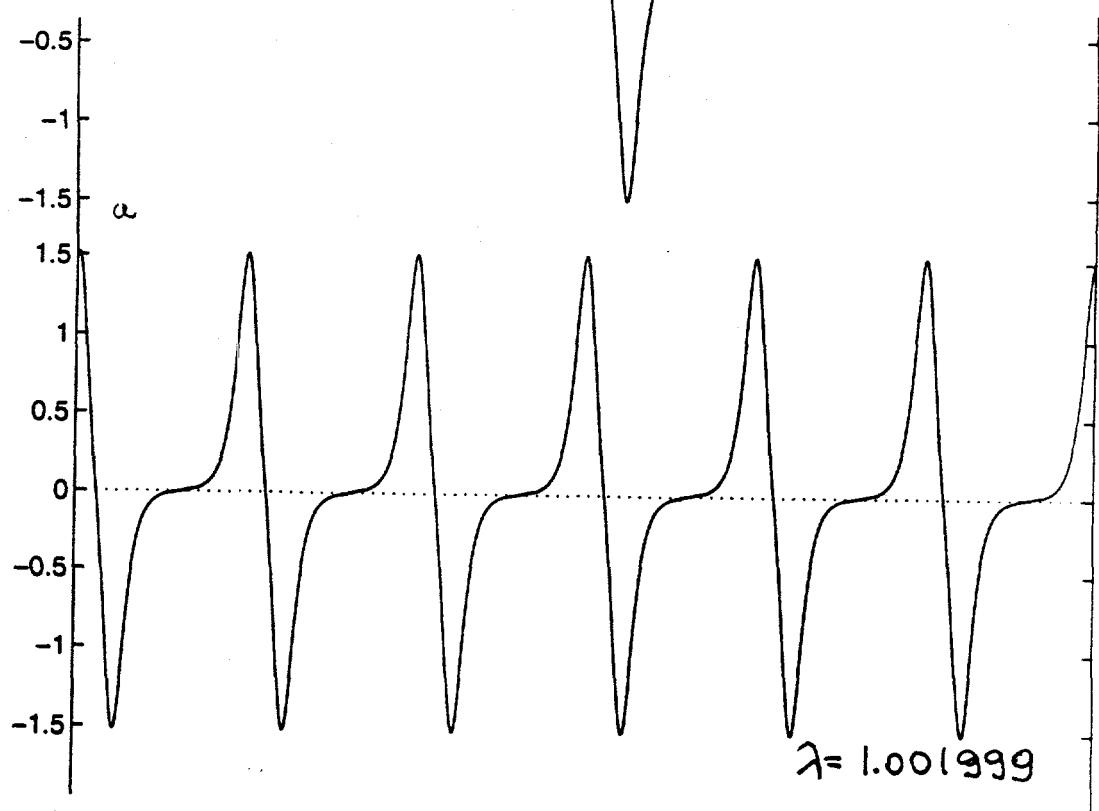
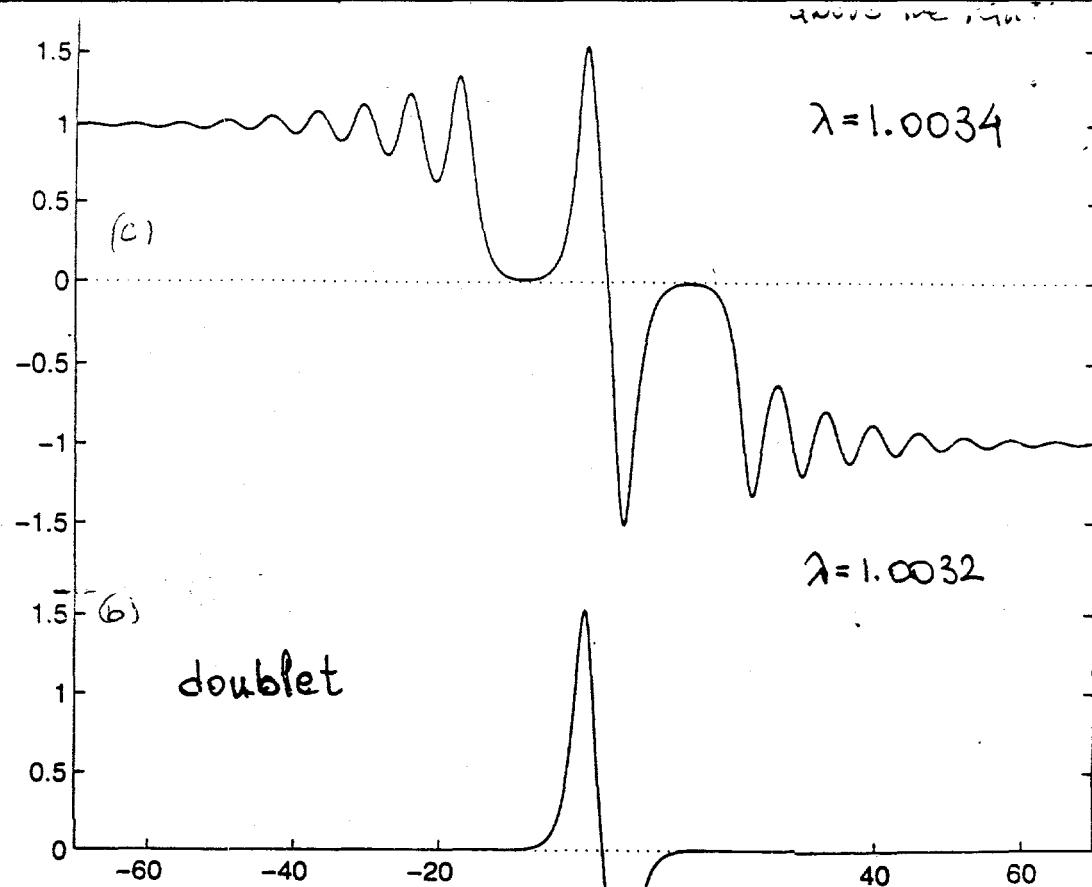
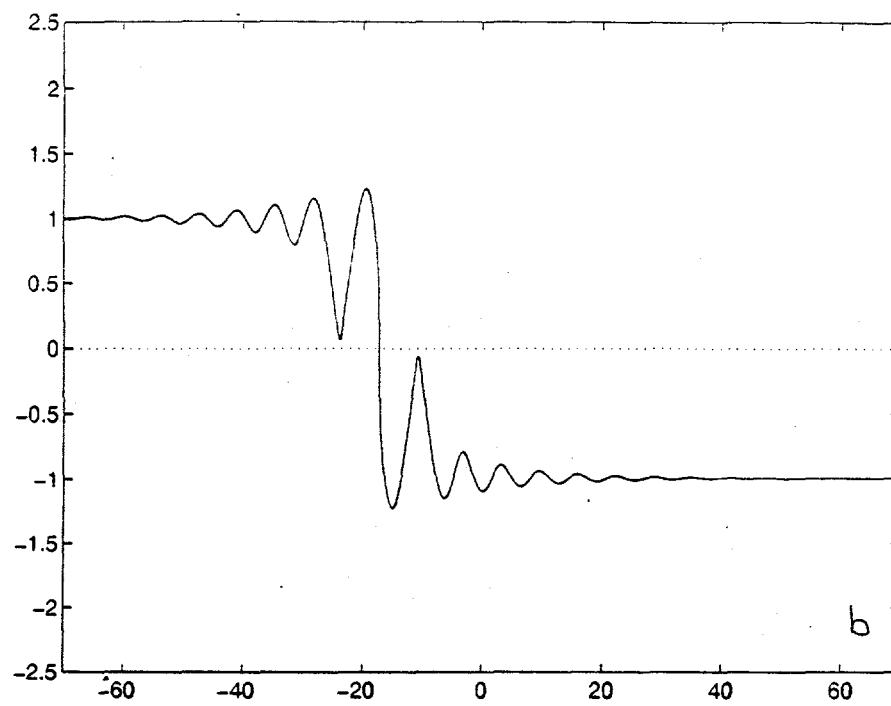
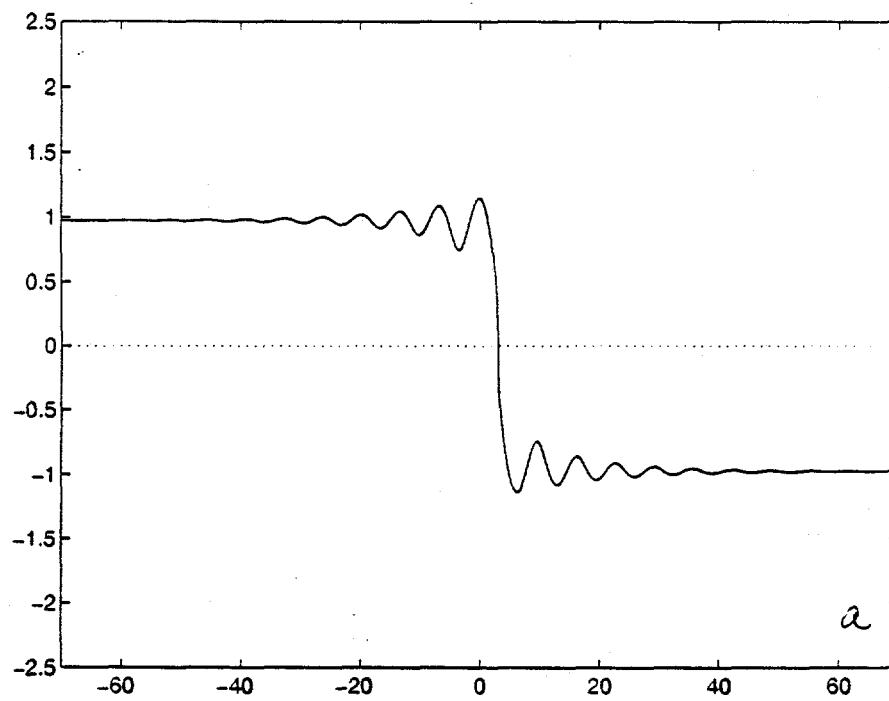


Fig 9

$n=5$



b



a

Fig. 6

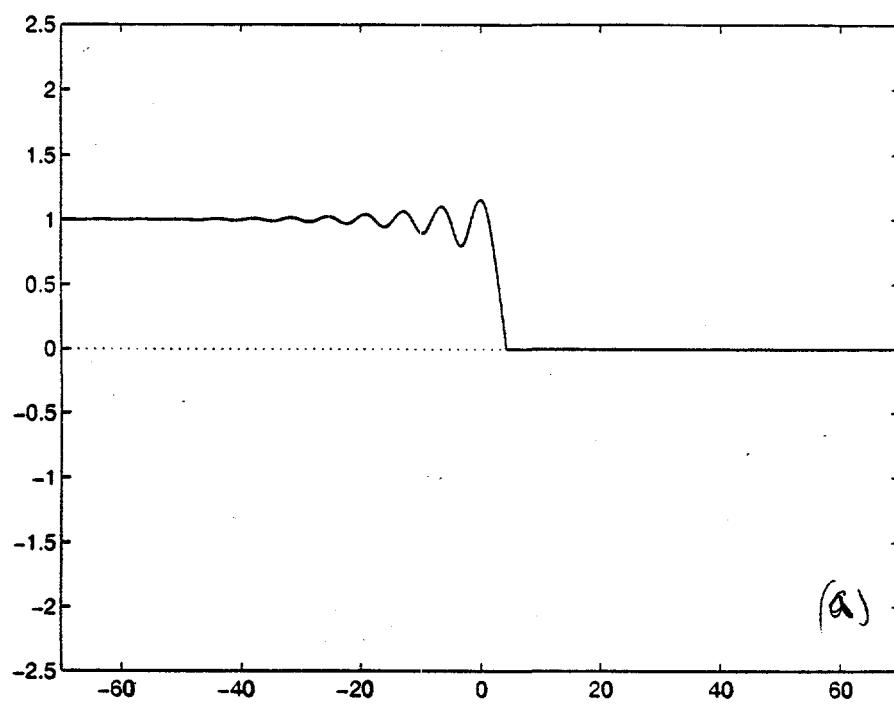
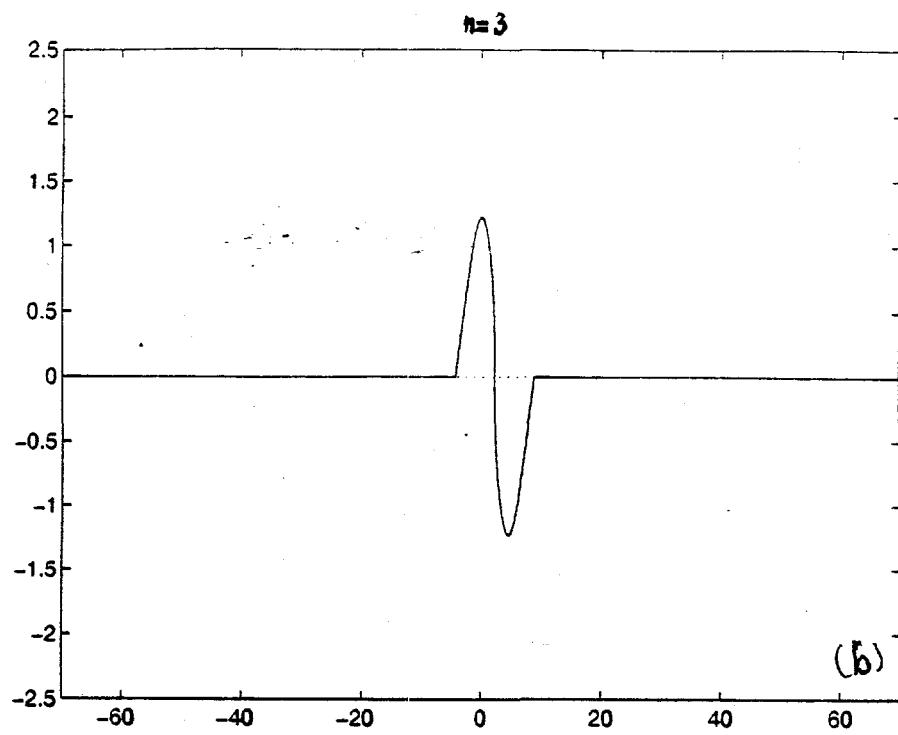


Fig 11

