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THE POWER-LAW THERMAL MODEL FOR BLACKBODY SOURCES

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ABSTRACT

The spectral radiant emittance W_E from a blackbody at a temperature kT for photons at energies E above the spectral peak (2.82144 kT) varies as $(kT)^{E/kT}$. This power-law temperature dependence, an approximation of Planck's radiation law, may have applications for measuring the emissivity of sources emitting in the soft x-ray region.

INTRODUCTION

It has long been evident that radiation emitted by a blackbody is more sensitive to temperature at the higher energies. A power-law thermal model describing this temperature dependence has been used to measure emissivity in earth-surface materials.[1,2] The model is based on an approximation of Planck's radiation equation, and the scale-factor behavior of this equation allows its use in the soft x-ray region. At energies above the spectral peak, E > 2.82144 kT (keV), the radiant emittance varies as kT raised to the power E/kT. We give the derivation of this power-law thermal model below.

THE POWER-LAW THERMAL MODEL

Planck's law describes the spectral distribution of the radiation from a blackbody as

$$W_\lambda d\lambda = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{ch/\lambda kT}-1} d\lambda \quad , \tag{1}$$

which with the substitution $E = hc/\lambda = nkT$ we may rewrite as

$$W_E dE = \frac{2\pi}{h^3c^2}\left[\frac{E^3}{e^{E/kT} - 1}\right] dE \tag{2}$$

$$W_n dn = \frac{2\pi (kT)^4}{h^3c^2}\left[\frac{n^3}{e^n - 1}\right] dn \tag{3}$$

where

W_λ = spectral radiant emittance (W $cm^{-2}\mu^{-1}$),

λ = wavelength (μ),

h = Planck's constant = 6.626176×10^{-34} W s^2

T = absolute temperature (K),

c = velocity of light = $2.99792458 \times 10^{10}$ cm^{-1},

k = Boltzmann's constant = 1.380662×10^{-23} W s K^{-1}

E = hc/λ, photon energy (keV),

n = E/kT (dimensionless).

By integrating Eq. 3 as follows, we obtain the Stefan-Boltzmann equation.

$$\frac{15}{\pi^4}\int_0^\infty \frac{n^3}{e^n - 1}\, dn = 1 \tag{4}$$

$$\int_0^\infty W_n dn = \sigma T^4 = \frac{\sigma}{k^4}(kT)^4 \tag{5}$$

where

$\sigma = (2\pi^5/15)k^4/h^3c^2 = .67032 \times 10^{-12}$ W cm^{-2} K^{-4},

$\sigma/k^4 = 1.0283 \times 10^{17}$ W cm^{-2} keV^{-4},

kT = temperature (keV).

The spectral distribution from Eq. 3 and 4 is shown in Fig. 1, in which the fractional intensity relative to σT^4 is given by W. This distribution peaks at n_{max} = 2.82144, a value found by setting the first derivative of the distribution equal to zero at n_{max}, as shown below:

$$n^3(e^n - 1)^{-1}\left[\frac{3}{n} - \frac{e^n}{e^n - 1}\right] dn = 0. \tag{6}$$

We set the bracketed expression equal to zero and rearranged the terms to obtain

$$e^n(3 - n) = 3 \tag{7}$$

at n_{max}, which has the solution given above.

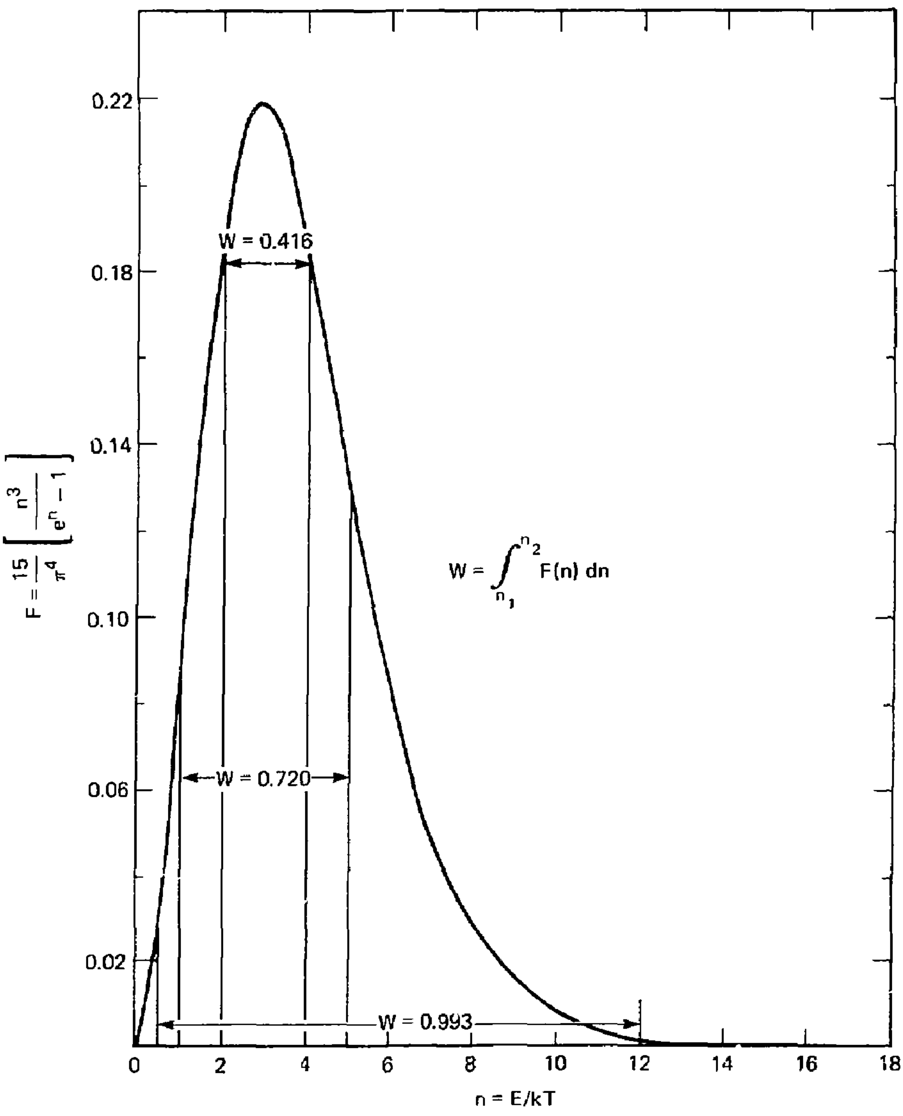

FIG. 1. Incident energy spectrum for blackbody radiation. The function F is normalized, such that the area under the curve is equal to 1.

Absorption experiments provide a way of hardening the spectrum shown in Fig. 1. This is useful for determining the source temperature.[3] Beryllium has an attenuation coefficient approximately equal to A/E^3, where A is a constant (1014 cm^{-1} keV^3, or equivalently 2.576 $mils^{-1}$ keV^3). This approximation reproduces the attenuation coefficients found in the McMaster compilation[4] to within ±10% at energies of 1-10 keV.

The attenuated spectrum for a blackbody source with an emissivity ε, attenuated by a beryllium filter with a thickness of x mils is

$$W_E dE = \frac{15}{\pi^4} \epsilon \frac{\sigma}{k^4} \left[\frac{E^3 e^{-Ax/E^3}}{e^{E/kT} - 1} \right] dE \tag{8}$$

$$W_n dn = \frac{15}{\pi^4} \epsilon \sigma T^4 \left[\frac{n^3 e^{-z/n^3}}{e^n - 1} \right] dn \tag{9}$$

where $z = Ax/(kT)^3$. This distribution has a peak which we locate by setting the first derivative equal to zero at n_{max}:

$$n^3 (e^n - 1)^{-1} e^{-z/n^3} \left[\frac{3}{n} - \frac{e^n}{e^n - 1} + \frac{3z}{n^4} \right] dn = 0 . \tag{10}$$

By setting the above bracketed expression equal to zero and rearranging terms, we obtain

$$z = n_{max}^3 \left[\frac{(n_{max}/3) e^{n_{max}}}{e^{n_{max}} - 1} - 1 \right] . \tag{11}$$

Solutions of Eq. 11 are given in Table 1. Figure 2 shows a plot of z versus n_{max}. The beryllium filter thickness required to shift the spectral peak to $E_{max}(z)$, where $E_{max} = n_{max} kt$, is $x = z(kT)^3/2.576$ mils.

There is an approximate power-law temperature dependence at energies above the peak for continuous radiation distributions, and this is described by Eq. 8. We determine the dependence by differentiating W_E with respect to T, holding E constant:

TABLE 1. Solutions of Eq. 11 for blackbody spectra which peak at $E_{max} = n_{max}kT$. The spectra are hardened by various thicknesses ($x = z(kT)^3/2.576$ mils) of beryllium. The data is also illustrated in Fig. 2.

n_{max}	z	Beryllium thickness (x) mils for designated temperature (kT):				
		0.2 (keV)	0.4 (keV)	0.6 (keV)	0.8 (keV)	1.0 (keV)
3	1.415	0.0	0.0	0.1	0.3	0.5
4	22.92	0.1	0.7	1.9	4.6	8.9
5	84.75	0.3	2.1	7.1	17	33
6	217.1	0.7	5.4	18	43	84
8	853.8	2.6	21	72	170	332
10	2334	7.2	58	196	464	906
15	13500	42	335	1130	2680	5240

$$dW_E = \frac{15}{\pi^4}\,\varepsilon\left[\frac{\sigma}{k^4}\right]E^3 e^{-Ax/E^3}\,\frac{(E/kT)\,e^{E/kT}}{(e^{E/kT}-1)^2}\,\frac{dT}{T} \quad . \tag{12}$$

Dividing dW_E by W_E, we obtain

$$\frac{dW_E}{W_E} = \frac{E}{kT}\left[\frac{e^{E/kT}}{e^{E/kT}-1}\right]\frac{dT}{T} \quad . \tag{13}$$

The bracketed expression is both multiplied and divided by $e^{-E/kT}$ and then expanded in a binomial series:

$$\frac{e^{E/kT}}{e^{E/kT}-1} = \frac{1}{1-e^{-E/kT}} = \sum_{m=0}^{\infty}\left[e^{-E/kT}\right]^m \quad . \tag{14}$$

The first term of the expansion is unity. Other terms are negligible when $E > n_{max}kT$, since $e^{-n_{max}} = 0.060 \ll 1$. Consequently the bracketed expression of Eq. 13 is taken as unity, giving

$$\frac{dW_E}{W_E} = \frac{E}{kT}\,\frac{dT}{T} \quad , \tag{15}$$

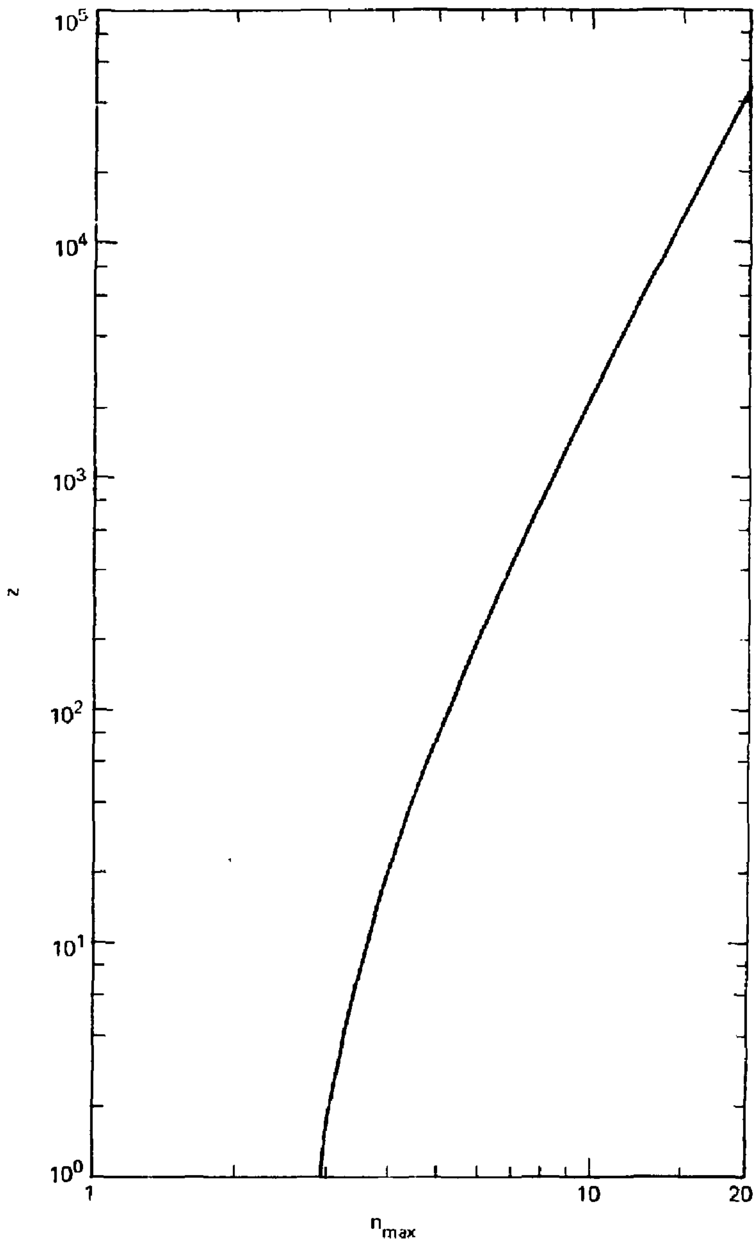

FIG. 2. Shift of blackbody spectral peak to $n_{max} = E_{max}/kT$. Spectra are hardened by beryllium filters. The ordinate $z = Ax/(kT)^3$, with A = 2.576 $mils^{-1}$ keV^3. The curve is obtained from values shown in Table 1.

which when integrated gives

$$\log_e W_E \Bigg|_{W_E(T_0)}^{W_E(T)} = -\frac{E}{kT}\Bigg|_{T_0}^{T} \tag{16}$$

$$\log_e \frac{W_E(T)}{W_E(T_0)} = -\frac{E}{kT_0}\left[\frac{T_0}{T} - 1\right] \tag{17}$$

from whence we obtain

$$\frac{W_E(T)}{W_E(T_0)} = \left[e^{\frac{T_0}{T} - 1}\right]^{-\frac{E}{kT_0}} \quad . \tag{18}$$

For small temperature excursions where $| (T_0/T) - 1 | << |$, the exponential term above is expanded. Second and higher-order terms of the expansion are negligible, giving

$$\frac{W_E(T)}{W_E(T_0)} = \left[1 + \frac{T_0}{T} - 1\right]^{-\frac{E}{kT_0}} \quad . \tag{19}$$

We may rewrite Eq. 19 for $T = T_0(1 \pm 0.1)$ as

$$\frac{W_E(T)}{W_E(T_0)} = \left[\frac{T}{T_0}\right]^{\frac{E}{kT_0}} \quad . \tag{20}$$

This power-law temperature dependence describes the characteristics of blackbody radiation spectra at energies above $2.82144(kT_0)$. A blackbody with a temperature of 1 keV produces signals at 6, 8, and 10 keV that vary respectively as the 6th, 8th, and 10th power of the temperature. Beryllium filters of approximately 84, 332 and 906 mils in thickness would harden the spectrum such that it peaks at about 6, 8, and 10 keV for a blackbody source at a temperature of 1 keV.

Spectral hardening increases the temperature sensitivity of radiation signals according to this power-law thermal model. Non-Planckian sources are not expected to have the behavior described by this model for a blackbody source. The model may have applications for measuring the emissivity of sources emitting in the soft x-ray region.

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