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PLASMA BEAT-WAVE ACCELERATOR

ROBERT J. NOBLE*
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Summary

We perform an analytic study of some quantities relevant to the plasma beat-wave accelerator (PBWA) concept. We obtain analytic expressions for the plasma frequency, longitudinal electron velocity, plasma density and longitudinal plasma electric field of a nonlinear longitudinal electron plasma oscillation with amplitude less than the wave-breaking limit and phase velocity approaching the speed of light. We also estimate the luminosity of a single-pass e^+e^- linear PBWA collider assuming the energy and collision beamstrahlung are fixed parameters.

Introduction Since the original proposal by Tajima and Dawson,[1] the plasma beat-wave accelerator (PBWA) has received increased attention as a possible ultra-high energy particle accelerator because of the very high gradients thought to be possible. In the PBWA, longitudinal electron plasma oscillations with phase velocities near the speed of light trap and accelerate charged particles to high energies. The electron plasma oscillations are resonantly excited by two collinear beating lasers whose frequency difference is the electron plasma frequency. Gradients of 1 GeV/cm seem possible in a plasma of density 10^{18} cm^{-3}.

In this paper we obtain analytic expressions for various quantities relevant to the PBWA. The fluid theory of plasmas is used with the plasma assumed to be cold and collisionless and the ions forming a stationary, neutralizing background. Finite plasma effects are omitted here by taking the plasma as infinite. We will be concerned with the steady-state properties of the electron plasma oscillation and not its generation.

Nonlinear Waves in Plasmas The equations describing nonlinear waves in a cold, collisionless relativistic plasma have been previously given by Akhiezer et al.[2,3] The fluid equations for the electron velocity $\bar{v}$, electron density n, and the fields $\bar{E}$ and $\bar{B}$ are (barred symbols represent 3-vectors)

$$\frac{\partial \bar{p}}{\partial t} + (\bar{v}\cdot\nabla)\bar{p} = -e\bar{E} - \frac{e}{c}(\bar{v}\times\bar{B})$$

$$\nabla\cdot\bar{E} = 4\pi e(n_0 - n)\ ,\ \nabla\times\bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \qquad (1)$$

$$\nabla\cdot\bar{B} = 0\ ,\ \nabla\times\bar{B} = -\frac{4\pi}{c}en\bar{v} + \frac{1}{c}\frac{\partial\bar{E}}{\partial t}$$

where $\bar{p}$ is the electron momentum

$$\bar{p} = \frac{m\bar{v}}{\sqrt{1-v^2/c^2}}\ , \qquad (2)$$

and n_0 is the equilibrium electron density.

The wave motion is a function of the single variable $\hat{i}\cdot\bar{r} - v_{ph}t$, where $\hat{i}$ is a constant unit vector in the direction of propagation, and v_{ph} is the phase velocity. Taking the vector $\hat{i}$ along the z axis and defining $\bar{\rho} = \bar{p}/mc$ and $\bar{u} = \bar{v}/c$, Akhiezer et al.[3] obtain the following equations for the electron density and electron motion from Eqs. (1),

$$n = \frac{n_0\beta_{ph}}{\beta_{ph} - u_z}\ , \qquad (3)$$

$$\frac{d^2\rho_x}{d\tau^2} + \frac{\omega_p^2\beta_{ph}^2}{\beta_{ph}^2 - 1}\,\frac{\beta_{ph}u_x}{\beta_{ph} - u_z} = 0\ , \qquad (4)$$

$$\frac{d^2\rho_y}{d\tau^2} + \frac{\omega_p^2\beta_{ph}^2}{\beta_{ph}^2 - 1}\,\frac{\beta_{ph}u_y}{\beta_{ph} - u_z} = 0\ , \qquad (5)$$

$$\frac{d}{d\tau}\left[(u_z - \beta_{ph})\frac{d\rho_z}{d\tau} + u_x\frac{d\rho_x}{d\tau} + u_y\frac{d\rho_y}{d\tau}\right] = \omega_p^2\frac{\beta_{ph}^2 u_z}{\beta_{ph} - u_z}\ , \qquad (6)$$

where

$$\beta_{ph} = \frac{v_{ph}}{c}\ ,\ \tau = t - \frac{z}{v_{ph}}\ ,\ \omega_p^2 = \frac{4\pi e^2 n_0}{m}\ . \qquad (7)$$

Longitudinal Plasma Oscillations Equation (6) with $u_x = u_y = 0$ describes longitudinal nonlinear waves in a cold, collisionless relativistic plasma,

$$\frac{d}{d\tau}\left[(u - \beta_{ph})\frac{d\rho}{d\tau}\right] = \frac{\omega_p^2\beta_{ph}^2 u}{\beta_{ph} - u}\ , \qquad (8)$$

where $u = u_z$ and $\rho = \rho_z$. Rewriting this in terms of the velocity u alone yields

$$\frac{d^2}{d\tau^2}\,\frac{1 - \beta_{ph}u}{\sqrt{1-u^2}} = \frac{\omega_p^2\beta_{ph}^2 u}{\beta_{ph} - u}\ , \qquad (9)$$

with corresponding first integral

$$\frac{1}{2}\left(\frac{d}{d\tau}\,\frac{1-\beta_{ph}u}{\sqrt{1-u^2}}\right)^2 = \beta_{ph}^2\omega_p^2\left[C - \frac{1}{\sqrt{1-u^2}}\right]\ , \qquad (10)$$

where C is an integration constant. Setting $C = (1-u_m^2)^{-1/2}$ it is clear that u oscillates in the range $-u_m \le u \le u_m$, where u_m is the amplitude of the longitudinal electron oscillation velocity.

For a longitudinal oscillation, $\bar{E} = E\hat{z}$ and $\bar{B} = 0$. From Eq. (3) the electron density is given by

$$n(\tau) = \frac{\beta_{ph}n_0}{\beta_{ph} - u(\tau)}\ . \qquad (11)$$

The electric field is found from the first of Eqs. (1) to be

$$E = \frac{mc}{e\beta_{ph}}(u - \beta_{ph})\frac{d\rho}{d\tau}\ . \qquad (12)$$

Using Eq. (10), this can be rewritten as

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$$E(\tau) = \pm\sqrt{2}\frac{m\omega_p c}{e}\left[\frac{1}{\sqrt{1-u_m^2}} - \frac{1}{\sqrt{1-u^2(\tau)}}\right]^{1/2}\ . \qquad (13)$$

Equations (11) and (13) give the density and electric field once $u(\tau)$ is known. In the wave-breaking limit $u_m \to \beta_{ph}$, the density develops a singularity. Physically the electric field wave steepens and breaks at this point, leading to turbulence.

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The solution of Eq. (10) for arbitrary β_{ph} has been reduced to quadrature by Akhiezer and Polovin[2] and Cavaliere.[4] For the PBWA we are interested in the limiting case $\beta_{ph} \to 1$. In this limit, $u(\tau)$ can be calculated exactly and expressed in terms of the inverse of the elliptic integral of the second kind. For $\beta_{ph} \to 1$, Eq. (9) becomes

$$\frac{d^2}{d\tau^2}\sqrt{\frac{1-u}{1+u}} = \frac{\omega_p^2 u}{1-u} \quad . \tag{14}$$

Introducing the new variable

$$x = \sqrt{\frac{1-u}{1+u}} \quad , \tag{15}$$

Equation (14) transforms to

$$\frac{d^2x}{d\tau^2} = \frac{\omega_p^2}{2}\left(\frac{1}{x^2} - 1\right) \quad . \tag{16}$$

The first integral of this equation is

$$\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 = \frac{\omega_p^2}{2}\left[\frac{2}{\sqrt{1-u_m^2}} - x - \frac{1}{x}\right] \quad , \tag{17}$$

corresponding to Eq. (10) with $\beta_{ph} \to 1$.

The variable x oscillates in the range $a \geq x \geq b$, where

$$a = \sqrt{\frac{1+u_m}{1-u_m}} \text{ and } b = \sqrt{\frac{1-u_m}{1+u_m}} \quad . \tag{18}$$

Integrating Eq. (17) yields

$$\pm\omega_p\tau = \int^x \sqrt{\frac{x}{(a-x)(x-b)}}\,dx \quad . \tag{19}$$

Choosing the initial conditions such that $x = a$ at $\tau = \tau_0 = 0$, Eq. (19) becomes[5]

$$\omega_p\tau = \int_x^a \sqrt{\frac{x}{(a-x)(x-b)}}\,dx = 2\sqrt{a}\,E(\psi,k) \quad , \tag{20}$$

where $E(\psi,k)$ is the incomplete elliptic integral of the second kind and

$$\psi = \sin^{-1}\sqrt{\frac{a-x}{a-b}} \ , \ k^2 = \frac{a-b}{a} = \frac{2u_m}{1+u_m} \quad . \tag{21}$$

This choice of initial conditions corresponds to $u = -u_m$ with $E = 0$ and $dE/d\tau < 0$ at $\tau = 0$.

From Eq. (20) the plasma frequency (2π/period) is

$$\omega = \frac{\pi}{2}\left(\frac{1-u_m}{1+u_m}\right)^{1/4} \frac{\omega_p}{E\left(\sqrt{2u_m/(1+u_m)}\right)} \quad , \tag{22}$$

where $E(k)$ is the complete elliptic integral of the second kind, and ω_p is defined by Eq. (7). In Fig. 1 the ratio ω/ω_p is shown as a function of the longitudinal electron velocity amplitude u_m.

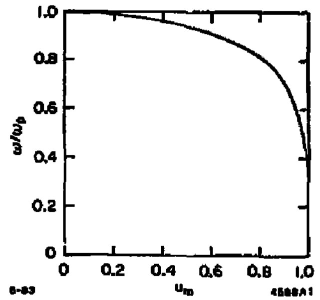

Fig. 1. The oscillation frequency ratio ω/ω_p as a function of the longitudinal electron velocity amplitude u_m.

We will denote the amplitude ψ of $E(\psi,k)$ by E^{-1}, the inverse of the elliptic integral of the second kind. Equation (20) can be inverted for $x(\tau)$ to yield

$$x(\tau) = \left(\frac{1+u_m}{1-u_m}\right)^{1/2} - \left[\left(\frac{1+u_m}{1-u_m}\right)^{1/2} - \left(\frac{1-u_m}{1+u_m}\right)^{1/2}\right] \cdot \sin^2\left[E^{-1}\left(\left(\frac{1-u_m}{1+u_m}\right)^{1/4}\frac{\omega_p\tau}{2}\ ,\sqrt{\frac{2u_m}{1+u_m}}\right)\right] \tag{23}$$

where $\tau = t - z/c$ ($v_{ph} \to c$). From Eq. (15) the electron oscillation velocity is found to be

$$u(\tau) = \frac{v_z(\tau)}{c} = \frac{1-x^2(\tau)}{1+x^2(\tau)} \quad . \tag{24}$$

The electron density is from Eq. (11)

$$n(\tau) = \frac{n_0}{1-u(\tau)} = \frac{n_0}{2}\left(1 + \frac{1}{x^2(\tau)}\right) \quad , \tag{25}$$

and the longitudinal electric field is from Eq. (13)

$$E(\tau) = \pm\sqrt{2}\,\frac{m\omega_p c}{e}\left[\frac{1}{\sqrt{1-u_m^2}} - \frac{1}{2}\left(x(\tau) + \frac{1}{x(\tau)}\right)\right]^{1/2} \quad , \tag{26}$$

where $E < 0$ for $0 < \tau < \pi/\omega$ and $E > 0$ for $\pi/\omega < \tau < 2\pi/\omega$.

The velocity u, normalized density n/n_0 and normalized electric field $-eE/m\omega_p c$ are shown in Figs. 2, 3 and 4, respectively, for the case $u_m = 0.6$. This case corresponds to the numerical simulations shown in Figs. 7 and 12 of Sullivan and Godfrey.[6] Comparison indicates reasonable agreement between the analytical and numerical results.

Luminosity One figure of merit for any high-energy acceleration technique is luminosity. Here we estimate the luminosity of a single-pass e^+e^- linear PBWA collider assuming the energy E and collision beamstrahlung σ_E/E are fixed parameters.

Neglecting the pinch effect, the luminosity for round Gaussian beams is given by

$$L_0 = N^2 f/4\pi\sigma^{*2} \quad , \tag{27}$$

where N is the number of particles per bunch, f is the collision frequency of the bunches, and σ^* is the beam width at the collision point.

In the PBWA a bunch consists optimally of N particles in each plasma wavelength λ_p. If the length of the string of bunches being accelerated by the plasma wave is ℓ, then the total number of bunches is ℓ/λ_p. Denoting the repetition rate of the lasers exciting the plasma wave by f_ℓ the collision frequency in Eq. (27) is

$$f = f_\ell \cdot \ell/\lambda_p \ . \tag{28}$$

Here we are assuming that one bunch from one beam interacts with only one bunch of the opposing beam and is then disrupted. If this does not occur, one might achieve an enhancement in the luminosity due to multiple collisions of successive bunches. This effect is not considered here.

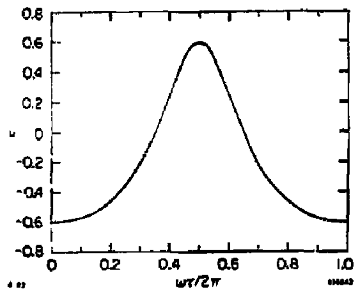

Fig. 2. Longitudinal electron velocity $u = v_z/c$ as a function of $\omega\tau/2\pi$, where ω is given by Eq. (22) and $\tau = t - z/c$ ($\beta_{ph} = 1$), for the case $u_m = 0.6$.

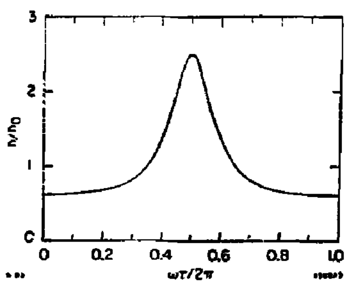

Fig. 3. Normalized electron density n/n_0 as a function of $\omega\tau/2\pi$, where n_0 is the equilibrium electron density, for the case $u_m = 0.6$.

When two bunches collide, the resulting electromagnetic fields deflect the particle trajectories causing the particles to emit synchrotron radiation. This "beamstrahlung" increases the energy spread in the beam, which the experimentalist would like kept to some minimum for the purpose of interpreting his results. For round Gaussian beams of energy $E = \gamma mc^2$, the fractional energy loss resulting from the collision is[7]

$$\frac{\sigma_E}{E} = 0.325 \frac{2r_e^3}{3} \frac{N^2\gamma}{\sigma^{*2}\sigma_z} \ , \tag{29}$$

where $r_e = e^2/mc^2$ is the classical electron radius, and σ_z is the bunch length.

In terms of σ_E/E the luminosity is

$$L_0 = \frac{1}{4\pi} \frac{3}{2r_e^3} \frac{1}{0.325} \frac{\sigma_E}{E} \frac{\sigma_z}{\gamma} f \ . \tag{30}$$

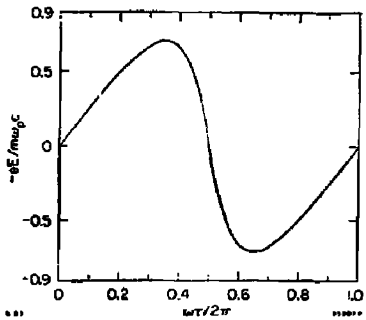

Fig. 4. Normalized longitudinal electric field $-eE/m\omega_p c$ as a function of $\omega\tau/2\pi$ for the case $u_m = 0.6$.

For the PBWA, $\sigma_z \simeq \lambda_p/2$ and f is given by Eq. (28). The luminosity is then

$$L_0(PBWA) = \frac{1}{0.325} \frac{3}{16\pi r_e^3} \frac{\sigma_E}{E} \frac{f_\ell \cdot \ell}{\gamma} \ . \tag{31}$$

The luminosity is seen to be independent of laser and plasma wavelength for fixed γ and σ_E/E.

If f_ℓ is expressed in sec^{-1} and ℓ in cm, then numerically

$$L_0(PBWA) = 8.2 \times 10^{38} \frac{\sigma_E}{E} \frac{f_\ell \cdot \ell}{\gamma} \ cm^{-2} sec^{-1} \ . \tag{32}$$

For a 2 TeV C.M. e^+e^- collider with a σ_E/E of 10% and strings of accelerated bunches of length 3 cm driven by lasers pulsed once per second, $L_0 \simeq 10^{30}\ cm^{-2}\ sec^{-1}$.

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