

SAND-97-1479C  
CONF-971164--**CORRECTING NUMERICAL INTEGRATION ERRORS CAUSED BY SMALL  
ALIASING ERRORS**

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Small sampling errors can have a large effect on numerically integrated waveforms. An example is the integration of acceleration to compute velocity and displacement waveforms. These large integration errors complicate checking the suitability of the acceleration waveform for reproduction on shakers. For waveforms typically used for shaker reproduction, the errors become significant when the frequency content of the waveform spans a large frequency range. It is shown that these errors are essentially independent of the numerical integration method used, and are caused by small aliasing errors from the frequency components near the Nyquist frequency. A method to repair the integrated waveforms is presented. The method involves using a model of the acceleration error, and fitting this model to the acceleration, velocity, and displacement waveforms to force the waveforms to fit the assumed initial and final values. The correction is then subtracted from the acceleration before integration. The method is effective where the errors are isolated to a small section of the time history. It is shown that the common method to repair these errors using a high pass filter is sometimes ineffective for this class of problem.

**INTRODUCTION**

In the development and use of a transient control program for electrodynamic shakers it was noticed that the numerically integrated acceleration occasionally differed greatly from the analytically determined velocity and displacement. This difference was significant because the velocity and displacement waveforms are used to determine the suitability of the acceleration waveform for shaker reproduction. This paper grew out of the effort to understand this difference. The study is limited to time histories sampled at equal increments of time. The paper is organized into four parts. The first part will demonstrate a problem exists. The second part will show that the problem is essentially an aliasing problem, and is independent of the numerical

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integration method used. The third part presents two possible solutions for the problem, and the fourth part illustrates the problem with corrections using two examples.

### ILLUSTRATION OF THE PROBLEM

Consider an acceleration waveform composed of the sum of exponentially decaying sinusoids of the form

$$a(t) = \sum_{i=1}^n U(t - \tau_i) A_i \exp(-\zeta_i \omega_i (t - \tau_i)) \sin(\omega_i (t - \tau_i)) \quad (1)$$

where  $U(t)$  is the unit step function.  $\zeta_i$  and  $\omega_i$  are restricted to positive numbers. This waveform has been used with great success for synthesizing waveforms for reproduction on shakers which match a specified shock response spectra. The waveform has the advantage of resembling many field environments which are essentially the impulse response of a structure.

The first component is called the compensating pulse, and the values of  $A_1$ ,  $\tau_1$ ,  $\omega_1$ ,  $\zeta_1$  are chosen to force the final acceleration, velocity, and displacement to zero [Smallwood and Nord, 1974, Smallwood, 1986]. The velocity and displacement of this waveform can be found analytically by integrating the equation for the acceleration twice.

$$v(t) = \sum_{i=1}^n U(s) \frac{-A_i}{\omega_i (\zeta_i^2 + 1)} \left\{ \exp(-\zeta_i \omega_i s) [\zeta_i \sin(\omega_i s) + \cos(\omega_i s)] + 1 \right\} \quad (2)$$

$$d(t) = \sum_{i=1}^n U(s) \left\{ \frac{A_i \exp(-\zeta_i \omega_i s)}{\omega_i^2 (\zeta_i^2 + 1)^2} [(\zeta_i^2 - 1) \sin(\omega_i s) + 2\zeta_i \cos(\omega_i s)] + \frac{A_i s}{\omega_i (\zeta_i^2 + 1)} - \frac{2A_i \zeta_i}{\omega_i^2 (\zeta_i^2 + 1)^2} \right\}$$

where  $s = t - \tau_i$ .

The analytical acceleration, velocity, and displacement waveforms for a two component waveform are shown as Figure 1. The parameters for the waveform are given in Table 1.

Table 1 Parameters for waveform in Figure 1, waveform sampled at a rate of 12,000 s/sec.

| Index<br>i | $A_i$<br>(g) | $\omega_i$<br>(Hz) | $\zeta_i$<br>(fraction of critical) | $\tau_i$<br>(s) |
|------------|--------------|--------------------|-------------------------------------|-----------------|
| 1          | -0.049505    | 100                | 1.0                                 | 0.0             |
| 2          | 1.0          | 4000               | 0.01                                | 0.0015837       |

The acceleration waveform was also integrated with a rectangular rule using a sample rate of 12000 samples/second and the resulting velocity and displacement are also shown in Figure 2. As can be seen, a significant error in the numerically integrated velocity and displacement waveform exists. The displacement error is so large that it obscures the true displacement. Figure 3 shows the results of a rectangular integration at a sample rate of 96000 samples/second. The results are improved, but a noticeable error in the displacement still exists. This example is an extreme case,

but as shown in the next section, the error will be present in any sampled waveform where the correct final velocity is not equal to the sum of the acceleration samples multiplied by the sample interval. And even if the sum of the acceleration samples is correct, the displacement can be in error if the sum of the velocity samples multiplied by the sample interval is not equal to the correct final displacement. The errors will also exist for other commonly used waveforms for shaker reproduction [Smallwood, 1986].

Significant errors in the numerical integration are most evident when there are components in the signal whose frequencies differ greatly and where the signal is sampled at not much more than twice the highest component frequency. The lowest frequency components require long waveforms relative to the duration of the high frequency components, and the long durations exaggerate the sampling errors of the high frequency components. The errors in the integrated velocity and displacement are caused by aliasing and are essentially independent of the integration method used, as will be demonstrated in the next section.

Accurate integration of the acceleration for the velocity and displacement waveforms is important in transient testing on shakers, because the velocity and displacement waveforms are used to estimate the shaker capability for a particular waveform. Small errors can be tolerated, but errors in velocity or displacement which obscure the true requirements are a problem.

One solution is to use the analytical expressions for the velocity and displacement. This is acceptable if the analytical solution is known for the waveform. Another solution would be to sample at a much higher rate before integration. However, sometimes these solutions are not available and another correction method must be used. An example is the reproduction of a field derived acceleration waveform.

### THE ERROR IN THE FINAL VALUE OF A NUMERICAL INTEGRATION IS INDEPENDENT OF THE NUMERICAL METHOD USED

For convenience in the discussion the original function will be called acceleration,  $a$ , the first integral the velocity,  $v$ , and the second integral the displacement,  $d$ . The acceleration waveform will be sampled with a sample interval  $T$ , giving the samples  $a_i$ . The samples  $a_i$  will be assumed to be zero outside the interval  $0 \leq i \leq N-1$ . The numerical integration of sampled waveforms can usually be written in the form

$$v_{i+1} = v_i + \sum_{k=k_1}^{k_2} b_k a_{i+k} \quad (3)$$

This is the form of a FIR (finite impulse response) filter. To start the integration an initial value,  $v_0$ , must be chosen. A common assumption is to assume that  $v_0$  is zero.  $k_1$  and  $k_2$  will span all the non-zero values of  $b_k$ . For example, the rectangular rule is

$$v_{i+1} = v_i + T a_{i+1} \quad (4)$$

and the trapezoidal rule is

$$v_{i+1} = v_i + T(0.5a_i + 0.5a_{i+1}) \quad (5)$$

A common property of all these methods is that

$$\sum_{k=k_1}^{k_2} b_k = T \quad (6)$$

This assures that the zero frequency response is correct.

After N steps the value for the integral is

$$v_N = \sum_{i=1}^N \sum_{k=k_1}^{k_2} b_k a_{i+k} \quad (7)$$

Let  $i+k=j$  then

$$v_N = \sum_{j=1+k}^{N+k} \sum_{k=k_1}^{k_2} b_k a_j \quad (8)$$

Interchanging the order of summation gives, letting  $a_j$  be zero for  $j \leq 0$  and  $j > N$ , gives

$$v_N = \sum_{k=k_1}^{k_2} b_k \sum_{j=1+k}^{N+k} a_j = T \sum_{j=1}^N a_j \quad (9)$$

Also

$$v_{N+j} = v_N \text{ for } j \geq 1, \quad (10)$$

since the values for  $a_{N+j}$  are assumed to be zero.

Thus the final value of the sum that approximates the integral is the same for all the commonly used integration formulas, and is given by the sample interval multiplied by the sum of all the input sample values. If an error exists in the final value using a simple rectangular rule, the same error will exist for all other commonly used integration formulas. This same error will exist even when an ideal Fourier reconstruction is used before the integration, since a Fourier reconstruction has the same form as Eq. 3 [Stearns, 1975, Eq. 5-44]. Since the ideal Fourier reconstruction is exact for strictly band-limited functions the errors must be due to aliasing errors in the signal,  $a$ , which has a finite time duration, and hence an infinite bandwidth.

All further discussion will use the rectangular rule without a loss of generality for other integration rules.

### HIGH-PASS FILTERING OF THE DATA

A first glance it would seem that simply high pass filtering the resulting integration's would correct the problem. Several filters were tried. One of the best results is shown as Figure 4. The acceleration was integrated with a rectangular rule to give the velocity. The velocity was integrated with a rectangular rule to give the displacement. The corrected waveform is much improved over the original rectangular integration. However a significant error in the displacement still exists because of the slow response of the filter.

Increasing the number of poles in the filter only made the results worse, as the filter 'ringing' increased. The high-pass cut-off frequency was chosen at half the frequency of the lowest component (50 Hz). Increasing the cut-off frequency will start to remove the important frequency components. Since the high pass-filter spreads the correction over a time period close to the period of the high-pass filter cut-off frequency, lowering the cut-off frequency spreads the integration errors over a longer time frame.

### A PROPOSED CORRECTION

In general, since the errors are aliasing errors, the waveform cannot be corrected without additional information. It will be assumed here that the desired waveforms will have initial and final values for the acceleration, velocity, and displacement of zero. Therefore, if a correction is subtracted from  $a$  which has the identical values at the final sample  $a_{N-1}$ ,  $v_{N-1}$  and  $d_{N-1}$ , the corrected waveform will have a final acceleration, velocity, and displacement of zero. First calculate the final values  $(a_f, v_f, d_f)$  by numerically integrating the acceleration. A correction to the acceleration waveform of the form

$$\hat{a}(t) = A_1 J(t-v) + A_2 \delta(t-v) + A_3 U(t-v) \quad (11)$$

can be used.  $J(t)$  is the unit jerk function,  $\delta(t)$  is the unit delta function,  $U(t)$  is the unit step function, and  $v$  is a delay. A jerk is defined as a function whose integral is a delta function. The velocity and displacement of the correction is given by

$$\hat{v}(t) = A_1 \delta(t-v) + U(t-v)(A_2) + U(t-v)(A_3(t-v)) \quad (12)$$

$$\hat{d}(t) = U(t-v)(A_1) + U(t-v)(A_2(t-v)) + U(t-v)\left(\frac{A_3}{2}t^2 - A_3vt\right) \quad (13)$$

The important point is that for times greater than  $v$  only  $A_3$  will determine the final acceleration. Once  $A_3$  is set,  $A_2$  will control the velocity for times greater than  $v$ . Similarly, once  $A_3$  and  $A_2$  are set  $A_1$  will control the displacement. If the acceleration, velocity, and displacement, are known at a time greater than  $v$  the constants  $A_1$ ,  $A_2$ , and  $A_3$  can be uniquely determined. In practice the functions will be evaluated at the sample points. A jerk can be approximated by two unit amplitude samples one sample apart, the first one positive and the second negative. A unit impulse can be approximated by a single unit sample.

The location in time of the error is still unknown. An optimum delay,  $v$ , is chosen to minimize an error function, as for example, the square error between the numerically derived displacement waveform and the displacement correction

$$e = \sum_{i=0}^{N-1} (d_i - \hat{d}_i)^2 \quad (14)$$

A similar approach was taken by Trujillo and Carter (1981) except they did not include the delay. If the errors are small an equation of the form of Eq. 11 can be used. In practice the amplitudes of the corrections can become unacceptably large. The correction is subtracted from the original waveform to give a corrected waveform.

$$\{\tilde{a}\} = \{a\} - \{\hat{a}\} \quad (15)$$

Because the corrections look like impulses and steps, energy will be added at all frequencies. It would be better if a function could be used which has the characteristics of Eq. 11, but with a better frequency distribution. The corrections will then be confined to the low frequencies. The subtraction of the correction from the original waveform will act like a parametrically designed high pass filter. If a function with only steps in the acceleration is used, the contributions of the corrections will roll off like  $1/f$ . If the correction is continuous, but has discontinuities in slope, the frequency content will roll off like  $1/f^2$ .

A useful continuous function with the desired properties is given by

$$\begin{aligned} \hat{a}(t) &= A_1 t \exp(\alpha t) && \text{for } t < 0 \text{ and } \alpha > 0 \\ \hat{a}(t) &= A_2 t \exp(-\alpha t) + A_3 (1 - \exp(-\alpha t)) && \text{for } t \geq 0 \text{ and } \alpha > 0 \end{aligned} \quad (16)$$

This will be called the TEXP correction. A companion paper (Smallwood and Cap, 1997) discusses other forms which could be used. A delay can be easily introduced by substituting  $t - v$  for  $t$  in Eq. 16. The waveform in Eq. 16 is sampled at  $t = Ti$ , giving

$$\hat{a}(i) \quad \text{for } i = 0:N-1 \quad (17)$$

The correction for a sampled waveform  $\{a\}$  with a length of  $N$  samples is found as follows:

1. The samples  $a(i)$  are numerically integrated twice giving  $v(i)$  and  $d(i)$ ,  $i = 0:N-1$ .
2. Initial values for the decay rate,  $\alpha$ , and delay,  $v$ , are chosen.
3. The final values (the value at the  $N$ th sample) for the velocity and displacement are determined by numerically integrating the acceleration. To reduce the effects of noise, the final value for the acceleration is often set as the mean of the acceleration for the last few values. These values will be called  $a_f$ ,  $v_f$ , and  $d_f$  respectively.
4. The final values for the correction with unity amplitudes are then determined.

$$\hat{a}_{1f} = \hat{a}(N) \Big|_{A_1=1, A_2=0, A_3=0} \quad \hat{a}_{2f} = \hat{a}(N) \Big|_{A_1=0, A_2=1, A_3=0} \quad \hat{a}_{3f} = \hat{a}(N) \Big|_{A_1=0, A_2=0, A_3=1}$$



$$\begin{aligned} \hat{v}_{1f} &= \hat{v}(N) \Big|_{A_1=1, A_2=0, A_3=0} & \hat{v}_{2f} &= \hat{v}(N) \Big|_{A_1=0, A_2=1, A_3=0} & \hat{v}_{3f} &= \hat{v}(N) \Big|_{A_1=0, A_2=0, A_3=1} \\ \hat{d}_{1f} &= \hat{d}(N) \Big|_{A_1=1, A_2=0, A_3=0} & \hat{d}_{2f} &= \hat{d}(N) \Big|_{A_1=0, A_2=1, A_3=0} & \hat{d}_{3f} &= \hat{d}(N) \Big|_{A_1=0, A_2=0, A_3=1} \end{aligned} \quad (16)$$

4. The amplitudes,  $A_1$ ,  $A_2$ , and  $A_3$  are then found solving the equation

$$\begin{bmatrix} \hat{a}_{1f} & \hat{a}_{2f} & \hat{a}_{3f} \\ \hat{v}_{1f} & \hat{v}_{2f} & \hat{v}_{3f} \\ \hat{d}_{1f} & \hat{d}_{2f} & \hat{d}_{3f} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} a_f \\ v_f \\ d_f \end{Bmatrix} \quad (17)$$

5. Values for the decay rate,  $\alpha$ , and delay,  $v$ , are then changed iteratively to minimize the error

$$e = \sum_{i=0}^{N-1} (d(i) - \hat{d}(i))^2$$

6. The correction is subtracted from the original waveform to give the corrected waveform.

$$\{\tilde{a}\} = \{a\} - \{\hat{a}\}$$

### EXAMPLES

The calculations in this paper were performed in MATLAB™. The following functions were used: *butter* was used to design the high pass filters, *filter* was used to filter the waveforms, *fmin* was used to minimize a function of a single variable, *fmins* was used to minimize a function of several variables.

The method discussed above was applied the example of Figure 1. The results are shown as Fig. 5. The parameters for the correction were as follows:  $A = [0.3996e-3 \quad 0.0055e-3 \quad 0.0055e-3]$  g's,  $\alpha = 0.0322$  (1/samples), and  $v = 50.7188$  samples. Note that the required correction is quite small. The correction is imperceptible on the acceleration plots. Three curves are shown for the acceleration, velocity, and displacement. The velocity and displacement were computed with the rectangular rule. The corrected velocity and displacement do not match the analytical result, but are much closer to the analytical result than the original numerical integration. The magnitude of the FFTs of the acceleration waveforms are also shown in Fig. 6. Four curves are shown: the waveform sampled at 12000 samples/second, the waveform sampled at 96000 samples/second, the waveform corrected with the high-pass filter, and the waveform corrected using TEXP. The curve for the waveform sampled at 96,000 samples/sec should give a good approximation of the shape of the Fourier transform of the analytical waveform except at the lowest frequencies. As can be seen all the corrections are essentially high-pass filters. The Butterworth filter removes the low frequency energy, but not as effectively as the correction given by TEXP. The horizontal tail at about 10 Hz on the FFT of the waveform sampled at 96,000 s/sec is also a deviation from the analytical result caused by aliasing, but is about 2 orders of magnitude less than the error in the waveform sampled at 12,000 s/sec. The spectrum of the analytical waveform will decrease proportional to  $f^2$ .

For a second example a 3500 Hz exponentially decaying sinusoid delayed 0.01 seconds with respect to the 4000 Hz sinusoid (Fig. 7) is added to the waveform of the first example. This is a long delay compared with the duration of the waveform. The two pulses at 3500 and 4000 Hz are quite distinct as can be seen in Fig. 7. This should be a severe test for the method. The waveform integrated with the rectangular rule is shown as Fig. 8. For this example, the best correction would probably be a separate correction near each of the two high frequency components. However, the method will not resolve these two times. The corrected waveform using the Butterworth filter is shown as Fig. 9. The corrected waveform using TEXP and the parameters  $A = [0.0031 \quad 0.0025 \quad 0.0025]$  g's,  $\alpha = 0.1428$  (1/samples), and  $v = 271.5693$  samples is shown as Fig. 10. The magnitude of the FFT of the waveforms is shown as Fig. 11. In this case the TEXP correction performed somewhat better than the high pass filter.

## CONCLUSIONS

Small errors in sampled acceleration waveforms can lead to large errors in the numerically integrated velocity and displacement waveforms. When the sampling errors in acceleration result in aliasing, the errors in velocity and displacement are largely independent of the integration method used. If the errors are concentrated in a narrow time window, the insertion of a correction based on an exponential function can significantly improve the results. In some cases a well chosen high pass filter can also result in satisfactory corrections.

The best method to avoid this problem is to integrate the waveforms analytically. The next best method is to over-sample the waveforms before numerical integration. If neither of these methods are available, the TEXP method works for many waveforms synthesized for shaker reproduction, because the errors are caused by a few high frequency components which have their peak values near the same point in time. The method should also work for other waveforms for which the final values of the waveform and its integrals are known or can be deduced from other data, and where the significant errors are localized in time. If the errors are not localized in time a well chosen high pass filter can improve the results.

## REFERENCES

- Smallwood, D. O., 1986, *Shock Testing on Shakers by Using Digital Control*, IES Technical Monograph, Institute of Environmental Sciences and Technology, 940 E. Northwest Highway, Mount Prospect, IL 60056.
- Smallwood, D. O., and Cap, J. S., 1997, "Salvaging Transient Data with Overloads and Zero Offsets," 68<sup>th</sup> Shock and Vibration Symposium.
- Smallwood, D. O., and Nord, A. R., 1974, "The Use of Exponentially Decaying Sinusoids to Duplicate Shock Spectra on Shaker Systems," S&V Bulletin, No. 44, Part 3, pp 43-56.
- Stearns, S. D., 1975, *Digital Signal Analysis*, Hayden Book Co. Rochelle Park, NJ.
- Trujillo, D. M., and Carter, A. L., 1982, "A New Approach to the Integration of Accelerometer Data," *Earthquake Engineering and Structural Dynamics*, Vol. 10, 529-535.

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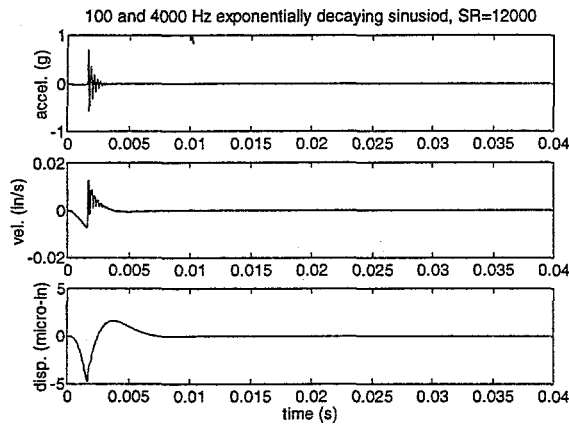


Figure 1 First example, analytical integrated, sampled at 12000 samples/sec

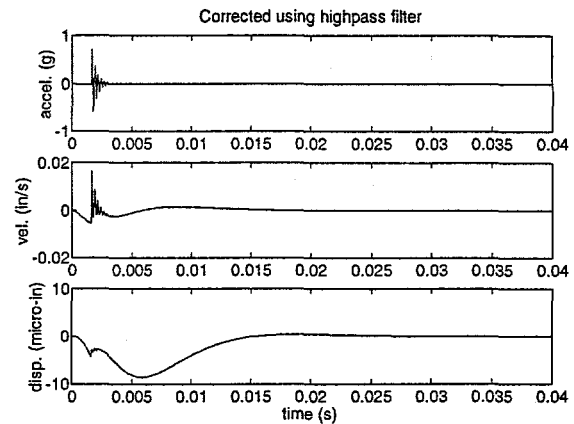


Figure 4 First example, corrected with a 2 order Butterworth high pass filter with a 50 Hz cutoff

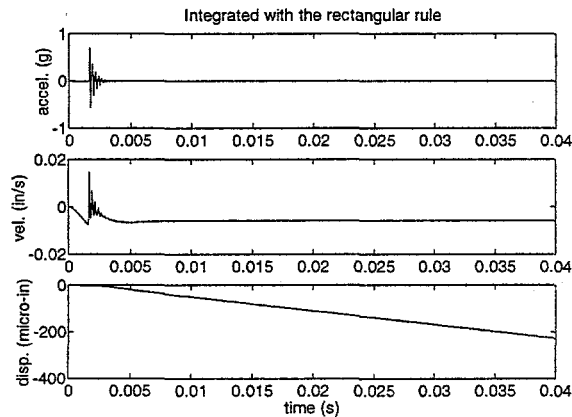


Figure 2 First example, integration with a rectangular rule, 12000 samples/sec

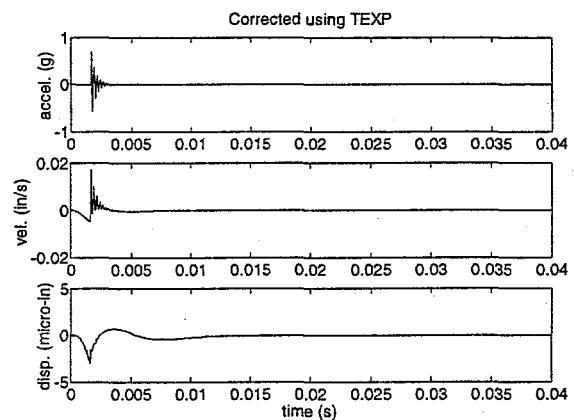


Figure 5 First example, corrected with of Eq. (14)

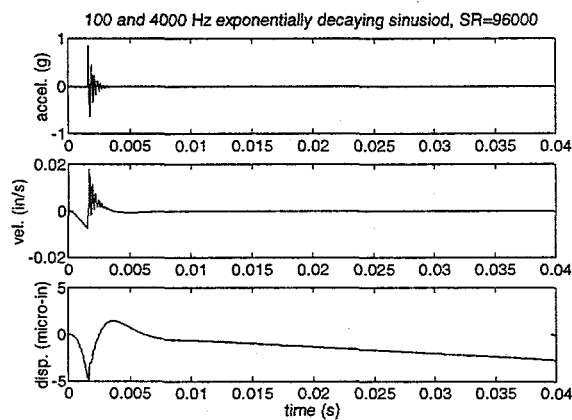


Figure 3 First example, integration with a rectangular rule, 96000 samples/sec

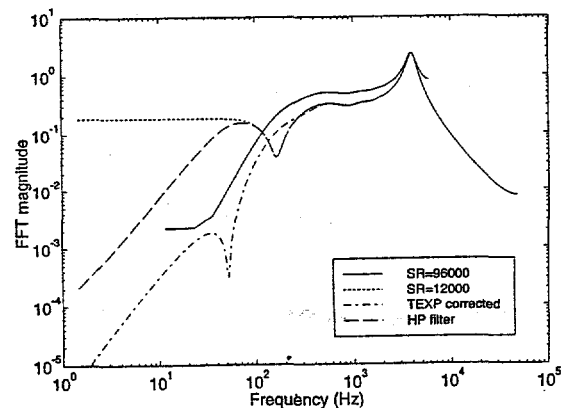


Figure 6 First example, FFT magnitude

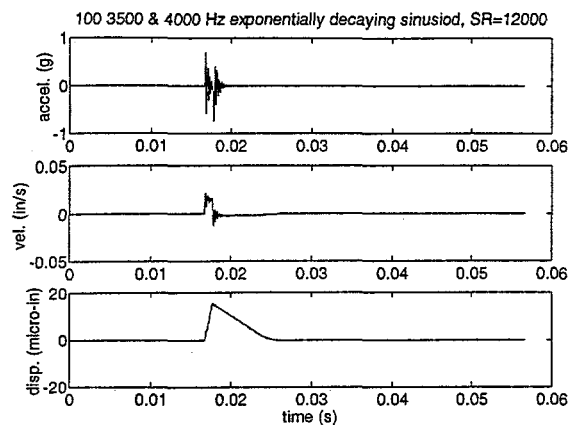


Figure 7 Second example, analytical integrated, sampled at 12000 samples/sec

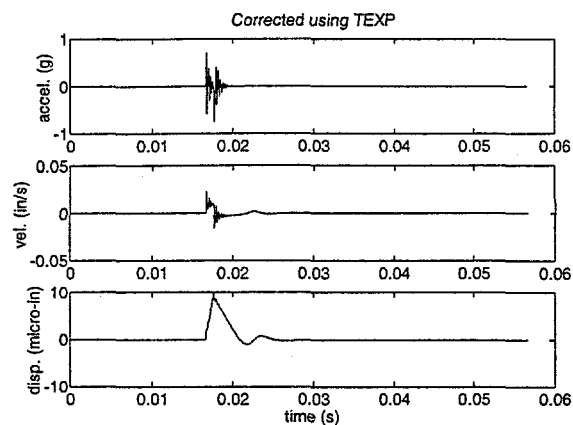


Figure 10 Second example, corrected with Eq. (14)

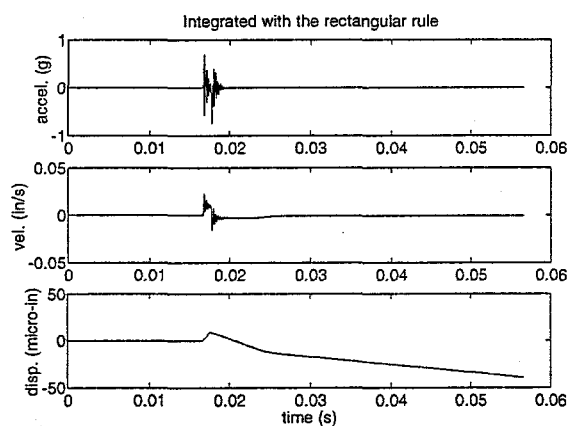


Figure 8 Second example, integration with a rectangular rule, 12000 samples/sec

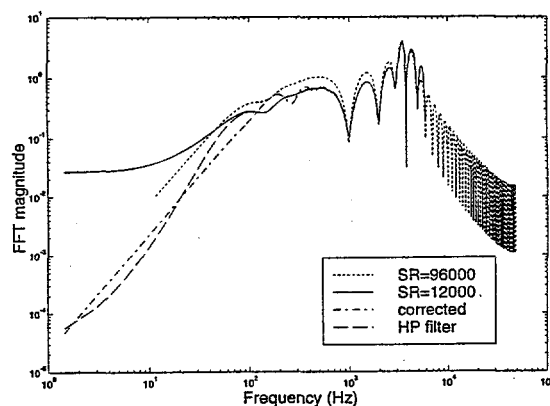


Figure 11 Second example, FFT magnitude

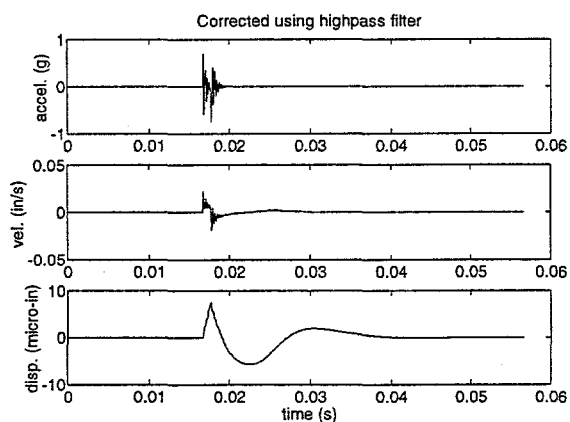


Figure 9 Second example, corrected with a 2 order Butterworth high pass filter with a 50 Hz cutoff