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LONGITUDINAL DYNAMICS IN STORAGE RINGS

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The single-particle equations of motion are derived for charged particles in a storage ring. Longitudinal space charge is included in the potential assuming an infinitely conducting circular beam pipe with a distributed inductance. The framework uses Hamilton's equations with the canonical variables ϕ and W. The Twiss parameters for longitudinal motion are also defined for the small amplitude synchrotron oscillations. The space-charge Hamiltonian is calculated for both parabolic bunches and "matched" bunches. A brief analysis including second-harmonic rf contributions is also given. The final sections supply calculations of dynamical quantities and particle simulations with the space-charge effects neglected.

1. Introduction

The goal of this work is to acquaint the reader with operations in longitudinal phase space. The approach taken is to assume a sinusoidal rf waveform averaged over the circumference. The treatment is two dimensional. We develop the single particle equations of motion using Hamilton's equations for synchrotron oscillations. Longitudinal space charge effects are included in the Hamiltonian. We next develop a matrix approach for small amplitude synchrotron oscillations; this treatment leads to the matched Twiss parameters α_s and β_s for longitudinal motion.

After defining the bunch shapes, we develop the single-particle Hamiltonian for different assumed charge distributions. This analysis is repeated assuming the addition of a higher harmonic rf voltage to the fundamental. We next derive the synchrotron quantities from the Hamiltonian. Finally we present simulations of adiabatic capture, 90° phase rotation, and creation of voids in continuous beams.

2. Equations of Motion

The single-particle dynamics uses the two canonically-conjugate variables ϕ and W. These quantities are (i) ϕ in units of rf phase and (ii) $W = \Delta E/\Omega$ where ΔE is the departure from the ideal or central energy and $\Omega = \beta c/R$ is the synchronous particle revolution frequency in the assumed machine of circumference $2\pi R$. We define the equations for synchrotron motion [1] assuming a sinusoidal waveform

$$\frac{dW}{dt} = \frac{eV}{2\pi}(\sin\phi - \sin\phi_s) \tag{1}$$

and

$$\frac{d\phi}{dt} = \frac{h\eta\Omega c W}{pcR} . \tag{2}$$

In these equations V is the maximum rf voltage, ϕ_s is the phase of the synchronous particle relative to the zero value of the (increasing) rf voltage wave, h is the harmonic number, $h = 2\pi f_{rf}/\Omega$, where f_{rf} is the rf frequency, $\eta = \gamma_t^{-2} - \gamma^{-2}$, where γ_t is the transition gamma of the machine and $\gamma = E/mc^2$. If operation is in the storage ring mode then $\phi_s = 0$ below transition or $\phi_S = \pi$ above transition. The momentum p is expressed in eV/c units. <u>For the remainder of this work, we assume $\phi_S = 0$ and $\gamma < \gamma_T$.</u>

Longitudinal space-charge effects influence the voltage seen by the beam. The space-charge field is given by [2]

$$E_z = -e \frac{\partial \lambda}{\partial z} \left(\frac{g_o}{4\pi \, \varepsilon_o \, \gamma^2} - \frac{dL}{dz} \beta^2 c^2 \right) \tag{3}$$

where the bunch has charge per unit length $e\lambda$, $dz = -R d\phi/h$, and $g_o = 1 + 2 \ln (b/a)$ where a is the beam radius and b is the beam-pipe radius. We assume the beam pipe is a perfectly conducting cylinder but has a distributed inductance dL/dz per unit length [2]. The voltage seen by a particle per turn, $V_s = \int E_z \, dz$, can be written

$$V_s = e\beta c R \frac{\partial \lambda}{\partial z} \, \mathrm{Im} \left(\frac{Z_e}{n} \right) \tag{4}$$

where

$$\mathrm{Im} \left(\frac{Z_e}{n} \right) = \Omega L - \frac{g_0 Z_0}{2\beta\gamma^2} \tag{5}$$

and the impedance of free space $Z_0 = 120 \, \pi$ ohms. The resulting Hamilton equation (1) is rewritten

$$\frac{dW}{dt} = \frac{eV}{2\pi} \sin\phi \, [1 - r] \tag{6}$$

where

$$r = -\frac{e\beta cR}{V\sin\phi}\frac{\partial\lambda}{\partial z}\,\mathrm{Im}\left(\frac{Z_e}{n}\right) \tag{7}$$

within the bunch and $r = 0$ outside. We can derive the Hamiltonian of the motion from Eqs. (2) and (6) via

$$\frac{\partial H}{\partial W} = \frac{d\phi}{dt} \text{ and } \frac{\partial H}{\partial \phi} = -\frac{dW}{dt} \quad .$$

We obtain

$$H(\phi,W) = GW^2 - \frac{eV}{2\pi}\int_0^{\phi}(1 - r)\sin\psi\, d\psi \tag{8}$$

where

$$G = h\eta\Omega c/(2pcR) \quad .$$

3. Derivation of the Matched Quantities

The goal is to write the equations using a matrix approach that is equivalent to that widely known for transverse phase space. This procedure works for small angles. We wish to transform the (ϕ,W) vector

$$\begin{pmatrix}\phi\\ W\end{pmatrix}_f = M\begin{pmatrix}\phi\\ W\end{pmatrix}_i \tag{9}$$

where

$$M = \begin{pmatrix}\cos\theta + \alpha_s\sin\theta & \beta_s\sin\theta \\ -\gamma_s\sin\theta & \cos\theta - \alpha_s\sin\theta\end{pmatrix} \quad . \tag{10}$$

In Eq. (10) β_s, α_s, and γ_s are the so-called matched functions and θ represents the phase shift for the cell. The functions repeat from cell to cell. We take the approximation $W_f = W_i + b\phi_i$, and $\phi_f = \phi_i + aW_i$. Then referring to Eqs. (2) and (6) we obtain

$$a = \frac{h\eta\Omega c\Delta t}{pcR} \tag{11}$$

and

$$b = \frac{eV}{2\pi}\,\Delta t\,(1-r) \tag{12}$$

where Δt is the transit time through the cell and V is the rf voltage seen by the particle through the cell.

The simplest example is to assume one rf cavity in a machine; then we evaluate the matched functions in the cavity center. The cell (or machine) consists of one-half cavity, a drift, and one-half cavity, respectively. Written in matrix form

$$M = \begin{pmatrix} 1 & 0 \\ b/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b/2 & 1 \end{pmatrix} \tag{13}$$

or

$$M = \begin{pmatrix} 1 + \frac{ab}{2} & a \\ b + \frac{ab^2}{4} & 1 + \frac{ab}{2} \end{pmatrix} . \tag{14}$$

Referring to Eq. (10), the matched functions are given in terms of a and b $\cos\theta = 1 + ab/2$, $\beta_s = a/\sin\theta$, and $\alpha_s = 0$. Since θ is small, we obtain θ

$\sim \sqrt{-ab}$ and $\beta_s = \sqrt{-a/b}$. The small-angle synchrotron tune $\nu_s = \theta/(2\pi)$ with $\nu_s = \nu_{so}\sqrt{1-r}$ where

$$\nu_{so} = \frac{1}{\beta}\left[\frac{-eV\eta h}{2\pi E}\right]^{1/2} \tag{15}$$

where β is the usual v/c (ν_s is the number of synchrotron oscillations per revolution). The units of β_s are in $(eVs)^{-1}$; it is defined by $\beta_s = \beta_{so}/\sqrt{1-r}$ where

$$\beta_{so} = 2\pi\left[\frac{-\eta f_{rf} c}{pcR\ eV}\right]^{1/2} . \tag{16}$$

4. Beam Sizes

Since the matched $\alpha_s = 0$, the longitudinal phase space is that of an upright ellipse with semi-axes (ϕ_M, W_M). The semi-axes are given by

$$\phi_M = \left(\frac{\beta_s h\ \varepsilon_\ell}{\pi}\right)^{1/2} \tag{17}$$

and

$$W_M = \left(\frac{h\varepsilon_\ell}{\pi\beta_s}\right)^{1/2} \tag{18}$$

where ε_ℓ is the longitudinal phase space area of a single bunch in eVs, $\varepsilon_\ell = \oint W d\phi/h$. The peak dp/p is obtained from Eq. (18)

$$\left(\frac{dp}{p}\right)_M = \frac{W_M \Omega}{\beta pc} \quad . \tag{19}$$

The full bunch length ℓ (in meters) is given by

$$\ell = \frac{2R\phi_M}{h} \quad . \tag{20}$$

The bunch length ℓ_o in the absence of space-charge effects is defined by $\ell/\ell_o = (\beta_s/\beta_{so})^{1/2} = (1-r)^{-1/4}$ so the bunch lengthens as r is decreased.

If the bunches are taken to have a parabolic density distribution then

$$\lambda(z) = \frac{6N}{\ell^3}\left(\frac{\ell^2}{4} - z^2\right) \quad \text{and} \quad \frac{\partial\lambda}{\partial z} = -\frac{12N}{\ell^3}\, z \tag{21}$$

where N is the number of particles in the bunch. We can rewrite Eq. (7) in the short bunch approximation as

$$r = -\frac{12R^2\, eN\beta c}{\ell^3 hV}\, \mathrm{Im}\left(\frac{Z_e}{n}\right) \tag{22}$$

and the equation for the bunch length becomes

$$\ell^4 + \frac{12R^2 eN\beta c}{hV}\, \mathrm{Im}\left(\frac{Z_e}{n}\right)\ell - \ell_o^4 = 0 \quad . \tag{23}$$

We obtain the corresponding Hamiltonian from Eq. (7), (8), and (21)

$$H(\phi,W) = GW^2 + \frac{eV}{2\pi}\left[\cos\phi - 1 + \frac{6e\beta cR^2N}{V\ell^3h}\,\mathrm{Im}\left(\frac{Z_e}{n}\right)\phi^2\right] \quad . \tag{24}$$

This equation is valid for $|\phi| \le \phi_M$. The space-charge term in Eq. (24) is zero for $|\phi| > \phi_M$.

If the beam is "matched to the bucket" then

$$\lambda(z) = \frac{Nh}{\pi R}\cos^2\frac{\phi}{2} \text{ and } \frac{\partial\lambda}{\partial z} = \frac{Nh^2}{2\pi R^2}\sin\phi \tag{25}$$

and the Hamiltonian [Eq. (8)] becomes

$$H(\phi,W) = GW^2 + \frac{eV}{2\pi}(\cos\phi - 1)(1 - r) \tag{26}$$

where

$$r = -\frac{e\Omega Nh^2}{2\pi V}\,\mathrm{Im}\left(\frac{Z_e}{n}\right) \quad . \tag{27}$$

5. Superposition of Harmonic Voltages

Higher harmonic voltages are sometimes used to control ν_{so} and to increase the spread in synchrotron frequencies in a bunch in order to provide Landau damping against longitudinal coupled-bunch instabilities in a synchrotron. In Fig. 1, I show an example of a fundamental with V = 1, plus 40% (d = 0.4) second harmonic (P = 2); the superposition (dotted) flattens out near the origin. The synchrotron tune ν_{so} is related to the slope of the voltage $dV/d\phi$ at the origin - in the illustrated case we see ν_{so} is reduced with this polarity of the second harmonic.

Analytically we rewrite Eq. (1) in a form analogous to Eq. (6)

$$\frac{dW}{dt} = \frac{eV}{2\pi} \sin\phi \; (1-\alpha) \tag{28}$$

with $\sin\phi_s = 0$ where

$$\alpha = \frac{d\sin P\phi}{\sin\phi} \quad . \tag{29}$$

In small-angle approximation $\alpha \sim 1 - dP$. The treatment above is still valid, i.e., $\nu_s = \nu_{so}\sqrt{1-\alpha}$ and $\beta_s = \beta_{so}/\sqrt{1-\alpha}$. For positive α, the synchrotron tune is reduced and the bunch length is increased. It is understood that for stability $\alpha < +1$.

We obtain the Hamiltonian from Eq. (2), (24), and (28)

$$H(\phi,W) = GW^2 + \frac{eV}{2\pi}\left[\cos\phi - 1 - \frac{d}{P}\left(\cos P\phi - 1\right)\right] \quad . \tag{30}$$

If we also consider a parabolic bunch (i.e., space-charge effects), the Hamiltonian becomes

$$H(\phi,W) = GW^2 + \frac{eV}{2\pi}\left[\cos\phi - 1 + \frac{6e\beta cR^2N}{V\ell^3h}\,\mathrm{Im}\left(\frac{Z_e}{n}\right)\phi^2 - \frac{d}{P}\left(\cos P\phi - 1\right)\right] \quad . \tag{31}$$

More information on this subject is contained in an earlier report [4]; the treatment includes acceleration.

6. Synchrotron Quantities for a Fundamental RF System

A given particle travels on a trajectory in (ϕ,W) phase space with extrema $\pm\,\phi_a$ and $\pm\,W_a$. The Hamiltonian is constant along the trajectory. Thus we obtain from Eq. (8)

$$W_a^2 = -\frac{eV}{2\pi G}\int_o^{\phi_a}(1-r)\,\sin\psi\,d\psi \tag{32}$$

and

$$W^2 = \frac{eV}{2\pi G}\int_{\phi_a}^{\phi}(1-r)\,\sin\psi\,d\psi\quad . \tag{33}$$

The area traced out by the trajectory in (ϕ,W) phase space is simply

$$A = 4\int_o^{\phi_a} W d\phi\quad . \tag{34}$$

For $\phi_a = \pi$, we refer to the result as the bucket area; the trajectory is called the separatrix. In Fig. 2 we show trajectories in (ϕ,W) space for $\phi_a = \pi/2$ and $\phi_a = \pi$; the space-charge term r is set to zero in these cases.

It is of interest to calculate relative quantities; still assume no space-charge effects are present. First we wish to know the relative areas for a particle with maximum ϕ_a to that of the separatrix $(\phi_a = \pi)$ - this is given by

$$R_A(\phi_a) = \frac{\int_o^{\phi_a}(\cos\phi - \cos\phi_a)^{1/2}\,d\phi}{2\sqrt{2}}\quad . \tag{35}$$

Next we wish to know the ratio of the corresponding W_a (or dp/p). This is given as

$$R_W(\phi_a) = \sin\left(\frac{\phi_a}{2}\right) \quad . \tag{36}$$

Finally we wish to understand the dependence of synchrotron tune ν_s upon ϕ_a. We know ν_{so} from Eq. (15) so we write this in the form of a ratio

$$R_{\nu_s}(\phi_a) = \frac{\nu_s}{\nu_{so}} = \frac{\pi}{2}\left[\int_0^{\phi_a} \frac{d\phi}{(\cos\phi - \cos\phi_a)^{1/2}}\right]^{-1} \quad . \tag{37}$$

I have carried out the calculations of the above ratios Eqs. (35)-(37) and they are listed in Table 1 vs ϕ_a; they are also plotted in Fig. 3 vs ϕ_a. The ratios (35) and (37) were carried out with a program. They can be approximated for $\phi_a \leq \pi/2$ by [3]

$$R_A(\phi_a) \approx \frac{\pi}{8}\,\phi_a \sin\frac{\phi_a}{2} \tag{38}$$

and

$$R_{\nu_s}(\phi_a) \approx \left[1 + \frac{\phi_a^2}{16} + \frac{11}{3072}\,\phi_a^4\right]^{-1} \quad . \tag{39}$$

The bucket area reduction due to space charge forces is particularly simple if the charge distribution is matched to the potential [Eq. (25)]; then $A/A(r = 0) = (1 - r)^{1/2}$ with r given by Eq. (27).

7. Operations

There are a number of procedures that are commonplace in storage rings and accelerators. Basically they involve changing the shape of the phase-space contour that encloses the particles in the longitudinal plane. Generally, if one raises the rf voltage, the result is to bunch the beam - similarly the beam debunches as the rf voltage is decreased. If the bunch does not have the proper β_s value [see Eq. (16)] then it is not matched - the shape changes with time as it rotates like a rigid body in the phase plane. For constant voltage, a matched bunch maintains its shape as long as its length occupies no more than 120° of rf phase. Another phenomenon is the filamentation of phase space: as is evident from Fig. 3, the synchrotron frequency changes with ϕ_e amplitude. Small amplitude particles spiral faster than those at large amplitudes.

Some gymnastics have been developed, especially for injection, extraction, and matching between rings. One is adiabatic capture of a beam injected from a linac. In this case the rf is off during the injection period; then the rf is raised slowly to capture the beam. In Fig. 4 I show an example of five turn H injection into the so-called SSC bottom-energy booster. The machine considered had a circumference of 250 m, transition gamma of 10.4, and the harmonic number was 48. The injection was studied at a kinetic energy of 500 MeV. To study the injection, we wrote a program that directly integrated the synchrotron equations of motion [Eqs. (1) and (2)] turn-by-turn. A second order Runge Kutta integrator was used for the tracking. Longitudinal space charge was not included because it is not significant at these levels. Figure 4(a) shows the linac pulses for one period of the ring rf period. We inject ten microbunches, each assumed to be a uniformly filled ellipse of half widths $\Delta\phi$ .005 and $\Delta(dp/p)$

.88 x 10^{-3}. The coordinate ϕ represents phase relative to the synchronous particle. The bunches were centered at ± π, ± 0.777 π, ± 0.557 π, ± 0.33 π, and ± 0.111 π in ϕ assuming a 440/9 MHz rf system. The rf voltage is zero during the process, which we continue for five turns. Figure 4(a) shows the injected phase space after five turns with 50 particles per microbunch (2500 total particles). Next, we raised the rf voltage slowly; for 200 turns the voltage was raised linearly at 1 kV/turn to a final value of 200 kV. Figure 4(b) shows the beam phase space along with the separatrix. The beam has been captured and is now ready for acceleration.

Another trick is the 90° phase-rotation. Suppose it is desired to extract a very short bunch from an accelerator. First you reduce the rf voltage to a very small value; then the bunch lengthens to a flat pencil in phase space. The voltage is then raised abruptly to a very large value. Since the resident bunch is mismatched to the new conditions, it just rotates as a rigid body. After a 90° rotation (or 1/4 of a synchrotron oscillation) it reaches its shortest extent in time. The corresponding number of turns is $1/(4\ \nu_{so})$ with ν_{so} given by Eq. (15). Figure 5(a) shows the example of Fig. 4 but with a matched bunch ϕ_M = π/4 and $(dp/p)_M$ = 0.106%. The voltage is abruptly reduced from 200 kV to 40 kV; this reduces ν_{so} to 0.0125. We show in Fig. 5(b) the state after 20 turns, which is $1/(4\ \nu_{so})$. We then raise the voltage to 1 MV/turn and track for five turns the phase space is shown in Fig. 5(c). The bunch is indeed very short. This technique has been used at FNAL to extract a very short proton bunch for TeV I experiments [5].

Another way to lengthen a bunch without changing the voltage is to (i) shift the stable phase ϕ_s by π to the unstable fixed point for a preset number of turns so the beam debunches; (ii) then the stable phase is shifted back by π. The mismatched bunch will just rotate - when it reaches the desired length then the beam can be, e.g., extracted [6].

The above operations are straightforward and can be found described elsewhere. One topic which has piqued my curiosity is the barrier bucket that was described by Jim Griffin in the 1983 PAC [7]. You use a single pulsed rf cavity that is just fired at the revolution frequency of the synchronous particle. With this bucket you can create and maintain a gap in a coasting beam. Mathematically this is done by just changing the sign of the rf voltage. Figure 6(a) shows a coasting beam with momentum spread $dp/p = \pm$ 0.1%. The ϕ range represents the whole storage ring. Figure 6(b) shows the situation after 1000 turns if you use a fourth harmonic barrier bucket. A gap length of 1/10 the machine has been created. The voltage was -3 kV; we use the same example as above.

8. Conclusions

There are still further operations in storage rings that I have not discussed for lack of space and energy. Nevertheless, I have tried to describe how one treats the single-particle motion via Hamilton's equations. The formalism has been laid out for the treatment of complex rf waveforms, and intense beam effects. I have not discussed the instabilities of bunched beams. Actually these do tend to limit the stored intensities. The extension to electron storage rings requires compensation for losses due to synchrotron radiation. The treatment becomes somewhat more complicated for acceleration or deceleration; in this case $\sin \phi_s \neq 0$.

TABLE I. Ratios vs ϕ_a

ϕ_a (deg)	R_A	R_W	R_{ν_s}
10	0.006	0.087	1.000
20	0.024	0.174	0.995
30	0.053	0.259	0.986
40	0.093	0.342	0.972
50	0.144	0.423	0.955
60	0.203	0.500	0.934
70	0.271	0.574	0.910
80	0.345	0.643	0.882
90	0.424	0.707	0.850
100	0.506	0.766	0.814
110	0.589	0.819	0.774
120	0.672	0.866	0.731
130	0.751	0.906	0.683
140	0.825	0.940	0.629
150	0.891	0.966	0.569
160	0.945	0.985	0.500
170	0.984	0.996	0.412
180	1.000	1.000	0.138

References

[1] B. W. Montague, "RF Acceleration," in Theoretical Aspects of the Behavior of Beams in Accelerators and Storage Rings, CERN 77-13 (1977).

[2] S. Hansen et al., IEEE Trans. Nuc. Sci. NS-22, 1381 (1975).

[3] See e.g., J. B. Marion, "Classical Dynamics," Second Edition, Academic Press (1970).

[4] E. P. Colton, IEEE Trans. Nuc. Sci. NS-32, 2570 (1985).

[5] J. Griffin et al., IEEE Trans. Nuc. Sci. NS-32, 2359 (1985).

[6] See e.g., E. Colton, "RF Manipulations for Matching between Synchrotrons," Los Alamos National Laboratory internal note LAMPF II 85-002, unpublished (1985).

[7] J. Griffin et al., IEEE Trans. Nuc. Sci. NS-32, 2359 (1985).

Figure Captions

Figure 1. RF voltage waveforms, "F" is fundamental, "P = 2" represents 40% second harmonic and "sum" represents the sum of the above two waveforms.

Figure 2. Trajectories of constant Hamiltonian calculated from Eq. (33) for the indicated values of ϕ_a. The axes are ϕ and the relative momentum spread dp/p. We assume the fundamental rf system with $r = 0$.

Figure 3. Ratios plotted vs. ϕ_a. R_A is defined by Eq. (35) and is the ratio of the area [Eq. (34)] for the trajectory with limiting $\phi = \phi_a$ to the bucket area. R_{ν_s} is defined by Eq. (37) and is the ratio of the synchrotron tune for $\phi = \phi_a$ to the small-amplitude value ν_{so} defined in Eq. (15).

Figure 4. Simulation of the adiabatic capture process for a sample ring with circumference 250 m, $\gamma_t = 10.4$, harmonic number $h = 48$, and proton kinetic energy 500 MeV: (a) Phase-space plot after five turns for ten linac microbunches injected per turn. (b) Situation after 200 turns with the rf voltage increased linearly by 1 kV/turn.

Figure 5. Simulation of the 90° phase rotation technique which is used to form a very short bunch. (a) Initial matched beam with $\phi_M = \pi/4$ for $V = 200$ kV/turn; (b) beam after reducing voltage to 40 kV/turn and tracking 20 turns; (c) beam after raising voltage to 1 MV/turn and tracking 5 turns.

Figure 6. Illustration of the formation of a gap in a coasting beam by use of a barrier bucket. (a) Phase space of coasting beam with $dp/p = \pm 0.1\%$. The ϕ range represents the whole storage ring. (b) Phase space after 1000 turns if you use a h = 4 single pulsed rf cavity that is just fired at the revolution frequency of the synchronous particle. The rf voltage is -3 kV/turn.

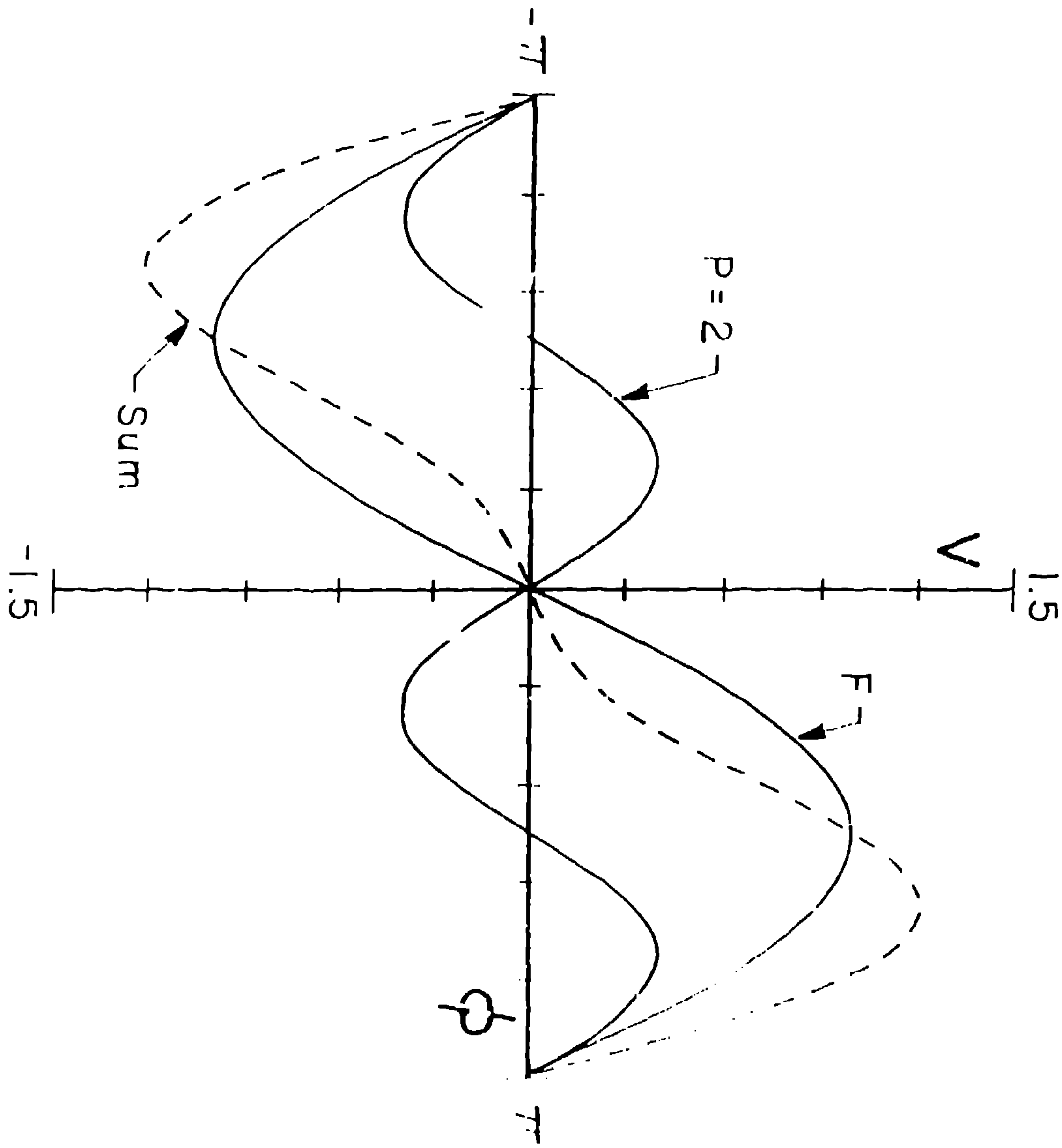
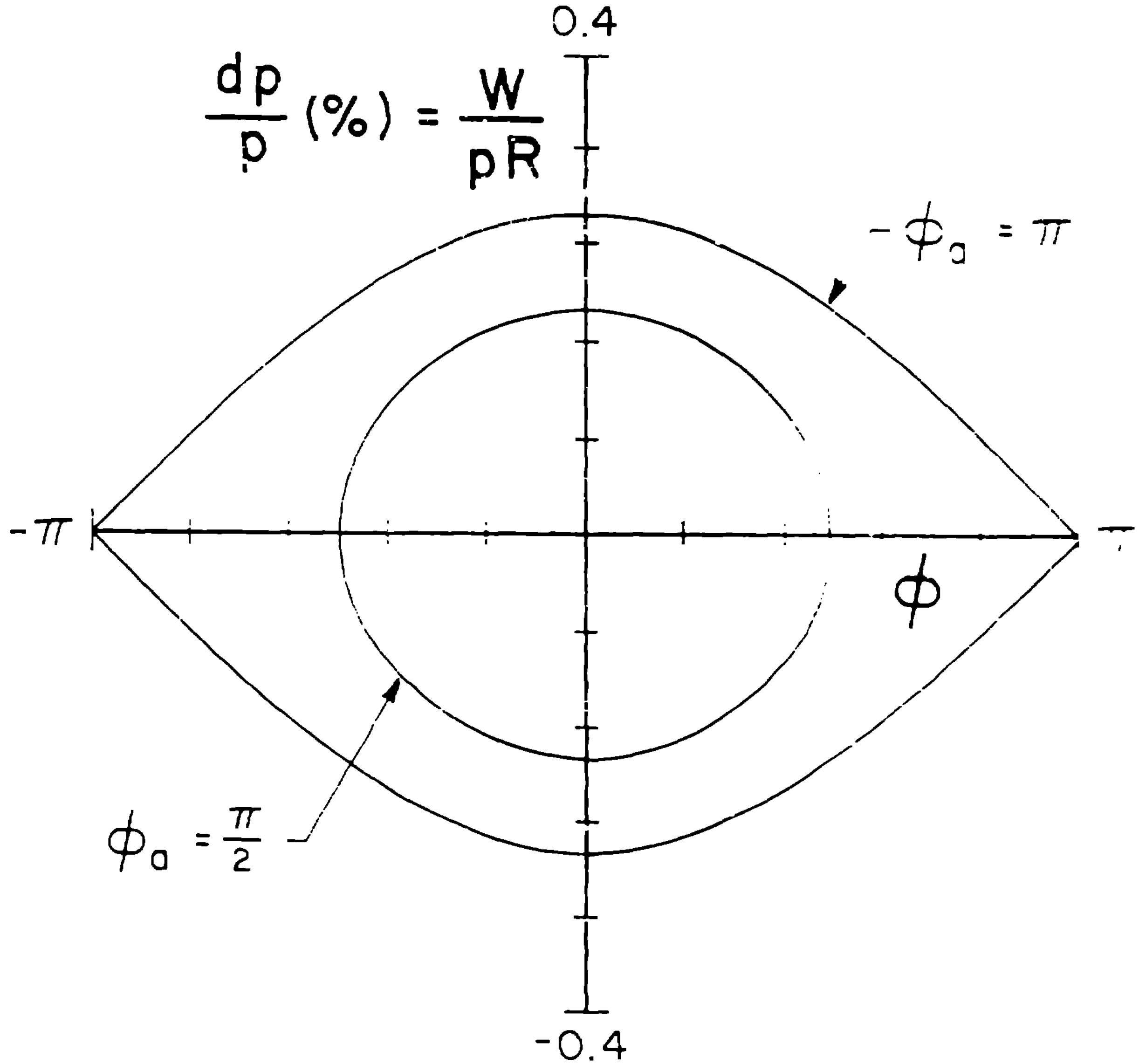
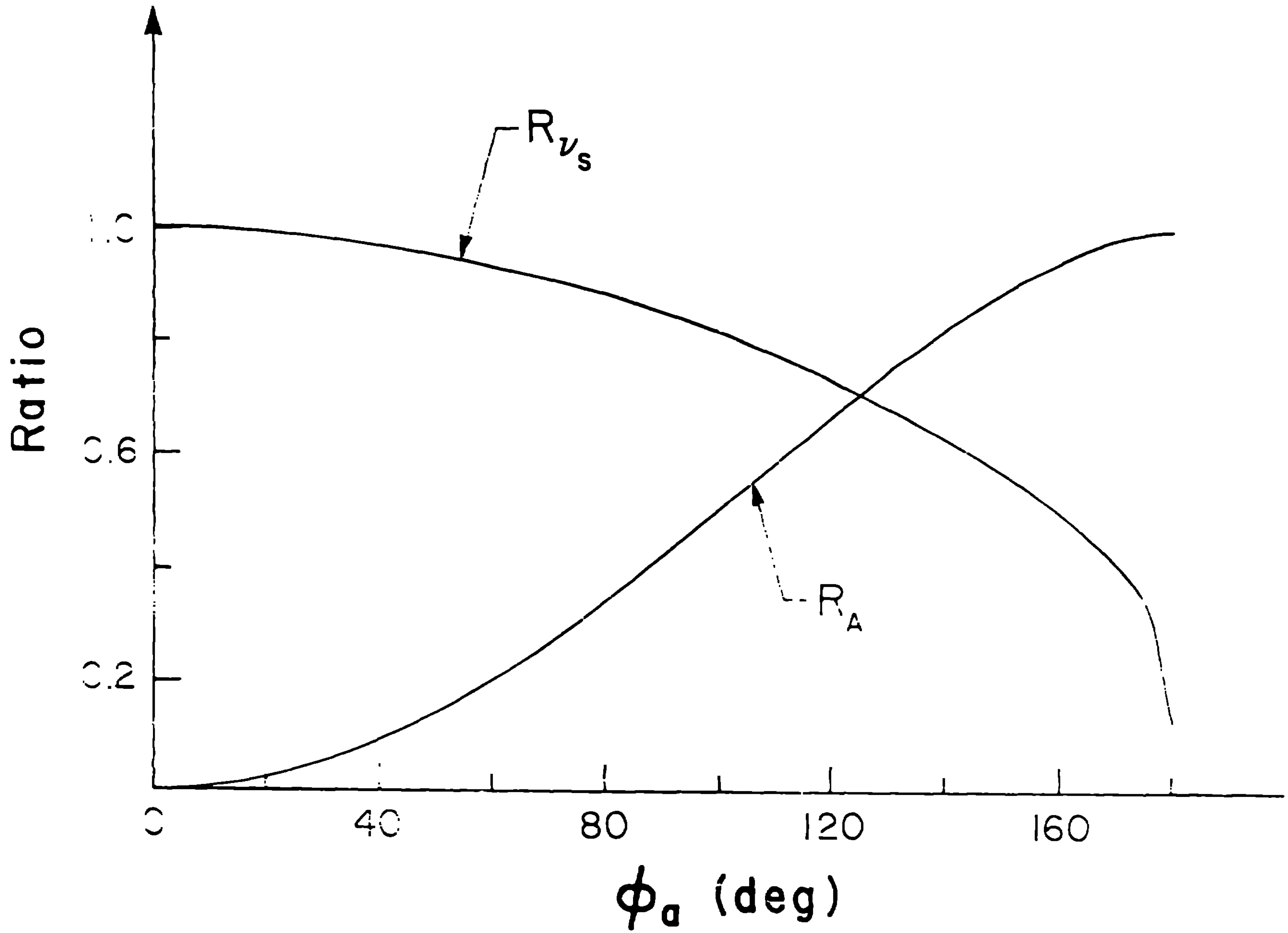
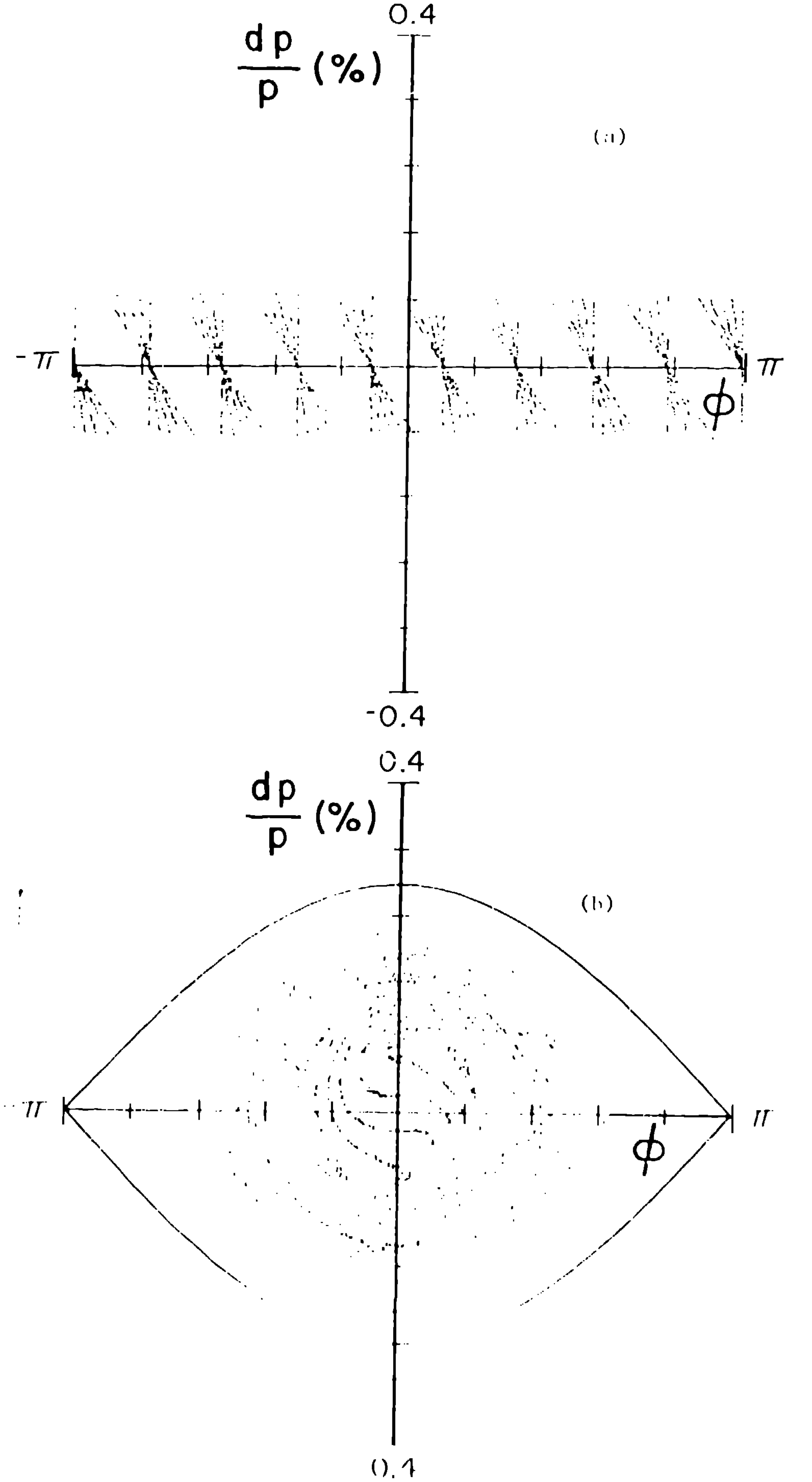


Figure 1

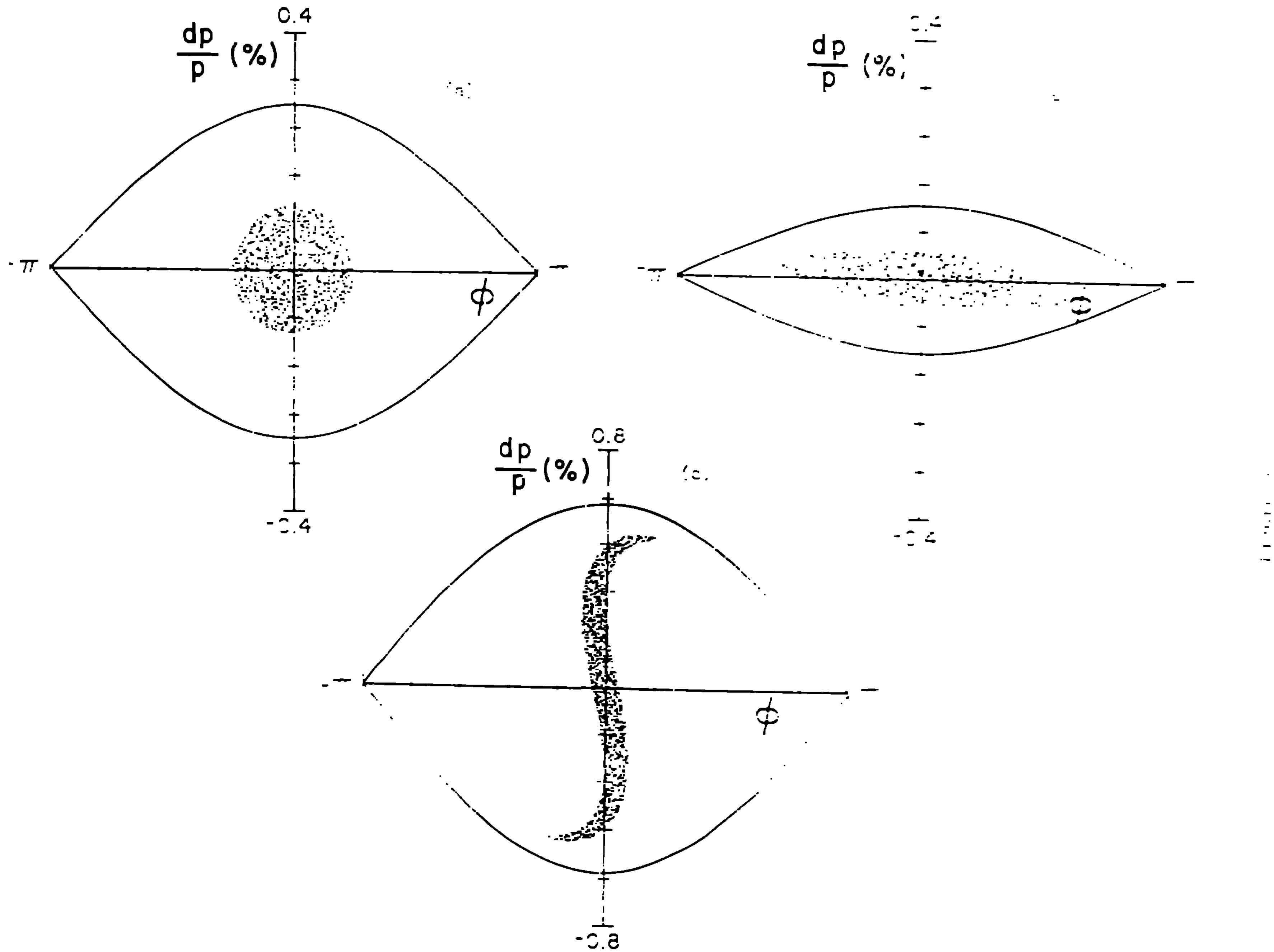

0.4
dp/p (%) = W/pR
$-\phi_a = \pi$
$-\pi$
ϕ
$\phi_a = \frac{\pi}{2}$
-0.4



Ratio
R_{ν_s}
R_A
1.0
0.6
0.2
0
40
80
120
160
ϕ_a (deg)



0.4
$\frac{dp}{p}$ (%)
(a)
-π
π
φ
-0.4
0.4
$\frac{dp}{p}$ (%)
(b)
π
φ



dp/p (%)
0.4
-0.4
-π
φ
dp/p (%)
dp/p (%)
0.8
-0.8
φ

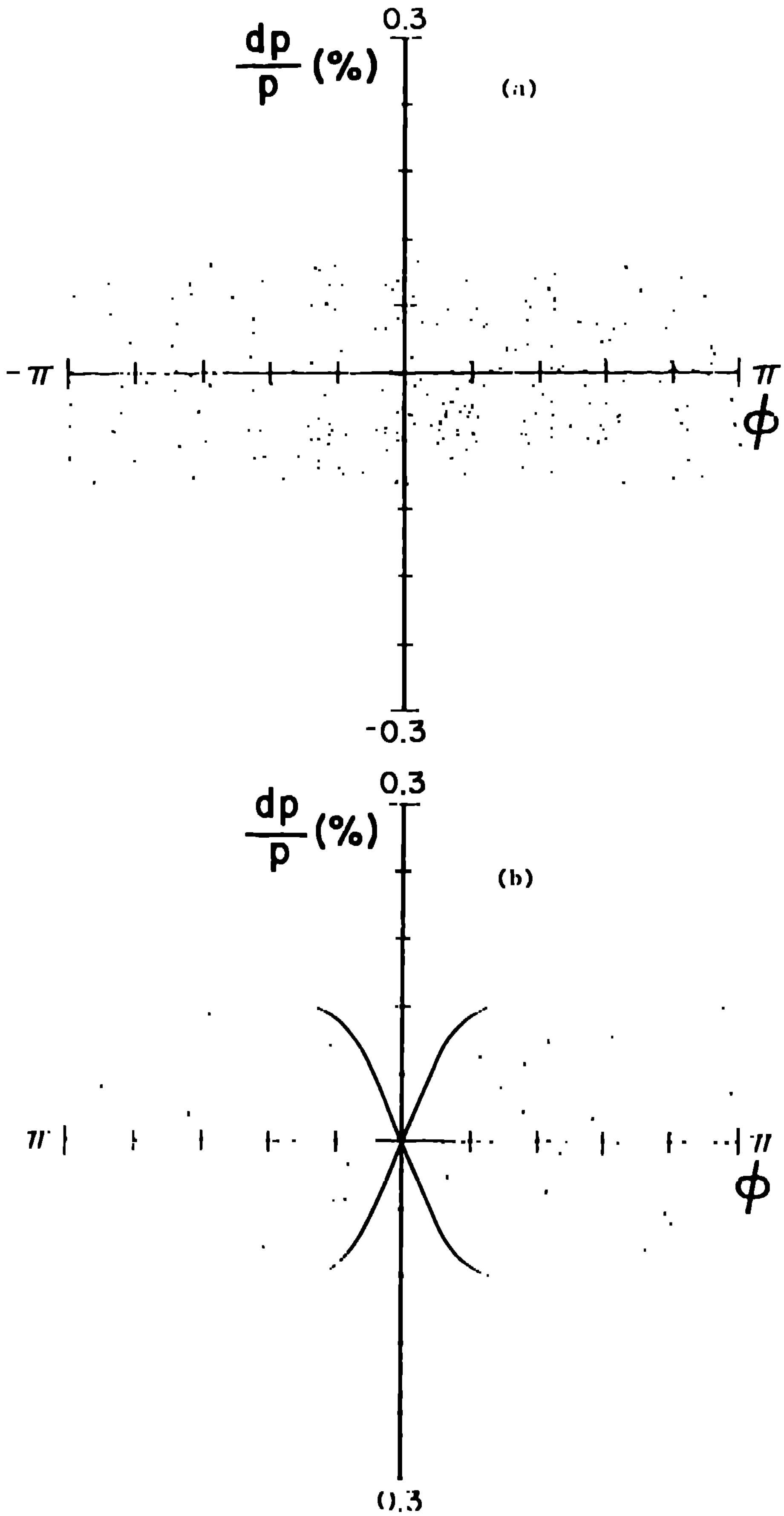


Figure 6