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WIND ENERGY CONVERSION

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FREE WAKE ANALYSIS OF WIND TURBINE AERODYNAMICS

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LIST OF SYMBOLS

C : blade chord

C_l : lift coefficient

C_d : drag coefficient ($C_d = C_{do} + C_{dk}\alpha^2$)

C_p : power coefficient $C_p = \frac{p}{\rho\pi R^2 V^3}$

C_t : thrust coefficient $C_t = \frac{t}{\rho\pi R^2 V^2}$

D : drag D_p : in-plane component of the drag

D_t : thrust component of the drag

F : Prandtl's correction factor for a finite number of blades

i : radial direction index

j : azimuthal direction vortex

L : lift L_p : in-plane component of the lift

L_t : thrust component of the lift

L_T : thrust coefficient $L_T = \frac{T}{\rho\pi R^2 (\Omega R)^2}$

L_P : power coefficient $L_P = \frac{P}{\rho\pi R^2 (\Omega R)^3}$

N : number of blades

n : number of positions

- First subscript : n : near wake and blade
  - i : intermediate wake
- Second subscript: c : centers
  - n : nodes
- Third subscript : r : radial direction
  - a : azimuthal direction

O	:	center of the rotor
P	:	power (extracted by the wind turbine)
R	:	radius of the rotor
r	:	radial position of a point
r_r	:	radial position of the root of a blade
T	:	thrust on the rotor
U	:	wind velocity relative to a blade section
V	:	free stream velocity
W	:	induced velocity

Subscript x : x component (or radial component if relative to a blade).

Subscript y : y component (or tangential component if relative to a blade

Subscript z : z component (axial component)

Subscript t : tangential component

Subscript r : radial component

Subscript b : blade

α	:	attack angle
α_s	:	stall angle
α_1	:	stall corrected attack angle

Γ	: - circulation	
	- concentrated vortex strength	
	subscripted 1 or 2 :	lumped strength of a sheet element
	t :	strength of the tip vortex
	r :	strength of the root vortex
	B :	bound circulation
λ	: inflow angle	
η	: radial position ($\eta = r/R$)	
	subscripted n :	near wake
	i :	intermediate wake
	v :	node position
	c :	center position
	t :	tip of the blade position
	r :	root of the blade position
$\Delta\eta$	: distance between two radial positions	
σ	: solidity of the rotor ($\sigma = \frac{Nc}{\pi R}$)	
θ	: pitch angle	
ρ	: air density	
ψ	: azimuth	
$\Delta\psi$	: increment in azimuthal direction	
μ	: advance ratio ($N = \frac{V}{\Omega R}$)	
Ω	: angular velocity of the rotor	

Section 1

INTRODUCTION

The precise study of the aerodynamics and the determination of the performances of a wind turbine are becoming necessary because of the appearence of large wind turbine for power generation.

A few years ago, these problems could only be approached using simplified models derived from the momentum theory. These models become very innacurate in the domaine of the small advance ratios, which is in fact the most interesting domaine for power generation.

They cannot be used, except with empirical corrections in the vortex ring conditions (reverse flow on some sections), and the influence of a finite number of blades can only be estimated using the semi-empirical corrections.

A first approach for the study of these problems, using computers, is the semi-rigid model [1]. The results obtained using this model show better agreement with experimental data than the model obtained using momentum theory.

The semi-rigid wake model is based on an approximation of the geometry of the wake ; a free wake model, because it works with a more precise definition of the wake, is therefore interesting when this geometry differs substantially from the approximation used in the semi-rigid wake model. This is the case for the low advance ratios, where the spirals of the wake are near each other, and for the case of a highly loaded rotor, where the induced velocities are large.

The free wake model must also be used to predict blade-wake interactions, for which a precise location of the wake must be determined. This last problem is important for computing the effects of blade-wake interactions for structural analysis.

The aerodynamic study of a horizontal axis wind turbine, with rotor of low solidity is a problem similar in many ways to the study of helicopter rotors in vertical flight and of aircraft propellers. [2] and [4]

For these problems, the velocity at any point in space is determined by the local inflow to which is added the pertubation due to the presence of the rotor.These pertubations are the velocities induced by the vorticity, which is located only on the blades and in the wake. The vorticity in the wake is determined by simple derivation of the distribution of circulation along the blades, which itself is determined from the velocities induced on the blade.
The blades can be modelled using lifting line theory or lifting surface theory. As this study is restricted to rotors of low solidity, the lifting line theory have been used.

The effects of viscosity are neglected, except for the drag on the blades and the existence of a core for vortices.

A solution of the problem can therefore be composed of the actual distribution of circulation along the blades and of the actual geometry of the wake (location of the points of the wake in space). From these results, the evaluation of the performances, or the determination of the velocity at any other point in space (if needed) is straightforward.

Compared to the propeller problems, the wind turbine slows the flow (to extract power) instead of accelerating it. Therefore, for similar inflows, the wake tends to stay near the rotor plane instead of moving away from it and the wake is expanding instead of contracting.
A propeller is optimized to get a maximum thrust per unit of shaft power (L_t coefficient), but the wind turbine is optimized to extract the maximum power (shaft power) per unit of wind kinetic energy (C_p coefficient).

Also Mach corrections are unnecessary for wind turbines.

The study of the aerodynamics of a wind turbine in its full generality is a very complex problem, Although this study is theoretically possible, it would involve not only the development of very large computer programs but mainly prohibitively long computer run times.

The simplest computer study uses a semi-rigid wake model, applied to the steady state calculation of rotor aerodynamics in a uniform inflow. This problem is time independent and has symmetry with respect to rotation.

The next generalisations of this problem is a semi-rigid wake model applied to a non uniform (shear flow, tower interference) but constant inflow, [1] and a free wake model applied to a uniform radial and constant inflow [3].

Compared to the first problem, the semi-rigid wake analysis applied to a non uniform inflow loses its symmetry of rotation, and is periodic in time, although time does not have to appear explicitely.

A free wake model applied to a constant uniform and radial inflow, which is the subject of this study, is time independent, has a symmetry of rotation, but involves the calculation of the induced velocities not only on the blades but also on the whole wake.

A natural generalisation of these two problems is a free wake analysis applied to a time independant non uniform inflow

All these problems can be treated using non-dimensional quantities. The linear dimensions are non-dimensionalized with respect to the radius of the rotor, and the velocities with respect to the blade tip velocity.

For the resolution of these time independent problems, two procedures can be used, one of which is time dependent, the other time-independent.

A time dependant procedure, starts with a rotor initially at rest. The rotor is started, and builds up its wake. The velocities induced by the blades and the new geometry are then calculated for each time or azimuth increment. The calculation is stopped when a steady state solution is achieved

It is very easy to understand that in order to reach a steady state, the calculation must be performed for a large number of increments (corresponding to the construction of a few spirals of the wake) and is therefore very time consuming. On the other hand, any instability or oscillating regime can be easily detected.

A time independent procedure, which is used in this work, starts from a guess of the induced velocities and of the geometry, of then computing the circulation along the blades and the distribution of vorticity on the wake. The velocities induced on the blades and on the wake (for a free wake model) are determined and the geometry recomputed by integration of the induced velocities (for a free wake model). Finally, the old and the new distributions of induced velocities and the old and new geometries are compared. When these are identical within a relative accuracy, we have the solution, if not, we have to refine the first estimate and repeat the process.

This procedure is relatively easy to program and can be very cost efficient. Theoretically, the results have to be identical with those of the first procedure, but this may not be the case. The last solution is the real one if: the solution is unique, the real solution is a fixed, steady state solution. However it is possible that the solution is oscillating. In that case the procedure is unable to predict a mean solution.

Other problems arise from this procedure such as numerical convergence. These problems are unrelated to the existence or not of an unstable regime. Because of this, a special study became necessary to force the numerical convergence.

Section 2

MODELLING OF THE ROTOR

2.1 The wind turbine considered is a horizontal axis machine, with a rotor composed of N identical blades regularly spaced.

The blades have a constant chord c and may be straight or twisted, each blade extends radialy from r_r to R in the radial direction.

The rotor, assumed to be rigid, is turning at a constant angular velocity Ω, in a steady axial uniform sream (velocity V).

Using non-dimensional quantities, we assume that the radius of the rotor (R) and the angular velocity (Ω) are equal to 1. The inflow can therefore be represented by the advance ratio :

$$\mu = \frac{V}{\Omega R}$$

The position of a point at a radius r on a blade is given by η

$$\eta = \frac{r}{R}$$

The radial position η ranges from η_r to 1

$$\eta_r = \frac{r_r}{R}$$

2.2 System of coordinates

The position of a point is given by its rectangular or cylindrical coordinates. The system of coordinates is fixed with respect to the rotor, the origin (O) is located at the center of the rotor, the X axis along one of the blade (named first blade), the Z axis is the axis of the rotor, oriented downstream.

The direction of rotation is such that the wake leaves the first blade in the positive Y direction (fig. 1)

2.3 Decomposition of the blade into sections

The decomposition of the blade into sections is given by a first set of points called nodes, distributed along the blade.

The nodes are numbered in increasing radial positions.

The radial location of the first node $\eta_{nv}(1)$ corresponds to the root of the blade (η_r) and the last node (numbered n_{nvr}) is located at the tip of the blade ($\eta_{nv}(n_{nvr})=1$).

There are $n_{nvr}-1$ sections, the i^{th} sections extends from $\eta_{nv}(i)$ to $\eta_{nv}(i+1)$.

A second set of points, called centers is also defined.

Each center is located at the middle of a section, and the i^{th} center is located between the nodes i-1 and i.

(This particular numbering arises from the modelling of the wake).

(fig. 2)

2.4 Modelling of the blade sections (fig. 3)

Each section is modelled using lifting line theory.

All the quantities relative to the i^{th} section are given for the corresponding center (located at a radius $\eta = \eta_{nc}(i+1)$).

The components of the velocity induced at $\eta_{nc}(i+1)$ being W_y (tangential component) and W_z (axial component), the local velocity perpandicular to the blade section is $\vec{U}$ (module U) of components

$$\vec{U} \begin{vmatrix} 0 \\ W_y + \eta \\ W_z + \mu \end{vmatrix} \qquad (2.1)$$

The local inflow angle is λ

$$\text{with} \quad tg\lambda = \frac{W_z + \mu}{W_y + \eta} \qquad (2.2)$$

The attack angle is $\alpha = \lambda - \theta$, where θ is the local pitch angle.

To account for stall effects, a corrected value of the attack angle (α_1) is used to compute the bound circulation. (fig 4)

$$\begin{aligned} \alpha_1 &= \alpha \quad \text{if } |\alpha| < \alpha_s \\ \alpha_1 &= \alpha_s \text{ if } |\alpha| > \alpha_s \end{aligned} \qquad (2.3)$$

Where α_s is the stall angle.

The bound circulation is then, using $C_1 = 2\pi\,\alpha_1$

$$\Gamma = \pi c U \alpha_1$$

Where c is blade chord

Assuming that the blades extend to the origin, the chord is a function of the rotor solidity

$$\sigma = \frac{Nc}{\pi R}$$

The bound circulation can then be written :

$$\Gamma = \frac{\pi^2 U \sigma \alpha_1}{N} \qquad (2.4)$$

2.5 Forces on the blade section. (fig. 5)

The lift per unit span (L), is perpendicular to $\vec{U}$ and expressed as :

$$L = \rho \Gamma U$$

where ρ is the air density.

The lift is decomposed into its thrust component L_t and its in-plane component L_f

$$L_t = L \cos \lambda$$

$$L_f = -L \sin \lambda$$

The profile drag (D) is given by

$$D = \frac{1}{2} \rho C_d U^2 c$$

The drag coefficient (Cd) is assumed to have the form

$$C_d = C_{do} + \alpha^2 C_{dk}$$

Where : C_{do} is the minimum drag coefficient

C_{dk} is a constant

(α is expressed in radians)

The drag is decomposed into its thrust component D_t and its in-plane component D_f

$$D_t = D \sin \lambda$$

$$D_f = D \cos \lambda$$

The total unit force per unit span has for thrust component :

$$L \cos \lambda + D \sin \lambda$$

and for in-plane component :

$$- L \sin \lambda + D \cos \lambda$$

The thrust is then : $$T = NR \int_{\zeta_2}^{1} (L \cos\lambda + D \sin\lambda)\, d\zeta$$

and is computed as $$T = NR \sum (L \cos\lambda + D \sin\lambda) \Delta\zeta$$

The power, derived from the in-plane components is

$$P = \Omega R^2 N \int_{\eta_r}^{1} (-L \sin\lambda + D \cos\lambda)\, \eta \, d\eta$$

and computed as :

$$P = \frac{\Omega R^2 N}{2} \sum (-L \sin\lambda + D \cos\lambda)\, \Delta(\eta^2)$$

2.6 Performance coefficients

The performance coefficients relative to the thrust are :

$$C_T = \frac{T}{\rho \pi R^2 V^2} \quad \text{(based on the wind velocity)}$$

$$L_T = \frac{T}{\rho \pi R^2 (\Omega R)^2} \quad \text{(based on the tip velocity)}$$

The performance coefficents relative to the power are :

$$C_P = \frac{P}{\rho \pi R^2 V^3} \quad \text{(based on the wind velocity)}$$

$$L_P = \frac{P}{\rho \pi R^2 (\Omega R)^3} \quad \text{(based on the tip velocity)}$$

The most important coefficient is C_P, which is a direct measure of the efficiency of the rotor, it is the ratio of the power extracted to twice the flow of kinetic energy through the disk area.

Section 3

MODELLING OF THE WAKE

To define the wake we need :

a) The positions in space of a given number of control points, located on the wake (called geometry)

b) The distribution of vorticity on the wake.

As the computation time increases as the square of the number of control points, it is necessary to keep that number to a minimum.

We need a large number of control points (small mesh size) near the blade, to have a good representation of the wake shape.

As we consider sections of the wake away from the blade from which the wake originates, the velocities induced on the blade become smaller, and the predicted location of the wake becomes less accurate. It is therefore interesting to consider a larger mesh size for these sections.

The wake has been decomposed in three sections named :

- near wake
- intermediate wake
- far wake

The two first sections have the same kind of definition, only the mesh size differs. The far wake has been modelled by semi-infinite cylinders.

3.1 Near wake (fig. 6)

The near wake begins at the blade and may extends in the azimuthal direction over 50 to 100 degrees, it is defined by 2-D mesh, the node noted (i,j) is located at the radial position (i) and at the azimuthal position (j).

The nodes of a given radial position are on the same streamline, and those of the same azimuthal position are approximatly at the same azimuth (this is not a requirement). The azimuthal position can be separated by 5 to 10 degrees.

Note : As the rotor operates in steady conditions, the vortex lines and the streamlines are the same.

The near wake is defined radially by the blade sections.

The nodes of radial position (i) are on the streamline originating from the i^{th} node of the blade, located at $n_{nv}(i)$. The nodes of the first azimuthal position of the near wake coincide with the nodes of the blade. The number of nodes in the radical direction is therefore n_{nvr} and n_{nva} in azimuthal direction.

The near wake is decomposed in surface elements, each element (i,j) is defined by its four corners which are the nodes : (i,j),(i+1,j),(i,j+1), (i+1,j+1).

Each element is approximated by a rectangular element, for which the formulation of the induced velocities exists in a close form.

The near wake is also bounded on the first (i=1) and the last ($i=n_{nvr}$) radial position by two concentrated vortices (root and tip vortices).

These vortices are modelled by straight segment elements, the j^{th} element (azimuthal position) being limiting by two nodes (1,j) and (1,j+1) for the root vortex, (n_{nvr},j) and (n_{nvr},j+1) for the tip vortex.

The induced velocities cannot be computed at the points which define the elements (the two end points for a segment element, the four corner points for a rectangular element), because of the singularities arising in the formulation of the induced velocities. It is therefore necessary to compute the induced velocities on different points, called centers. The velocity induced at a given node will be obtained by linear interpolation of the velocities induced on the four centers around it.

It is necessary, in order to interpolate the velocities at a given node located on the root or the tip vortex to have centers placed symetrically with respect to the segment element, therefore the mesh of the centers ($n_{ncr} = n_{nvr}+1$), the radials positions 1 and n_{ncr} are located outside the wake.

In azimuthal direction there are n_{nca} ($n_{nca} = n_{nva}+1$) position for the centers and the azimuthal position (1) extends before the blade.

The center numbered (i,j) is located at the middle of the four corners numbered (i-1,j-1),(i,j-1),(i-1,j) and (i,j).

3.2 <u>Intermediate wake</u> (fig. 7)

The intermediate wake is similar to the near wake. It extends from the near wake, to the far wake.

In the azimuthal direction the intermediate wake may span from one spiral to two spirals in elements of 15 to 25-30 degrees. This last value is limited by necessity to keep the sheet elements as undistorted as possible. The radial position i of the intermediate wake is located on the streamline originating from $n_{iv}(i)$.

There are n_{ivr} radial positions. As the radial positions 1 and n_{ivr} correspond to the root and tip vortices, it is necessary that :

$$n_{iv}(1) = n_{nv}(1)$$

$$n_{iv}(n_{ivr}) = n_{nv}(n_{nvr}) = 1$$

The choice of the distribution of the values $n(i)$ for $i = 2$ to $n_{ivr}-1$ is independent of the distribution $n_{nv}(i)$.

There are also n_{iva} azimuthal positions for the nodes, $n_{icr}(n_{icr} = n_{ivr}+1)$ radial positions for the centers, and $n_{ica}(n_{ica} = n_{iva}+1)$ azimuthal positions for the centers.

3.3 Transition wake

Because of the singularities in the formulation of the velocities induced by an element, it has not been possible to make the first azimuthal position of the intermediate wake coincide with the last azimuthal position of the near wake. Because of the difference of size, the centers of one wake may be located on or very near the edges or the corners of the element of the other wake ; and unrealistic values of the velocities arise.

To avoid this problem a transition wake have been implemented ; the principle is the following : (See fig. 8)

The end of the near wake and the beginning of the intermediate wake overlap composing the transition wake, over a distance equal to two times the size of an element of the intermediate wake.

The near and intermediate wake are composed of three regions :

1 - The near wake, transition wake not included (noted A)

2 - The transition wake noted B, when defined by the near wake,

and B' when defined by the intermediate wake

3 - The intermediate wake, transition wake not included (noted B)

The induced velocities relative to the wakes will be computed as follow :

- Region A induces velocities on the centers of A,B,C
- Region C induces velocities on the centers of A,B',C
- The transition wake will induce velocities :

on the centers of A using the definition B

on the centers of C using the definition B'

on the centers of B using the definition B

To interpolate the velocities at the nodes of the transition wake, the following procedure have been used :

- Interpolation of the velocities induced at the centers of B on the nodes of B.
- Interpolation and addition at these nodes of the velocities induced at the centers of B'.
- Interpolation of the velocities at the nodes of B on the nodes of B'

For the intermediate wake, the transition wake is composed of the first two element rows, and of the centers j = 1,2,3 and 4.

The near wake, transition wake not included goes from the nodes $i = 1$, to $i = n_{tva}$, the centers from $i = 1$ to $i = n_{tca} = n_{tva}+1$.
The first azimuthal position for the nodes of the intermediate wake coincide with the azimuthal position $j = n_{tva}$ for the nodes of the near wake.

3.4 Far wake

The far wake begins at the end of the intermediate wake and extends to $z = +\infty$.
After some distance downward, the influence of a finite number of blades disappear, and the wake is fully expanded.
Because of the symmetry, the vortex lines are regularly spaced spirals, of constant radius. (We assume that the far wake is stable, as may not be the case of the real wake, however the induced velocities are small, and the errors are acceptable). The system of vortices may then be replaced by a vortex cylinder (Appendix B) and a series solution obtained.
A convenient procedure consists in the precalculation of values from the series at given locations; the velocities induced at any point may then be determined by interpolation.

Section 4

CALCULATION OF THE STRENGTHS OF THE ELEMENTS

4.1 Sheet elements (rectangular elements)

The determination of the strengths of the elements is based on the law of conservation of circulation. (fig. 9)

For a sheet vortex of strength γ, of width $d\eta'$, originationg from a segment of length $d\eta$ on the blade.

We can write : $\Gamma_B(\eta + d\eta) = \Gamma_B(\eta) + \gamma\, d\eta'$

where

$$\gamma = \frac{d\eta}{d\eta'} \left(\frac{d\Gamma_B}{d\eta}\right)$$

Assuming that the expansion of the sheet $\left(\frac{d\eta'}{d\eta}\right)$ bounded by the two trailing vortices originating from η_1 and $\eta_2 = \eta_1 + \Delta\eta$ is constant and equal to $\Delta\eta'/\Delta\eta$

we can write : $\gamma(\eta) = \frac{\Delta\eta}{\Delta\eta'} \frac{d\Gamma_B(\eta)}{d\eta}$

and $$\int_{\eta_1}^{\eta} \gamma(\eta)\, d\eta = \frac{\Delta\eta}{\Delta\eta'} \left[\Gamma_B(\eta) - \Gamma_B(\eta_1)\right]$$

Using sheet element bounded by the line vortices (1) and (2) (fig. 10), γ can be determined only when the expansion $\Delta\eta'/\Delta\eta$ is known.

As the sheet elements have been modelled by rectangular elements, which formulation allows us a linear form for $\gamma(\eta)$, we introduce the two following quantities called lumped strength of the element

$$\Gamma_1 = \gamma(\eta_1)\,\Delta\eta' = \frac{d\Gamma_B}{d\eta}(\eta_1)\,\Delta\eta$$

$$\Gamma_2 = \gamma(\eta_2)\,\Delta\eta' = \frac{d\Gamma_B}{d\eta}(\eta_2)\,\Delta\eta$$

Γ_1 and Γ_2 have the dimensions of concentrated vortices

For an element of width $\Delta\eta'$, the strength will be :

$$\gamma(\eta) = \left[\Gamma_1(\eta_2-\eta) + \Gamma_2(\eta-\eta_1)\right] / \left[\Delta\eta' \cdot \Delta\eta\right]$$

the following intermediate values have also been used :

$\Gamma_m = (\Gamma_1+\Gamma_2)/2$ constant lumped strength component

$\Delta\Gamma_m = (\Gamma_2-\Gamma_1)/2$ variable lumped strength component

The quantities Γ_1 and Γ_2 depend only on the distribution of circulation along the blade.

4.2 Modelling of the root and tip vortices

At the tip and the root of the blade, the circulation decreases very rapidly to zero, and two concentrated vorticies appear on the inner and the outer boundaries of the wake.

These vortices have been modelled by segments of line vortices.

As it is necessary to make a distinction between these segment elements and the rectangular elements (used for vortex sheets), the strength of the tip vortex is taken as -K times the circulation on the last center of the blade, the strength of the root vortex as +K times the circulation on the first center of the blade (j=2).

The value K can vary from 0 to 1 ; in the computer program, this value is specified as an input (variable COEFF)

$\Gamma_t = -K\,\Gamma_B(n_{ncr} - 2)$ strength of the tip vortex

$\Gamma_r = K\,\Gamma_B(2)$ strength of the root vortex

4.3 Calculation of Γ_1 and Γ_2 (near wake)

For the nodes $i = 2\ n_{nvr}-1$, located between the centers i and $i+1$

The value $\frac{d\Gamma}{d\eta}$ is evaluated by

$$\frac{d\Gamma}{d\eta}(i) = \frac{\Gamma(i+1) - \Gamma(i)}{\eta_{nc}(i+1) - \eta_{nc}(i)}$$

$\Gamma(i)$ circulation at the center located at $\eta_{nc}(i)$

For the node $i = 1$, located on the root vortex, the value of

is dependent on the choice of K.

For the concentrated root vortex of strength $K\,\Gamma_b(1)$, the bound circulation at $\eta = \eta_{nv}(1)$ will be :

and $$\frac{d\Gamma}{d\eta}(1) = (1-K)\frac{\Gamma_B(1)}{\Delta\eta_c}$$

with

$$\Delta\eta_c = \eta_c(2) - \eta_{nv}(1) = \frac{1}{2}\left[\eta_{nc}(2) - \eta_{nc}(1)\right]$$

$$\frac{d\Gamma}{d\eta}(1) = \frac{2(1-K)\,\Gamma_B(1)}{\eta_{nc}(2) - \eta_{nc}(1)}$$

A similar calculation is done for the tip vortex

Once the distribution $\frac{d\Gamma}{d\eta}(i)$ is determined for all the positions $i = 1$ to n_{nvr}, $\Gamma_1(i)$ and $\Gamma_2(i)$ for the i^{th} element ($i = 1$ to $n_{nvr}-1$) are given by

$$\Gamma_1(i) = \frac{d\Gamma}{d\eta}(i) \quad \Delta\eta_n$$

$$\Gamma_2(i) = \frac{d\Gamma}{d\eta}(i+1) \quad \Delta\eta_n$$

$$\Delta\eta_n = \eta_{nv}(i+1) - \eta_{nv}(i)$$

4.4 Case of the intermediate wake

The same procedure has been used for the intermediate wake ; a previous interpolation of the circulation at pseudo centers of the blade is necessary.

These centers are those obtained by decomposition of the blade in sections following the distribution of η_{iv}, instead of η_{nv}

4.5 Method used to determine the strengths of the elements

The strength of the elements ($\Gamma_1(i)$, $\Gamma_2(i)$, Γ_t, Γ_r) are linear combinations of the values of the circulation at the centers ; we can write for example :

$$\Gamma_1(i) = \sum_{j=1}^{n_{ncr}-2} H_1(i,j)\,\Gamma_B(j+1)$$

where $H_1(i,j)$ is the strength $\Gamma_1(i)$ for a distribution of circulation:

$$\Gamma_B(j+1) = 1$$

$$\Gamma_B(k+1) = 0 \qquad k \neq j$$

This method has been used, as the H coefficients have been calculated for the determination of the influence coefficients. (Section 5.4)

Section 5

ITERATIVE PROCEDURES

5.1 The solution of the problem can be defined by the knowledge of :

a) The geometry of the wake (locations of a given number of points on the wake, noted $\underset{\sim}{G}$)

b) The distribution of tangential and axial induced velocities along the blade (noted $\underset{\sim}{W}_B$)

Once these quantities are determined, the calculation of the other quantities is a straightforward problem.

(Example : - determination of the velocity field

- determination of the bound circulation, of the strength of the elements
- determination of the forces and performance coefficients)

An iterative procedure is necessary to determine ($\underset{\sim}{G}$, $\underset{\sim}{W}_B$).

From a guess ($\underset{\sim}{G}_1$, $\underset{\sim}{W}_{B1}$), we recompute these quantities ($\underset{\sim}{G}_2$, $\underset{\sim}{W}_{B2}$) if these quantities are identical, within a relative accuracy, then the solution is obtained ; if not, ($\underset{\sim}{G}_1$, $\underset{\sim}{W}_{B1}$) and ($\underset{\sim}{G}_2$, $\underset{\sim}{W}_{B2}$) are used to refine the guess, and another iteration is performed.

5.2 Case of the semi-rigid wake model

The special case of the semi-rigid wake model is used to determine the first guess for the free wake procedure. Its iterative procedure is a simplification of the free wake case.

In the semi-rigid wake analysis, the geometry is determined completely by the distribution of inflow angle ($\underset{\sim}{\lambda}$) along the blade.
From a first guess ($W_{\theta 1}$) $\underset{\sim}{\lambda}$ and $\underset{\sim}{\gamma}$ are computed, then by calculation of the induced velocities, the quantities $\underset{\sim}{W}_{\theta 2}$ are obtained and compared to $\underset{\sim}{W}_{\theta 1}$ (fig. 11)

5.3 <u>Case of the free wake model, direct method</u>. (fig 12)
In order to recompute the geometry, we need the velocities induced on the control points of the wake ($\underset{\sim}{W}$), which can be computed from a guess of the geometry ($\underset{\sim}{G}_1$) and a guess of the strengths of the elements, derived from the distribution of bound circulation ($\underset{\sim}{\gamma}$), itself derived from ($\underset{\sim}{W}_{\theta 1}$).
The velocities are added to the free stream velocity, and integrated to obtain the new geometry ($\underset{\sim}{G}_2$).
At the same time, the velocities induced on the blades ($\underset{\sim}{W}_{\theta 2}$) are also computed.
The comparison can be done between $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$, or $\underset{\sim}{W}_1$ and $\underset{\sim}{W}_2$
This comparison is difficult and long, if $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ are compared (a larger test value must be used for the points far from the blades) ; it can be reduced to the comparison of $\underset{\sim}{W}_{\theta 1}$ and $\underset{\sim}{W}_{\theta 2}$ as in the semi-rigid wake case, assuming that if $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ pass the test, $\underset{\sim}{W}_{\theta 1}$ and $\underset{\sim}{W}_{\theta 2}$ also pass the test.

5.4 Free wake model, use of the influence coefficients

The last procedure cannot converge faster than the procedure used in the semi-rigid wake case.

By introducing influence coefficients, we arrive at a two loop procedure which, as it allows us to determine a better guess, can converge faster.

The influence coefficients ($\underset{\sim}{T}$) are two arrays : T_{wy} and T_{wz}," which the element (i,j) is the velocity induced on the i^{th} segment of the blade (centered at $\eta_{nc}(i+1)$) by the j^{th} segment only of the wake of current geometry. The distribution of strength is derived from the following distribution of circulation:

$\Gamma(j+1) = 1$ or unit circulation on the j^{th} segment of the blade

$\Gamma(k+1) = 0$ for $k \neq j$

T_{wy} is tne influence coefficient for the tangential components and T_{wz} for the axial components.

These influence coefficents allow the complete determination of the induced velocities on the blade, and of the corresponding distribution of circulation, for a given geometry.

These calculations are performed in an internal loop, called loop on the circulation. (fig. 13)

From the arrays T_{wy} and T_{wz} , we can write :

$$W_y(i) = \sum_{j=1}^{n_{ncr}-2} T_{wy}(i,j)\ \Gamma(j+1)$$

$$W_z(i) = \sum_{j=1}^{n_{ncr}-2} T_{wz}(i,j)\ \Gamma(j+1)$$

then applying (2.1) to (2.4) to the i^{th} section we arrive at a new distribution of circulation.

We iterateup to the moment when all $W_y(i)$ and $W_z(i)$ are equal to their preceding value in the last iteration.

The procedure is of interest in that each iteration of the loop on the circulation takes a very short time compared to the calculation of the induced velocities on the wake performed for each iteration of the outer loop (called loop on the geometry). To obtain convergence a smaller number of iterations is needed.

5.5 Weighting techniques. (fig.14)

The procedure by which the new guess at each iteration is given directly by the new values of the induced velocities, never converges.

It is necessary to use as a new guess a combination of the old guess and of the resultant values, defined by weighting factors.

In the semi-rigid wake model, only the components of the induced velocities are weighted. In the free wake model weighting has to introduced at various locations in the procedure:

a) Weighting of the induced velocities on the wake.
   Three factors have been used for the three components (radial, tangential and axial components)
b) Weighting of the influence coefficients
c) Weighting of the distributions of circulation, as determined from the loop on the circulation

The initial guess of the distribution of circulation, for the inner loop is the one obtained from the last iteration.

The new values for the velocities induced along the blade are obtained from the new distribution of circulation and the current influence coefficients.

Weighting of the induced velocities is also used in the loop on the circulation.

Section 6

CONVERGENCE OF THE SOLUTION

Problems of numerical convergence have been experienced by Humes [1] and Scully [2] with similar procedures. These problems range from a slow convergence to a catastrophic divergence. By choosing weighting factors on a trial and error basis, it is always possible to avoid divergence, at the cost of a slow convergence. It was therefore interesting to find optimum weighting factors, evaluated by the program itself.

The case of the semi-rigid wake model is the most interesting, where convergence has usually been successfull.

6.1 Case of the momentum theory applied to a blade element.

The semi-rigid wake procedure has many similarities to the blade element momentum theory.

According to the momentum theory, the axial and tangential velocities can be written :

$$W_z = -\frac{1}{2} \frac{\Gamma_B \, N}{2\pi F \eta \, \mathrm{tg}\lambda} \qquad (6.1)$$

$$W_u = -W_z \, \mathrm{tg}\lambda$$

where : F is the Prandtl's correction factor which can be written :

$$F = \frac{2}{\pi} \arccos e^{-f}$$

$$f = \frac{N}{2} \frac{\sqrt{1+\mu^2}}{\mu} (1-\eta) \qquad (6.2)$$

The equations (6.1) and (6.2) can be derived from those given in Volume II in which we neglect the drag. The Prandtl's factor F is an approximate correction for the effect of a finite number of blades.

Associating (6.1) with the equations (2.1) to (2.4) we arrive at a system of equations which can be solved for W_y and W_z using an iterative procedure.

Comparing this momentum theory model and the semi-rigid wake model (6.1) takes the place of the calculation of the induced velocities on the blade, given λ, and Γ.

It can be noted that $W_z(\eta)$ in function only of the bound circulation at η, therefore all the blade sections are independent in the momentum theory.

In the semi-rigid wake model, the sections are not independent.

A finite number of blade introduces some coupling terms.

(Note : In the free wake model, this effect can be visualized by the influence coefficients ; the values $T_{wz}(i,j)$ for $i \neq j$ (coupling terms) are much smaller than the values $T_{wz}(i,i)$).

Neglecting the variations of W_y in (6.1), (2.1) to (2.4), we can reduce the system of equation to a form

$$W_z = f\,(W_z) \qquad (6.3)$$

Assuming that the solution of (6.3) is W_z^*, we can write, for a small variation δW around W_z^* :

$$f(W_z^* + \delta W) \approx W_z^* + A\,\delta W = W_z^* + \frac{\partial f}{\partial W_z}\,\delta W \qquad (6.4)$$

The value of A determines the behavior and the rate of convergence. In the iterative procedure $W_{zi+1} = f(W_{zi})$, a variation δW becomes $A^n\ \delta W$ after n iterations.

Therefore it diverges for $|A| > 1$, converges with a sinusoidal behavior for $A < 0$, and with an exponential behavior for $A > 0$.

From the value of A we can compute the optimum weighting factors.

For one iteration $W_{z1} = W_z^* + \delta W$

becomes : $W_{z2} = W_z^* + A\delta W$

Calling f_n the weighting factor for the new solution. (W_{z2})

we can write :

$$W_z^* = f_n\, W_{z2} + (1-f_n)\, W_{z1} = f_n\,(W_z^* + A\,\delta W) + (1-f_n)(W_z^* + \delta W)$$

therefore:

$$f_n\, A\,\delta W + (1-f_n)\,\delta W = 0$$

$$f_n = \frac{1}{1-A}$$

An approximation of A can be derived from (6.3), (2.1) to (2.4).

From (6.3) and (2.4) :

$$W_z = -\frac{\sigma\pi}{4}\,\frac{U\alpha_1}{2\,tg\lambda} \tag{6.5}$$

Neglecting the variations of W_y, and using small angles assumptions

$$A = -\frac{\sigma\pi}{4}\left[\frac{1}{tg\lambda}\,\frac{\partial\alpha_1}{\partial W_z} + \alpha_1\frac{\partial}{\partial W_z}\left(\frac{1}{tg\lambda}\right)\right]$$

if the blade section is stalled : $\alpha = \alpha_s = Ct$

and $$A = -\frac{\sigma\pi}{4}\,\frac{\partial}{\partial W_z}\left(\frac{1}{tg\lambda}\right) \approx \frac{-W_z}{2+W_y}\left(\frac{1}{\lambda}\right) \tag{6.6}$$

if the blade section is not stalled then $\frac{\partial\alpha_1}{\partial W_z} = \frac{\partial\lambda}{\partial W_z}$

and $$A = +\frac{W_z}{2+W_y}\left(\frac{1}{\alpha_1} - \frac{1}{tg\lambda}\right) \approx \frac{W_z}{2+W_y}\left(\frac{\theta}{\alpha\lambda}\right) \tag{6.7}$$

6.2 Case of the semi-rigid wake model

From (6.6) and (6.7) we can draw the following conclusions :

- the behavior of the convergence for a stalled section is very different from that of a non stalled section
- a weighting factor optimum for a section, will induce a slow convergence, or divergence on other sections
- convergence problems can be expected when A is large in absolute value (highly loaded rotor, small inflow angle, large pitch angle)

These previous conclusions are in agreement with those drawn from the examination of a large number of cases, but an attempt to use (6.6) and (6.7) in the semi-rigid wake program failed completly for the following reasons :

- the coupling between the sections was neglected
- the use of a finite number of sections has a strong influence on the effective value of A
- the effect of the axial component of the induced velocity is not negligible
- non linearities have been neglected

As the previous results from the momentum theory cannot be used quantitavely, it was necessary to evaluate A directly ; this can be done using the two last iterations.

Supposing that at a given blade section the axial induced velocity goes from W_{zo} to W_{zn} for the iteration (n-1) and from W'_{zo} to W'_{zn} for the iteration n.

Calling W_z^* the solution , we can write :

$$(W_{zn} - W_{zn}^*) = A\ (W_{zo} - W_z^*)$$

$$(W'_{zn} = W_{zn}^*) = A\ (W'_{zo} - W_z^*)$$

which can be solved for A :

$$A = -\frac{W'_{zn} - W_{zn}}{W'_{zo} - W_{zo}}$$

The guess for the next iteration will be :

$$W_{zo}'' = f_n W_{zn}' + (1 - f_n) W_{zo}'$$

where f_n is the weighting factor for the new values

$$f_n = \frac{1}{1-A} = \frac{W_{zo}' - W_{zo}}{W_{zo}' - W_{zo} + W_{zn}' - W_{zn}}$$

This last formulation have been successfull, but with the following modifications :

- limitation of f_n to the range (.1, .9)
- weighting of f_n itself with the factor used in the precedent iteration

In the free wake analysis this formulation have been used for the loop on the circulation, with f_n computed from the average of the velocities induced on all the sections.

6.3 Influence of the tangential component

Assuming that the blade sections are independent, we have in fact a two variables system of equations :

$$W_y = f_1 (W_y, W_z)$$

$$W_z = f_2 (W_y, W_z)$$

The concept of weighting factors base on a single value A, is valid only if the variations of W_y are negligible, and this is not the case for the inner sections. Instead of exhibiting a smooth pseudo-sinusoidal convergence, with a period of two iterations, the convergence of some inner sections was based on a period of four iterations. A small correction on W_y, could restore the period of two iterations necessary for the appropriate estimation of A and f_n ; it consits, after weighting, to multiply W_y by $tg\lambda_1$ / $tg\lambda$ wher $tg\lambda_1$ is the tangent of the inflow angle based on the weighted values of W_y and W_z, and $tg\lambda$, the tangent of the inflow angle used in the precedent iteration.

6.4 Case of the free wake model

The free wake model iterative procedure is based on two loops :

- the loop on the geometry
- the loop on the circulation

Qualitatively it may be noted, as expected, that the convergence difficulties which arise in the semi-rigid wake model have been in the loop on the circulation. Therefore, because of the small cost of an iteration on the circulation, the necessity of a fast convergence is not so crucial, and the utilisation of the influence coefficients is justified.

The loop on the geometry converges rapidly, but it has not been possible to estimate the optimum weighting factor for the velocities induced on the wake and for the influence coefficients, as the convergence of the geometry depends mainly on the roll-up and the deflection of the wake.

Section 7

RESULTS

Only a few cases could be obtained, mainly because of the very long computer time necessary to have one case.

The detailed results of four cases are presented in tables 1 to 8, comparing for each case, the semi-rigid wake analysis and the free wake analysis.

The input for the casesis given in table 9.

7.1 Power coefficients

The power coefficients obtained from the free wake analysis show a small increase ranging from 1.5 to 5% as compares to the semi-rigid wake analysis.

7.2 Wake distortion

Three effects were expected and observed (fig.15)

a) Expansion of the wake (table 9)

The vortex lines are moving outward, for increasing azimuth.

The exact measure of the expansion is difficult because of the roll-up of the wake and the tip vortex radial position is not the best measure of it.

b) Roll up of the wake (fig. 16)

Because of the small number of sections at the tip and the root, the exact shape is not well represented ; we observe in fact a progressive folding of the wake, and after an azimuth ranging from 45 to 110 degrees. (table 10), the vortex line (5) is at a radial position longer than that of the tip vortex.

c) Deflection of the wake (fig.17)

Because of the bound circulation, the wake has a tendency to remain in the rotor plane when it leaves the blade, then to move downward when it comes near the last blade.Only a weak deflection has been predicted.

Appendix A

Formulation of the wake elements

We have considered three types of elements for the decomposition of the wake.

- A semi-infinite cylinder element (See Appendix B)
- Segment elements
- Rectangular elements.

When computing the induced velocities at large distances from the segment or rectangular elements, the elements can be assumed to be points of concentrated vorticity. It is convenient to introduce this assumption into the formulations.

NOTATIONS :

$M(X, Y, Z)$	: Point where the induced velocity is to be evaluated
$M(X', Y', Z')$	: Point on the element
R_0	: Distance from M to the center of the element
R	: Distance from M to M'
r	: Ratio R/R_o
$\vec{\Omega}(\Omega, 0, 0)$	: Mean or reference vorticity
$\vec{W}(W_x, W_y, W_z)$	: Velocity induced at M by the element (Subscripted : o for the limiting point element; s for the segment element; c for the constant vorticity rectangular element; v for the linearly varying vorticity rectangular element)
M_1, M_2, M_3, M_4	: Points limiting the segment or rectangular element
ΔX	: Half length of the segment element or half side (X direction) of the rectangular element
ΔY	: Half side (Y direction) of the rectangular element
ΔS	: Length of the segment element ($S = 2\,\Delta X$) or surface of the rectangular element ($\Delta S = 4\,\Delta X\,\Delta Y$)
η, ζ	: Reduced coordinates of M' ($\eta = \frac{X'}{\Delta X}$ $\zeta = \frac{Y'}{\Delta Y}$)

Thickness of the wake effect

ε_1 : core thickness of a rectangular element

ε_2 : core radius of a segment element

Z_c : corrected Z coordinate of the point

Ω_c : corrected vorticity

Velocities induced by segment elements, case of the semi-rigid wake routine.

η_p : position of the point, where the velocity is calculated

η_v : radial position of the centers of the element

ψ : azimuth of the element

$\Delta\psi$: azimuth increment (size of the element)

ΔA : projected length of the element (on XOY)

ΔS : length of the element

λ : angle of the segment with the plane (XOY)

A) <u>Segment elements</u> (fig. A-1)

The segment element is used to model concentrated vortices. We consider an element of length $\Delta S = 2\ \Delta X$, which is bounded by the points : M_1 ($-\Delta X$, 0, 0) and M_3 ($+\Delta X$, 0, 0).

The velocity induced at the point M (X, Y, Z) is given by the Biot-Savart law :

$$\vec{W}_s(M) = -\frac{1}{4\pi}\int_{-\Delta X}^{+\Delta X} \frac{\overrightarrow{M'M} \times \vec{\Omega}}{R^3}\, dx \qquad (A\text{-}1)$$

Defining $\eta = \frac{X'}{\Delta X}$ which ranges from -1 to +1,

M' has the coordinates ($\eta\ \Delta X$, 0 , 0) and R is written as :

$$R = \sqrt{(X - \eta\,\Delta X)^2 + Y^2 + Z^2}$$

It then follow that :

$$\vec{W}_s(M) = \left|\begin{array}{l} W_{xs} = 0 \\ W_{ys} = -\dfrac{1}{4\pi}\,\Delta S\, Z\, \Omega \displaystyle\int_{-1}^{+1} \frac{1}{2R^3}\, d\eta \\ W_{zs} = +\dfrac{1}{4\pi}\,\Delta S\, Y\, \Omega \displaystyle\int_{-1}^{+1} \frac{1}{2R^3}\, d\eta \end{array}\right. \qquad (A\text{-}2)$$

If we consider R to be constant equal to R_0, in the integration we a arrive at the limiting point element which induces the velocity $\vec{W}_0(M)$

$$\vec{W}_0(M) = \left| \begin{array}{l} W_{X0} = 0 \\ W_{Y0} = -\frac{1}{4\pi} \frac{\Delta S \; Z \; \Omega}{R_0^3} \\ W_{Z0} = +\frac{1}{4\pi} \frac{\Delta S \; Y \; \Omega}{R_0^3} \end{array} \right. \qquad \text{(A-3)}$$

where: $$R_0 = \sqrt{X^2 + Y^2 + Z^2}$$

The componants of $\vec{W}_s$ can be optained by multiplying the componants of $\vec{W}_0$ by I_0, where:

$$I_0 = \frac{1}{2} \int_{-1}^{+1} \frac{R_0^3}{R^3} \, d\eta$$

where: $$R = \sqrt{(X - \eta \, \Delta X)^2 + Y^2 + Z^2}$$
$$= R_0 \sqrt{1 - 2\eta \frac{X \Delta X}{R_0^2} + \eta^2 \left(\frac{\Delta X}{R_0}\right)^2}$$

Then I_0 can be written:

$$I_0 = \int_{-1}^{+1} \frac{d\eta}{2\left[1 - 2\eta \frac{X\Delta X}{R_0^2} + \eta^2 \left(\frac{\Delta X}{R_0}\right)^2\right]^{3/2}} \qquad \text{(A-4)}$$

Any I_0 is then function of only two parameters:

$$A = -\frac{X \Delta X}{R_0^2} \quad \text{and} \quad \alpha = \frac{\Delta X}{R_0}$$

It follows, after integration:

$$I_0(A, \alpha) = \frac{1}{2(\alpha^2 - A^2)} \left[\frac{\alpha^2 + A}{\sqrt{1 + 2A + \alpha^2}} + \frac{\alpha^2 - A}{\sqrt{1 - 2A + \alpha^2}} \right]$$

(A-5)

Using X,Y,Z,ΔX I_0 can also be written :

$$I_0 = \frac{R_0^3}{2\Delta X\,(Y^2+Z^2)}\left[\frac{X+\Delta X}{R_1} - \frac{X-\Delta X}{R_3}\right] \quad \text{(A-7)}$$

where R_1 and R_3 are the distances from M to the two end points of the segment (M_1 and M_3)

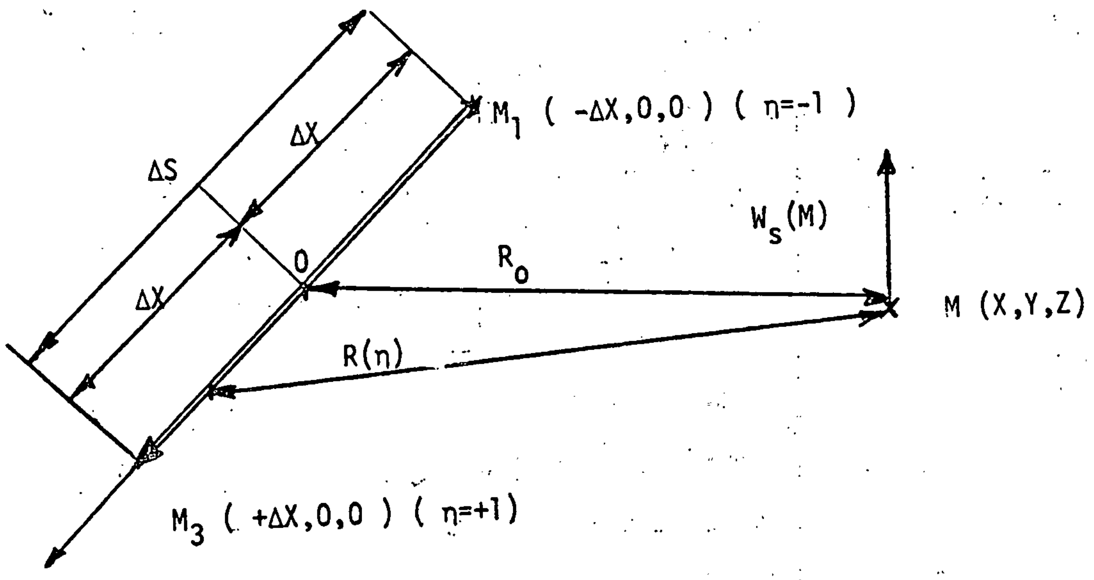
$$R_1 = \sqrt{(X+\Delta X)^2 + Y^2 + Z^2}$$

$$R_3 = \sqrt{(X-\Delta X)^2 + Y^2 + Z^2}$$

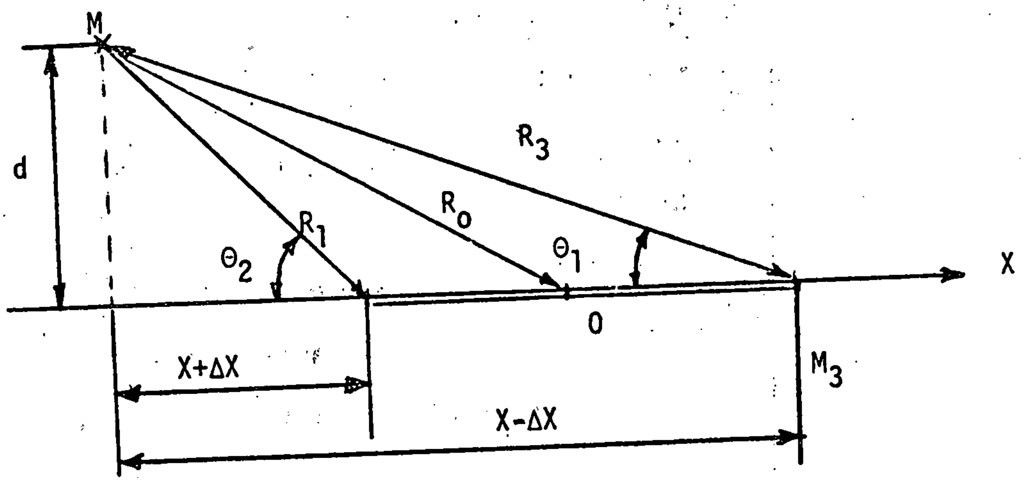
The geometrical interpretation of I_0 is shown in fig. [A-2]
It should be noted that Y^2+Z^2 is the squared distance from M to the line supporting the segment, and $\frac{X + \Delta X}{R_1}$ (or $\frac{X - \Delta X}{R_3}$) is the cosine of the angle formed by the intersection of MM_1 (or MM_3) and the segment.

It can be seen that I_0 rapidly approaches 1 as the distance from M to the segment becomes large compared to ΔX.

In that case W_{yo} and W_{zo} can be substituted in place of W_{ys} and W_{zs}. The simpler formulations for W_{ys} and W_{zs} are even more appropriate in that case, as the evaluation of I_0 in a computer program is very imprecise because of round-off errors.



(figure A-1) geometry of the segment element



(figure A-2) interpolation of I_0

B) Rectangular elements

We consider a local system of coordinates (O , X, Y, Z), such that the rectangle is on the plane (X O Y), centered on O, the direction of vorticity aligned with the (O X) axis. (See figure A-3).

The four points limiting the rectangle are M_1 to M_4, of coordinates (± ΔX, ± ΔY, 0).

The vorticity may vary linearly in Y direction, and for simplicity we can consider the rectangle element as a linear combination of a constant vorticity element (subscript c) and a linearly varying vorticity element (subscript v) with zero mean vorticity.

The vorticity at M' (X', Y', 0) is (Ω, 0, 0) for the constant vorticity element and ($\Omega \frac{Y'}{\Delta Y}$, 0, 0) for the linearly varying vorticity element.

The velocity induced at M by the constant vorticity element is $\vec{W}_c(M)$

$$\vec{W}_c(M) \quad \left| \begin{array}{l} W_{xc} = 0 \\ W_{yc} = -\frac{1}{4\pi} Z \Omega \iint_S \frac{dx'dy'}{R^3} \\ W_{zc} = +\frac{1}{4\pi} Y \Omega \iint_S \frac{(1 - Y'/Y)}{R^3} dx'dy' \end{array} \right. \qquad \text{(A-7)}$$

For the linearly varying vorticity element we have $\vec{W}_v(M)$:

$$\vec{W}_v(M) \quad \left| \begin{array}{l} W_{xv} = 0 \\ W_{yv} = -\frac{1}{4\pi} \Omega Z \iint_S \frac{Y'}{\Delta Y R^3} dx'dy' \\ W_{zv} = +\frac{1}{4\pi} Y \Omega \iint_S \frac{(1 - Y'/Y) Y'}{\Delta Y R^3} dx'dy' \end{array} \right. \qquad \text{(A-8)}$$

We then consider the reduced coordinates for M':

$$\eta = \frac{X'}{\Delta X} \qquad \zeta = \frac{Y'}{\Delta Y}$$

and the reference velocity $\vec{W}_o$ (0, W_{yo}, W_{zo}) given by the point element limit of the constant vorticity rectangular element.

$$\vec{W}_o = -\frac{1}{4\pi}\,\frac{\vec{OM}\times\vec{\Omega}}{R_o^3}\,\Delta S = \left|\begin{array}{l} 0 \\ W_{Yo} = -\dfrac{1}{4\pi}\,\dfrac{Z\Omega}{R_o^3}\,\Delta S \\ W_{Zo} = +\dfrac{1}{4\pi}\,\dfrac{Y\Omega}{R_o^3}\,\Delta S \end{array}\right.$$

We then define the factors I_1 to I_4 such that:

$$W_{yc} = W_{yo} * I_1$$

$$W_{zc} = W_{zo} * I_2$$

$$W_{yv} = W_{yo} * I_3$$

$$W_{zv} = W_{zo} * I_4$$

It follows:

$$I_1 = \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1}\frac{d\eta\, d\zeta}{r^3} \qquad \text{(A-9)}$$

$$I_2 = \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1}\left(1-\frac{\Delta Y}{Y}\zeta\right)\frac{d\eta\, d\zeta}{r^3}$$

$$I_3 = \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1}\frac{\zeta\, d\eta\, d\zeta}{r^3}$$

$$I_4 = \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1}\zeta\left(1-\frac{\Delta Y}{Y}\zeta\right)\frac{d\eta\, d\zeta}{r^3}$$

with :

$$r = R/R_o$$

$$R = \sqrt{(X-\eta\Delta X)^2 + (Y-\zeta\Delta Y) + Z^2}$$

$$R_o = \sqrt{X^2 + Y^2 + Z^2}$$

The integration of the four I coefficients can be derived from the following integrals :

$$J_1 = \int^{u}\int^{v} \frac{du\,dv}{(1+u^2+v^2)^{3/2}} = \operatorname{arc\,tg}\left(\frac{u\,v}{\sqrt{1+u^2+v^2}}\right)$$

$$J_2 = \int^{u}\int^{v} \frac{v\,du\,dv}{(1+u^2+v^2)^{3/2}} = -\ln\left(u+\sqrt{1+u^2+v^2}\right)$$

$$J_3 = \int^{u}\int^{v} \frac{v^2\,du\,dv}{(1+u^2+v^2)^{3/2}} = u\ln\left(v+\sqrt{1+u^2+v^2}\right) + 2\operatorname{arc\,tg}\left(u+v+\sqrt{1+u^2+v^2}\right)$$

The reduction of these integrals to more common forms is developed at the end of this appendix.

The general form for a I coefficient is :

$$I = \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1} \frac{a\zeta^2+b\zeta+c}{r^3}\,d\eta\,d\zeta \qquad \text{(A-10)}$$

with : $r=R/R_0$

$$r=\sqrt{1-2\eta\frac{X\Delta X}{R_0^2}+\eta^2\frac{\Delta X^2}{R_0^2}-2\zeta\frac{Y\Delta Y}{R_0^2}+\zeta^2\frac{\Delta Y}{R_0^2}}$$

We introduce the intermediate notations :

$$A = -\frac{X\Delta X}{R_o^2} \qquad \alpha = \frac{\Delta X}{R_o} \qquad B = -\frac{Y\Delta Y}{R_o} \qquad \beta = \frac{\Delta Y}{R_o}$$

Using these notations r can be written :

$$r = \sqrt{1 + 2A\eta + \alpha^2\eta^2 + 2B\zeta + \beta^2\zeta}$$

If Z is different from zero we can make the following change of variables to get a form ($1 + u^2 + v^2$) at the denominator.

$$u = (\alpha\eta + A/\alpha)/E$$

$$v = (\beta\zeta + B/\beta)/E$$

with $E = \sqrt{1 - (A/\alpha)^2 - (B/\beta)^2} = |Z|/R_o$

The general form of the I coefficient is :

$$I = \frac{1}{4\alpha\beta E}\left\{a\frac{E^2}{\beta^2} J_3(u,v) + \left[\frac{bE}{\beta} - \frac{2aBE}{\beta^3}\right] J_2(u,v) + \left[c - \frac{bB}{\beta^2} + \frac{aB^2}{\beta^4}\right] J_1(u,v)\right. \Bigg|_{v^-}^{v^+}\Bigg|_{u^-}^{u^+} \qquad \text{(A-11)}$$

Where u± and v± are the new limits at which the integrals must be evaluated.

We get :

$$u_\pm = (A/\alpha \pm \alpha)/E$$

$$v_\pm = (B/\beta \pm \beta)/E$$

The square root becomes :

$$\sqrt{1+U_\pm^2+V_\pm^2} = \frac{1}{E}\sqrt{E^2+(B/\beta \pm \beta)^2+(A/\alpha \pm \alpha)^2}$$

or, using X, Y, Z, ΔX, ΔY :

$$\frac{1}{|Z|}\sqrt{(X\mp\Delta X)^2+(Y\mp\Delta Y)^2+Z^2}$$

then :

$$J_1(u,v)\Big|_{U_-}^{U_+}\Big|_{V_-}^{V_+} = (\pm)(\pm)\ \mathrm{arc\,tg}\left[\frac{(X\mp\Delta X)(Y\mp\Delta Y)}{|Z|\sqrt{(X\mp\Delta X)^2+(Y\mp\Delta Y)^2+Z^2}}\right]$$

$$J_2(u,v)\Big|_{U_-}^{U_+}\Big|_{V_-}^{V_+} = (\pm)(\pm)\ -\ln\left[\frac{-(X\mp\Delta X)+\sqrt{(X\mp\Delta X)^2+(Y\mp\Delta Y)^2+Z^2}}{|Z|}\right]$$

$$J_3(u,v)\Big|_{U_-}^{U_+}\Big|_{V_-}^{V_+} = (\pm)(\pm)\left\{-\frac{X\mp\Delta X}{|Z|}\ln\left[\frac{-(Y\mp\Delta Y)+\sqrt{(X\mp\Delta X)^2+(Y\mp\Delta Y)^2+Z^2}}{|Z|}\right]\right.$$

$$\left.+2\ \mathrm{arc\,tg}\left[\frac{-(X\mp\Delta X)-(Y\mp\Delta Y)+\sqrt{(X\mp\Delta X)^2+(Y\mp\Delta Y)^2+Z^2}}{|Z|}\right]\right\}$$

$$\frac{1}{4\alpha\beta E} = 1\Big/\left[4\ \frac{\Delta X}{R_o}\ \frac{\Delta Y}{R_o}\ \frac{|Z|}{R_o}\right] = \frac{R_o^3}{|Z|\Delta S}$$

$$\frac{a\,E^2}{\beta^2} = \left(\frac{Z}{\Delta Y}\right)^2 a \qquad \text{(A-12)}$$

$$\frac{b\,E}{\beta} - 2a\,\frac{B\,E}{\beta^2} = \frac{b|Z|}{\Delta Y} + \frac{2a\,Y\,|Z|}{\Delta Y^2}$$

$$c - \frac{b\beta}{\beta^2} + a\frac{\beta^2}{\beta^4} = c + \frac{Y}{\Delta Y} b + \frac{Y^2}{\Delta Y^2} a$$

Using J_1, J_2 and J_3 defined in (A-12)

we can write the I coefficients :

I_1 (a=0,b=0,c=1)

$$I_1 = \frac{R_o^3}{|Z|\,\Delta S} J_1 \qquad \text{(A-13)}$$

I_2 ($a=0, b=-\frac{\Delta Y}{Y}, c=1$)

$$I_2 = \frac{-R_o^3}{Y \cdot \Delta S} J_2 \qquad \text{(A-14)}$$

I_3 (a=0,b=1,c=0)

$$I_3 = \frac{R_o^3}{\Delta Y\,\Delta S}\left[\frac{Y}{|Z|} J_1 + J_2\right] \qquad \text{(A-15)}$$

I_4 ($a=-\frac{\Delta Y}{Y}, b=1, c=0$)

$$I_4 = \frac{-R_o^3}{\Delta S\,\Delta Y}\left[J_2 + \frac{|Z|}{Y} J_3\right] \qquad \text{(A-16)}$$

At this point we introduce some properties of the formulas.

a) $|Z|$ appears in J_1 and J_3 and at the same time in I_1, I_3, and I_4, in such a way that every $|Z|$ can be replaced by Z.

b) In J_2 which is the sum of four logarithms, $|Z|$ can be removed, as the constants in $|Z|$ cancel each other, two by two.

c) Because of the symetry all ($X \mp \Delta X$) and ($Y \mp \Delta Y$) can be changed into ($X \pm \Delta X$) and ($Y \pm \Delta Y$)

d) Appling twice the transformation :

$$\text{arc tg } a + \text{arc tg } b = \text{arc tan} \left(\frac{a + b}{1 - ab} \right)$$

One can verify that :

$$(\pm)(\pm) \text{ arc Tg} \left[\frac{-(X \pm \Delta X) - (Y \pm \Delta Y) + \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}}{Z \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}} \right]$$

$$= (\pm)(\pm) \text{ arc tg} \left[\frac{(X \pm \Delta X) + (Y \pm \Delta Y) + \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}}{Z \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}} \right]$$

e) Using the property (valid for any a,b,c) :

$$\ln \left(\frac{-a + \sqrt{a^2 + b^2}}{-c + \sqrt{c^2 + b^2}} \right) = - \ln \left(\frac{a + \sqrt{a^2 + b^2}}{c + \sqrt{a^2 + b^2}} \right) \qquad \text{(A-13)}$$

We can transform J_2 and J_3 :

$$J_2 = (\pm)(\pm) \ln \left[\frac{(X \pm \Delta X) + \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}}{|Z|} \right]$$

$$J_3 = (\pm)(\pm) \left\{ \frac{X \pm \Delta X}{Z} \ln \left[\frac{Y \pm \Delta Y + \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}}{|Z|} \right] \right.$$

$$\left. + 2 \text{ arc Tg} \left[\frac{(X \pm \Delta X) + (Y \pm \Delta Y) + \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + Z^2}}{Z} \right] \right.$$

The functions I_1 to I_4 are interesting only for comparisons and verifications. It can be shown that I_1 and I_2 go to 1 . I_3 and I_4 go to 0 as the point M goes far from the element ; furthermore the calculation of J_1, J_2, J_3 is always lengthy and becomes very imprecise because of round-off errors in the same conditions. It is then appropriate to substitute the formulation $\vec{W}_o$ instead of $\vec{W}_{yc}$ and to neglect $\vec{W}_{yv}$ when R_o is larger than a predetermined value.

This value which can be expressed in term of the square root of the element area is a compromise between the accuracy desired, the round-off errors, and the CPU time in the computer program.

Verifications : (for $R_0 >> \Delta S$)

As I_1 goes to 1 : $J_1 \approx Z \Delta S / R_0^3$

I_2 goes to 1 : $J_2 \approx -Y \Delta S / R_0^3$

J_2 and J_3 verify I_3 goes to 0

I_4 goes to 1 : $J_3 \approx -\frac{Y}{Z} J_2 \approx \frac{Y^2 \Delta S}{Z R_0^3}$

Final formulations (wake thickness effect non included).

$$R_1 = \sqrt{(X + \Delta X)^2 + (Y + \Delta Y)^2 + Z^2}$$

$$R_2 = \sqrt{(X - \Delta X)^2 + (Y + \Delta Y)^2 + Z^2}$$

$$R_3 = \sqrt{(X + \Delta X)^2 + (Y - \Delta Y)^2 + Z^2}$$

$$R_4 = \sqrt{(X - \Delta X)^2 + (Y - \Delta Y)^2 + Z^2}$$

$$J_1 = \text{arc tg}\left[\frac{(X + \Delta X)(Y + \Delta Y)}{R_1 Z}\right] - \text{arc tg}\left[\frac{(X - \Delta X)(Y + \Delta Y)}{R_2 Z}\right] - \text{arc tg}\left[\frac{(X + \Delta X)(Y - \Delta Y)}{R_3 Z}\right] + \text{arc tg}\left[\frac{(X - \Delta X)(Y - \Delta Y)}{R_4 Z}\right]$$

$$J_2 = \ln\left[\frac{(X+\Delta X)+R_1}{(X-\Delta X)+R_2} \times \frac{(X-\Delta X)+R_4}{(X+\Delta X)+R_3}\right]$$

$$J_3 = \frac{1}{Z}\left[(X+\Delta X)\ln\left(\frac{Y+\Delta Y+R_1}{Y-\Delta Y+R_3}\right) - (X-\Delta X)\ln\left(\frac{Y+\Delta Y+R_2}{Y-\Delta Y+R_4}\right)\right]$$

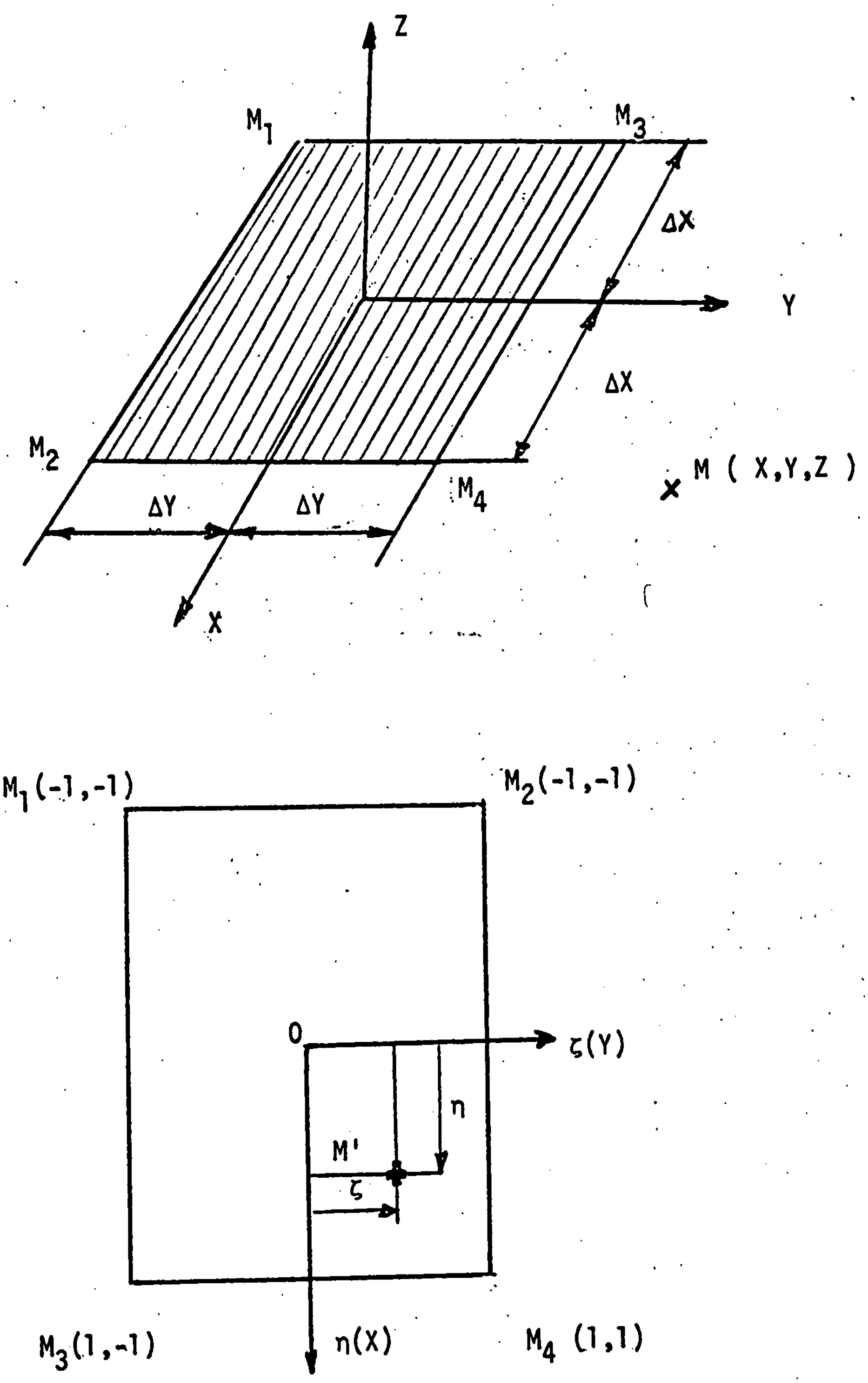
$$+\,2\left[\operatorname{arctg}\left(\frac{X+\Delta X+Y+\Delta Y+R_1}{Z}\right) - \operatorname{arctg}\left(\frac{X-\Delta X+Y+\Delta Y+R_2}{Z}\right) - \operatorname{arctg}\left(\frac{X+\Delta X+Y-\Delta Y+R_3}{Z}\right) + \operatorname{arctg}\left(\frac{X-\Delta X+Y-\Delta Y+R_4}{Z}\right)\right]$$

$$W_{yc} = -\frac{\Omega}{4\pi} J_1$$

$$W_{zc} = -\frac{\Omega}{4\pi} J_2$$

$$W_{yv} = -\frac{\Omega}{4\pi\Delta Y}\left(Y J_1 + Z J_2\right)$$

$$W_{zv} = -\frac{\Omega}{4\pi\Delta Y}\left(Y J_2 + Z J_3\right)$$



(figure A-3) geometry for a rectangular element

Singularities :

They occur when the point M where the induced velocities are computed is located on the plane (XOY) or Z = 0.

If the point is outside the element, there is theoritically no singularity, in fact when they occur they cancel each other in the formulations.

a) Case of J_1 :

J_1 which is composed only of arc tangents is bonded by -2π and 2π, but the arguments of all the arc tangents go to infinite.

By symmetry $W_{yc}(X, Y, Z) = -W_{zc} (X, Y, - Z)$. If M is outside the element it is continuous, therefore $J_1 = 0$ for Z = 0. I

If M is inside the element, the induced velocity experiences a discontinuity of intensity Ω when M crosses the surface of the element. Therefore the strength of the discontinuity is 4π for J_1, but by symmetry we can still consider that $J_1 = 0$ for Z = 0. This result is also obtained if one performs the direct integration of J_1 for the case Z = 0.

b) Case of J_2

J_2 is a logarithmic form, therefore it can go to infinite, for this, at least one of the forms $X \pm \Delta X + \sqrt{(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2}$

must go to zero, this happens when ($X \pm \Delta X$) is negative and when ($Y \pm \Delta Y$) goes to zero, or on the two vortex lines bounding the rectangular element ($Y = \pm \Delta Y$)

For a point M near the vortex line $Y = \Delta Y$ and of coordinates X, $Y = \Delta Y + \delta Y$, $Z = \delta Z$ (δY and $\delta Z << 1$)

$$J_2 = \ln\left(\frac{X + \Delta X + \sqrt{(X+\Delta X)^2 + (2\Delta Y + \delta Y)^2 + \delta Z^2}}{X - \Delta X + \sqrt{(X-\Delta X)^2 + (2\Delta Y + \delta Y)^2 + \delta Z^2}}\right)$$

$$- \ln\left(\frac{X + \Delta X + \sqrt{(X+\Delta X)^2 + \delta Y^2 + \delta Z^2}}{X - \Delta X + \sqrt{(X-\Delta X)^2 + \delta Y^2 + \delta Z^2}}\right)$$

The first part is finite, the second part is also finite when X is larger than ΔX. If X is smaller than $-\Delta X$, the numerator and the denominator go to zero together, but using the transformation (A-13), one can see that the argument of the logarithm is still finite. If one must evaluate induced velocities for $Z = 0$ outside the element, he must, in that case, apply the transformation to avoid overflows and round-off errors in a computer program.

If $-\Delta X < X < \Delta X$ (M located along the segment bounding the element then the numerator is finite but, as $X - \Delta X$ is negative, the denominator is equal to $|X-\Delta X| \left(\sqrt{1 + \frac{\delta Y^2 + \delta Z^2}{(X-\Delta X)^2}} - 1\right)$

and $$J_2 \approx \ln \frac{\sqrt{\delta Y^2 + \delta Z^2}}{|X - \Delta X|}$$

$\sqrt{\delta Y^2 + \delta Z^2}$ is the distance from the point to the segment bounding the element.

This singularity is cancelled by the next element if the two element match perfectly.

c) Case of J_3

The logarithmic part is divided by Z; therefore J_3 cannot be defined, but the value ZJ_3 which in fact appear in I_4 has no singularity.

although , the individual logarithm goes to the infinite on the two segments bounding the element, perpendicular to the vortex lines, they are multiplied by ($X \pm \Delta X$) and the products go to zero ; but this case should also be very carefully avoided in numerical calculations.

While the arguments of the individual arc tangent go to infinity the arc tangent part of ZJ_3 is bounded by $- 2\pi Z$ and $+ 2\pi Z$ therfore we can conclude that ZJ_3 has no singularity.

Thickness of the wake effect.

In the physical wake, there are no singularity, the velocities are always finite and continuous, because of viscous effects. The physical wake is not a zero thickness surface, but the velocity is distributed around the mid-surface of the wake.

Viscous effects have to be considered not only because of their reality but mainly as a convenient mean to eliminate the singularities which arise from non matching elements.

It can be shown that even a large value of the artificial thickness does not change significantly the final values, except that the vortex core size of the tip and root vortices may have an effect on the predicted roll-up of the wake.

A first approach to the problem consists of multiplying the induced velocity components by the correction factor, this factor has to be close to one for the points far from the segment (for a segment element) or the mid-surface of a rectangular element.

For example : $1/(1+\varepsilon_2^2/d^2)$

where ε_2 = core radius

d = distance from the point to the line supporting the element.

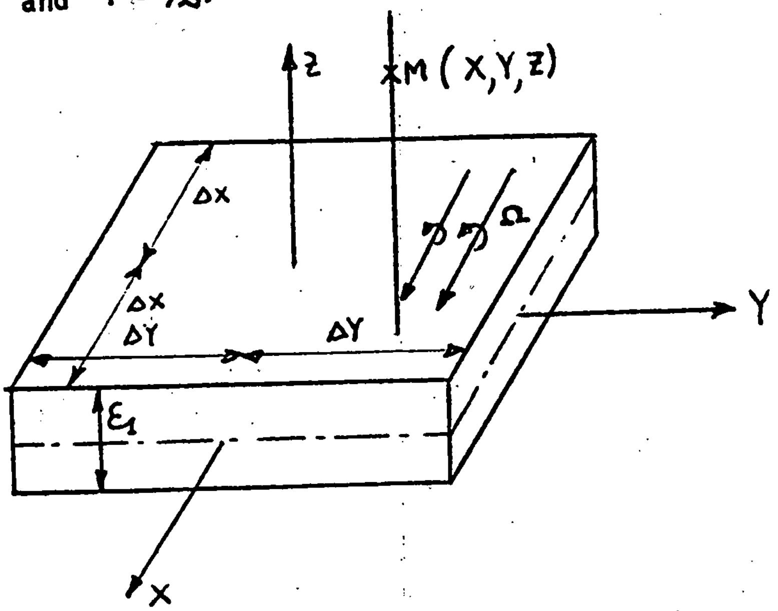
This factor has been choosed for the segment elements.

This formulation has been satisfactory for the segment elements, as it is very easy to choose control points away from the line supporting the element (where the singularities occur for this element). For the rectangular elements, however although reasonable velocity components were obtained, the problem of the singularities was not solved as many control points, even far from the elements, are very near their mid-planes ($Z \simeq 0$). This last case had to be detected and a special procedure used.

Futhermore, for each point, the case had to be reduced to $X > 0$ and $Y > 0$ with all the transformations involved, in order to avoid numerical errors and round-off errors.

A new approach, introducing physical considerations, can solve the problem of the singularities, and is very efficient in the computer program as the test for $Z = 0$ and the transformations are eliminated.

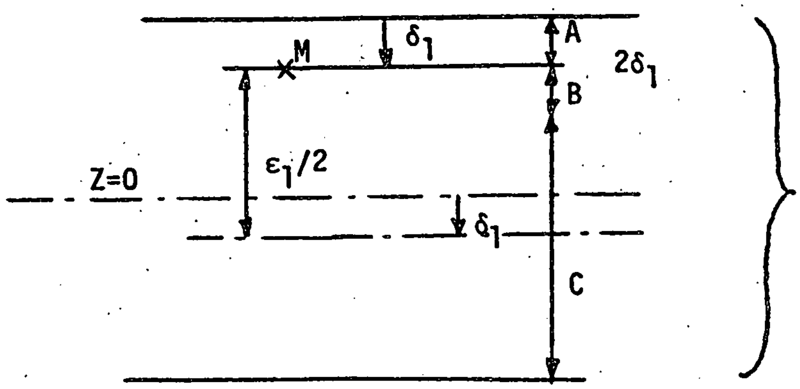
We consider a rectangular element of constant vorticity, to which we add thickness by distributing the vorticity uniformly along Z, between $-\frac{\varepsilon_1}{2}$ and $+\varepsilon_1/2$.



For a point M crossing the original zero thickness element, the W_{yc} velocity profile is symetric and has a discontinuity of strength 4π at $Z = 0$.

The velocity profile for a real wake of thickness ε_1 has no discontinuity but experiences a rapid change in the region $|Z| <- \varepsilon_1/2$ the modified element gives such a profile.

If we consider a point inside the element ($Z = \varepsilon_1/2 - \delta$) the contribution of the part A and B of the element, to W_{yc} cancel each other exactly, therefore W_{yc} induced by the whole element is equivalent to W_{yc} induced by the remaining part (C). This equivalent element is centered at a constant distance $\varepsilon_1/2$ of the point but has a residual strength in the ratio of the thickness ($\Omega * \frac{\varepsilon_1 - 2\delta}{\varepsilon_1}$)



Considering the equivalent element, we can formulate a corrected Z coordinate (Z_c), and a corrected strengh Ω_c.

$$Z_c = \begin{cases} Z & \text{for } Z > \varepsilon_1/2 \\ \varepsilon_1/2 & \text{for } 0 < Z < \varepsilon_1/2 \\ -\varepsilon_1/2 & \text{for } -\varepsilon_1/2 < Z < 0 \\ Z & \text{fo } Z < -\varepsilon_1/2 \end{cases}$$

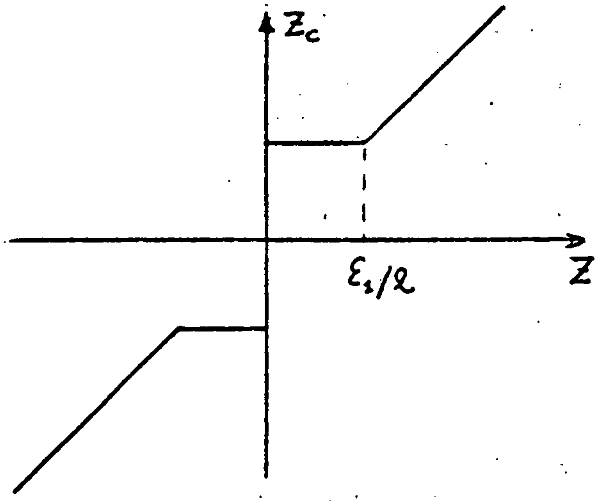
$$\Omega_c = \Omega \text{ for } |Z| > \varepsilon_1/2$$

$$\Omega_c = \Omega * \frac{|Z|}{\varepsilon_1/2} \text{ for } |Z| < \varepsilon_1/2$$

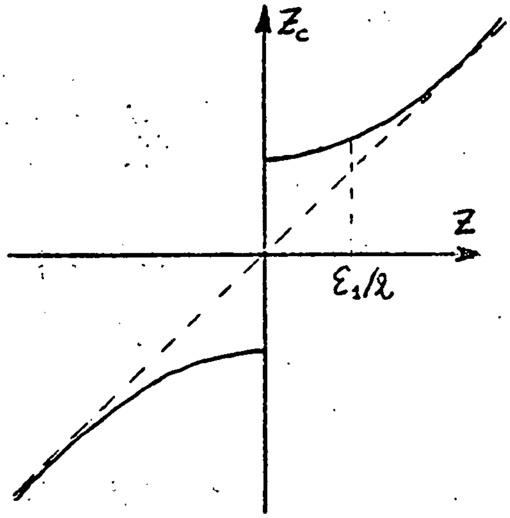
As the vorticity of a real element is not uniformaly distributed from $-\varepsilon_1/2$ to $+\varepsilon_1/2$ but is more concentrated around the mid-plane by viscous effects, smoother formulations are more appropriate.

$$Z_c = \begin{cases} \sqrt{Z^2 + (\varepsilon_1/2)^2} & (Z>0) \\ -\sqrt{Z^2 + (\varepsilon_1/2)^2} & (Z<0) \end{cases} \qquad \text{(A-14)}$$

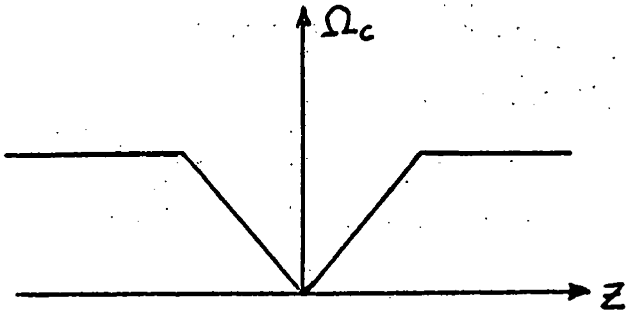
$$\Omega_c = \frac{Z^2}{Z_c^2}\,\Omega$$



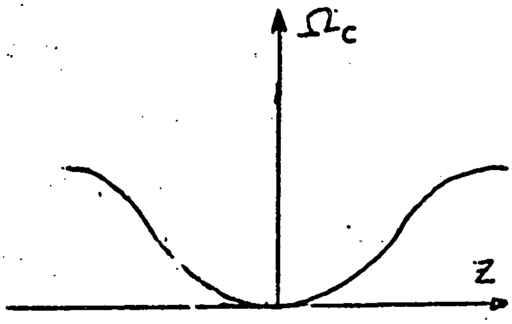
$Z_c(Z)$ for a uniformly distributed vorticity element



$Z_c(Z) = \pm\sqrt{Z^2 + (\varepsilon_1/2)^2}$



$\Omega_c(Z)$ for a uniformly distributed vorticity element



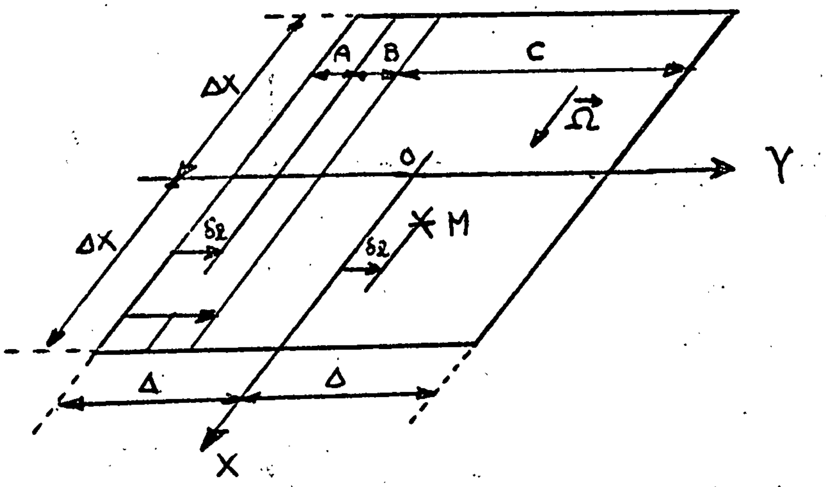
$\Omega_c(Z) = Z^2 / Z_c^2$

This concept of a modified coordinate, can be applied also to the W_{zc} component. Physically the W_{zc} component does not vary significantly when the point M moves around $Z = 0$.

We consider a point moving on the surface of the element W_{zc} is equal to zero for $Y = 0$; for $Y = \delta_2$ the contributions of the parts A' and B' of the zero thickness element cancel each other exactly by symetry.

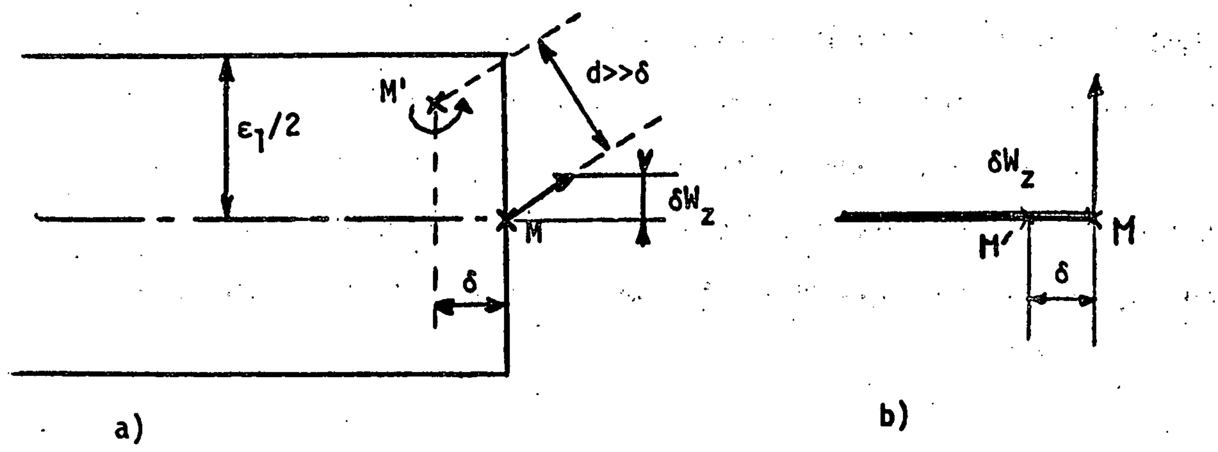
W_{zc} induced by the whole element is equivalent to W_{zc} induced by the remaining part (C').

As δ_2 goes to ΔY_2 the parts A' and B' disappear and W_{zp} approaches the singularity.



The element then contributes principally by its circulation, near the boundary for $Y \simeq \Delta Y$. This contribution, for zero thickness elements is cancelled only by the next element, if this one is on the same plane, but because of non matching elements and some angle between the elements, the cancellation is not exact, futhermore the singularity in the formulas are not avoided.

If we consider an element with thickness, the contribution of the circulation located near the edge and particularly at $Y \geq \approx \Delta Y - \varepsilon_1/2$, is gradually reduced (and effectively nullified for $Y = \Delta Y$), for a point M located at $Z = 0$ and $Y = \Delta Y$, as the circulation is distributed along the Z direction. To cancel the contribution of the circulation near the edge as M approaches it, we can consider an equivalent zero thickness C' element with edge position moved up to a distance $\varepsilon_1/2$ from the edge as M moves to $Y = \Delta Y$



(elementary contributions of a point M' (X', ΔY, Z') to a point M (X, ΔY, 0): a) for an element of thickness $\varepsilon_1/2$
b) for a zero thickness element

This can be implemented in J_2 by transforming the value $(Y \pm \Delta Y)^2$ (squared distance from the point to the edges) into $(Y \pm \Delta Y)^2 + (\varepsilon_1/2)^2$
The radical becomes : $(X \pm \Delta X)^2 + (Y \pm \Delta Y)^2 + (\varepsilon_1/2)^2 + Z^2$
and one can identify $(\varepsilon_1/2)^2 + Z^2$ as Z_c^2 which was used for W_{yp}, Therefore the viscous effects (thickness of the wake) can be reduced to a single transformation on the Z component. It can be noted that all the problem arising from the singularities in the computer program, disappear as the corrected Z value is never zero.

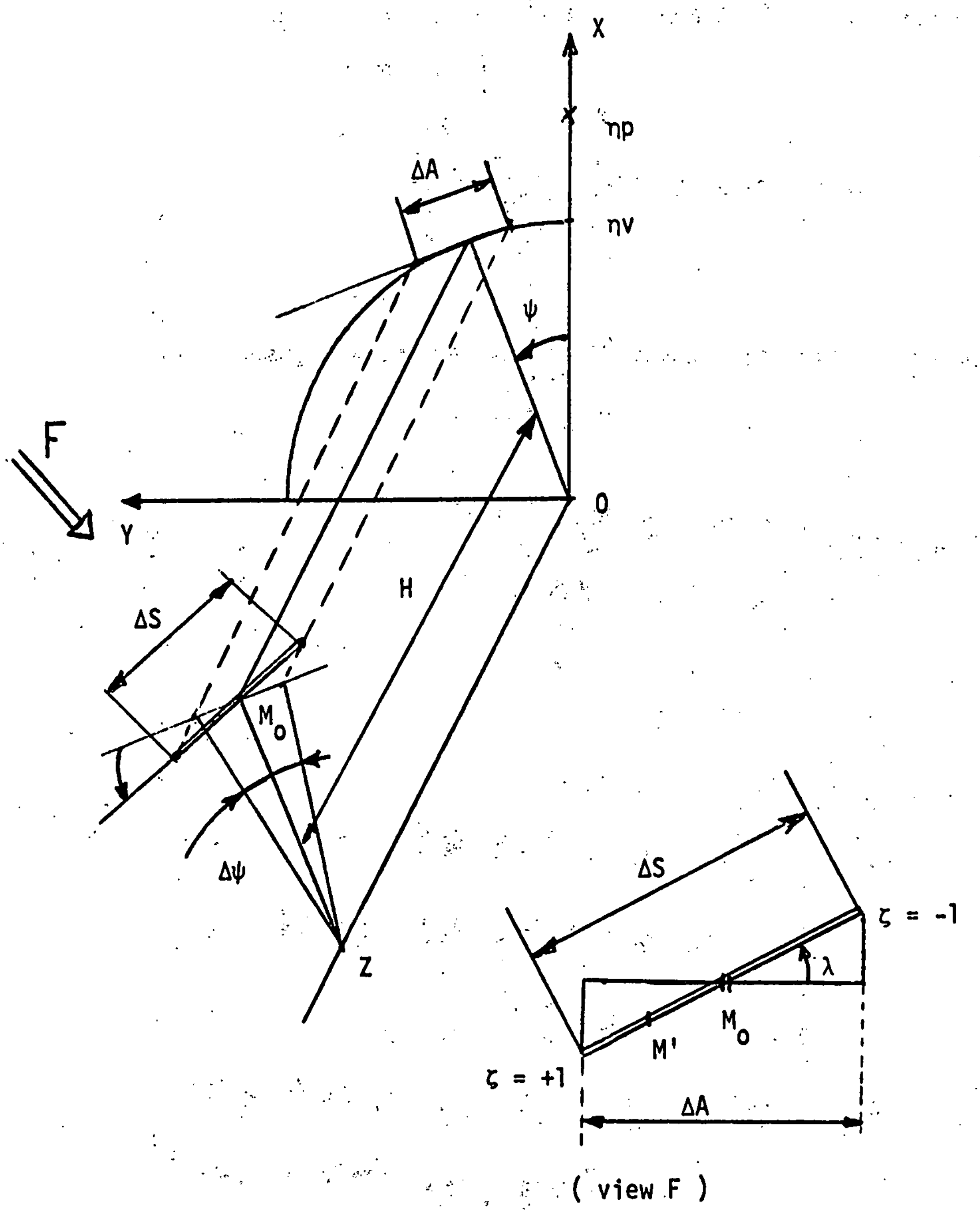
Velocities induced by segment elements.

Case of the semi-rigid wake routine. (figure A-4)

Because of the particular geometry of the wake, and of the positions of the points where the induced velocities are computed, a special formulation is more appropriate, as it can reduce the computation time.

The only points where the induced velocities are computed, are on the blade. Therefore, the position of M is given by η_p.

The elements are drawn on a cylinder of axis Z and radius η_v, the positions of their centers are given by the cylindrical coordinates (η_p, ψ, H) ; they make an angle λ with the (XOY) plane, and their projections on the plane spans over an angle $\Delta\psi$ = 10°, small angles assumptions are made.



(figure A-4) geometry for a segment element, case of the semi-rigid wake routine

The projected length of the segment is $\Delta A = \eta_v \, \Delta\psi$ (small angle assuption).

The components of $\vec{\Delta S}$ are then : $\vec{\Delta S}$:

$$\vec{\Delta S} \left| \begin{array}{l} -\Delta A \sin\psi \\ \Delta A \cos\psi \\ \Delta A \,\mathrm{tg}\,\lambda \end{array} \right.$$

Rectangular coordinates of M_0' :

$$M_0' \left| \begin{array}{l} \eta_v \cos\psi \\ \eta_v \sin\psi \\ H \end{array} \right.$$

The coordinates of a point M' on the segment are given by

$$\vec{OM'} = \vec{OM_0'} + \zeta \frac{\vec{\Delta S}}{2}$$

$$M' \left| \begin{array}{l} \eta_v \cos\psi - \zeta \frac{\Delta A}{2} \sin\psi \\ \eta_v \sin\psi + \zeta \frac{\Delta A}{2} \cos\psi \\ H + \zeta \frac{\Delta A}{2} \,\mathrm{tg}\,\lambda \end{array} \right.$$

Then the components of $\vec{M'M}$ are :

$$\vec{M\Omega'} \left| \begin{array}{l} \eta_p - \left(\eta_v \cos\psi - \zeta \frac{\Delta A}{2} \sin\psi\right) \\ -\left(\eta_v \sin\psi + \zeta \frac{\Delta A}{2} \cos\psi\right) \\ -\left(H + \zeta \frac{\Delta A}{2} \,\mathrm{tg}\,\lambda\right) \end{array} \right.$$

and

$$R^2 = M\Omega'^2 = \eta_p^2 + \eta_v^2 - 2\eta_p \eta_v \cos\psi + H^2$$
$$+ 2\zeta \left(\frac{\Delta A}{2}\right) \left(\eta_p \sin\psi + \mathrm{tg}\,\lambda \; H\right)$$
$$+ \zeta^2 \left(\frac{\Delta A}{2}\right)^2 \left(1 + \mathrm{tg}\,\lambda\right)$$

The distance from the center M'_0 to the point M is R_0

$$R_0 = MM'_0 = R(\zeta = 0)$$

$$R_0 = \sqrt{\eta_p^2 + \eta_v^2 - 2\eta_p \eta_v^2 \cos\Psi + H^2}$$

$$R = R_0 \sqrt{1 + 2\zeta\left(\frac{\Delta A}{2R_0}\right)\left(\frac{\eta_p \sin\Psi + H \operatorname{tg}\lambda}{R_0}\right) + \zeta^2\left(\frac{\Delta A}{2R_0}\right)^2\left(1 + \operatorname{tg}^2\lambda\right)}$$

$$= R_0 \sqrt{1 + 2B\zeta + A\zeta^2}$$

with :

$$A = \left(\frac{\Delta A}{2R_0}\right)^2 \left(1 + \operatorname{tg}^2\lambda\right)$$

$$B = \left(\frac{\Delta A}{2R_0}\right)\left(\frac{\eta_p \sin\Psi + H \operatorname{tg}\lambda}{R_0}\right) \qquad \text{(A-15)}$$

The velocity induced at M by the element is $\vec{W}$ given by the Biot-Savart law.

$$\vec{W} = -\frac{\gamma_t}{4\pi} \int_{\zeta=-1}^{\zeta=+1} \frac{\overrightarrow{MM'} \times \overrightarrow{dS}}{R^3}$$

$$= -\frac{\gamma_t}{4\pi} \int_{\zeta=-1}^{\zeta=+1} \frac{\overrightarrow{MM'_0} \times \overrightarrow{dS}}{R^3}$$

as $\overrightarrow{M'_0M'} \times \overrightarrow{dS} = 0$

$\vec{W}$ can be written : $\vec{W} = I_0 \, \vec{W}_0$

$$\vec{W}_0 = -\frac{\gamma_t}{4\pi R_0^3} \left(\overrightarrow{MM'_0} \times \overrightarrow{\Delta S} \right) = + \frac{\gamma_t \Delta A}{4\pi R_0^3} \left| \begin{array}{l} \xi_v \sin\psi \, tg\lambda - H \cos\psi \\ tg\lambda \left[\xi_\rho - \xi_v \cos\psi \right] - H \sin\psi \\ \xi_v - \xi_\rho \cos\psi \end{array} \right.$$

$$I_0 = \frac{1}{2} \int_{\zeta=-1}^{\zeta=+1} \frac{R_0^3}{R^3} \, d\zeta$$

I_0 which is identical to (A-5) is written ; using (A-15) :

$$I_0 = \frac{1}{2(A - B^2)} \left[\frac{A + B}{\sqrt{1 + 2B + A}} + \frac{A - B}{\sqrt{1 - 2B + A}} \right]$$

I_0 goes to 1 as R_0 / (ΔS) goes to infinite.

In the semi-rigid wake analysis, I_0 has been included for the near wake and neglected for the far wake.

Derivation of the integrals J_1, J_2 and J_3.

a) J_1 :

$$J_1(\eta,\zeta) = \int^{\eta}\int^{\zeta} \frac{d\eta\, d\zeta}{(1+\eta^2+\zeta^2)^{3/2}}$$

Change of variable : $\quad X = \zeta/\sqrt{1+\eta^2}$

$$\int^{\zeta} \frac{d\zeta}{(1+\eta^2+\zeta^2)^{3/2}} = \int^{X} \frac{dX}{(1+\eta^2)\,(1+X^2)^{3/2}}$$

We use the common form : $\displaystyle \int \frac{dX}{(1+X^2)^{3/2}} = \frac{X}{\sqrt{1+X^2}}$

Therefore $\displaystyle \int^{\zeta} \frac{d\zeta}{(1+\eta^2+\zeta^2)^{3/2}} = \frac{\zeta}{(1+\eta^2)\sqrt{1+\eta^2+\zeta^2}}$

$$J_1(\eta,\zeta) = \int^{\eta} \frac{\zeta\, d\eta}{(1+\eta^2)\sqrt{1+\eta^2+\zeta^2}}$$

Change of variable : $\quad \eta = \sqrt{1+\zeta^2}\ \operatorname{sh} t$

$$\left(d\eta = \sqrt{1+\zeta^2}\ \operatorname{ch} t\ dt \right)$$

$$J_1 = \int^{t} \frac{\zeta\, dt}{\left[1+(1+\zeta^2)\operatorname{sh}^2 t\right]}$$

Which is of the form ;

$$\int \frac{dx}{p^2 + q^2 \sinh^2 ax} = \frac{1}{ap\sqrt{q^2-p^2}} \operatorname{atan}\left[\frac{\sqrt{q^2-p^2}\tanh ax}{p}\right]$$

with p = 1, $q = (1 + \zeta^2)$ and a = 1

$$\int \frac{\zeta\, dt}{[1+(1+\zeta^2)\,sh^2 t]} = \frac{\zeta}{\sqrt{1+\zeta^2-1}} \operatorname{atan}\left[\sqrt{1+\zeta^2-1}\tanh t\right]$$

$$= \frac{\zeta}{|\zeta|} \operatorname{atan} |\zeta| \frac{\eta}{\sqrt{1+\eta^2+\zeta^2}}$$

$$= \operatorname{atan} \frac{\eta\zeta}{\sqrt{1+\eta^2+\zeta^2}}$$

b) $J_2(\eta,\zeta)$

$$J_2(\eta,\zeta) = \int^{\eta}\!\!\int^{\zeta} \frac{\zeta\, d\eta\, d\zeta}{(1+\eta^2+\zeta^2)^{3/2}}$$

$$\int^{\zeta} \frac{\zeta\, d\zeta}{(1+\eta^2+\zeta^2)^{3/2}} = \frac{1}{2}\int^{\zeta} \frac{d(\zeta^2)}{(1+\eta^2+\zeta^2)^{3/2}} = -\frac{1}{\sqrt{1+\eta^2+\zeta^2}}$$

Then :

$$J_2(\eta,\zeta) = -\int^{\eta} \frac{d\eta}{\sqrt{1+\eta^2+\zeta^2}} = -\ln\left(\eta + \sqrt{1+\eta^2+\zeta^2}\right)$$

c) $J_3(\eta,\zeta)$

$$J_3(\eta,\zeta) = \int^{\eta}\int^{\zeta} \frac{\zeta^2\, d\eta\, d\zeta}{(1+\eta^2+\zeta^2)^{3/2}}$$

Integration with respect to ζ :

We have the form

$$\int^{x} \frac{x^2\,dx}{(a^2+x^2)^{3/2}} = -\frac{x}{\sqrt{a^2+x^2}} + \ln\left(x+\sqrt{a^2+x^2}\right)$$

$$\int^{\zeta} \frac{\zeta^2\,d\zeta}{(1+\eta^2+\zeta^2)^{3/2}} = -\frac{\zeta}{\sqrt{1+\eta^2+\zeta^2}} + \ln\left(\zeta+\sqrt{1+\eta^2+\zeta^2}\right)$$

Integration with respect to η : we apply again

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right)$$, to the first part

therefore

$$\int^{\eta} \frac{(-\zeta)\,d\eta}{\sqrt{1+\eta^2+\zeta^2}} = -\zeta \ln\left(\eta+\sqrt{1+\eta^2+\zeta^2}\right)$$

Integration of : $\ln\left(\zeta+\sqrt{1+\eta^2+\zeta^2}\right)$

Change of variable :

$$\eta = \sqrt{1+\zeta^2}\,\operatorname{sh} t \qquad \left(d\eta = \sqrt{1+\eta^2}\,\operatorname{ch} t\, dt\right)$$

$$\int^{\eta} \ln\left(\zeta+\sqrt{1+\eta^2+\zeta^2}\right) =$$

$$\int^{t} \ln\left(\zeta+\sqrt{1+\zeta^2}\,\mathrm{ch}\,t\right)\sqrt{1+\zeta^2}\,\mathrm{ch}\,t\,dt$$

Change of variable ;

$$u=\sqrt{1+\zeta^2}\,\mathrm{ch}\,t \qquad \left[\begin{array}{l} du=\sqrt{1+\zeta^2}\,\mathrm{sh}\,t\,dt \\ dt=\dfrac{du}{\sqrt{u^2-(1+\zeta^2)}} \end{array}\right]$$

Then :

$$A=\int^{\eta} \ln\left(\zeta+\sqrt{1+\eta^2+\zeta^2}\right)d\eta=\int^{u} \ln(\zeta+u)\frac{u\,du}{\sqrt{u^2-(1+\zeta^2)}}$$

Integration by parts of A :

$$x=\ln(\zeta+u) \qquad dx=\frac{du}{\zeta+u}$$

$$dy=\frac{u\,du}{\sqrt{u^2-(1+\zeta^2)}} \qquad y=\sqrt{u^2-(1+\zeta^2)}$$

$$A=\sqrt{u^2-(1+\zeta^2)}\,\ln(\zeta+u)-\int\frac{\sqrt{u^2-1+\zeta^2}}{\zeta+u}\,du$$

$$A=\eta\ln\left(\zeta+\sqrt{1+\eta^2+\zeta^2}\right)-B$$

with :

$$B=\int\frac{\sqrt{u^2-(1+\zeta^2)}}{\zeta+u}\,du$$

Change of variables : $z = \zeta + \upsilon$

$$B = \int^{z} \frac{\sqrt{z^2 - 2tz - 1}}{z}\, dz$$

We have the form :

$$\int \frac{\sqrt{ax^2+bx+c}}{x}\, dx = \sqrt{ax^2+bx+c} + \frac{b}{2}\int \frac{dx}{\sqrt{ax^2+bx+c}} + c\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

therefore :

$$B = \sqrt{z^2-2tz-1} - \zeta\int \frac{dz}{\sqrt{z^2-2tz-1}} - \int \frac{dz}{z\sqrt{z^2-2tz-1}}$$

$$\sqrt{z^2-2tz-1} = \sqrt{\upsilon^2-(1+\zeta^2)} = +\eta$$

$$\int \frac{dz}{\sqrt{z^2-2tz-1}} = \ln\left(\eta + \sqrt{1+\eta^2+\zeta^2}\right)$$

$$J_3 = -\zeta \ln\left(\sqrt{1+\eta^2+\zeta^2} + \eta\right) + \eta \ln\left(\zeta + \sqrt{1+\eta^2+\zeta^2}\right)$$
$$- \eta - (-\zeta)\ln\left(\eta + \sqrt{1+\eta^2+\zeta^2}\right)$$
$$+ \int \frac{dz}{z\sqrt{z^2-2tz-1}}$$

$$J_3 = \eta \ln\left(\zeta + \sqrt{1+\eta^2+\zeta^2}\right) - \eta + \int \frac{dz}{z\sqrt{z^2 - 2tz - 1}}$$

integration of $C = \int \frac{dz}{z\sqrt{z^2 - 2tz - 1}}$

$$C = \int \frac{du}{(u+\zeta)\sqrt{u^2 - (1+\zeta^2)}} = \int \frac{dt}{\sqrt{1+\zeta^2}\,\mathrm{ch}\,t + \zeta}$$

$$\mathrm{ch}\,t = \frac{e^t + e^{-t}}{2}$$

$$C = \int \frac{2\,e^t\,dt}{\sqrt{1+\zeta^2}\,(e^{2t}+1) + 2\zeta e^t}$$

change of variable : $e^t = X$

$$C = \int \frac{2\,dx}{\sqrt{1+\zeta^2}\,(x^2+1) + 2\zeta X}$$

which is of the form : $\int \frac{dz}{1+z^2}$, $z = \sqrt{1+\zeta^2}\left(x - \frac{\zeta}{\sqrt{1+\zeta^2}}\right)$

therefore :

$$C = 2 \,\mathrm{arc\,tg}\left[\sqrt{1+\zeta^2}\left(x - \zeta/\sqrt{1+\zeta^2}\right)\right]$$

$$= 2 \,\mathrm{arc\,tg}\left[\sqrt{1+\zeta^2}\left(e^t - \zeta/\sqrt{1+\zeta^2}\right)\right]$$

As $\eta = \text{Ash}\, t$ we have $$e^{t} = \frac{\eta}{\sqrt{1+\eta^2}} + \sqrt{\frac{\eta^2}{(1+\eta^2)} + 1}$$

$$C = 2 \operatorname{atan} \left(\eta + \zeta + \sqrt{1 + \eta^2 + \zeta^2} \right)$$

$$J_3(\eta, \zeta) = \eta \ln\left(\zeta + \sqrt{1+\eta^2+\zeta^2}\right) - \eta + 2 \operatorname{arc\,tg}\left(\eta + \zeta + \sqrt{1+\eta^2+\zeta^2}\right)$$

Note : The term $-\eta$ can be dropped, it will disappear in the double integration :

Appendix B

Velocities induced by a semi-infinite cylinder

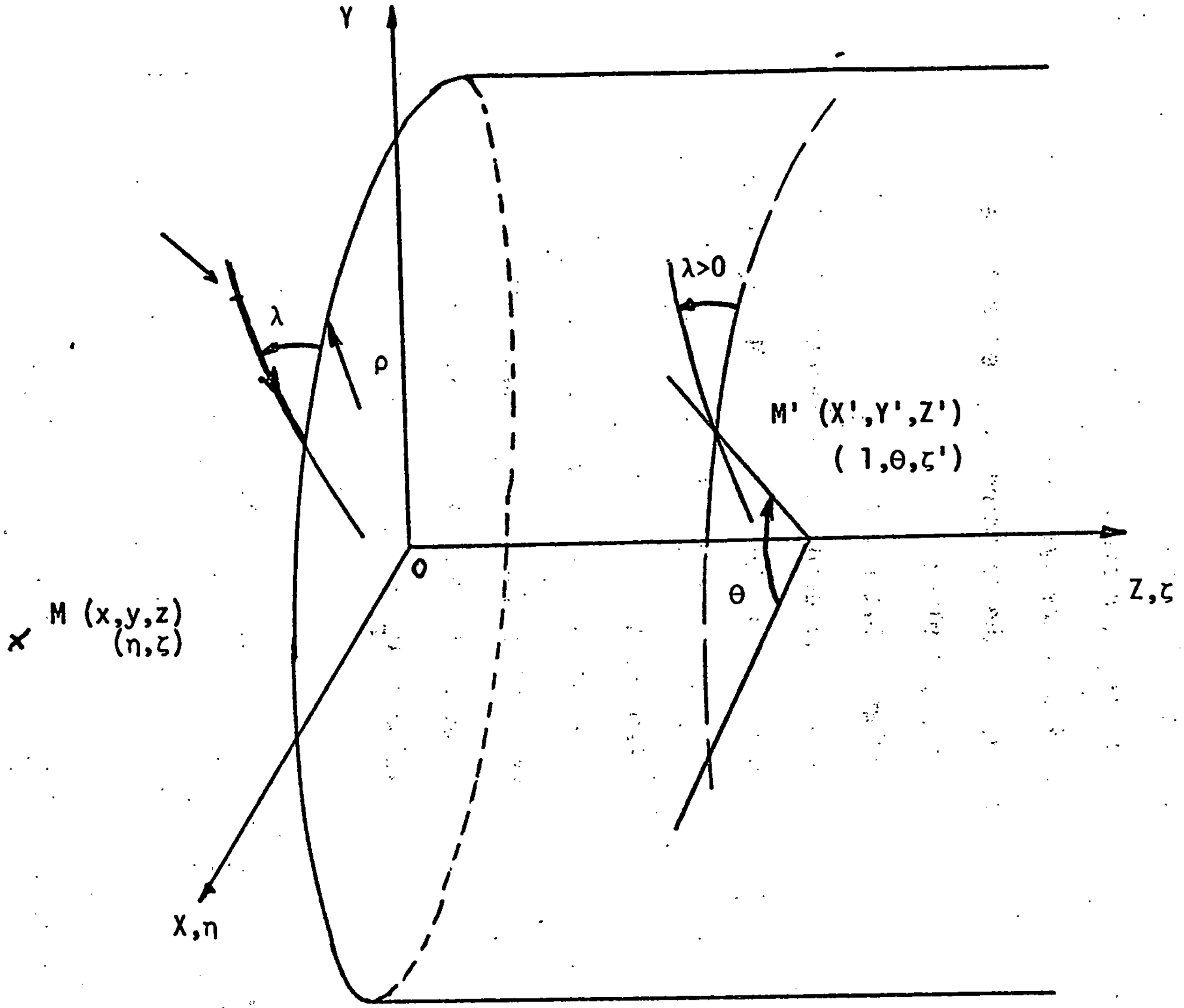
If a stable geometry of a wake exists for a constant inflow, once the expansion of the wake is finished and the azimuthal influence of a finite number of blades becomes negligible, the trailing vortices take the shape of spirals of the constant angle and diameter.

The amount of calculation for the induced velocities can be greatly reduced by using a semi-infinite cylinder of vorticity to model this part of the wake (called far wake). This model can be constructed by continuity from the incoming trailing vortices by spreading the vorticity of each helicoid over an angle of $2\pi/n$, where n is the number of incoming trailing vortices.

The velocities at any point in space by the semi-infinite cylinder can be evaluated by series or by interpolation in an array of precomputed values.

NOTATIONS :

$M\ (x, y, z)$	: Point where the induced velocity is calculated.
$M'\ (x', y', z')$	: current point on the cylinder.
R	: distance M to M'.
(ρ, θ, z')	: Cylindrical coordinates of M'.
ρ	: Radius of the cylinder.
λ	: Angle of the incoming vortices with the (x,o,y) plane.
n	: Number of blades.
Ω	: Strength of each incoming vortex.
$\vec{\omega}$	: Vorticity on the cylinder at M' (module ω).
η, ζ	: Reduced parameters of the problem : $\eta = x/\rho$ and $\zeta = z/\rho$.
ζ', r	: Reduced variables $\zeta' = z'/\rho$, $r = R/\rho$.
W_x^*	: Value of W_x for $\cos\lambda = 1$ and $\omega = 1$
W_y^*	: Value of W_y for $\sin\lambda = 1$ and $\omega = 1$
W_z^*	: value of W_z for $\cos\lambda = 1$ and $\omega = 1$



geometry of the semi-infinite cylinder

The velocity induced at a point M is given by the Bio-Savart law ;

$$\vec{W}(M) = -\frac{1}{4\pi}\iint_{(S)} \frac{\vec{M'M} \times \vec{\omega}}{R^3}\, dS$$

where (S) is the surface of the cylinder.

The problem can be reduced because of the symmetry of rotation. Only the point M on the (x,0,z) plane will be considered.

Using cylindrical coordinates (ρ,θ,z) for the point M',

one gets :

$$\vec{M'M} \; (x - \rho\cos\theta,\ -\rho\sin\theta,\ z - z')$$

$$\vec{\omega} \; (-\omega\sin\theta\cos\lambda,\ \omega\cos\theta\cos\lambda,\ \omega\sin\lambda)$$

It becomes, for the components of $\vec{W}$ (M)

$$W_x(M) = \frac{\rho\omega\cos\lambda}{4\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\cos\theta\,(z-z')}{R^3}\, dz'd\theta$$

$$W_y(M) = \frac{\rho\omega\sin\lambda}{4\pi}\int_0^{2\pi}\int_0^{\infty} \frac{(x-\rho\cos\theta)}{R^3}\, dz'd\theta$$

(B-1)

$$W_z(M) = \frac{\rho\omega\cos\lambda}{4\pi}\int_0^{2\pi}\int_0^{\infty} \frac{(\rho-x\cos\theta)}{R^3}\, dz'd\theta$$

with :

$$R = \sqrt{x^2 + \rho^2 + (z'-z)^2 - 2\rho x \cos\theta}$$

The induced velocity $\vec{W}$ is dependent of the parameters : $x, z, \lambda, \rho, \omega$.

The problem can be further reduced to only two parameters η and ζ by using reduced coordinates for x, z, z' and defining W_x^* and W_z^* as W_x and W_z for $\omega = 1$ and $\cos\lambda = 1$, W_y^* as W_y for $\omega = 1$ and $\sin\lambda = 1$

$\zeta = z/\rho$; $\zeta' = z'/\rho$

$\eta = x/\rho$

$r = R/\rho$

It becomes :

$$W_x^* = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{(\zeta-\zeta')\cos\theta \; d\zeta' d\theta}{r^3}$$

$$W_y^* = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{(\eta-\cos\theta) \; d\zeta' d\theta}{r^3} \qquad \text{(B-2)}$$

$$W_z^* = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{(1-\eta\cos\theta) \; d\zeta' d\theta}{r^3}$$

with

$$r = \sqrt{1 + \eta^2 + (\zeta'-\zeta)^2 - 2\eta\cos\theta}$$

Integration in azimuthal direction.

The integration of (B-2)can be reduced to the two forms :

$$I_1(A) = \int_0^{2\pi} \frac{d\theta}{(1 - A\cos\theta)^{3/2}}$$

and

$$I_2(A) = \int_0^{2\pi} \frac{\cos\theta \, d\theta}{(1 - A\cos\theta)^{3/2}}$$

These integrals can be evaluated only by series.

If one considers the development of (1+B)^k in a power serie of B :

$$(1+B)^k = 1 + kB + k(k-1)\frac{B^2}{2!} + \cdots$$

$$\cdots + k(k-1)\cdots(k-n+1)\frac{B^n}{n!} + \cdots$$

one gets for the development of (1-B)$^{3/2}$

$$(1-B)^{-3/2} = 1 + \frac{3}{2}B + \frac{3}{2}\times\frac{5}{2}\frac{B^2}{2!} + \cdots$$

$$\cdots + \frac{3}{2}\times\frac{5}{2}\times\cdots\times\left(\frac{2n+1}{2}\right)\frac{B^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{B^n\,(2n+1)!}{4^n\,(n!)^2}$$

I_1 and I_2 can then be transformed into :

$$I_1(A) = \sum_{n=0}^{\infty} \frac{A^n (2n+1)!}{(n!)^2 \, 4^n} \int_0^{2\pi} \cos^n \theta \, d\theta \qquad \text{(B-3)}$$

$$I_2(A) = \sum_{n=0}^{\infty} \frac{A^n (2n+1)!}{(n!)^2 \, 4^n} \int_0^{2\pi} \cos^{n+1} \theta \, d\theta$$

$\int_0^{2\pi} \cos^n \theta \, d\theta$ can be evaluated by successive integrations by parts and yields :

$$2\pi \frac{(2k)!}{4^k (k!)^2} \qquad \text{for} \quad n = 2k$$

$$0 \qquad \text{for} \quad n = 2k+1$$

By using n=2k for I_1 and n=2k+1 for I_2 one gets :

$$I_1(A) = 2\pi \sum_{k=0}^{\infty} \frac{A^{2k} (4k+1)!}{(2k)! \, (k!)^2 \, 4^{3k}}$$

(B-4)

$$I_2(A) = 2\pi A \sum_{k=0}^{\infty} \frac{A^{2k} (4k+1)!}{(2k)! \, (k!)^2 \, 4^{3k}} \left(\frac{4k+3}{4k+4} \right)$$

These results can be applied to :

$$r = \sqrt{1 + \eta^2 + (\zeta' - \zeta)^2 - 2\eta\cos\theta}$$

which can be written : $C\sqrt{1 - A\cos\theta}$

with : $C = \sqrt{1 + \eta^2 + (\zeta' - \zeta)^2}$

and : $A = 2\eta / [1 + \eta^2 + (\zeta' - \zeta)^2]$

It becomes for W_x^* :

$$\frac{1}{2} \sum_{n=0}^{\infty} \frac{(4n+1)!}{(2n)!\,(n!)^2\, 4^{3n}} \left(\frac{4n+3}{4n+4}\right) J_1(2n+1) \qquad \text{(B-5)}$$

where : $$J_1(i) = \int_0^{\infty} \frac{(\zeta - \zeta')\, A^i\, d\zeta'}{C^3}$$

$$W_Y^* = \eta S_1 - S_2$$

$$W_Z^* = S_1 - \eta S_2$$

with : $$S_1 = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(4n+1)!}{(2n)!\,(n!)^2\, 4^{3n}} J_2(2n)$$

$$S_2 = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(4n+1)!}{(2n)!\,(n!)^2\, 4^{3n}} \left(\frac{4n+3}{4n+4}\right) J_2(2n+1)$$

$$J_2(i) = \int_0^{\infty} \frac{A^i}{C^3}\, d\zeta' \qquad \text{(B-6)}$$

Integration in ζ direction.

a) Evaluation of J_1

$$J_1(i) = \int_0^\infty \frac{(\zeta-\zeta')A^i}{C^3}\,d\zeta' = \int_0^\infty \frac{(\zeta-\zeta')(2\eta)^i}{\left[1+\eta^2+(\zeta'-\zeta)^2\right]^{i+3/2}}\,d\zeta'$$

an anti-derivative exists for this form, one gets :

$$J_1(i) = \frac{-(2\eta)^i}{(2i+1)\,(1+\eta^2+\zeta^2)^{i+1/2}} \qquad \text{(B-7)}$$

b) Evaluation of J_2

$$J_2(i) = \int_0^\infty \frac{A^i}{C^3}\,d\zeta' = \int_0^\infty \frac{(2\eta)^i\,d\zeta'}{\left[1+\eta^2+(\zeta'-\zeta)^2\right]^{i+3/2}}$$

which is of the form $\displaystyle\int \frac{dx}{(a^2+x^2)^n}$

this form can be integrated by parts and yields the reduction formula :

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2(n-1)\,a^2\,(x^2+a^2)^{n-1}} + \frac{2n-3}{a^2(2n-2)}\int \frac{dx}{(x^2+a^2)^{n-1}}$$

(B-8)

using n=3/2+i this reduction formula becomes :

$$\int \frac{dx}{(a^2+x^2)^{3/2+i}} = \frac{x}{(2i+1)\, a^2\, (x^2+a^2)^{i+1/2}}$$

$$+ \frac{2i}{a^2(2i+1)} \int \frac{dx}{(x^2+a^2)^{3/2+(i-1)}}$$

The final integration will be :

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}$$

Using the notations T_i, A_n, C_k

such that : $$T_i = \int \frac{dx}{(x^2+a^2)^{3/2+i}}$$

and after k integrations by parts T_i being written :

$$T_i = \sum_{n=1}^{k} A_n + C_k\, T_{i-k}$$

the reduction formula can be written

$$T_i = \frac{x}{(2i+1)\, a^2\, (x^2+a^2)^{1/2+i}} + \frac{2i}{a^2\,(2i+1)}\, T_{i-1} \qquad \text{(B-9)}$$

or, for i-k :

$$T_{i-k} = \frac{x}{[2(i-k)+1]\, a^2 \, (x^2+a^2)^{1/2+i-k}} + \frac{2(i-k)}{a^2\,[2(i-k)+1]} T_{i-(k+1)}$$

using

$$T_i = \sum_{n=1}^{k} A_n + C_k \frac{x}{[2(i-k)+1]\, a^2 \, [x^2+a^2]^{1/2+(i-k)}} + C_k \frac{2(i-k)}{a^2[2(i-k)+1]} T_{i-(k+1)} \qquad \text{(B-10)}$$

By identification it becomes :

$$A_{k+1} = C_k \frac{x}{[2(i-k)+1]\, a^2 \, (x^2+a^2)^{1/2+(i-k)}}$$

$$C_{k+1} = C_k \frac{2(i-k)}{a^2[2(i-k)+1]} \qquad \text{(B-11)}$$

with $C_0 = 1$

$$C_k = \frac{[2i][2(i-1)][2(i-2)]\cdots[2(i-(k+1))]}{a^{2k}[2i+1][2(i-1)+1]\cdots[2(i-(k-1))+1]}$$

$$C_k = \frac{4^k (i!)^2 [2(i-k)+1]!}{a^{2k} [(i-k)!]^2 (2i+1)!} \qquad \text{(B-12)}$$

$$A_{k+1} = \frac{2x\, 4^{k+1} (i!)^2\, 2\, [i-(k+1)]!\, [2(i-k)+1](i-k)[2(i-k)-1]}{(2i+1)!\, a^{2k} ([i-(k+1)]!)^2\, 4(i-k)^2 [2(i-k)+1]\, (x^2+a^2)^{3/2+i-(k+1)}}$$

this last formula can be reduced and for the subsripts k one gets :

$$A_k = \frac{x\, 4^{k+1} (i!)^2 [2(i-k)]! [2(i-k)+1]}{(2i-1)!\, a^{2k} [(i-k)!]^2 [2(i-k)+2]\, (x^2+a^2)^{3/2+i-k}}$$

The development of (B-8) to the order k is then :

$$\int \frac{dx}{(x^2+a^2)^{3/2+i}} = \sum_{n=1}^{k} \frac{x(i!)^2 4^n [2(i-n)]! [2(i-n)+1]}{(2i+1)!\, a^{2n} [(i-n)!]^2 [2(i-n)+2]\, (x^2+a^2)^{3/2+i-k}}$$

$$+ \frac{4^k (i!)^2 [2(i-k)+1]!}{a^{2k} [(i-k)!]^2 (2i+1)!} \int \frac{dx}{(x^2+a^2)^{3/2+i-k}}$$

By replacing x^2 by $(\zeta'-\zeta)^2$ and a^2 by $1+\eta^2$ and evaluating the integrals from zero to infinite,. we get:

$$\left. \frac{\zeta'-\zeta}{[1+\eta^2+(\zeta'-\zeta)^2]^{3/2+i-n}} \right|_0^{\infty} = \frac{\zeta}{(1+\eta^2+\zeta^2)^{3/2+i-n}}$$

and for k=i

$$\int \frac{d\zeta'}{[1+\eta^2+(\zeta'-\zeta)^2]^{3/2}} = \frac{1}{(1+\eta^2)} \left[1 + \frac{\zeta}{\sqrt{1+\eta^2+\zeta^2}} \right]$$

(B-13)

Then :

$$\int_0^{\infty} \frac{d\zeta'}{\left[1+\eta^2+(\zeta'-\zeta)^2\right]^{3/2+i}} =$$

$$\frac{\zeta (i!)^2}{(2i+1)!} \left\{ \sum_{n=1}^{i} \frac{4^n \left[2(i-n)\right]! \left[2(i-n)+1\right]}{\left[(i-n)!\right]^2 \left[2(i-n)+2\right] (1+\eta^2)^n \left(1+\eta^2+\zeta^2\right)^{3/2+i-n}} \right\}$$

$$+ \frac{4^i (i!)^2}{(2i+1)!\,(1+\eta^2)^i} \left\{ \frac{1}{(1+\eta^2)} \left[1 + \frac{\zeta}{\sqrt{1+\eta^2+\zeta^2}} \right] \right\} \qquad \text{(B-14)}$$

The terms can be rearranged by changing n into i+1-n.

$$\int \frac{d\zeta'}{\left[1+\eta^2+(\zeta'-\zeta)^2\right]^{3/2+i}} =$$

$$\frac{\zeta (i!)^2}{(2i+1)!} \left\{ \sum_{n=1}^{i} \frac{4^{i-n+1} (2n)!\,(2n+1)}{(n!)^2 (2n+2) (1+\eta^2)^{i+1-n} \left(1+\eta^2+\zeta^2\right)^{1/2+n}} \right\}$$

$$+ \frac{4^i (i!)^2}{(2i+1)!\,(1+\eta^2)^i} \left[\frac{1}{(1+\eta^2)} \left(1 + \frac{\zeta}{\sqrt{1+\eta^2+\zeta^2}} \right) \right]$$

$$J_2(i) = \frac{4^i (i!)^2}{(2i+1)!\,(1+\eta^2)^{i+1}} \left[1 + \zeta \sum_{n=0}^{i} \frac{(2n)!\,(1+\eta^2)^n}{(n!)^2\, 4^n \left(1+\eta^2+\zeta^2\right)^{1/2+n}} \right]$$

(B-15)

The expansions of S_1 and S_2 can then be derived from

$$S_1 = \frac{1}{2(1+\eta^2)\sqrt{1+\eta^2+\zeta^2}} \times$$

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\, 4^k} \left(\frac{2\eta}{1+\eta^2}\right)^{2k} \left[\sqrt{1+\eta^2+\zeta^2} + \zeta \sum_{i=0}^{2k} \frac{(2i)!}{(i!)^2\, 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i\right]$$

$$S_2 = \frac{1}{2(1+\eta^2)\sqrt{1+\eta^2+\zeta^2}} \times \qquad \text{(B-16)}$$

$$\sum_{k=0}^{\infty} \frac{(2k+1)!}{(k!)^2\, 4^k (2k+2)} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1} \left[\sqrt{1+\eta^2+\zeta^2} + \zeta \sum_{i=0}^{2k+1} \frac{(2i)!}{(i!)^2\, 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i\right]$$

the final formulation for W_x^*, W_y^* and W_z^* are then :

$$W_x^* = \frac{-1}{2\sqrt{1+\eta^2+\zeta^2}} \sum_{k=0}^{\infty} \frac{(4k+1)!}{(2k)!\,(k!)^2\, 4^{3k}(4k+4)} \left(\frac{2\eta}{1+\eta^2+\zeta^2}\right)^{2k+1}$$

$$W_y^* = \eta S_1 - S_2$$

$$W_z^* = S_1 - \eta S_2 \qquad \text{(B-17)}$$

Mathematical properties and numerical evaluation of the series.

From (B-16) one can consider the following decomposition for S_1 and S_2 :

$$S_1 = \frac{1}{2(1+\eta^2)}\left[D_1 + \frac{\zeta E_1}{\sqrt{1+\eta^2+\zeta^2}}\right]$$

$$S_2 = \frac{1}{2(1+\eta^2)}\left[D_2 + \frac{\zeta E_2}{\sqrt{1+\eta^2+\zeta^2}}\right]$$

with :

$$D_1 = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\, 4^k} \left(\frac{2\eta}{1+\eta^2}\right)^{2k}$$

$$D_2 = \sum_{k=0}^{\infty} \frac{(2k+1)!}{(k!)^2\, 4^k} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1} \qquad \text{(B-18)}$$

$$E_1 = \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\, 4^k} \left(\frac{2\eta}{1+\eta^2}\right)^{2k} \sum_{i=0}^{2k} \frac{(2i)!}{(i!)^2\, 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^{i}$$

$$E_2 = \sum_{k=0}^{\infty} \frac{(2k+1)!}{(k!)^2\, 4^k (2k+2)} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(2i)!}{(i!)^2\, 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^{i}$$

There are closed forms for D_1 and D_2 :

$$\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2\, 4^k} A^k$$ is the development in power series of $(1-A)^{-1/2}$

thus $$D_1 = \left[1-\left(\frac{2\eta}{1+\eta^2}\right)^2\right]^{-1/2} = \sqrt{\frac{1+2\eta^2+\eta^4}{1-2\eta^2+\eta^4}} = \frac{1+\eta^2}{|1-\eta^2|}$$

$$D_2 = \sum_{k=0}^{\infty} \frac{(2k+1)!}{4^k (k!)^2} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left(1 - \frac{1}{2k+2}\right) \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1}$$

$$= \left(\frac{2\eta}{1+\eta^2}\right) \left\{ \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1} \right\}$$

$$- \left(\frac{1+\eta^2}{2\eta}\right) \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+2)} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+2}$$

One can recognize the left part as $\left(\frac{2\eta}{1+\eta^2}\right) D_1$

and transform $\frac{1}{2k+2}\left(\frac{2\eta}{1+\eta^2}\right)^{2k+2}$ into

$$D_2 = \frac{2\eta}{|1-\eta^2|} - \left(\frac{1+\eta^2}{2\eta}\right) \int \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2} \left(\frac{2\eta}{1+\eta^2}\right)^{2k+1} d\left(\frac{2\eta}{1+\eta^2}\right)$$

$$= \frac{2\eta}{|1-\eta^2|} - \left(\frac{1+\eta^2}{2\eta}\right) \int \frac{1+\eta^2}{|1-\eta^2|} \frac{2\eta}{(1+\eta^2)^2} d\eta$$

Two cases have to be considered.

1st case $|\eta| > 1$

$$D_2 = \frac{2\eta}{|1-\eta^2|} - \frac{1+\eta^2}{\eta}\left(\frac{1}{1+\eta^2} + Ct\right)$$

since D_2 goes to 0 for infinite η, Ct=0

$$D_2 = \frac{2\eta}{|1-\eta^2|} - \frac{1}{\eta}$$

2^{nd} case $|\eta| < 1$

$$D_2 = \frac{2\eta}{|1-\eta^2|} + \frac{1+\eta^2}{\eta^2}\left(\frac{1}{1+\eta^2} + Ct\right)$$

as $D_2=0$ for $\eta=0$, $Ct=-1$

$$D_2 = \frac{2\eta}{|1-\eta^2|} - \eta$$

The two last formulations can be merged into :

$$D_2 = \left(1+\eta^2\right)\left[\frac{(1+\eta^2) - |1-\eta^2|}{4\eta\,|1-\eta^2|}\right] \qquad \text{(B-19)}$$

this last formulation is valid for any η different of ± 1.

These results can be applied to W_y^* and W_z^*

$$W_y^* = \eta S_1 - S_2 = \frac{1}{2(1+\eta^2)}\left[(\eta D_1 - D_2) + \zeta\frac{(\eta E_1 - E_2)}{\sqrt{1+\eta^2+\zeta^2}}\right]$$

$$= \frac{1}{4\eta}\left[1 + \frac{\eta^2-1}{|1-\eta^2|}\right] + \frac{\zeta}{2(1+\eta^2)}\left[\frac{\eta E_1 - E_2}{\sqrt{1+\eta^2+\zeta^2}}\right]$$

The left part $\left(\frac{1}{4\eta}\left[1+\frac{\eta^2-1}{|1-\eta^2|}\right]\right)$ is equal to 0 for $|\eta| < 1$
and to $\frac{1}{2\eta}$ for $|\eta| > 1$

As the right part is odd with respect to ζ, it becomes :

$W_y^*(\eta,-\zeta) = - W_y^*(\eta,\zeta)$ for $|\eta| < 1$

$W_y^*(\eta,-\zeta) = \frac{1}{\eta} - W_y^*(\eta,\zeta)$ for $|\eta| > 1$

If the serie S_1 and S_2 are evaluated according to (B-16) there is divergence only for $\eta=\pm 1$ and ζ positive, the discontinuities of D_1, D_2, E_1, E_2 concelling each other for negative values of ζ.

This behavior can be explained by considering the development of $\sqrt{1+\eta^2+\zeta^2}$ in a power serie of $\left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)$

$$\sqrt{1+\eta^2+\zeta^2} = \frac{|\zeta|}{\sqrt{\frac{\zeta^2}{1+\eta^2+\zeta^2}}} = |\zeta| \left[1 - \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)\right]^{-1/2}$$

which can be developed into :

$$|\zeta| \sum_{i=0}^{\infty} \frac{(2i)!}{(i!)^2 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i$$

The following term which appears in S_1

$$\sqrt{1+\eta^2+\zeta^2} + \zeta \sum_{i=0}^{2k} \frac{(2i)!}{(i!)^2 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i$$

can be written

$$2\sqrt{1+\eta^2+\zeta^2} - \zeta \sum_{i=2k+1}^{\infty} \frac{(2i)!}{(i!)^2 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i \quad \text{for } \zeta \text{ positive}$$

(B-20)

$$|\zeta| \sum_{i=2k+1}^{\infty} \frac{(2i)!}{(i!)^2 4^i} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i \quad \text{for } \zeta \text{ negative}$$

Similarly for W_z^*

$$W_z^* = S_1 - \eta S_2 = \frac{1}{2[1+\eta^2]}\left[(D_1 - \eta D_2) + \zeta \frac{(E_1 - \eta E_2)}{\sqrt{1+\eta^2+\zeta^2}}\right]$$

$$= \frac{1}{4}\left[\frac{1-\eta^2+|1-\eta^2|}{|1-\eta^2|}\right] + \frac{\zeta}{2(1+\eta^2)}\left[\frac{E_1 - \eta E_2}{\sqrt{1+\eta^2+\zeta^2}}\right] \quad \text{(B-21)}$$

the left term is equal to 0 for $|\eta| > 1$

and to 1/2 for $|\eta| < 1$

Then it becomes :

$W_z^*(\eta,-\zeta) = -W_z^*(\eta,\zeta)$ for $|\eta| > 1$

$W_z^*(\eta,-\zeta) = 1-W_z^*(\eta,\zeta)$ for $|\eta| < 1$

for W_x^* we have $W_x^*(\eta,-\zeta) = W_x(\eta,\zeta)$

Convergence of the series.

The four series D_1, D_2, E_1 and E_2 converge for any value of η and ζ, except on the two lines $\eta=\pm 1$ which are the intersections of the cylinder with the plane (x, o, z).

The close forms for D_1 and D_2 indicates that these series go to infinity when $\eta^2=1$. For positive values of ζ there is discontinuity of the components W_y^* and W_z^* when the point M crosses the surface of the cylinder, explaining the divergence, but for the negative values of ζ there is no discontinuity of W_y^* and W_z^* but the four series still diverge.

The term (B-20) is bounded by $\sqrt{1+\eta^2+\zeta^2}$ and $2\sqrt{1+\eta^2+\zeta^2}$ for ζ positive, therefore S_1 converges like D_1 which diverges for $\eta=\pm 1$, When ζ is negative (B-20) is bounded by :

$$|\zeta| \frac{(4k+2)!}{[(2k+1)!]^2 4^{2k+1}} \sum_{i=2k+1}^{\infty} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^i$$

$$= |\zeta| \frac{(4k+2)!}{[(2k+1)!]^2 4^{2k+2}} \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)^{2k+1} \frac{1+\eta^2+\zeta^2}{\zeta^2}$$

S_1 is then bounded by a convergent geometric serie and always converge when ζ is negative.

Therefore, to have a rapid convergence for the numerical evaluation of S_1 and S_2, D_1 and E_1 (respectively D_2 and E_2) must not be evaluated separately or the closed forms of D_1 and E_2 must be used. Furthermore if one needs to evaluate the series for positive values of ζ, he has to evaluate them for ($\eta,-\zeta$) and use the transformations.

Evaluations of W_x, W_y, W_z at M (x,y,z) from W_x^*, W_y^*, W_z^*.

From the modelling of n concentrated vortices by a semi-infinite cylinder, we can consider that n segments of concentrated vortices of the length $\Delta\ell$ and strength Ω are spread over a surface $\Delta S = 2\pi\,\rho\,\Delta\ell\,\sin\lambda$ on the cylinder.

then :

$$\omega = \frac{n\,\Omega}{2\pi\,\rho\,\sin\lambda}$$

To obtain the components of the velocities induced at a point M (x,y,z) by the semi-infinite cylinder, one must evaluate $\eta = \sqrt{x^2+y^2}/\rho$ and $\zeta = z/\rho$, then compute the components for the point $M_o\left(\sqrt{x^2+y^2},\,0,\,z\right)$

$$W_{xo} = \frac{n\,\Omega}{2\pi\,\rho\,\mathrm{tg}\,\lambda}\,W_x^*(\eta,\zeta)$$

$$W_{yo} = \frac{n\,\Omega}{2\pi\,\rho}\,W_y^*(\eta,\zeta)$$

$$W_{zo} = \frac{n\,\Omega}{2\pi\,\rho\,\mathrm{tg}\,\lambda}\,W_z^*(\eta,\zeta)$$

then one has to make a change of variable :

$$W_x = (W_{xo} * X - W_{yo} * Y)/\sqrt{X^2+Y^2}$$

$$W_y = (W_{yo} * X + W_{xo} * Y)/\sqrt{X^2+Y^2}$$

$$W_z = W_{zo}$$

Particular values of the induced velocities

a) Velocities induced at points located on the plane limiting the cylinder ($\zeta = 0$).

For $\zeta = 0$ in [B-17], we get : $W_y^*(\eta, 0) = 0$ for $\eta<1$ and

$W_y^*(\eta, 0) = 1/\eta 2$ for $|\eta|>1$

Similarly, for $\zeta = 0$ in [B-17], we get : $W_z^*(\eta, 0) = 0$ for $\eta>1$

$W_z^*(\eta, 0) = .5$ for $\eta<1$

b) Velocities induced on the axis of the cylinder

$W_x^* = 0$

$W_y^* = 0$

from [B-2] $$W_z^* = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty}\frac{d\zeta' d\theta}{r^3} = \frac{1}{2}\int_0^{\infty}\frac{d\zeta'}{r^3}$$

with

$$r = \sqrt{1+(\zeta'-\zeta)^2}$$

after integration

$$W_z^* = \frac{1}{2}\left[1 + \frac{\zeta}{\sqrt{1+\zeta^2}}\right]$$

Numerical evaluation of the series

Due to the complexity of the terms involved in the calcualation of W_x^*, W_y^*, W_z^*, it is interesting, in order to limit the computer time, to evaluate the terms by recurrence. Using this method we limit the number of elementary operations to a minimum.

a) Calculation of W_x^*

From (B-17), W_x^* can be written

$$W_x^* = \frac{C}{2\sqrt{1+\eta^2+\zeta^2}} \sum_{k=0}^{\infty} T_k$$

with :
$$T_k = \frac{(4k+1)!}{(2k)!(k!)^2\, 4^{3k}(4k+4)} (C^2)^k$$

$$C = \frac{2\eta}{1+\eta^2+\zeta^2}$$

we get the recurrence relation ;

$$T_k = T_{k-1} + C^2\left(\frac{k^2-1/8}{k(k-1)}\right)$$

The first term is $T_0 = \frac{1}{4}$.

b) Calculation of S_1 and S_2 .

It is convenient to compute S_1 and S_2 together :

We first introduce the following notations :

$$U = \left(\frac{2\eta}{1+\eta^2}\right)$$

$$H' = \left(\frac{1+\eta^2}{1+\eta^2+\zeta^2}\right)$$

$$F = \frac{1}{2\sqrt{1+\eta^2+\zeta^2}}$$

$$T_n = \sqrt{1+\eta^2+\zeta^2} + \sum_{i=0}^{n} H_i$$

$$H_i = \zeta \frac{(2i)!}{(i!)^2 4^i} (H')^i$$

We can then rewritte S_1 and S_2 :

$$S_1 = \frac{F}{1+\eta^2} \sum_{k=0}^{\infty} C_{1k}$$

$$S_2 = \frac{F}{1+\eta^2} \sum_{k=0}^{\infty} C_{2k}$$

$$C_{1k} = \frac{(2k)!}{(k!)^2 4^k} U^{2k} T_{2k} = C_{1k}^* T_{2k}$$

$$C_{2k} = \frac{(2k+1)!}{(k!)^2 4^k (2k+2)} U^{2k} T_{2k+1} = C_{2k}^* T_{2k+1}$$

The calculation is initialized by :

$$T_0 = \sqrt{1+\eta^2+\zeta^2} + \zeta$$

$$T_1 = T_0 + \zeta \frac{H'}{2}$$

$$C_{10} = T_0$$

$$C_{20} = T_1/2$$

$$H_0 = \zeta$$

$$H_1 = \zeta \frac{H'}{2}$$

$$C_{10}^* = 1$$

The recurrence formulas are ;

$$H_n / H_{n-1} = \frac{2n-1}{2n} H'$$

$$T_{2k} = T_{2k-1} + H_{2k} = T_{2(k-1)+1} + H_{2k}$$

$$H_{2k} = H_{2(k-1)+1} * \left(\frac{k-1/4}{k}\right) H'$$

$$H_{2k+1} = H_{2k} * \left(\frac{k+1/4}{k+1/2}\right) H'$$

$$C^*_{1k} = C^*_{1(k-1)} * U^2 * \left(\frac{k-1/2}{k}\right)$$

$$C^*_{2k} = C^*_{1(k-1)} * \left(\frac{k+.5}{k+1}\right)$$

For each terms added to the series, C_{1k} is first recomputed from $C_{1(k-1)}$ then T_{2k} (used for S_1) is computed from $T_{2k-1} = T_{2(k-1)+1}$ (which was computed for S_2 in the last iteration) by adding $H_{2k} = H_n$ then T_{2k+1} used for S_2 can be computed from T_{2k} by adding $H_{2k+1} = H_n$

Appendix C

PROGRAM FWC

Generalities

The program FWC, which performs a free wake analysis of a wind turbine has been developed with the Time-Sharing-Option (TSO) of the IBM 370/168 system at MIT-IPC and with the IBM 360/65 batch system at MIT-LNS.

The program is written in FORTRAN IV, compiled with the IBM FORTRAN IV G level 21 compiler. It complies with the American National Standard (ANS) Fortran,X3.9-1966, except for the following extensions :

- NAMELIST statement
- ENTRY statement
- END and ERR parameters in I/O operations

The program has been linked with the IBM 128K design level F linkage editor (F128).

To be executed, the program needs approximately 158K bytes of core memory in a 360/370 System. Without the overlay structure, approximately 256K bytes are needed.

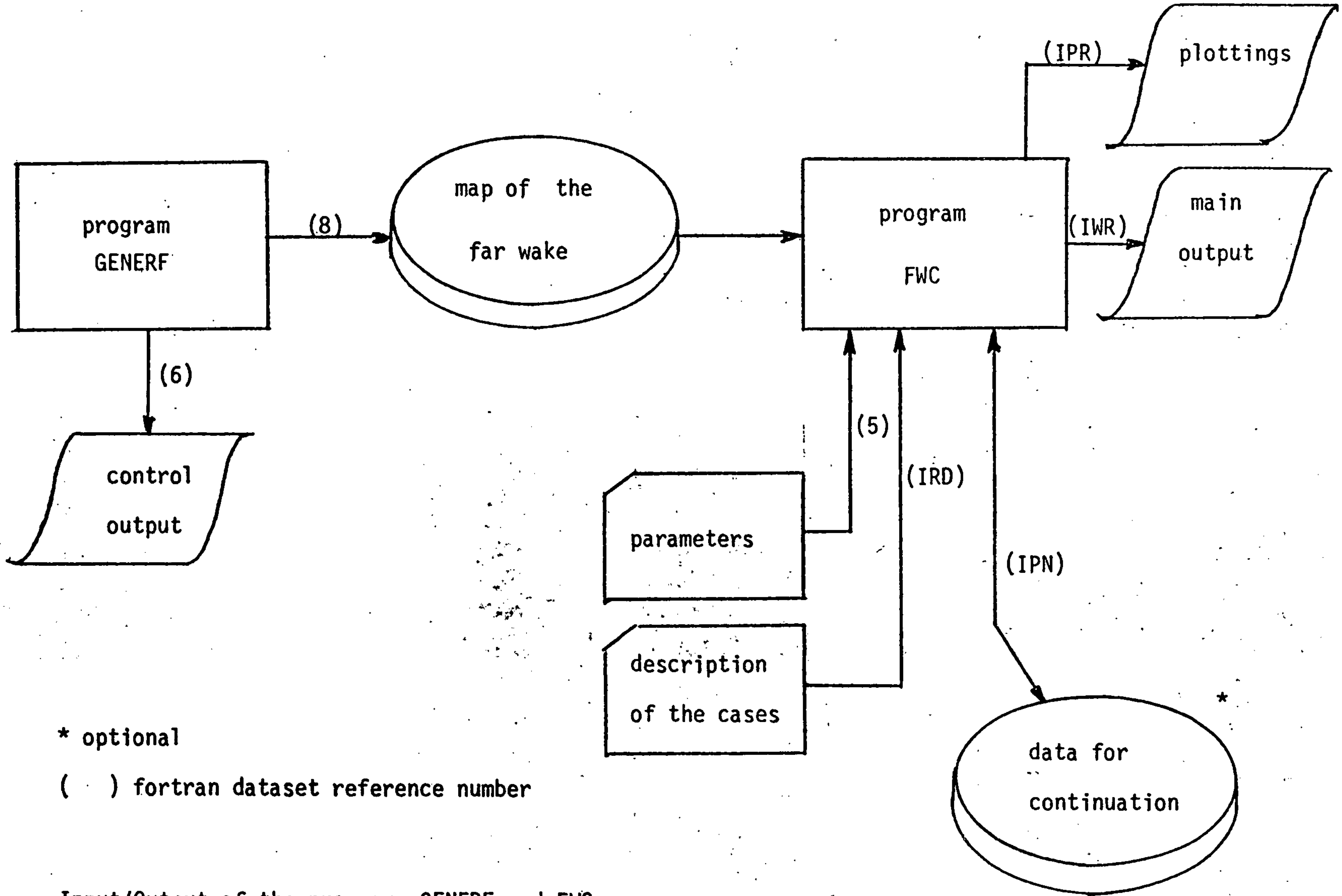
The program also includes an additional program (GENERF), for the generation of the map of the far wakes. GENERF has to be executed only once, since the map is written on a sequential file, which can be used any number of times.

The program can compute any number of cases, described sequentially in input, or continue on an unfinished case, from the point this case was discontinued by a preceding job, before computing other cases. This last option was necessary because of the limitations (CPU time and number of lines printed) at LNS.

Before performing any computation, the program reads parameters, which describe the datasets used, the levels of output desired, or command the optional executions of verification routines, and the optional continuation.

Because the user's input is read in using NAMELISTs, standard parameters are defined in the main program, and the user needs only to specify the parameters he wants to modify.

For the description of the cases, the user must describe completely, only the first case ; for all the others, he only needs to specify the data which differs from the preceding case.



* optional

( ) fortran dataset reference number

Input/Output of the programs GENERF and FWC

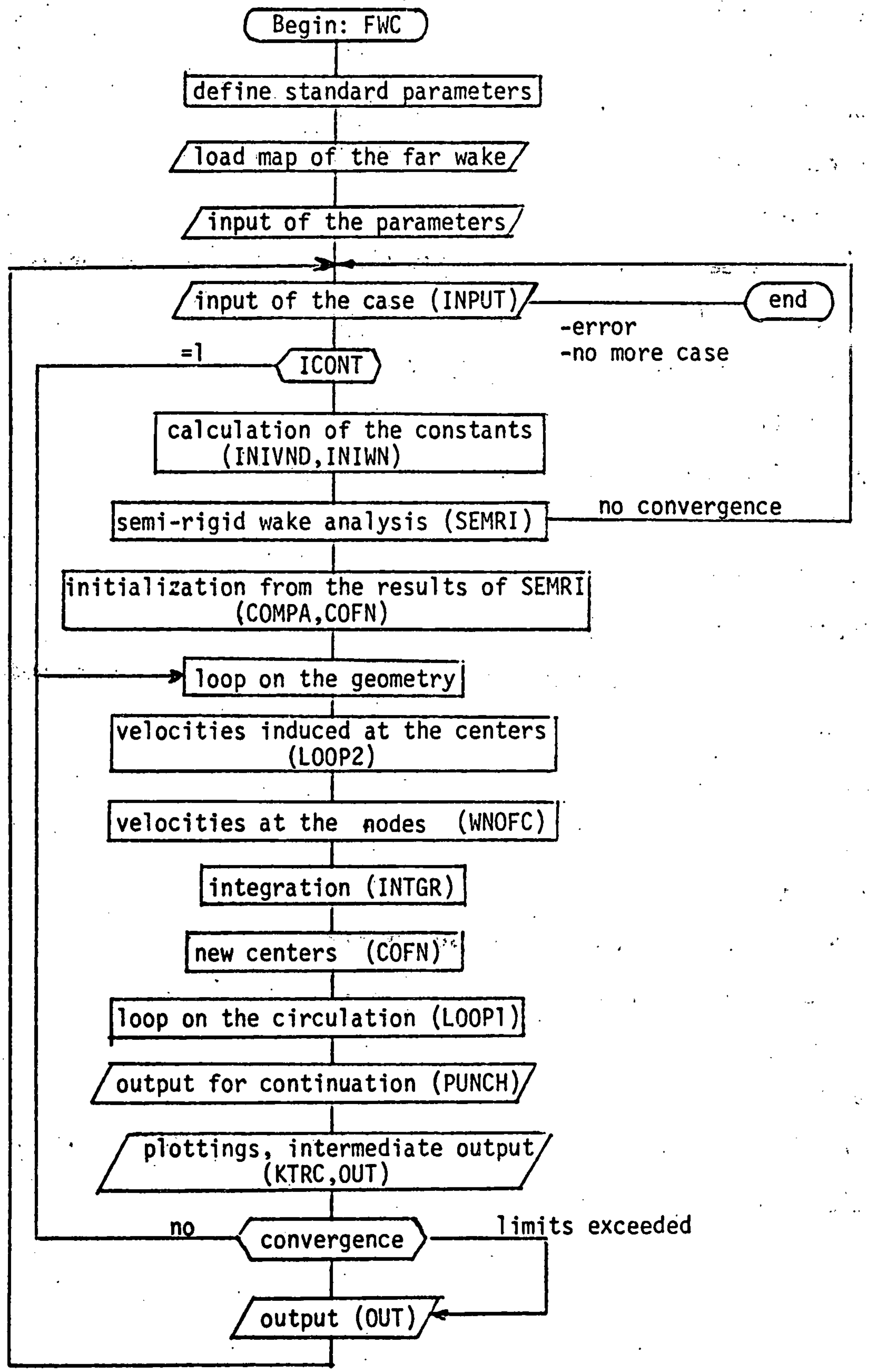
Notes :

- The input for the description of the cases can follow the parameters input, in that case, IRD=5.
- The plots and the main output can be directed to the same device (IWR=IRD).
- The input (parameters and description of the cases) comes, in general, from the card reader in a batch environment. They have been separated to allow the parameters and the cases to be described in different sequential files or from the terminal in a TSO environment.
- The main output and the plots are directed to a printer in a batch environment. In a TSO environment, the plots cannot be directed to the terminal, therefore they can be separated from the main output, since it is interesting to have it at the terminal.
- The map of the far wake and the data for continuation must reside on mass storage devices, since they are read and written without format.
- Any dataset reference number can be specifies for the I/O operations except 8, which is reserved for the map of the far wake.

Structure of the program FWC

The program is composed of a main program and fifteen subroutines (not included GENERF). The decomposition in subroutines is done on a functional basis and each module is independent from the others, and depends on this data structure reflected in the commons. The task of the main program (which size is kept to a minimum) is principely to command the execution of each module, it does not perform any calculation. The calling structure of the main program is reflected in the overlay structure. Each module may group a series of dependent or similar functions, each associated in general to a different entry point, all arguments with a few exceptions are transmitted by commons.

General flow chart of FWC



Overlay structure

All subroutines can be overlaid once exited, except :

- INDVEL between the call to the entry COORD and all the corresponding calls to the entry WXYZ.

- TRACE, between the call to the entry TRCINI and all the following calls, up to the final call to the entry TRCEXE.

The commons cannot be overlaid, except those holding work areas for some subroutines :

- WNDATA (used by WNOFC).
- WDATA (used by INTGR).
- OTDATA (used by OUT).
- TRCDA (used by KTRC).

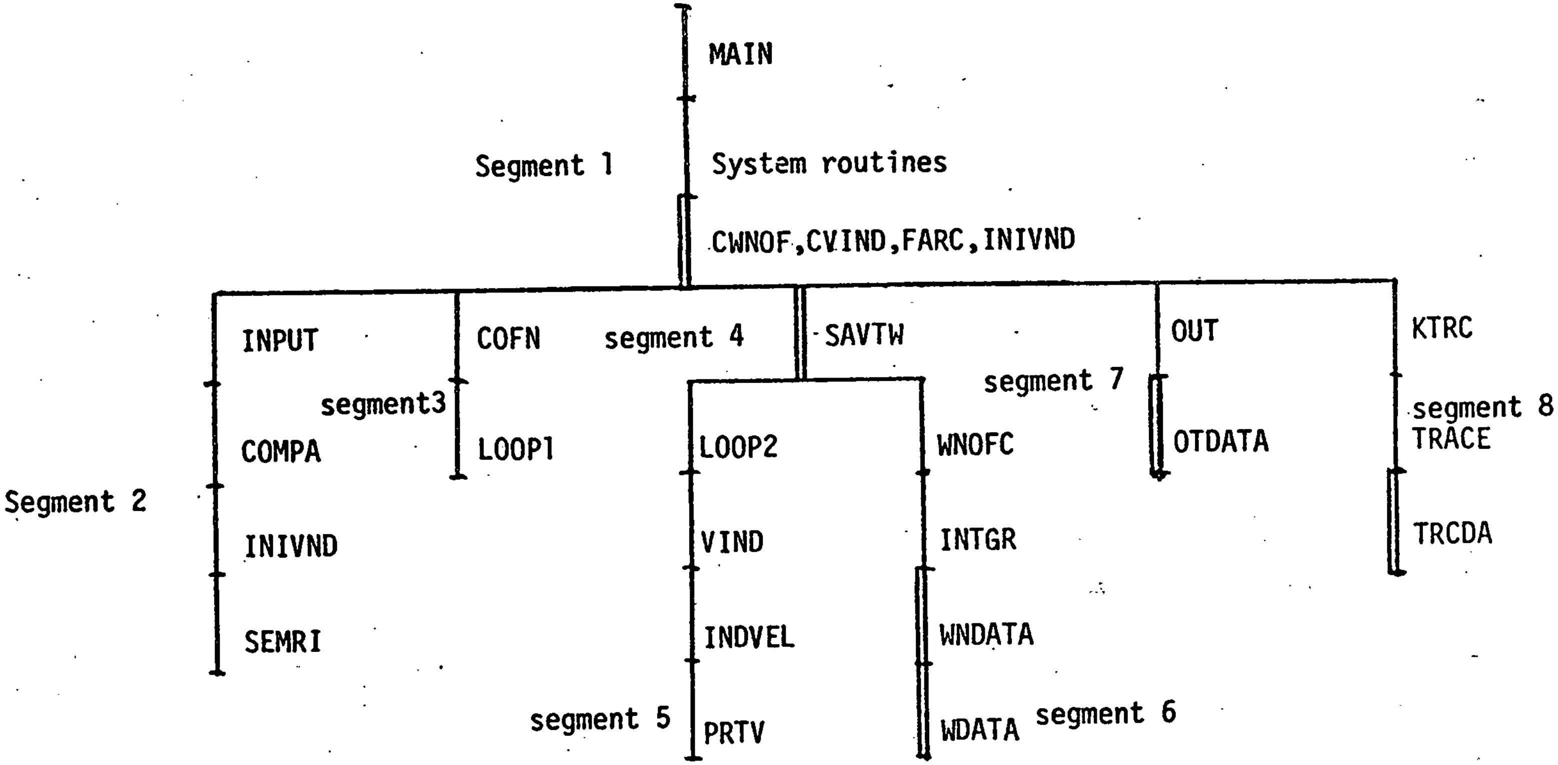
If the overlay structure is not used, all these commons can be renamed under an identical name.

Because they are used only in overlaid subroutines, but must be kept during all the execution, the following commons must be inserted explecitely in the root segment :

- CWNOF (precomputed constants used by WNOFC).
- CVIND (precomputed constants, created by INIVND, used by LOOP2 and VIND).
- VORT

The common SAVTW cannot be overlaid by any segment between the call to LOOP2 and the call to WNOFC.

OVERLAY STRUCTURE



F128 level linkage editor input.

Note :

Each subroutine is supposed to reside in a library, under a member name identical to the subroutine name.

```
ENTRY MAIN
INCLUDE SYSLMOD(MAIN)
INSERT CWNOF,CVIND,FARC,VORT
OVERLAY N1
INCLUDE SYSLMOD(INPUT,COMPA,INIVND,SEMRI)
OVERLAY N1
INCLUDE SYSMOLD(COFN,LOOP1)
OVERLAY N1
INSERT SAVTW
OVERLAY N2
INCLUDE SYSLMOD(LOOP2,VIND,INDVEL,PRTV)
OVERLAY N2
INCLUDE SYSLMOD(WNOFC,INTGR)
OVERLAY N1
INCLUDE SYSLMOD(OUT)
OVERLAY N1
INCLUDE SYSLMOD(KTRC,TRACE)
NAME FWC
```

List of subroutines.

- in parenthesis : entry points
- underlined : entries not used

MAIN

INPUT (PUNCH)

SEMRI

COMPA

INIVND

COFN

LOOP2

VIND (VINDB, VINDN, VINDI, EXECN, INICN)

INDVEL (COORD, WXYZ)

PRTV (PRTCEN, PRTNOD, PRTVD, CLV, PRTVN, PRTVI)

LOOP1

WNOFC (INIWN)

INTGR

OUT (OUTINT)

KTRC (TRCPT, TRCN, TRCVEL, TRCW, TRCT)

TRACE (TRCINI, TRCPTS, TRCLIN, TRCSEG, TRCCUB, TRCEXE)

Independent program :

GENERF

Description of the user's input

All input variables types follow the FORTRAN convention (real single precision A-H, O-Z integer single precision I-N).

A) Parameters

The parameters are described in the NAMELIST NPARM, this NAMELIST is read from the input dataset number 5, it must appear only once.

(in parenthesis, value given by default in the main program)

- IWR(6) : main output dataset reference number, must specify a printer or a TSO terminal.
- IRD(5) : description of the cases dataset reference number
- IPN(9) : input/output dataset reference number, used for the continuation option.
  (must specify a sequential file on disk or tape and must not be 8).
- IPR(6) : output dataset reference number for the plots ; it must specify a printer.
- ITRACE(0) : output level
  Range : 0 or 1
  0 : no output
  1 : the following are printed on (IWR)
  - The parameters (in MAIN)
  - The description of the current case (in INPUT)
  - The coefficients used for the calculation of the strength of the elements (in INIVND)
  - The interpolation coefficients for WNOFC (in INIWN)

- The values WZM and R3 (in COMPA)
- The positions of the nodes generated by COMPA (in MAIN)
- The return code of LOOP1 for each iteration (in LOOP1)
- A message each time the intermediate results are written on (IPN)

- LPSEM(1) : output level for the semi-rigid wake analysis (subroutine SEMRI)

  Range : 0 to 2

  0 : no output

  1 : output of the final results of SEMRI, includes : induced velocities and circulation distributions along the blade, forces and performance coefficients.

  2 : same as LPSEM=1, but the induced velocities and the circulation along the blades are printed for each iteration. (This option may be useful if convergence problems occur in this subroutine).

- NMES : not used

- ITRCL1(0) : output level for loop on the circulation (LOOP1)

  Range : 0 or 1

  0 : on output

  1 : for each iteration are printed :

  - The initial guess for the induced velocities at the blade level, the corresponding distribution of circulation.

- The final results and the return code of LOOP1.

This parameter is useful to examine the convergence of the circulation.

- ITRCL2(0) : output level for the subroutine LOOP2

  Range : 0 to 4

  0 : no output

  1 : the final velocities induced at the centers are printed

  2 : the induced velocities at the centers are printed after the contribution of each type of elements. This parameter is reset to 1 after the first call to LOOP2

  3 : same as ITRCL2=2, but the arrays containing the induced velocities are cleared after each output, so that the output represents exactly the contribution of each type of element. This parameter is then rest to 1 and the calculation of the induced velocities is reexecuted.

  4 : same as ITRCL2=2 but the parameter is not reset to 1

- ITRCG(0) : output level : strength of the elements for each iteration (in LOOP2)

  Range : 0 or 1

  0 : no output

  1 : the distribution of circulation and the strength of the elements are printed for each iteration of LOOP2

- ITRCNT(0) : control of the influence coefficients (see LOOP2 and VIND)

  Range : 0 or 1

0 : no control performed

1 : a simple verification of the calculation of the influence coefficients is performed ; the velocities induced at the blade level are computed by two different methods and then printed.

This parameter is reset to 0 after the first call to LOOP2.

- ITRCF(0) : control of the far wake modal (performed in LOOP2)

  Range : 0 or 1

  0 : no control

  1 : the velocities induced by the far wake using the semi-infinite cylinder are printed, then these velocities are recomputed, using line vortices and then printed.

- ITRCW(0) : output level for WNOFC

  Range : 0 to 2

  0 : no output

  1 : the velocities induced at the nodes and the influence coefficients after interpolation are printed.

  2 : same as ITRCW=1 but the velocities induced at the nodes are printed after each step of WNOFC.

- ITRTG(0) : output level for the integration (in INTRG)

  Range : 0 or 1

  0 : no output

  1 : for each azimuthal position, the result of the integration are printed, including the corrections.

- IVERGR(0) : control of the integration (in INTRG)

  Range : 0 or 1

  0 : no control performed

  1 : a simple verification of the integration is performed by integrating and printing the results twice, the results have to be identical.

- ICGEN(0) : output level, loop on the geometry

  Range : 0 to 2

  0 : no output after each iteration

  1 : output of the induced velocities on the blades, of the circulation, forces and performance coefficients for each iteration.

  2 : same as ICGEN=1, but the positions of the nodes are included.

- JPUNCH(0) : output of the results after each iteration, on mass storage (IPN)

  Range : 0 or 1

  0 : no output

  1 : output for each iteration

  This parameter must be set to 1 if the user wants to use the continuation option. In that case, a dataset must be supplied to the program.

- ICONT(0) : continuation option

  Range 0 or 1

  0 : no continuation

1 : the program will continue on the unfinished case at the point a precedent job left it. In that case valid data must be present in a file, referenced by (IPN), all the other parameters are ignored.

- IPLOT(0) : output level : plotting of the positions of the nodes in (R - Z) coordinates.

  Range : 0 to 2

  0 : no plotting

  1 : plotting at the end of the case

  2 : plotting for each iteration on the geometry

- IPLOTV(0) : output level : plotting of the induced velocities at the centers.

  Range : 0 to 2

  0 : no plotting

  1 : plotting at the end of the case

  2 : plotting for each iteration on the geometry

- IPLOTW(0) : output level :plotting of a side view of the wake

  Range : 0 to 2

  0 : no plotting

  1 : plotting at the end of the case

  2 : plotting for each iteration on the geometry

- ISW(1) : size parameter used for the plotting of a side view of the wake

  Range : 1 or 2

  1 : the plotting is done on a single width of listing

  2 : the plotting is done on a double width of listing

- IPLOTG(0) : output level : plotting of the distribution of circulation along the blade and of the strength of the elements.

  Range : 0 to 2

  0 : no plotting

  1 : plotting at the end of the case

  2 : plotting for each iteration on the geometry

- IPLOTT(0) : output level : plotting of the tip vortex versus the azimuth (first two spires, axial position and distance from the Z axis)

  Range : 0 to 2

  0 : no plotting

  1 : plotting at the end of the case

  2 : plotting for each iteration on the geometry

- ITRANS(1) : transition wake parameter

  Range : 0 or 1

  0 : no transition wake used

  1 : a transition wake is used

  In general, this parameter should not be modified.

  It can be set to 0 if the elements of the near wake and the elements of the intermediate wake have approximatly the same size.

Description of the cases

Each case is described in a namelist CASE.

The cases are read from the input dataset referenced by (IRD), if this parameter has not been modified in the parameters list, the namelist CASE follows immediately the namelist NPARM.

All variables are realsingle precision (4 bytes) or integer single precision (4 bytes).

In parenthesis : initial value given to the variable, when applicable.

- NBLDS1 : number of blades
  Type : integer
  Maximum value : 8
- LTWIST : twist parameter
  Type : integer
  Range : 0 or 1
  0 : the blades ara not twisted, the pitch angle is THETOD
  1 : the blades are twisted, the pitch angle distribution is given by THETAD
- SIGMA : rotor solidity (blade area / rotor area)
  Type : real
- KNNVR : number of nodes (blade and near wake)
  Type : integer
  Maximum value : 15
- NNVA : not used

- NTVA : number of nodes (azimuthal direction) for the near wake, transition wake not included.

  Type : integer

  Maximum value : NTVA+2*(DPSIID/DPSIND) must be smaller or equal to 18

- DPSIND : azimuthal size of the elements of the near wake (in degrees)

  Type : real

- ETAN : radial positionsof the nodes (blade and near wake)

  Type : array of reals, one dimension.

  They must be exactly KNNVR positions defined, in increasing values, the first position (ETAN(1)) correspondsto the root of the blade, the last position (ETAN(KNNVR)) corresponds to the tip of the blade and must be 1.

- KNIVR : same as KNNVR, but for the intermediate wake

  Maximum value : 6

- NIVA : same as NTVA, but for the intermediate wake, the transition wake is included.

  Maximum value : 50

- DPSIID : same as DPSIND, but for the intermediate wake

- ETAI : same as ETAN, but for the intermediate wake

  ETAI(1) must be equal to ETAN(1), and ETAI(KNIVR) must be equal to 1.

- THETOD : pitch angle (in degrees)

  Type : real

  This value must be defined only for untwisted blades (LTWIST=0), it is ignored when the blades are twisted

- FMU : advance ratio (inverse of the tip speed ratio)

  Type : real

- THETAD : pitch angle distribution (in degrees)

  Type : array of reals, one dimension

  THETAD (I) is the pitch angle at ETAN (I) ; there must be KNNVR elements defined. The array must be defined only for twisted blades (LTWIST=1), it is ignored when the blades are untwisted.

- ALPHAS : stall angle (in radians)

  Type : real

- CDO, CDK : definition of the drag coefficient

  Type : reals

  The drag coefficient C_d will be :

  C_d = CDO + α^2 CDK if the blade section is not stalled

  C_d = 2 CDO + α^2 CDK if the blade section is stalled

  (α : attack angle in radians)

- LIMS(30) : maximum number of iterations for the semi-rigid wake analysis routine (SEMRI)

  Type : integer

  If the convergence is not obtained within LIMS iterations, the case is terminated

- LIM1(20) : maximum number of iterations for the circulation loop (in LOOP1)

  Type : integer

  If convergence is not obtained within LIM1 iterations, the case is terminated

- LIM2(15) : maximum number of iterations for the loop on the geometry

  Type : integer

  If convergence is not obtained within LIM2 iterations, the case is terminated

- FPS1 : core size for the rectangular elements

  Type : real

  The core size is expressed in term of the radius of the rotor

- FPS2 : core size for the segment elements

  Type : real

  The core size is expressed in term of the radius of the rotor

- COEFF1(.5) : tip and root vortices coefficient

  Type : real

  The strength of the vortex is taken as COEFF1 times the circulation at the nearest center of the blade.

The following parameters, which appear in NPARM, can also be defined for the current case : IPLOT, IPLOTV, ITRANS, JPUNCH, IPLOTW, ISW, IPLOTG, IPLOTT.

Examples of input.

a) No modification of the parameters, 2 cases, the second differs from the first only by the advance ratio.

```
&NPARM &END
&CASE NBLDS1=2,SIGMA=.1061,
KNNVR=4,ETAI=.2,.3,.5,.8,.9,1.,NTVA=8,DPSIND=10
KNIVR=4,ETAI=.2,.4,.9,1.,NIVA=20,DPSIID=20
FMU=.1,ALPHAS=.2,CDO=.01,CDK=.5,
FPS1=.01,FPS2=.02, &END
&CASE FMU=.222, &END
```

b) continuation of the last case only :

```
&NPARM ICONT=1 &END
```

Description of the user's output

The user's output is composed of two parts :

a) Distribution of the quantities relative to the blade followed by the performance coefficients.

b) The coordinates of the nodes and the velocities induced at these points.

The user may have also intermediate results by specifying some parameters as input (Refer to subscetion " Description of the user's input ")

In part a) are included :

1 - The description of the case :

- the number of blades
- the rotor solidity (SIGMA)
- the mean pitch angle (THETAO)
- the advance ratio (MU)
- the drag coefficient formulation (CD)
- the stall angle

2 - For each blade section (refered by the position of its center ETA)

- the radial induced velocity component on the blade (WXC)
- the tangential induced velocity component on the blade (WYC)
- the axial induced velocity component on the blade (WZC)
- the total velocity relative to the blade (U)
- the circulation (GMC)
- the inflow angle (LAM) in radians
- the attack angle (ALP) in radians

- the pitch angle (THE) in radians
- the lift (LIP)
- the thrust component of the lift (TLIP)
- the in-plane component of the lift (FLIP)
- the drag (DP)
- the thrust component of the drag (TDP)
- the in-plane component of the drag (FDP)
- the thrust (TP)
- the in-plane force (FP)

3 - The performance coefficients

- $C_t = T/(\rho\pi R^2V^2)$
- $C_p = P/(\rho\pi R^2V^3)$
- $L_t = T/(\rho\pi\Omega^2R^4)$
- $L_p = P/(\rho\pi\Omega^3R^5)$

where : - T is the thrust
- P is the power extracted
- ρ is the air density
- R is the radius of the rotor
- V is the free stream velocity
- Ω is the rotational speed ot the rotor

Note : C_p and L_p are negative when power is extracted from the wind

Part b) : geometry of the wake

For each wake (near wake and intermediate wake) are printed :

1 - The distribution of the nodes on the blade, from which the streamlines originate.

2 - For each azimuthal position :

- the coordinates of the nodes (XNV,YNV,ZNV for the near wake, XIV,YIV,ZIV for the intermediate wake)
- the components of the induced velocities, in cylindrical coordinates (WRNV,WTNV,WZNV for the near wake, WRIV,WTIV,WZIV for the intermediate wake).
- the azimuth position of the nodes in degrees (PSI)
- the radial position of the nodes (R)
- the components of the induced velocities, in rectangular coordinates (WX* and WY*)

List of commons

In parenthesis : subroutines where the common is used.

PARM	(MAIN, INPUT, COMPA, INIVND, SEMRI, LOOP1, LOOP2, VIND INDVEL, PRTV, WNOFC, INTGR, OUT, KTRC)
GEOM	(MAIN, INPUT, COMPA, INIVND, SEMRI, COFN, LOOP1, LOOP2, VIND, PRTV, WNOFC, INTRG, OUT, KTRC)
FARC	(MAIN, INDVEL)
VEL	(INPUT, LOOP2, VIND, INDVEL)
RESUL	(INPUT, COMPA, SEMRI, COFN, LOOP1, LOOP2, VIND, PRTV, WNOFC, INTGR, OUT, KTRC)
CVIND	(INPUT, INIVND, LOOP2, VIND)
CWNOF	(INPUT, WNOFC)
VORT	(LOOP2, VIND, KTRC)
SAVTW	(LOOP2, WNOFC)
WNDATA	(WNOFC)
WDATA	(INTGR)
OTDATA	(OUT)
TRCDA	(KTRC)

List of variables in common

Common PARM

This common contains the parameters (see description of user's input), as read by the main program, they are not modified during the execution, except : IVERGR,ITRCL2,ITRCF, ITRCNT,ICONT.

The common contains also :

- NITER : current loop index for the loop on the geometry.
- ISAMEB,ISAMET : test values : comparison with the preceding case

  Type : integers

  Values : 0 or 1

  If the rotor described in the current case has the same number of blades as the rotor of the preceding case, then ISAMEB is set to 1.

  In the same way, if the distribution of the nodes, for the near wake and the intermediate wake, and COEFF are not modified, then ISAMET is set to 1.

  When one of thes values is equal to 1, then the calculation of the constants which depend on the number of blades or on the distribution of the control points, can be eliminated.
- ITEST : return code

  Type : integer

  This return code is set by SEMRI and LOOP1

Range : 0 or 1 (SEMRI), 0 to 2 (LOOP1)

0 : convergence is reached in SEMRI

1 : convergence is reached in SEMRI

0 : convergence is reached for the case

1 : convergence is not reached for the case, but the loop on the circulation has converged (LIM1 iterations)

2 : the loop on the circulation has not converged within LIM1 iterations.

Common FARC

This common holds the map of the far wake (velocities induced by a semi-infinite cylinder).

All variables are reals single precision.

- ETA(40) : single dimension array : positions of the points which appear in the map.
- WX(22,40) : double dimension array : X components of the velocities induced by the semi-infinite cylinder.
  WX(I,J) is $W_x^* / 2\pi$ for η = ETA(I) and ζ = ETA(J)
- WY(22,40) : same as WX, for Y component
- WZ(22,40) : same as WX, for Z component

Common GEOM

Description of the current case

- NBLDS1 : number of blades, as read by INPUT

  Type : integer

- NBLDS : number of blades

  Type : integer

- SIGMA : rotor solidity (σ)

  Type : real

- FMU : advance ratio (μ)

  Type : real

- ETAN(15) : distribution of nodes in radial direction (blade and near wake), as read by INPUT

  Type : array of reals, one dimension

- KNNVR : number of nodes in radial direction (blade and near wake), as read by INPUT

  Type : integer

- ETAI(6) : distribution of nodes in radial direction (intermediate wake), as read by INPUT

  Type : array of reals, one dimension

- KNIVR : number of nodes in radial direction (intermediate wake), as read by INPUT

  Type : integer

- LTWIST : twist parameter

  Type : integer

  Range : 0 or 1

0 : blades are not twisted

1 : blades are twisted

- THETAD(15) : distribution of pitch angle (in degrees)

  THETAD(15) is the pitch angle at ETA(I)

  Type : array of reals, one dimension

- THETA(15) : same as THETAD, but in radians

  Type : array of reals, one dimension

- THETAC(14) : same as THETA, but for the centers 2 to NNCR2 of the blade

  Type : array of reals, one dimension

  (THETA(I) is the pitch angle at the position ETANC(I+1))

- THETAO : pitch angle for an untwisted blade (in radians)

  Type : real

- THETOD : pitch angle for an untwisted blade (in degrees)

  Type : real

- ALPHAS : stall angle (in radians)

  Type : real

- CDO,CDK : definition of the drag coefficient

  Type : reals

- DPSIND : azimuthal size of the elements of the near wake (in degrees)

  Type : real

- DPSIN : same as DPSIND, but in radians

  Type : real

- DPSIID : azimuthal size of the elements of the intermediate wake (in radians)

Type : real

- DPSII : same as DPSIID, but in radians
- COEFF1 : tip and root vortices coefficient, as read by INPUT

  Type : real
- COEFF : tip and root vortices coefficient

  Type : real
- C : cosine of (2 π / number of blades)
- S : sine of (2 π / number of blades)
- BLADES : number of blades

  Type : real
- NNVR : number of nodes in radial direction for the blade and the near wake

  Type : integer
- NNVA : number of nodes in azimuthal direction for the near wake

  Type : integer
- NNVR1 : equal to NNVR-1, it is the number of elements of the near wake in radial direction
- NNVA1 : equal to NNVA-1, it is the number of element of the near wake in azimuthal direction
- NNCR : number of centers in radial direction, for the blade and the near wake (it is equal to NNVR+1)
- NNCA : number of centers in azimuthal direction, for the near wake (it is equal to NNVA+1)
- ETANV(15) : radial positions of the nodes (same as ETANV)
- ETANC(16) : radial positions of the centers for the blade and the near wake.

- NNCR1 : equal to NNCR-1, equivalenced to NNVR
- NNCA : equal to NNCA-1, equivalenced to NNVA
- NNCR2 : equal to NNCR-2, equivalenced to NNVR1

  It is the number of centers located between ETANV(1) and ETANV(NNVR)

The following variables have the same meanings and types as the variables which names are obtained by replacing I by N, but apply to the intermediate wake : NIVR, NIVA, NIVR1, NICR, NICA, ETANV(6), ETAIC(7), NICR1, NICA1, NICR2.

- NTVA : number of nodes in azimuthal direction for the near wake, transition wake not included.
- NTVA1 : equal to NTVA-1, it is the number of elements of the near wake, transition wake not included.
- NTCA : number of centers in azimuthal direction for the near wake, transition wake not included.
- FPS1 : core size for the rectangular elements
- FPS2 : core size for the segment elements

Common RESUL

This common contains all the arrays of induced velocities, of positions of the points of the near and intermediate wake, and of influence coefficients, and the actual distribution of circulation.

All arrays contain single precision real variables.

- GAMMC(16)	:	distribution of circulation along the blade GAMMC(I) is the circulation at the center ETANC(I) of the blade. GAMMC(1) and GAMMC(NNCR) are set to 0.
- TWX(14,14)	:	influence coefficients, X components. TWX(I,J) is X component of the velocities induced at the center ETANC(I+1) of the blade, assuming a unit circulation at the center ETANC(J+1) of the blade only.
- TWY(14,14)	:	same as TWX, Y components
- TWZ(14,14)	:	same as TWX, Z components
- WXC(14)	:	velocities induced on the blade, X component WXC(I) is the velocity induced at the center ETANC(I+1) of the blade
- WYC(14)	:	same as WXC, Y components
- WZC(14)	:	same as WXC, Z components
- WXNC(16,19)	:	velocities induced at the centers of the near wake, WXNC(I,J) is the velocity induced at the center of coordinates : XNC(I,J), YNC(I,J), ZNC(I,J)
- WYNC(16,19)	:	same as WXNC, Y components
- WZNC(16,19)	:	same as WXNC, Z components
- WXIC(7,51)	:	same as WXNC, but for the intermediate wake WXIC(I,J) is the X component of the velocity induced at the center of coordinates XIC(I,J), YIC(I,J),ZIC(I,J)
- WYIC(7,51)	:	same as WXIC, Y components
- WZIC(7,51)	:	same as WXIC, Z components

- WRNV(15,18) : radial components of the velocities induced at the nodes of the near wake.
  WRNV(I,J) is the radial component of the velocity induced at the node (I,J) of the near wake.
- WTNV(15,18) : same as WRNV, tangential components
- WZNV(15,18) : same as WRNV, axial components
- WRIV(6,50) : same as WRNV, intermediate wake
- WTIV(6,50) : same as WTNV, intermediate wake
- WZIV(6,50) : same as WZNV, intermediate wake
- XNC(16,19),
  YNC(16,19),
  ZNC(16,19) : rectangular coordinates of the centers of the near wake
  XNC(I,J) is the X coordinate of the center located on the streamline originating from ETANC(I), at the azimutal position (J).of the centers
- XIC(7,51),
  YIC(7,51),
  ZIC(7,51) : same as XNC, YNC, ZNC, for the intermediate wake
- XNV(15,18),
  YNV(15,18),
  ZNV(15,18) : rectangular coordinates of the nodes of the near wake
  XNC(I,J) is the X coordinate of the center located on the streamline originating from ETANV(I), at the azimutal position (J) of the nodes.
- XIV(6,50),
  YIV(6,50),
  ZIV(6,50) : same as XNV, YNV, ZNV, for the intermediate wake

Common VORT

Strengths of the elements

- GM1N(14) : first lumped stengths of the elements of the near wake
  GM1N(I) refers to the I^{th} element
- GM2N(14) : second lumped strengths of the elements of the near wake
- GM1I(5) : same as GM1N, for the intermediate wake
- GM2I(5) : same as GM2N, for the intermediate wake
- GAMMT1 : strength of the root vortex of the near wake
- GAMMT2 : strength of the tip vortex of the near wake
- GAMMT3 : strength of the root vortex of the intermediate wake
- GAMMT4 : strength of the tip vortex of the intermediate wake

Common CVIND

Coefficients for the determination of the strengths of the elements. These coefficients depend only on the distributions of the control points for the near wake and the intermediate wake, and on the factor COEFF

- HM1N(14,14) : coefficients for GM1N; by definition:
  GM1N(I) = ∑ HM1N(I,J) * GAMMC(J+1)
- HM2N(14,14) : coefficients for GM2N
- HM1I(5,14) : coefficients for GM1I
- HM2I(5,14) : coefficients for GM2I
- HAMMT1(14) : coefficients for GAMMT1
- HAMMT2(14) : coefficients for GAMMT2
- HAMMT3(14) : coefficients for GAMMT3
- HAMMT4(14) : coefficients for GAMMT4

This common contains the arguments for INDVEL (calculation of the velocity induced at M by the current element)

- X,Y,Z : used by WXYZ: coordinates of the point M, where the induced velocity is to be evaluated
- UX,UY,UZ : returned by WXYZ: components of the velocity induced at M (X, Y, Z) by the current element.
- X1,Y1,Z1 : used by COORD: coordinates of the first point defining a rectangular or a segment element.
- X2,Y2,Z2 : used by COORD: coordinates of the second point defining a rectangular element
- X3,Y3,Z3 : used by COORD: coordinates of the third point defining a rectangular element, or coordinates of the second point defining a segment element.
- X4,Y4,Z4 : used by COORD: coordinates of the fourth point defining a rectangular element.
- GM : returned by COORD, for rectangular elements: constant strength component of the element.
  used by WXYZ: constant strength component for a rectangular element, strength of the element for a segment element or a semi-infinite cylinder element
- GM1 : used by COORD, for a rectangular element: first lumped strength of the element
- GM2 : used by COORD, for a rectangular element: second lumped strength of the element.

- DGM : returned by COORD, and used by WXYZ: variable strength component of a rectangular element.
- IT : used by COORD and WXYZ: indicates the type of the current element.

  Type: integer single precision

  IT = 1 : the current element is a rectangular element

  IT = 2 : the cuurent element is a segment element

  IT = 3 : the cuurent element is a semi-infinite cylinder

  Note: only the values 1 or 2 may be used for a call to COORD
- RHO : used by WXYZ, for a semi-infinite cylinder element: radius of the cylinder
- ZPF : used by WXYZ, for a semi-infinite cylinder element: axial position of the plane limiting the element
- TL : used by WXYZ, for a semi-infinite cylinder element: tangent λ of the element.
- EPS1 : used by COORD, for a rectangular element: thickness of the element
- EPS2 : used by COORD, for a segment element: core radius of the element.
- STRE : not used

Common SAVTW

SAVTW contains a copy of the influence coefficients arrays TWY and TWZ of the precedent iteration. The copy is made by LOOP2 and used by WNOFC.

- TWYT(14,14) : copy of TWY
- TWZT(14,14) : copy of TWZ

Main program

Function : the main program does not perform any calculation it controls the excecution of the different steps of the calculation, by sequentially and conditionally calling the subroutines. It also defines the standard parameters, loads the map of the far wake, prints messages relative to the convergence.

Input : (directly by the main program) : map of the far wake, from unit (8)

Output : (directly by the main program) : on (IWR) : messages

Parameters used : ITRACE,ICGEN,ICONT,ITEST,ITRCW,ITRCL2,JPUNCH,IPLOT, IPLOTV,IPLOTW,IPLOTG,IPLOTT.

Subroutines called :
- INPUT(PUNCH)
- INIVND
- SEMRI
- COMPA
- COFN
- PRTV(PRTNOD)
- KTRC(TRCPT,TRCN,TRCG,TRCVEL,TRCW,TRCT)
- OUT(OUTINT)
- LOOP2
- LOOP1
- INTGR
- WNOFC(INIWN)

Notes : - In case of continuation : the parameter ICONT contained in common PARM and read at the beginning, must be saved (in JCONT), as it will be redefined immediatly after (in INPUT), it must also be cleared after its action (ICONT = 1) commands the skipping of the initialization section and the redifinition of the loop variable (JTER).

- The end of execution is performed by INPUT

Local variables :

(All variables are integers single precision)

- JCONT : saved value of ICONT
- NCAS : current number of cases processed during the job
- JTER : loop variable (loop on the geometry)

  1 : for a new case

  Set to NITER + 1, case of continuation, (NITER the number of iterations already done)

Subroutine INPUT

Function : - Input of the current case, verification of the data, calculation of the constants used by more than one subroutine, end of execution.

- Input/Output for continuation.

Entries : - INPUT

- PUNCH : output for continuation.

Arguments : all arguments are transmitted by the commons except NCAS : actual case number.

Parameters used : ITRACE, ICONT, ITRANS, current case, input performed.

Input : on (IWR) : - error messages

- description of the current case

on (IPN) : - content of commons PARM, GEOM, CVIND, CWNOF, RESUL, for continuation (entry PUNCH)

Local variables : - CONV : convertion factor degrees to radians

- TWOPI : value of 2π

- I : loop variable

Entry INPUT : Two cases occur :

- With ICONT = 1 in the parameter list, the user has specified that he wants to continue on a case left unfinished by a precedent Job. The content of the commons PARM, GEOM, CVIND, CWNOF, RESUL, are read from (IPN). The validity of the data is not verified.

- in the normal case, the description of the case is read using namelist CASE from the dataset referenced by (IRD). If no data is present (no more case submitted to the program) and End-of-file is encountered and the program is stopped.

To avoid the recalculation of the constants, the current case is compared to the preceding one ; the result of the comparison is reflected in the variable ISAMEB and ISAMET.

ISAMEB is set to 1 if the number of blades is identical.

ISAMET is set to 1 if the number of radial position , their distribution and the tip and root vortices factor (COEFF) are identical, and then the constants contained in the common (GEOM) will not be recomputed. The validity of the data is checked (limitations because of the actual size of the arrays, order of the radial positions, etc...).

If the parameter ITRACE is different from zero, the description of the current case is printed on (IWR).

Effect of an error :

- End-of-file on (IRD) : termination of the program
- Read error on (IRD) : message, termination of the program
- End-of-file on (IPN) : message, termination of the program
- Read error on (IPN) : message, termination of the program
- Invalid data : message is printed, output of the invalid data as it was specified by the user, termination of the program.

Entry PUNCH :

The content of the commons PARM,GEOM,CVIND,CWNOF,RESUL, are written on (IPN). For this, the commons PARM and GEOM are declared in equivalence with two arrays (TPARM and TGEOM) with identical size. Each of the commons CVIND,CWNOF,RESUL, which otherwise are not used in the subroutine is declared with a single array. (TVIND for CVIND, TWNOF for WNOFC, TRESUL for RESUL) each array is declared with the size of the corresponding common.

Notes :

The IBM Fortran real and integer variables have the same size (4bytes). The size of the arrays TPARM,TGEOM,TWNOF,TRESUL, must be updated if the program is executed on a different system or if the commons are modified The output is performed using two WRITE orders, as the IBM system limits the record size to 32K bytes (the actual size of RESUL is 32128 bytes). A REWIND order must br excecuted before the WRITE to place the dataset pointer at the beginning each time PUNCH is called.

A second REWIND order must also be excecuted after the second WRITE. The content of a buffer is effectively written on a dataset only then this one is full. If the job terminates by an ABEND (CPU time limitation or number of lines limitation) the last record is not written on the dataset.

The writting of the last buffer contend is forced by executing the second REWIND. The full execution of the output can be checked as a message is written on (IWR) if the output is successfull.

Subroutine SEMRI

Function : semi-rigid wake analysis of the wind turbine.
The results obtained are used to initialize the free wake analysis
This subroutine have been derived from the program SS15 developed by Thomas Humes. [1]

Entry : SEMRI

Arguments : transmitted by commons

Input : none

Output : on (IWR)

- if LPSEM = 1 : results of the semi-rigid wake analysis in a format identical to the output of the free wake analysis.
- if LPSEM = 2 : output obtained by LPSEM = 1, plus :
  - the Prandtl's correction factors for the blade sections (F)

  For each iteration :
  - the radial, tangential and axial distributions of the induced velocities along the blade (WX,WY,WZ)
  - the distribution of circulation (GAMM)
  - the distribution of weighting factors for the old values of the induced velocities (FACV)

- the tangential and axial distribution of induced velocities (WY,WZ) after weighting.
- error messages : - no convergence
  - reverse flow conditions

Parameters used : LPSEM

Return code : ITEST (in common PARM)

0 : convergence is reached within LIMS iterations

1 : no convergence

Local variables :

All variables are reals single precision except loop variables, labels and when specified.

- variables beginning by L : format labels used for the output
- PHI5 : constant : 5° in radians
- PHI10 : constant : 10° in radians
- PI : constant π
- FPI : constant $1/4\pi$
- TWOPI : constant 2π
- PI2 : constant π^2
- PSI : azimuth of the current blade
- PHIS : azimuth of the current element

  (PSI and PHIS are used in the calculation of SN and CN only, the azimuth is characterized after by it sine and cosine).
- SN(36,8),CN(36,8) : arrays of sine and cosine

  SN(J,N) and CN(J,N) are the sine and the cosine of the azimuth of the J^{th} element of a spiral (azimuthal direction), coming from the N^{th} blade.
- FS : constant term in "f"
- SMALLF : "f" in the formulation of the Prandtl's correction factor
- BIGF : "F" Prandtl's correction factor
- WI : absolue value fo the mean axial induced velocity

- Fl : value of the inflow angle at the beginning of any iteration
- CCR : value of Fl/λ (λ coming from the precedent iteration)

  Il and CCR are used to limit the effect of the variation of W_y on the convergence
- TGLAM : tangent of the inflow angle (tg λ)
- FLAMDA(14) : distribution of the inflow angle (λ)
- ALPHA(14) : distribution of the attack angle (α)
- ALPHA1 : equivalent attack angle (stall effect included)
- AAA : interpolation factor : velocity at the node N versus the velocities at the centers N-1 and N
- E : test value (limitation of the number of spirals of the far wake)
- GTMIN,GTMAX : minimum and maximum value of the far wake
- GTT : strength of the tip vortex of the far wake
- WM2T : current total axial velocity (free stream velocity plus axial component of the induced velocity)
- ETAV : radial position of the current element (η_v)
- DA : projected length of the current element (ΔA)
- GAMMAT : strength of the current element (γ_t)
- FF : intermediate factor $\dfrac{\Delta A \cdot \gamma_t}{4\pi}$
- ETAP : axial position of the point where the induced velocity is computed (η_p)
- WX2,WY2,WZ2 : radial, tangential and axial components of the velocities induced by the current serie of element at the current point.

- SP,CP : sine and cosine of the azimuth of the centers of the current element
- F : intermediate value : $\eta_P - \eta_V \cos\Psi$
- H : axial position of the centers of the current element
- RO : distance from the center of the current element to the point (R_0)
- THE : intermediate value : $\left(\frac{\Delta A}{2 R_o}\right)$
- AA,BB : intermediate values used in the calculation I_0 (A and B)
- FJ : integration factor (I_0)
- FU : intermediate value : $\frac{\Delta A \ \gamma_t \ I_o}{4\pi \ R_o^3}$
- ATEST : test value for the importance of the first spiral ATEST is the axial velocity induced at the current point by the current spiral
- HO : azimuth of the beginning of the current spiral minus 5° (used in the calculation of H for the far wake)
- NNN : first element of the current spiral

  NNN = 10 for the first spiral

  NNN = 1 for the others

  Type : integer single precision
- XO : intermediate value in the calculation of the weighting factors

- SAV1(14),SAV2(14) : saved values of the axial induced velocities of the precedent iteration
  SAV1 : value of WZC (old distribution)
  SAV2 : value of WZCT (new distribution)
- FACV(14) : distribution of the weigthing factor for the old distribution of the induced velocities at the centers.
- FACN : current value of the weighting factor for the new distribution of induced velocities at the centers
- WYCT(14) : new distribution of the tangential components of the induced velocities at the centers
- WZCT(14) : new distribution of the axial components of the induced velocities at the centers
- WYV(15) : distribution of the tangential components of the induced velocities at the nodes
- WZV(15) : distribution of the axial components of the induced velocities at the nodes
- U(14) : distribution of total velocities relative to the blade sections
- the quantities LIP(14),TLIP(14),FLIP(14),DP(14),FDP(14),TDP(14) TP(14),FP(14),LT(14),LT,LP,CD,CT,CTCD have the same signification as those used in the subroutine OUT

- loop variables ND,J,I,N,K,MM,NN,JJ,L
  - NB : loop on the blades
  - K : iteration loop (momentum theory approximation)
  - MM : loop on the elements (radial direction)
  - NN : loop on the points where the induced velocity is computed
  - JJ : loop on the elements (azimuthal direction)
  - L : loop on the spirals

Note : - arrays relative to the nodes : WYV,WYZ

- arrays relative to the centers : ALPHA,WYCT,WZCT,FLAMDA,U,LID, TLIP,FLIP,DP,FDP,TDP,TP,BIGF,SAV1,SAV2,FACV

Details of the calculations

One can refer to the subsection " Velocities induced by the segment elements case of the semi-rigid wake routine ", for the formulation of the induced velocities and to [1] for the concept of the semi-rigid wake model.

Wake model

In a semi-rigid wake analysis, the wake have been modelled by two wakes : the near wake and the intermediate wake.

The near wake is composed of trailing vortices originating from the nodes, modelled by segment element which projections span angles of 10° ; there are nine elements for each trailing vortices, therefore the near wake extends from 0 to 90°.

Thd far wake is composed of only two trailing vortices : the tip and the root vortices ; their strength are equal to the maximum value of the circulation (GTT : strength of the tip vortex = $-\gamma_{max}$; the strength of the root vortex is -GTT).

The far wake is decomposed in spirals, the L^{th} spiral range from the azimuth $2\pi(L-1)$ to the azimuth $2\pi L$, except the first one which ranges from $\pi/2$ to 2π.

The number of spiral is limited to 15, but if a spiral induces an axial velocity at a given point less than EPS (.0005 times the near axial induced velocity evaluated from the precedent iteration), the remaining spirals are neglected.

Each spiral is modelled by 36 elememts of 10°

Initialization of the semi-rigid wake analysis

The span wise momentum theory is used to generate the first guess of the induced velocities, the Prandtl's correction for the finite number of blades is included :

Formulations :

$$\operatorname{tg} \lambda = \frac{W_z + \mu}{\eta + W_y}$$

$$\alpha = \lambda - \theta$$

$$\alpha_1 = \alpha \text{ if } \alpha < \alpha_s$$

$$\alpha_1 = \alpha_s \text{ if } \alpha > \alpha_s$$

$$U = \sqrt{(W_z + \mu)^2 + (\eta + W_y)^2}$$

$$\gamma = \pi^2 \sigma U \alpha_1 / N$$

$$W_z = -\frac{\gamma N}{4\pi \eta \operatorname{tg} \lambda \, F} = -\frac{\pi}{4} \frac{\sigma U \alpha_1}{F \eta \operatorname{tg} \lambda}$$

$$W_y = -W_z \operatorname{tg} \lambda$$

with

$$F = \frac{2 \arccos e^{-f}}{\pi}$$

$$f = \frac{N}{2} \frac{\sqrt{1 + \mu^2}}{\mu}$$

Ten iterations are performed, and a weighting factor of 1/4 is used for the new value of W_z.

Convergence test

Convergence is reached when for a given iteration the resulting (new) axial and tangential components of the velocities induced at the centers (WZCT,WYCT) are equal (within E = .01WI, WI mean axial induced velocity) to the initial (old) velocities.

The convergence is not reached within LIMS iterations, a message is printed and the free wake analysis of the case is not performed.

Weighting functions

It is necessary in order to obtain convergence, to weight the old and the new distribution of axial and tangential induced velocities, before the next iteration.

For the first two iterations a weigthing factor of .8 is used for the old distribution ; for the remaining iterations, computed values of the weighting factors are used, based in the values of the axial induced velocities from the current and precedent iterations.

(Refer to subsection " convergence " for the formulations)

The computed values of the weighting factors are also weighted with the weighting factors from the precedent iteration, to damp the coupling effect of the different sections.

Note : to decrease the calculation time, the values of sin ψ and cos ψ are precomputed (arrays SN and CS)

SN(J,N) = sin $\psi_{j,n}$

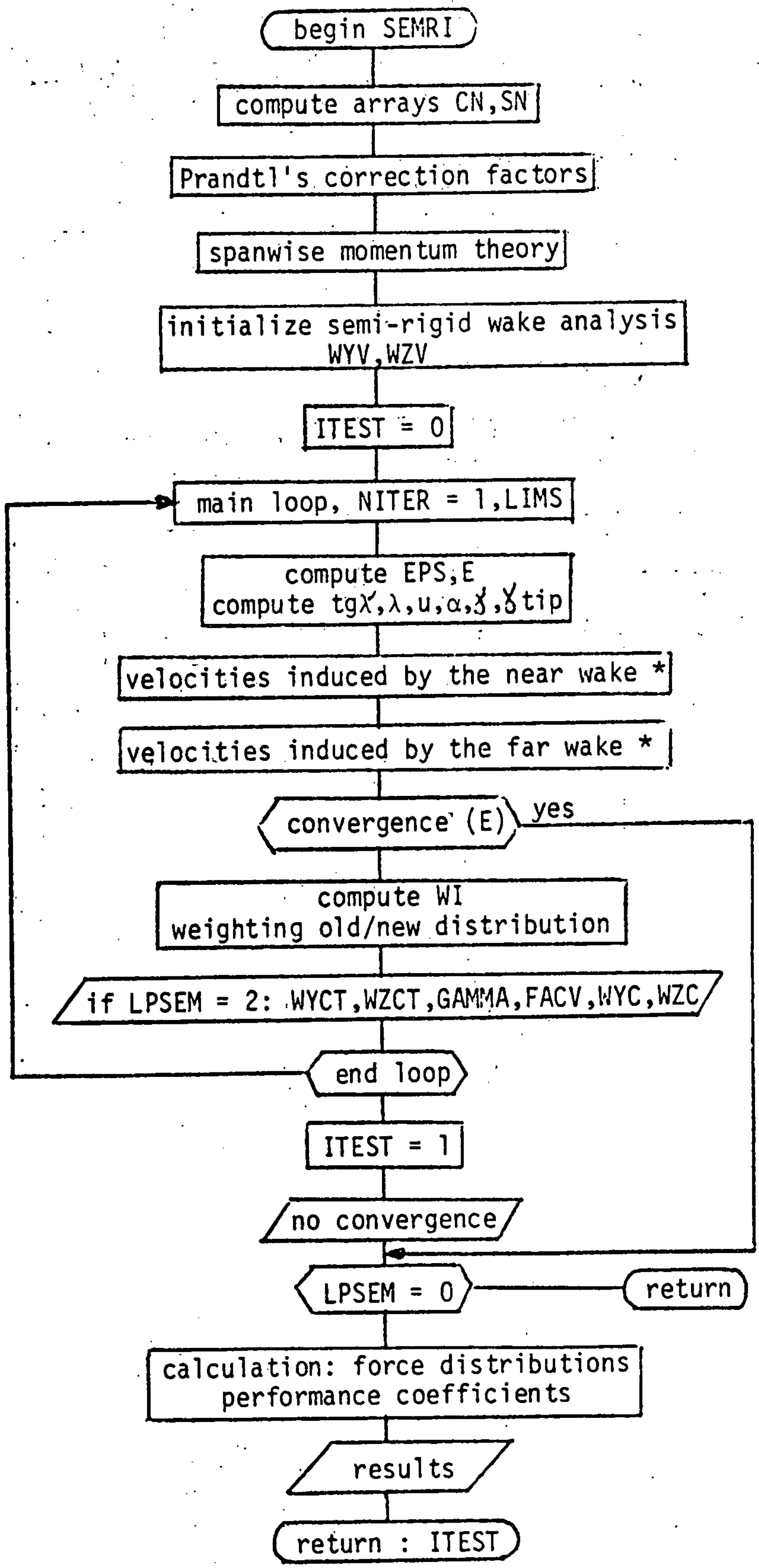
CN(J,N) = cos $\psi_{j,n}$

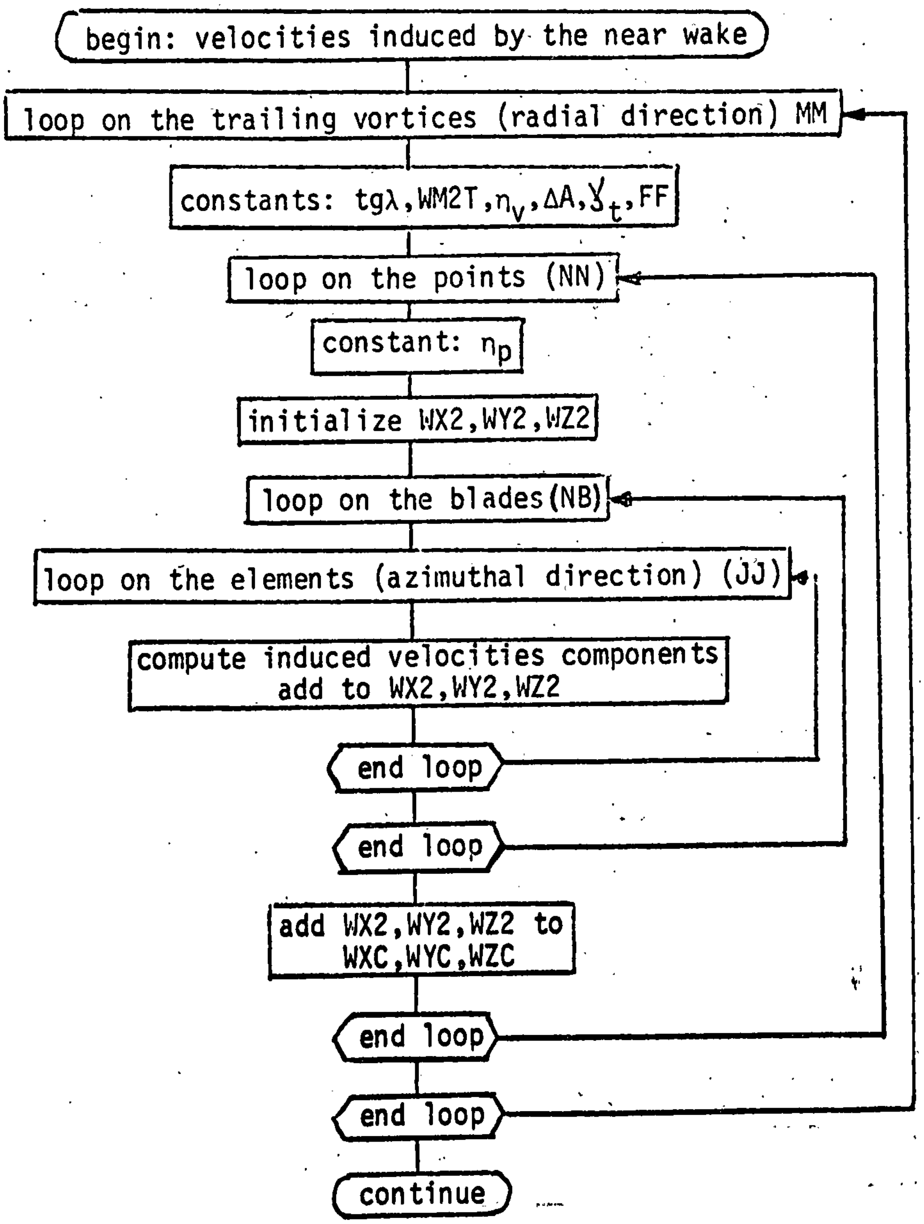
When $\psi_{j,n}$ is the azimuth of the center of the j^{th} element coming from the n^{th} blade

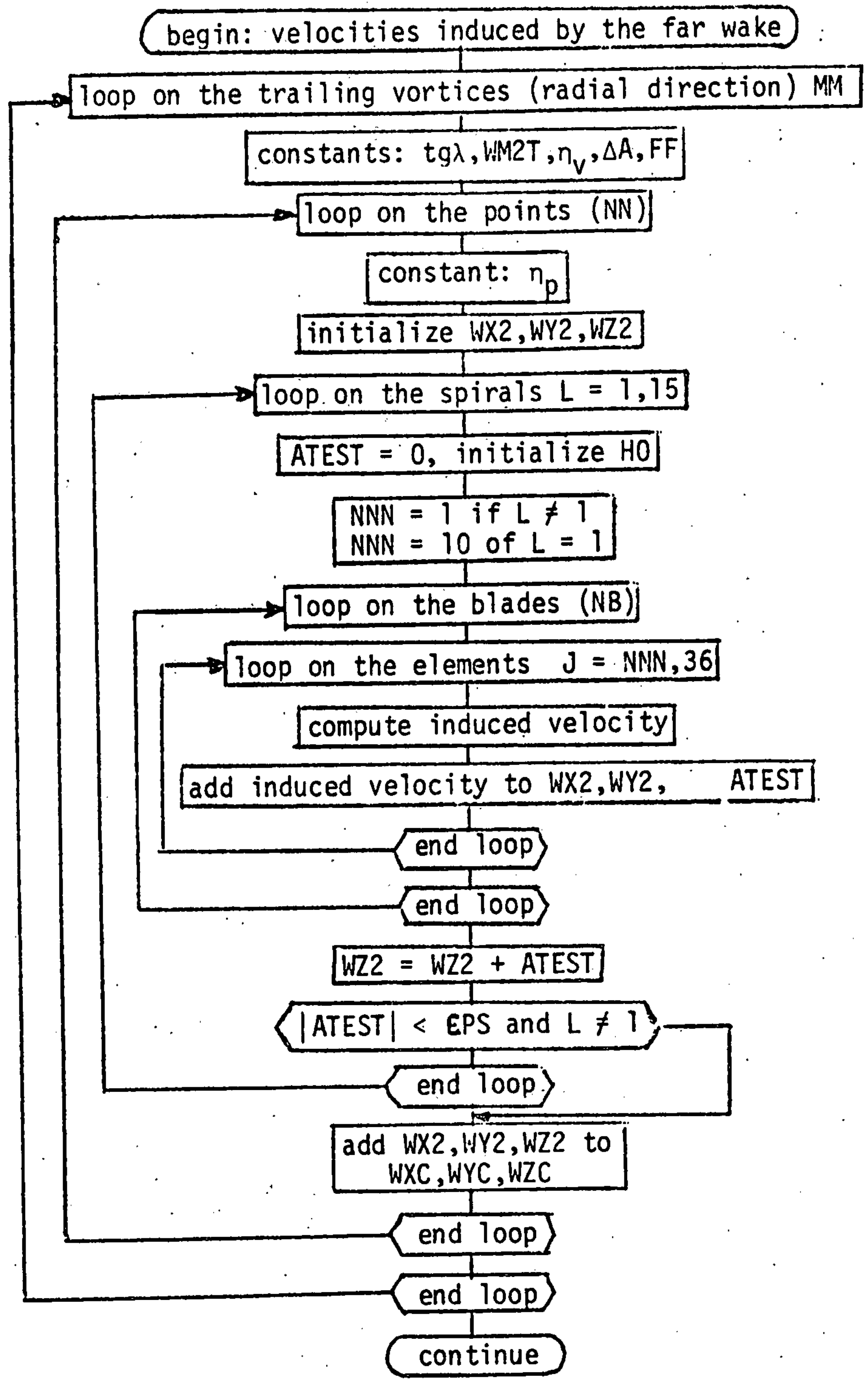
$$\psi_{j,1} = (j-1) * 10° + 5°$$

$$\psi_{j,n} = \psi_{j,1} + 2\pi(n-1)/N$$

(N the number of blades)


begin SEMRI
compute arrays CN,SN
Prandtl's correction factors
spanwise momentum theory
initialize semi-rigid wake analysis WYV,WZV
ITEST = 0
main loop, NITER = 1,LIMS
compute EPS,E
compute tgχ,λ,u,α,γ,γtip
velocities induced by the near wake *
velocities induced by the far wake *
convergence (E)
yes
compute WI
weighting old/new distribution
if LPSEM = 2: WYCT,WZCT,GAMMA,FACV,WYC,WZC
end loop
ITEST = 1
no convergence
LPSEM = 0
return
calculation: force distributions
performance coefficients
results
return : ITEST


begin: velocities induced by the near wake
loop on the trailing vortices (radial direction) MM
constants: tgλ, WM2T, n_v, ΔA, γ_t, FF
loop on the points (NN)
constant: n_p
initialize WX2, WY2, WZ2
loop on the blades (NB)
loop on the elements (azimuthal direction) (JJ)
compute induced velocities components add to WX2, WY2, WZ2
end loop
end loop
add WX2, WY2, WZ2 to WXC, WYC, WZC
end loop
end loop
continue


begin: velocities induced by the far wake
loop on the trailing vortices (radial direction) MM
constants: tgλ, WM2T, η_v, ΔA, FF
loop on the points (NN)
constant: η_p
initialize WX2, WY2, WZ2
loop on the spirals L = 1,15
ATEST = 0, initialize HO
NNN = 1 if L ≠ 1
NNN = 10 of L = 1
loop on the blades (NB)
loop on the elements J = NNN,36
compute induced velocity
add induced velocity to WX2, WY2, ATEST
end loop
end loop
WZ2 = WZ2 + ATEST
|ATEST| < EPS and L ≠ 1
end loop
add WX2, WY2, WZ2 to WXC, WYC, WZC
end loop
end loop
continue

Subroutine COMPA

Function : initialization of the free wake calculation

Entries : COMPA

Arguments : transmitted by commons

Parameters used : ITRACE

Input : none

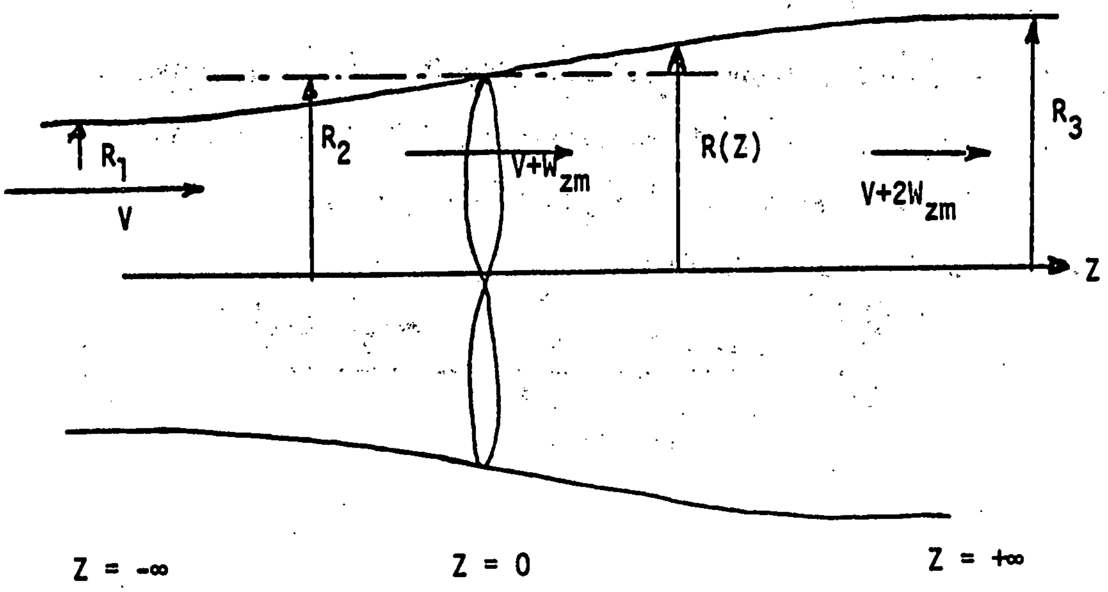
Output : R_3 and WZM (if ITRACE = 1)

Local variables :
- ETAM : mean radial position, to compute approximation of tangential velocity
- IK : center position nearest from ETAM
- WT : tangential induced velocities at the center of the blade located at ETANC(IK) (W_{tm})
- CPSI : factor used to compute the approximation of the azimuth from the nominal one
- WZM : near axial induced velocity (W_{zm})
- WWZ : near axial velocity (V_m)
- DRDZ : rate of expansion at the tip of the blade ($\frac{dR}{dZ}$)
- R3 : radius of the wake after full expansion (R_3)
- WZN(I) : axial velocity at the node i of the near wake originating from the position $\eta_{nv}(i)$
- WZI(I) : same as WZN(I), for the intermediate wake
- VR : radial induced velocities
- PSI : azimuth ψ
- AM : value DRDZ/(1.-AM)
- Z : current axial position of the nodes

- AEXP : current value of the radial position of the nodes
- CCC : cos ψ
- SSS : sin ψ
- I,J,I2 : loop variables
- I1 : index of the radial position of the near wake, nearest the current radial position of the intermediate wake
- ET : current radial position of the intermediate wake
- A : interpolation factor
- PSIIB : nominal azimuth of the first position of the nodes of the intermediate wake.

<u>Approximation after expansion</u>

A simple result of the momentum theory is used



If the axial induced velocity at the rotor plane level is c nstant and equal to W_{zm}, it will be $2W_{zm}$ at $z = +\infty$, after full expansion, and the wake expands from a radius $R_2 = 1$ at the rotor plane, to a radius R_3 at $z = +\infty$. R_3 can be determined from a mass conservation equation. The mass flow through the rotor is : $\rho \pi R_2^2 (V+W_{zm})$.

And can be written : $\rho \pi R_3^2 (V+2W_{zm})$ at $z = +\infty$

Therefore :

$$R_3 = \sqrt{\frac{V + W_{zm}}{V + 2W_{zm}}}$$

W_{zm} is taken as the mean induced velocity. The volume flow through the surface swept by the blades is :

$$\phi = \int_S (W_z + V)\, ds = S\,(W_{zm} + V)$$

$$S = \pi\,(\eta_t^2 - \eta_r^2)$$

On the blade segment between $\eta_{nv}(i)$ and $\eta_{nv}(i+1)$, the axial induced velocity is WZ(I) (induced at the center $\eta_{nc}(i+1)$). And the surface swept by the blade element is $\pi\,[\,\eta_{nv}^2(i+1) - \eta_{nv}^2(i)\,]$

The mean induced velocity is then :

$$W_{zm} = \sum_{i=1}^{n_{nvr}-1} \frac{W_{zc}(i)\,[\,\eta_{nv}^2(i+1) - \eta_{nv}^2(i)\,]}{\eta_t^2 - \eta_r^2}$$

At the tip of the blade the rate of expansion ($\frac{dR}{dz}$) is W_x / $(V + W_z)$

Where W_x and W_z are the radial and axial components of the velocity induced at the tip of the blade, V the free stream velocity.

The values are known for the last center position.

Therefore : $\frac{dR}{dz}$ = DRDZ = WXC(NNCR2) / (FMU + WZC(NNCR2)).

The expansion (radius of the wake versus the axial position) is approximated with the exponential form : $R(z) = b + c \exp(-a\, z)$

with : $R(0) = R_2 = 1$

$R(+\infty) = R_3$

and $\left(\frac{dR}{dz}\right)_{z=o}$ = DRDZ

therefore : $R(z) = R_3 - (R_3 - 1) \exp\left(- \frac{dR}{dz}\Big|_{z=o} * \frac{z}{R_3 - 1} \right)$

The corresponding radial induced velocities V_r

$$V_r = \frac{dR}{d\psi} = \frac{dR}{dz} \frac{dz}{d\psi}.$$

$$\frac{dR}{d\psi} = \frac{dR}{dz}\Big|_{z=o} * \exp\left(- \frac{dR}{dz}\Big|_{z=o} * \frac{z}{R_3 - 1}\right) * W_m$$

For an inner section i, the distance of a point to the Z axis is taken equal to $\eta_{nv}(i) * R(z)$

The approximation of the azimuth of the nodes is given by

$$\psi = \psi_n * \left(1 + \frac{W_{tm}}{\eta_m} \right)$$

Where ψ_n is the nominal azimuth (value obtained if there were no tangential induced velocities)

W_{tm} the tangential induced velocity at η_m, given by SEMRI η_m the center nearest from the radial position ETAM = $(\eta_r + \eta_t) / 2$

The azimuthal positions are therefore separeted of $\Delta\psi$ = DPSIN * $\left(1+\frac{W_{tm}}{\eta_m}\right)$ for the near wake and $\Delta\psi$ = DPSII * $\left(1+\frac{W_{tm}}{\eta_m}\right)$ for the intermediate wake

The axial position is given by $\psi_n * (W_z + \mu)$

W_z tha axial induced velocity obtained by interpolation of the values computed at the centers of the blade by SEMRI

Subroutine INIVND

Function : calculation of the coefficients for the determination of the strength of the elements (the constants are placed in the common CVIND)

Entries : INIVND

Arguments : transmitted by commons

Parameters used : ITRACE

Input : none

Output : on (IWR) : the coefficients are printed if ITRACE-1

These coefficients depend only on the distribution of the nodes of the bear and intermediate wake, and of the coefficient COEFF. They are evaluated only once at the beginning of the case.

The strength of an element (value in common VORT) is a linear combination of the values of the circulation at the centers along the blade.

A coefficient gives, for a unit circulation at the j^{th} center of the blade (located at $\eta_{nc}(j+1)$), the strength of the corresponding element

The name of a coefficient is given by substituing an H as first letter of the name of a strength component

Examples : HM1N : array of coefficients for the strengths GM1N

HAMMT1 : array of coefficients for the strength GAMMT1

HM1N,HM2N,HM1I,HM2I are arrays with two dimensions,

The first subscript corresponds to the subscript of GM1N,GM2N,GM1I,GM2I.

The second subscript corresponds to the J^{th} unit circulation.

HAMMT1,HAMMT2,HAMMT3,HAMMT4 are arrays with only one dimension, the subscriot corresponds to the J^{th} unit circulation.

Utilisation of the coefficients (examples)

GM1N(I) = $_{j}$ HM1N(I,J) * GAMMC(J+1)

GAMMT1(I) = $_{j}$ HAMMT1(J) * GAMMC(J+1)

for j = 1,NNCR2

Where GAMMC(J+1) is the circulation at the center ETANC(J+1)

Local variables

All variables are reals single precision, execpt loop variables and format labels.

- variables beginning by L: format labels
- HT1,HT2,HT3,HT4 : current values of HAMMT1(J),HAMMT2(J), HAMMT3(J),HAMMT4(J)
- ET : value of ETAIC(I), used for the interpolation of the values relative to the intermediate wake, given the values relative to the near wake
- K : index of the streamline of the near wake located immediatly before the current streamline of the intermediate wake

  Type : integer single precision
- I, J, N, KI, KJ, IK : loop variables

  J: used for the loop on the unit circulations

  I: used for the loop on the radial positions of the streamlines
- GNC(14) : pseudo-distribution of circulation (relative to the centers of the blade or of the near wake)
- GIC(6) : same as GNC, for the intermediate wake
- GNTV(14) : pseudo distribution of lumped strengths of the trailing vortices of the near wake
- GITV(14) : same as GNTV, for the intermediate wake
- HNTV(14) : pseudo-distribution of strengths of the sheet vortices of the near wake, at the nodes
- HITV(14) : same as HNTV, for the intermediate wake

Subroutine COFN

Function : calculation of the coordinates of the centers of the near and intermediate wake, using the coordinates of the nodes.

Entry : COFN

Arguments : transmitted by commons GEOM and RESUL

Input/Output : none

Parameters used : none

Local variables : I and J : loop variables.

Because of the need for lines of centers outside the wake, the streamline I of the mesh created by the centers is located between the streamlines I-1 and of I of the nodes, the azimuthal line J of the mesh created by the centers is located between the azimuthal lines J-1 and J of the nodes. The coordinates of the centers located inside the wake are given by single four points interpolation.

Calling $\vec{X}_c$ (i,j) the position vector of the center (I,J)

and $\vec{X}_n$ (i,j) the position vector of the node (I,J)

$$\vec{X}_c(i,j) = \frac{1}{4}\left[\vec{X}_n(i,j) + \vec{X}_n(i,j-1) + \vec{X}_n(i-1,j) + \vec{X}_n(i-1,j-1)\right]$$

For the near wake i runs from 2 to $N_{ncr}-1$ and j from 1 to $N_{nca}-1$.

The centers located outside the wake must be extrapolated.

For those centers, (except the corners), the middle of the segment joining two nodes on the edge, is constrained to be the middle of the segment joining the extrapolated center and the nearest inner center

Example for i = 1

$$\frac{1}{2}\left[\vec{X}_n(1, j-1) + \vec{X}_n(1, j)\right] = \frac{1}{2}\left[\vec{X}_c(1, j) + \vec{X}_c(2, j)\right]$$

or $$\vec{X}_c(1, j) = \vec{X}_n(1, j-1) + \vec{X}_n(1, j) - \vec{X}_c(2, j)$$

For the centers located at the corners of the wake, the corresponding corner node is constrained to be the middle of the four centers around it.

Example for i = 1 and j = 1

$$\vec{X}_n(1,1) = \frac{1}{4}\left[\vec{X}_c(1,1) + \vec{X}_c(1,2) + \vec{X}_c(2,1) + \vec{X}_c(2,2)\right]$$

or $$\vec{X}_c(1,1) = 4\,\vec{X}_n(1,1) - \vec{X}_c(1,2) - \vec{X}_c(2,1) - \vec{X}_c(2,2)$$

Note : The two streamlines bounding a wake section and the first azimuthal line of nodes of the near wake support concentrated vortices.

A straight concentrated vortex segment does not induce velocities on itself ; to have correct induced velocities at the nodes located on concentrated vortices (interpolated from the velocities of the four centers around it), the contributions of a vortex segment to the velocities induced at the two corresponding centers must cancel each other, therfore these centers have to be located symetrically with the vortex segment.

Subroutine LOOP2

Function : - calculation of the strength of the elements.

- loop on the elements to compute the induced vel velocities

Entry : LOOP2

Arguments : transmitted by commons

Subroutines called : INDVEL(COORD)

VIND(VINDB,VINDN,VINDI,INICN,EXECN)

PRTV(PRTV,PRTVD,CLV)

Parameters used : ITRCL2,ITRANS,ITRCF,ITRCNT,ITRCG

Input : none

Output : on (IWR)

- strength of the elements (ITRCG ≠ 0)
- messages
- velocities induced at the centers (performed by PRTV, following the value of ITRCL2)
- print out for verification (parameters ITRCF, ITRCN)

The subroutine performs principally three functions :

- calculation of the strength of the elements
- copy of the influence coefficients (used by WNOFC)
- loop on the elements (for each element the calculation of the induced velocities and of the influence coefficients is done by VIND)

The optional functions which are also performed or commanded by LOOP2 are :

- verification of the influence coefficients
- verification of the velocities induced by the far wake

Local variables :

- KIT,KI,I,J : loop variables, when used for the loops on the elements :

  I is used for the radial direction and J is used for the azimuthal direction

  Type : integers single precision

- ITT2 : value of ITRCL2 at it was specified when entering LOOP2

  Type : integer single precision

- IDEB : first azimuthal position of the centers of the intermediate wake for which the induced velocities will be computed. (transmitted as argument to VIND)

  Type : integer single precision

- S2,C2,TT2 : constants : sin (15°), cos(15°), 2 * tg(15°/2), usedfor the far wake

  Type : reals single precision

- T2 : axial increment used for the far wake

  Type : real single precision

- VX1,VY1,VZ1 : component of the vector used for the definition of (tg λ) of the far wake

  Type : real single precision

- ATL : λ of the far wake

  Type : real single precision

- variables beginning by L : format labels
- NK : number of segment elements used to model the beginning of the far wake for the velocities induced on the intermediate wake (NK = 48).

  Also : used for the verification of the far wake model (NK = 480)

Part 1 : Calculation of the strength of the elements

Are computed : GM1N,GM2N,GM1I,GM2I,GAMMT1,GAMMT2,GAMMT3,GAMMT4.

where : GM1N(I) and GM2N(I) are the two lumped strengths of all the I^{th} rectangular elements of the near wake (elements limited by the streamlines originating from $\eta_{nv}(I)$ and $\eta_{nv}(I+1)$.

- GAMMT1,GAMMT2 the strength of the root and the tip vortices bounding the near wake
- GM1I(I) and GM2I(I) are the two lumped strengths of all the I^{th} rectangular elements of the intermediate wake (elements limited by the streamlines originating from $\eta_{iv}(I)$ and $\eta_{iv}(I+1)$.
- GAMMT3 and GAMMT4 the strength of the root and the tip vortices bounding the intermediate wake.

The values GM1N,GM2N,GAMMT3 and GAMMT4 are used also for the far wake, as the elements of the far wake (semi-infinite cylinders) are derived from continuation of the intermediate wake.

All strengths are evaluated respectively from the coefficients HM1N, HM2N,HM1I,HM2I,HAMMT1,HAMMT2,HAMMT3,HAMMT4 and the distribution of circulation GAMMC (all these coefficients depend only on the distribution of the control points η along the blade and the value COEFF, they are calculated by INIVND).

Part 2 :

The Y and Z influence coefficients of the precedent iteration (TWY and TWZ) are saved in the arrays TWYT,TWZT of the common SAVTW

Part 3 : loop on the elements

Calculation of the induced velocities and of the influence coefficients. Different cases have to be considered following the output level ITRCL2 and the verification parameter (ITRCF).

This part is composed of seven different subsections :

A) Velocities induced by the circulation along the blades

B) Velocities induced by the rectangular elements of tha near wake

C) Velocities induced by the segment elements of the near wake

D) Velocities induced by the rectangular elemts of the intermediate wake

E) Velocities induced by the segment elements of the intermediate wake

F) Velocities induced by the far wake, in continuation of the rectangular elements of the intermediate wake.

G) Velocities induced by the far wake, in continuation of the segment elements of the intermediate wake

For each subsection :

- By a call to COORD, the variables depending only on the elements are evaluated by INDVEL

- By a call to VINDB, the velocities induced by the elements on the blades will be computed, (influence coefficients) where :

The first argument is the type of element :

1 : rectangular element of the near wake

2. : segment element of the near wake

3 : rectangular element of the intermediate wake.

(also : continuation of these elements by the far wake).

4 : segment element of the intermediate wake

(also : continuation of these elements by the far wake).

The second argument : the radial position of the element

- By a call to VINDN the velcoities induced on the centers of the near wake will be computed.

- By a call to VINDI the velocities induced on the centers of the intermediate wake will be computed.

For VINDN and VINDI, the velocities will be conputed for the nodes which azimuthal positions runs from the first to the second value transmitted as arguments.

- By call to PRTV the velocities at the centers and the influence coefficient will be printed.

- By a call to PRTVD the velocities at the centers will be printed, then the arrays are reinitialized.

- By a call to EXECN the verification of the ibfluence coefficients is performed

<u>Subsections A to E</u> :

- The type of element (IT) used by INDVEL is set to :

  1 : segment element

  2 : rectangular element

- A message is printed (ITRCL2 = 1)

- Inside the loop(s) on the elements (single loop for A, loop on radial positions of the elements, then on the azimuthal positions for B,C,D,E) :

  a) The strength(s) of the element (GM1,GM2 or GM) and the points defining the elements (X1,Y1,Z1,...Z4) are place in the common VEL.

  b) COORD is called

c) The velocities are computed (call to VINDR,VINDN,VINDI).

- at the end, PRTV,PRTVD,EXECN are called according to ITRCL2 and ITRCNT

Subsections F and G :

Velocities induced by the far wake.

The elements of the far wake are semi-infinite cylinders (calculation of the influence coefficient and of the velocities induced on the near wake). For the velocities induced on the intermediate wake, to avoid calculation of velocities induced very near the edge of a semi-infinite cylinder, (where the hypothesis is of a uniformaly distributed strength is innacurate) the first two spires are defined with 48 segments elements and only the remaining of the far wake is defined by a semi-infinite cylinder. Therefore after the calculation of the geometry of the cylinder, the influence coefficient and the velocities induced on the near wake are computed, then 48 segments elements (which geometry is given by the semi-infinite cylinder) are generated and their velocities induced on the intermediate wake computed.

Verification of the semi-infinite cylinder model for the far wake

The verification is performed when the parameter ITRCF is set to 1 when LOOP2 is entered.

The verification consists in the calculation of the velocities at the centers of the near wake and intermediate wake, by the two different definitions of the far wake (loop KTT)

a) Far wake using a semi-infinite cylinder

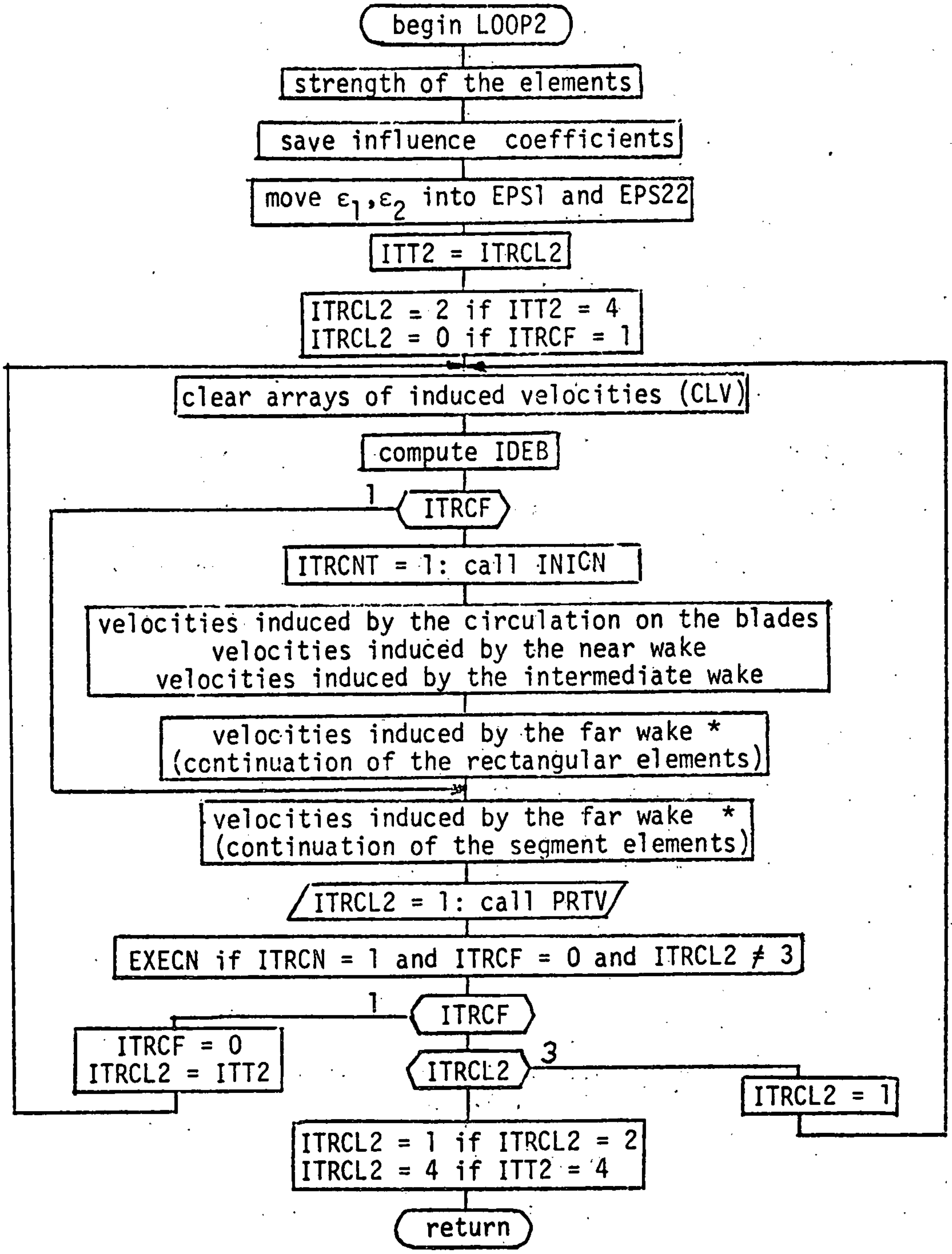
b) Far wake defined by 10 spirals of segment elements (each element spans an angle of 15°), equivalent to the semi-infinite cylinder element

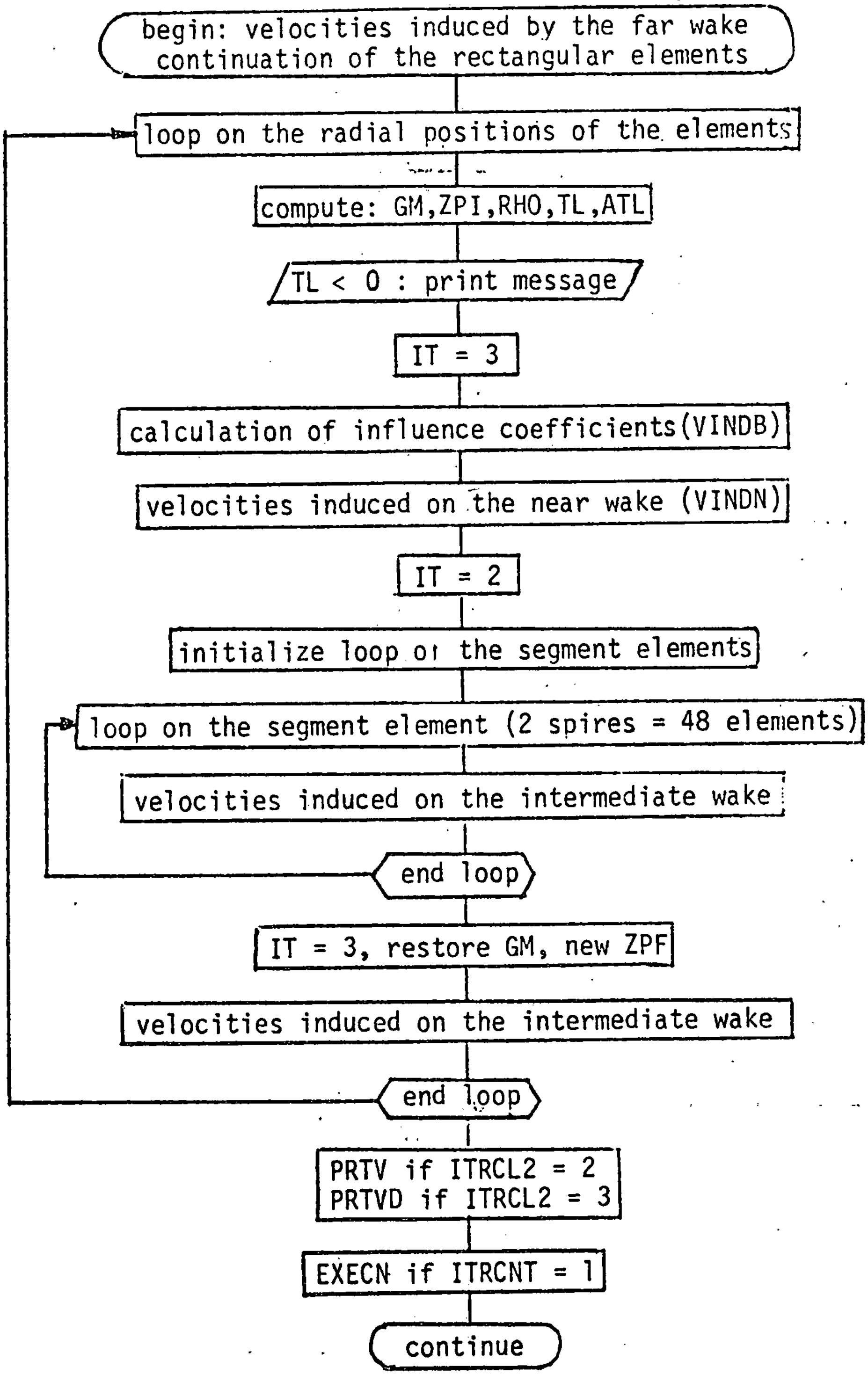
The induced velocities are calculated and printed for the two definitions, for the root and tip cylinders, they results have to be identical within the errors due to the models.

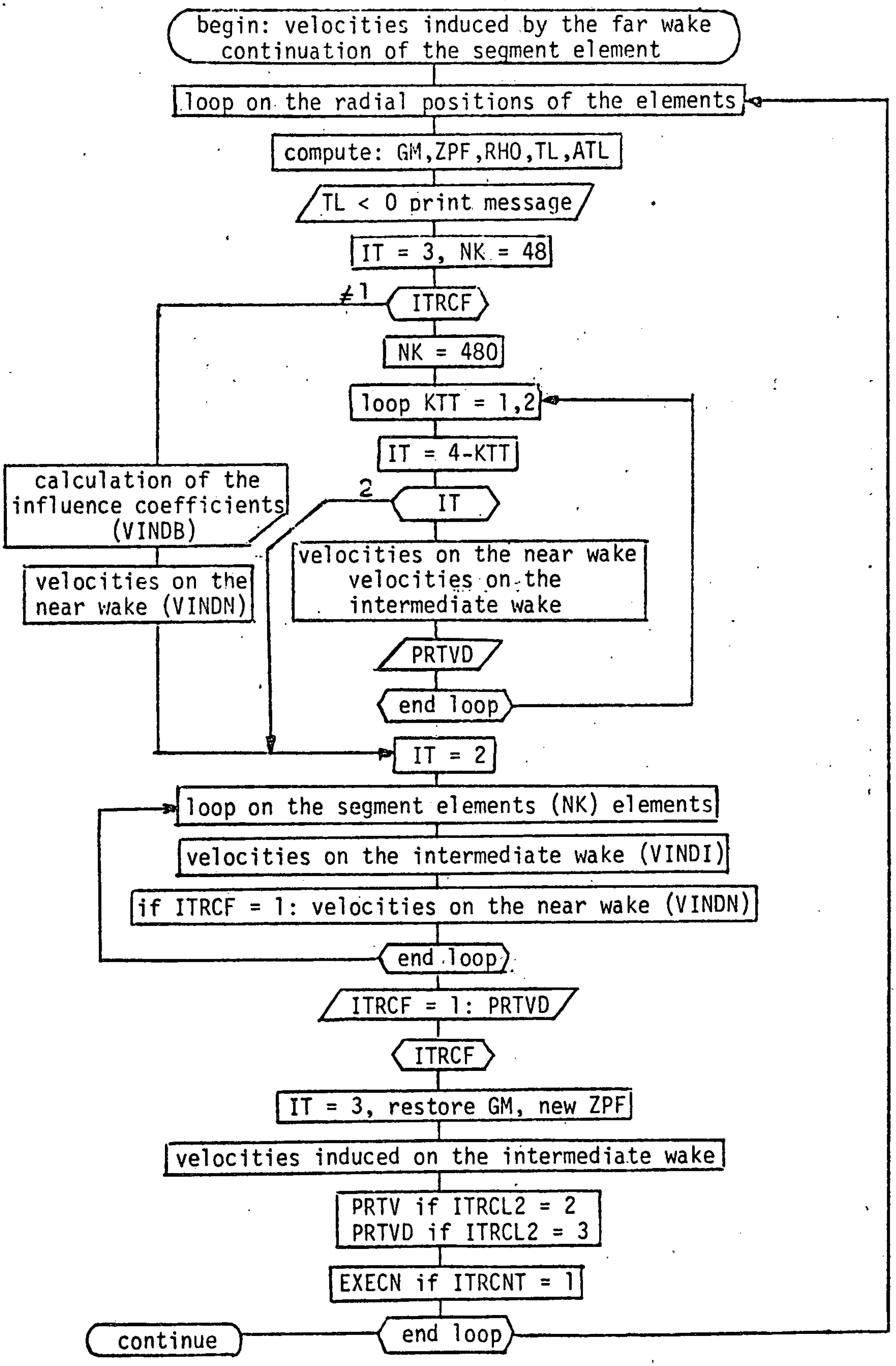
In case of verification of the far wake model, the subsections A to F are skipped, the verification performed, then parameter ITRCF is cleared and the normal calculation of the induced velocities resumed

Case fo ITRCL2 = 3 (calculation and print out of the velocities induced by each type of elements individually)

The calculation of the induced velocities is performed a first time with ITRCL2 = 3 then, as the arrays containing the induced velocities are destroyed, the parameter ITRCL2 is reset to 1, and the induced velocities recomputed.


begin LOOP2
strength of the elements
save influence coefficients
move ε1,ε2 into EPS1 and EPS22
ITT2 = ITRCL2
ITRCL2 = 2 if ITT2 = 4
ITRCL2 = 0 if ITRCF = 1
clear arrays of induced velocities (CLV)
compute IDEB
1
ITRCF
ITRCNT = 1: call INICN
velocities induced by the circulation on the blades
velocities induced by the near wake
velocities induced by the intermediate wake
velocities induced by the far wake *
(continuation of the rectangular elements)
velocities induced by the far wake *
(continuation of the segment elements)
ITRCL2 = 1: call PRTV
EXECN if ITRCN = 1 and ITRCF = 0 and ITRCL2 ≠ 3
1
ITRCF
ITRCF = 0
ITRCL2 = ITT2
ITRCL2
3
ITRCL2 = 1
ITRCL2 = 1 if ITRCL2 = 2
ITRCL2 = 4 if ITT2 = 4
return


begin: velocities induced by the far wake continuation of the rectangular elements
loop on the radial positions of the elements
compute: GM,ZPI,RHO,TL,ATL
TL < 0 : print message
IT = 3
calculation of influence coefficients(VINDB)
velocities induced on the near wake (VINDN)
IT = 2
initialize loop on the segment elements
loop on the segment element (2 spires = 48 elements)
velocities induced on the intermediate wake
end loop
IT = 3, restore GM, new ZPF
velocities induced on the intermediate wake
end loop
PRTV if ITRCL2 = 2
PRTVD if ITRCL2 = 3
EXECN if ITRCNT = 1
continue


begin: velocities induced by the far wake continuation of the segment element
loop on the radial positions of the elements
compute: GM,ZPF,RHO,TL,ATL
TL < 0 print message
IT = 3, NK = 48
ITRCF
≠1
NK = 480
loop KTT = 1,2
IT = 4-KTT
IT
2
velocities on the near wake velocities on the intermediate wake
PRTVD
end loop
calculation of the influence coefficients (VINDB)
velocities on the near wake (VINDN)
IT = 2
loop on the segment elements (NK) elements
velocities on the intermediate wake (VINDI)
if ITRCF = 1: velocities on the near wake (VINDN)
end loop
ITRCF = 1: PRTVD
ITRCF
IT = 3, restore GM, new ZPF
velocities induced on the intermediate wake
PRTV if ITRCL2 = 2
PRTVD if ITRCL2 = 3
EXECN if ITRCNT = 1
end loop
continue

Subroutine VIND

Function : loop on the points where the induced velocities are computed and on the blades

Also : verification of the influence coefficients method

Entries : - VINDB : velocities on the blade (calculation of the influence coefficients)

- VINDN : velocities induced on the near wake
- VINDI : velocities induced on the intermediate wake
- INICN : initialization of the internal arrays used for the verification of the influence coefficients
- EXEN : verification of the influence coefficients

Arguments : transmitted by commons

plus :

for VINDB(K,IKK)

- K : current type of element
  1 : rectangular element of the near wake
  2 : segment element of the near wake
  3 : rectangular element of the intermediate wake
  Also : continuation of these elements by the far wake

4 : segment element of the intermediate wake

Also : continuation of these elements by the far wake

- IKK : radial position of the current element

for VINDN(K1,K2)

- K1 : first azimuthal position of the centers of the near wake, for which the induced velocities are computed.

- K2 : last azimuthal position of the centers of the near wake, for which the induced velocities are computed

for VINDI(K1,K2)

- K1 and K2 have the same signification as for VINDN but apply for the azimuthal positions of the intermediate wake

INICN and EXECN have no arguments

Input : none

Output : on (IWR), by EXECN : print out of the components of the velocities induced at the inner centers of the blades, computed directly (WXC,WYC,WZC) and by the influence coefficients (WXCA,WYCA,WZCA), for the verification of the influence coefficients method. The two sets of values must be identical within the round-off errors

Parameters used : ITRCNT

Local variables :

- variables beginning by L : format labels used for the output of EXECN
- SAV1,SAV2 : saved values of GM and DGM during the calculation of the influence coefficients
- NBL : loop on the blades variable :

  NBL = NBLDS (number of blades) for rectangular and segment elements

  NBL = 1 for a semi-infinite cylinder element
- CC,SS : cosine and sine of the azimuth of the current blade
- SAV : intermediate value, during the calculation of the rotated coordinates of a point and during the recalculation of CC and SS
- J,KK,JJ,N,I1,J1,I,KI : loop variables

  KK : loop on the blade positions

  JJ : loop on the two first azimuthal positions of the centers of the near wake (in VINDB)

  N : loop on the blades

  I1 : loop on the centers (radial direction)

  J1 : loop on the centers (azimuthal direction)

  J : loop on the unit circulation along the blade

Details of the calculations

- Calculation of the influence coefficients

By definition : an influence coefficient $T_w(i,j)$ is the velocity induced at the inner center i (located at $\eta_{nc}(i+1)$) assuming a unit circulation on the blade section j (centered at $\eta_{nc}(j+1)$.

The actual strength of the element (GM,DGM) are saved, they must be restored afterward, then for the current element, a loop an (j) is performed. If the element has a non zero strength for a given unit circulation at (j), the velocities induced by this element on all the inner centers (loop on KK) are computed and added to the corresponding influence coefficients. The strength of the elements for a unit circulation at (j) are given by the H coefficients (HM1N,HM2N,HM1I,HM2I, HAMMT1,HAMMT2,HAMMT3,HAMMT4) the corresponding arguments for INDVEL (GM and DGM) are recomputed. If the strength of the element is zero then the K coefficients are zero (integer value) (KM1N,KM2N,KM1I, KM2I,KAMMT1,KAMMT2,KAMMT3,KAMMT4)

Special case of the rectangular elements of the near wake :

The centers of the blades are located on the boundaries of the rectangular elements (first azimuthal position), this situation must be avoided, therefore the induced velocities on a particular center of a blade is computed as the half sum of the velocities induced on the two centers of the near wake located symetrically with respect to the center of the blade. More precisely, the strength GM and DGM are devided by two and the velocities are simply added.

- Loop on the blades

The symetry properties of the wake are used to decrease the calculation time.

At any given point the induced velocity is the sum of the velocities induced by the B wakes originating from the B blades. A given element (on the first blade) has B-1 counterparts, obtained by successive rotations of $2\pi/B$. Instead of computed velocities induced by these B elements at a given point, (therefore defining B different elememts, and calling COORD B times), the point is rotated of $+2\pi n/B$, and the induced velocity vector, obtained by WXYZ, rotated of $-2\pi n/B$

Therefore, defining a rotation of $2\pi/B$ by C and S (cosine and sine of $2\pi/B$), and a rotation of $2\pi n/B$ by CC and SS (cosine and sine of $2\pi n/B$) the coordinates X_n and Y_n of a point (rotation of $2\pi n/B$) are obtained

by : $X_n = X_{n-1} * C - Y_{n-1} * S$

$Y_n = X_{n-1} * S + Y_{n-1} * C$

$(Z_n = Z_{n-1})$

and the velocity at the original point (W_x, W_y, W_z)

by : $W_x = UX * CC + UY * SS$

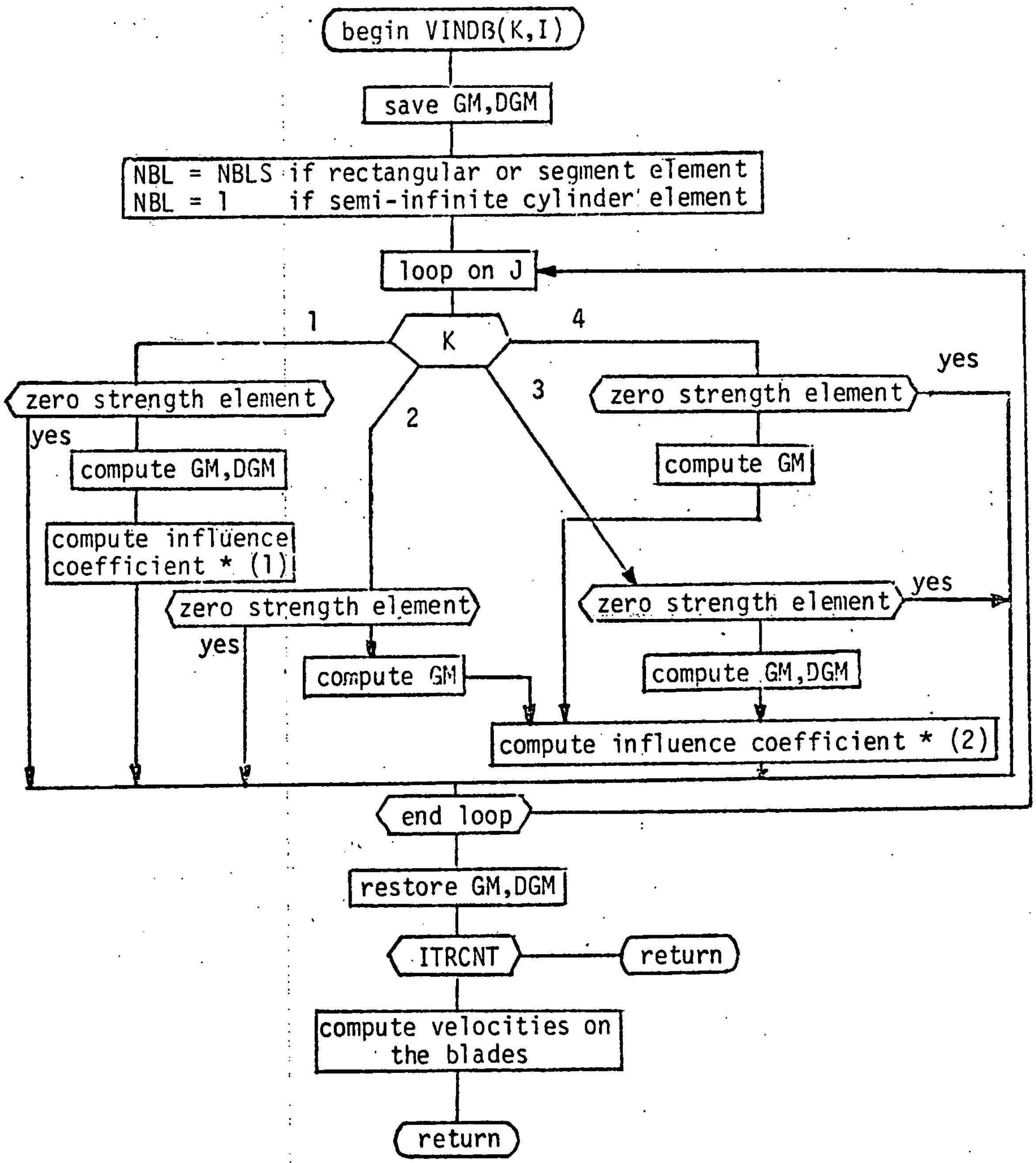
$W_y = UY * CC - UX * SS$

$(W_z = UZ)$

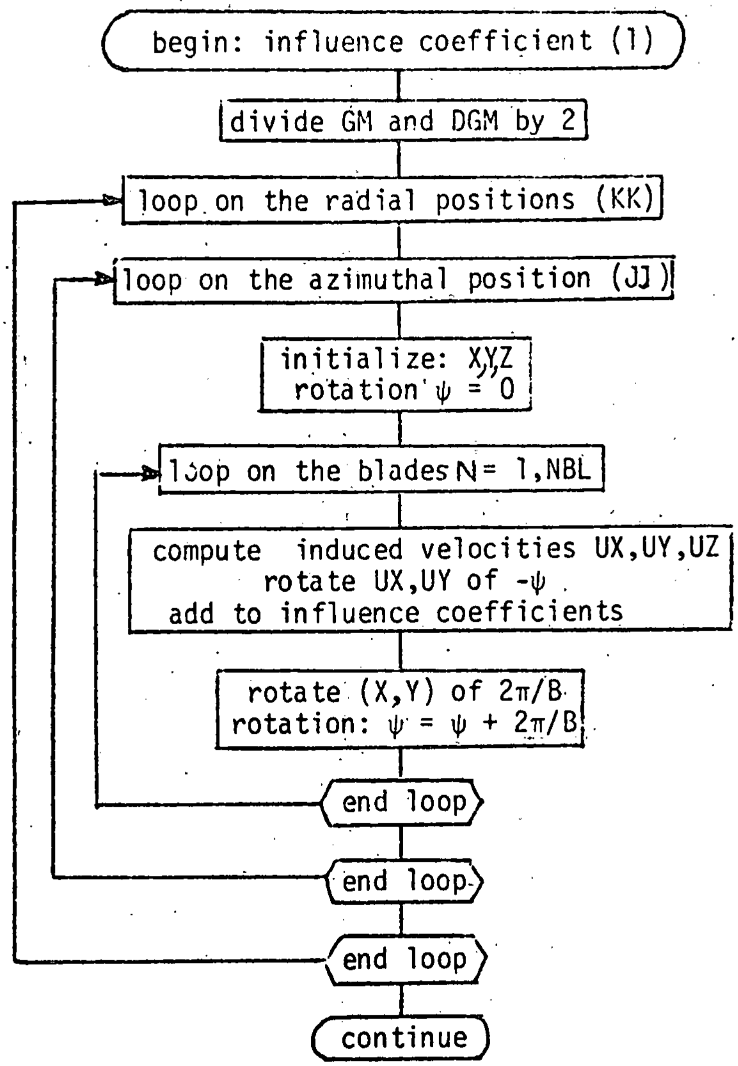
- (UX,UY,UZ ; velocities components returned by WXYZ)

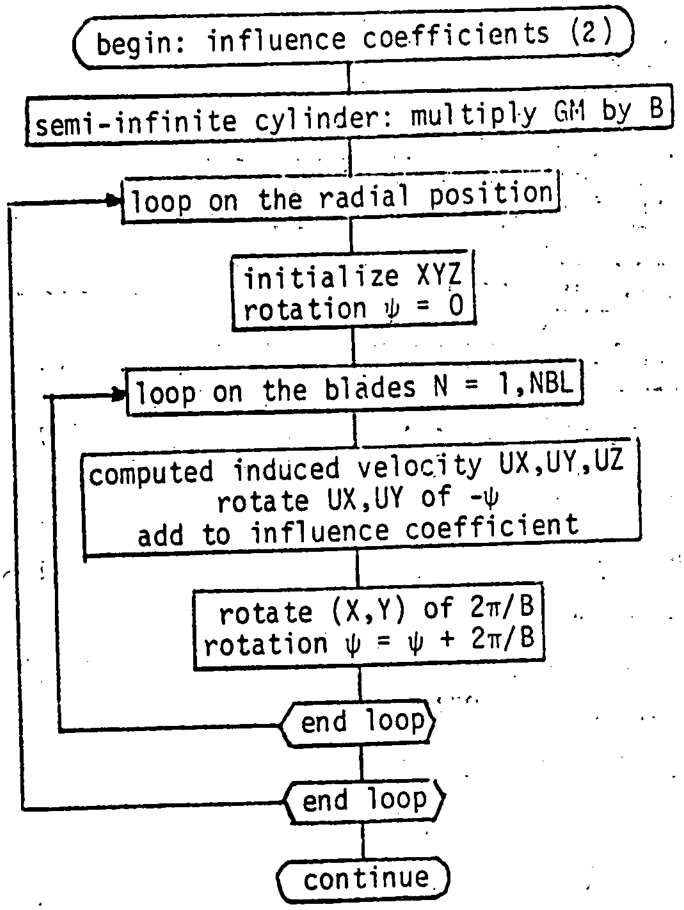
- Case of a semi-infinite cylinder element

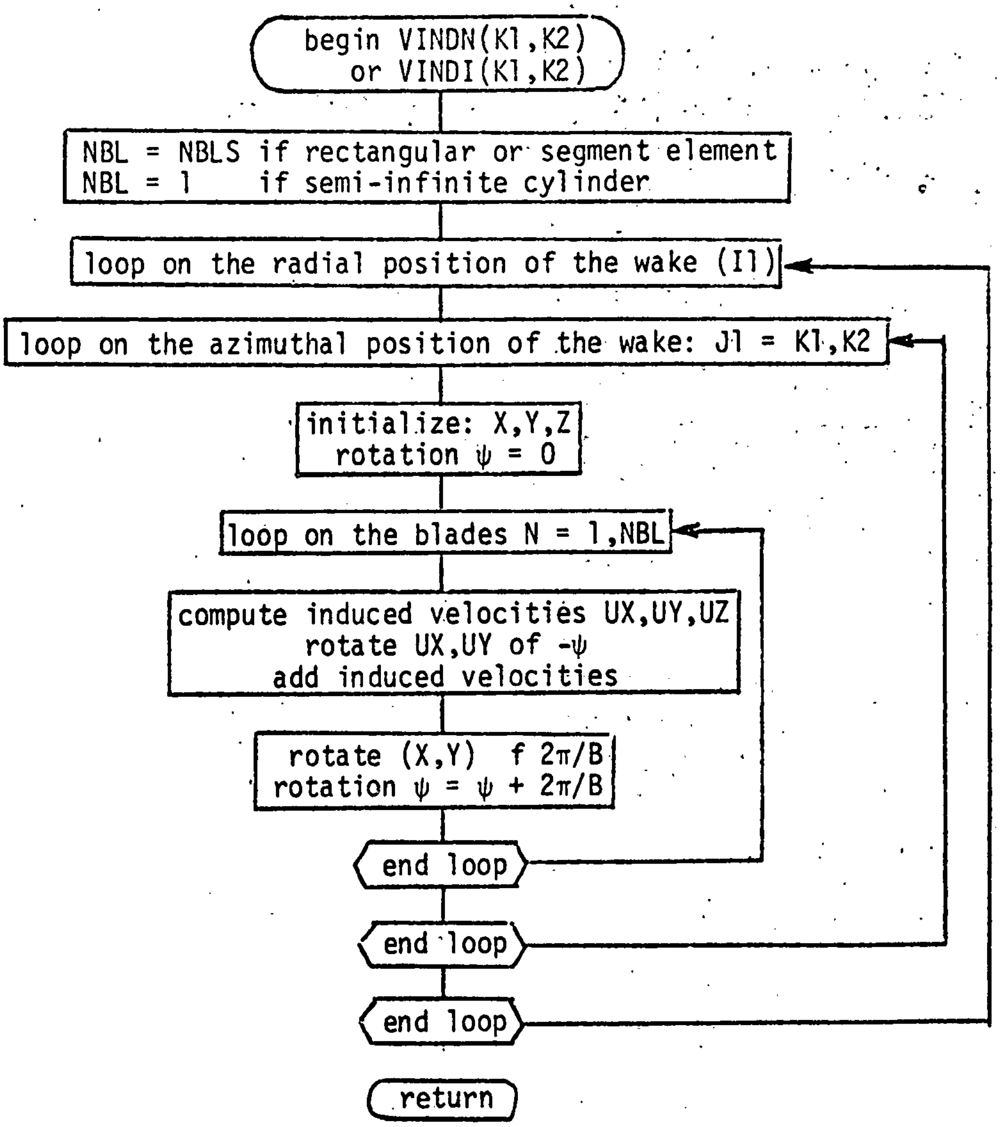
The induced velocity is independent of the azimuth, therefore one element can be considered for the B blades. The strength is multiplied by the number of blades in LOOP2 for the velocities induced on the near and intermediate wake and in VINDB for the calculation of the influer coefficients.



* (1 and 2) : detailde on next flow charts


begin: influence coefficient (1)
divide GM and DGM by 2
loop on the radial positions (KK)
loop on the azimuthal position (JJ)
initialize: X,Y,Z
rotation ψ = 0
loop on the blades N = 1,NBL
compute induced velocities UX,UY,UZ
rotate UX,UY of -ψ
add to influence coefficients
rotate (X,Y) of 2π/B
rotation: ψ = ψ + 2π/B
end loop
end loop
end loop
continue


begin: influence coefficients (2)
semi-infinite cylinder: multiply GM by B
loop on the radial position
initialize XYZ
rotation ψ = 0
loop on the blades N = 1,NBL
computed induced velocity UX,UY,UZ
rotate UX,UY of -ψ
add to influence coefficient
rotate (X,Y) of 2π/B
rotation ψ = ψ + 2π/B
end loop
end loop
continue


begin VINDN(K1,K2)
or VINDI(K1,K2)
NBL = NBLS if rectangular or segment element
NBL = 1 if semi-infinite cylinder
loop on the radial position of the wake (I1)
loop on the azimuthal position of the wake: J1 = K1,K2
initialize: X,Y,Z
rotation ψ = 0
loop on the blades N = 1,NBL
compute induced velocities UX,UY,UZ
rotate UX,UY of -ψ
add induced velocities
rotate (X,Y) f 2π/B
rotation ψ = ψ + 2π/B
end loop
end loop
end loop
return

Subroutine INDVEL

Function : calculation of the induced velocities for a rectangular, a segment or a semi-infinite cylinder element.

Entries : COORD, WXYZ

Note : INDVEL is not used as an entry point.

Arguments : all arguments are transmitted by the common VEL, but is also used, the common FARC which contains the map of the far wake (velocities induced by a semi-infinite cylinder).

Input/Output : none

Parameters used : none

This subroutine is independent of the data structure of the program FWC. It is the most important subroutine, as the quasi-totality of the CPU time is spent in the mathematical subroutines (SQRT, ALOG, ATAN) called in WXYZ. The subroutine is composed of two parts, each part associated to an entry point. For a rectangular or a segment element COORD computes all the constants which depend only on the element, then for each call to WXYZ the velocity induced by the current element is returned. For a semi-infinite cylinder element COORD must not be called.

Case of a rectangular element :

For the call to COORD the following variables must be defined :

- IT : set to 1, indicates that the current element is a rectangle

- $X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3, X_4, Y_4, Z_4$: the coordinates of the points defining the element in space.
- GM_1, GM_2 : the two lumped strength of the element.
- EPS1 : the thickness of the element

Are returned :

- GM : the constant lumped strength component of the element
- DGM : the variable lumped strength component of the element

For a call to WXYZ : must be defined.

- IT : set to 1
- X,Y,Z : coordinates of the point where the velocity induced by the current element is to be computed.
- GM,DGM : strength components of the element, returned by COORD. These strength components may be modified for the calculation of the influence coefficients.

Are returned :

- UX,UY,UZ : component of the velocity induced at (X,Y,Z)

<u>Case of a segment element</u> :

For the call COORD must be defined :

- IT : set to 2 : indicates that the current element is a segment element.
- $X_1, Y_1, Z_1, X_3, Y_3, Z_3$: coordinates of the two end points of the segment.
- EPS2 : core size of the element.

For a call to WXYZ must be defined :

- IT : set to 2
- X,Y,Z : coordinates of the points

Are returned :

- UX,UY,UZ : components of the velocity induced at (X,Y,Z).

<u>Case of a semi-infinite cylinder</u> :

Must be defined for a call to WXYZ

- IT : set to 3 indicates that the current element is a semi-infinite cylinder.
- RHO : radius of the cylinder.
- ZPF : position of the semi-infinite cylinder on the Z axis (the semi-infinite cylinder has OZ as axis and extends from ZPF to + ∞ on this axis).
- TL : tangent λ of the semi-infinite cylinder (the vortex lines make an angle λ with the (X O Y) plane; from the sign conventions of FWC, λ is positive)
- GM : lumped strength of the cylinder
- X, Y, Z : coordinates of the point where the induced velocity will be computed.

Are returned:

- UX, UY, UZ : components of the velocity induced at (X, Y, Z)

Details of the calculations

1 - COORD for a rectangular element.

a) Geometrically the element defined by the four points (M_1', M_2', M_3', M_4') which coordinates ($X_1, Y_1, Z_1, X_2, Y_2, Z_2, X_3, Y_3, Z_3, X_4, Y_4, Z_4$) are given by the calling program and are the corners of a sligthly distorted rectangle. A rectangular approximation of the element is then defined by a local system of coordinates, the half length (ΔX) and the half width (ΔY) of the rectangle.

In the local system of coordinates the four corrected points have the coordinates M_1 ($-\Delta X, -\Delta Y, 0$), M_2 ($-\Delta X, +\Delta Y, 0$), M_3 ($+\Delta X, -\Delta Y, 0$), M_4 ($+\Delta X, +\Delta Y, 0$) the points M_1' and M_3' , M_2' and M_4' are on the same vortex lines, these lines oriented from M_1' to M_3' and M_2' to M_4'.

The local system of coordinates is defined by the coordinates of the centers of the element (C_t) in the general system of coordinates and by the three unit vectors (V_1, V_2, V_3) perpendicular one to each other. The first vector (V_1) is oriented by the mean vortex line.

To limit the effect of the corrections on the induced velocities, the calculations of the mean vortex line and of the length of the element give more importance to the side on which the strength is the largest.

For this the mean vortex line is oriented by AB and the length of the element is given by the length AB, where A [B] is the weighted center of M_1 (γ_1) [M_3 (γ_1)] and M_2 (γ_2) [M_4 (γ_2)], γ_1 the surface strength of the segment M_1M_3 and γ_2 the surface strength of the segment M_2M_4.

To define the second unit vector $\vec{V}_2$, which is given approximately by the direction M_1M_2 (or M_3M_4), we use the vector $\vec{V}_2 = \overrightarrow{CD}$ (C center of M_1M_3 and D center of M_2M_4).

The length CD will give the width ($2\Delta Y$) of the element and $\vec{V}_2$ will be defined by a linear combination of $\vec{V}_2$ and $\vec{V}_1$, such that $\vec{V}_2$ is a unit vector perpandicular to $\vec{V}_1$.

V_2 is then given by the equations :

$$|\vec{V}_2|^2 = 1 \quad V_2 = a\,\vec{V}_1 + b\,\vec{V}_2' \quad \vec{V}_2 \cdot \vec{V}_2' > 0 \quad \vec{V}_2 \cdot \vec{V}_1 = 0$$

We arrive at

$$b + 1/\sqrt{\vec{V}_2' \cdot \vec{V}_2' - \vec{V}_1 \cdot \vec{V}_2}$$

$$a = -b(\vec{V}_1 \cdot \vec{V}_2')$$

the third unit vector ($\vec{V}_3$) is given by $\vec{V}_1 \times \vec{V}_2$

b) The value GM (mean lumped strength) and DGM (variable lumped strength) are computed to limit the calculation time in WXYZ. These values are respectively the half sum and the half difference of the lumped strength relative to the sides M_1M_3 (GM_1) and M_2M_4 (GM_2). These values are placed in the common VEL to allow their temporary modifications for the calculation of the influence coefficient.

c) A test value $RMAX_2$ (r^2_{max}) is also computed : as stated in appendix A, if the distance from M to the element is large compared to the element a shorter formulation can be used. This distance (r_{max}) is taken as twenty times the square root of the area of the rectangle.
By performing the test on the squared distance, a square root calculation can be saved in WXYZ.

d) EPS12 ($(\varepsilon_1/2)^2$) is also calculated, to save computer time

2 - WXYZ for a rectangular element.

(Refer to appendix A for the formulations).

Are first computed : the coordinates of the point M in the local system of coordinates and RO2 the squared distance from the point to the center of the element, if RO2 is larger than RMAX2 a simplified formulation is used.

If RO2 is smaller than RMAX2, the complete formulas are used. To limit the CPU time, the number of operation and the number of calls to mathematical function is left to a minimum by transforming a sum (difference) of logarithms into the logarithms of a product (quotient) and reducing the sum of four (in J_1 and J_3) into one arc tangent by applying twice the transformation arc tangents atan (a) - atan (b) = ($\frac{a - b}{1 + ab}$) + $k\pi$

where k = 0 for : a and b of the same sign

$a > 0,\ b < 0$ and $\frac{a - b}{1 + ab} > 0$

$a < 0,\ b > 0$ and $\frac{a - b}{1 + ab} < 0$

k = 1 for $a > 0,\ b < 0$ and $\frac{a - b}{1 + ab} < 0$

k =-1 for $a < 0,\ b > 0$ and $\frac{a - b}{1 + ab} > 0$

Note : the corrected strength $\Omega_c = \frac{Z^2}{Z_c^2}$ have been implemented by multiplying WYP by (Z12 - EPS12) / Z12.

At the end the components of the induced velocities (WYP and WZP) in the local system of coordinates are transformed into their component UX, UY, UZ) in the general system of coordinates.

3 - COORD for a segment element :

Are computed :

- XCT,YCT,ZCT : the coordinates of the centers of the segment
- DSX,DSY,DSZ : the components of the half segment vector
- RMAX2 : test value (r_{max}^2 with r_{max} = 10 times the length of the element
- FVA : value of ($\frac{2\ \varepsilon_2}{\Delta S}$)^2, precomputed constant

4 - WXYZ for the segment element

(refer to appendix A for the formulations)

Are first computed : the components of the vector M_oM (DXX,DYY,DZZ) with its squared module RO2 (R_o^2) ; this part is common with the beginning of WXYZ for rectangular elements.

- From RO2 : the value RO3 (R_o^3) (r_{max})
- RO2 is tested against RMAX2, if RO2 is larger than RMAX2, I_o is later equal to 1.
- The components of $\frac{\overrightarrow{MoM} \times \overrightarrow{\Delta S}}{2}$ are computed (DSMX,DSMY,DSMZ) and its squared module (DSM2)
- The core size correction is implemented by multiplying the intermediate value FACT by ($\frac{1}{1 + FVA / DSM2}$) which is equal to $\frac{1}{1 + \varepsilon_2^2 / d^2}$, d the distance from M to the line supporting the segment .
- UX,UY,UZ , the components of the velocities induced at M are computed and returned.

5 - WXYZ for a semi-infinite cylinder

(refer to appendix B for the formulations ; and the program GENERF for the concept and utilisation of the map of the far wake).

Note :

- WXYZ computes velocities for ζ negative or positive.
- The induced velocity is set to zero for points outside the wake.

Local variables

(All variables are reals single precision, except when indicated)

- PI : constant = π
- EPI : constant = $1/4\pi$
- TWOPI : constant = 2π
- AA, BB, CC,DD : constants used for the calculation of the position of a point in the map of the far wake.

Rectangular elements.

a) Variables used only in COORD

- AGM1 : absolute value of GM1
- AGM2 : absolute value of GM2
- ZMOD1 : intermediate value : length of the element multiplied by (AGM1 + AGM2)
- TZ1 : 1 / ZMOD1
- DOT : product $\vec{V}_1 . \vec{V}_2'$
- A, B : coefficients used for the transformation $\vec{V}_2'$ $\vec{V}_2$

b) Values computed by COORD, used by WXYZ

- EPS12 : $(\varepsilon_1/2)^2$
- XCT, YCT, ZCT : coordinates of the centers of the element
- ZMOD2 : width of the element
- DX : half length of the segment (ΔX)
- DY : half width of the segment (ΔY)
- DS : area of the element
- RMAX2 : test value = $(R_{max})^2$

- XX_1, YY_1, ZZ_1 : components of the first unit vector
- XX_2, YY_2, ZZ_2 : components of the second unit vector
- XX , YY , ZZ : components of the third unit vector

c) Variables used in WXYZ

- DXX, DYY, DZZ : components of the vector M M in the local system of coordinates.
- RO2 : squared distance MM_o (R_o^2)
- GC : constant component of the strength of the element (GC = GM / ZMOD2)
- GV : variable component of the strength of the element divided by ΔY (GV = DGM / (ZMOD2 * DY))
- XP, YP, ZP : coordinates of the point M in the local system of coordinates. (ZP is also the corrected Z coordinate of the point)
- F : intermediate value : $\frac{\Delta S}{4\pi} \frac{\Omega}{R_o^3}$
- X1P : value $X + \Delta X$
- X1M : value $X - \Delta X$
- Y1M : value $Y + \Delta Y$
- Y1M : value $Y - \Delta Y$
- X1P2 : value $(X + \Delta X)^2$
- X1M2 : value $(X - \Delta X)^2$
- Y1P2 : value $(Y + \Delta Y)^2$
- Y1M2 : value $(Y - \Delta Y)^2$
- Z12 : value $(Z_c)^2$

- R1 : distance $R_1 = MM_1$
- R2 : distance $R_2 = MM_2$
- R3 : distance $R_3 = MM_3$
- R4 : distance $R_4 = MM_4$
- ARG1 to ARG7 : intermediate values used for J_1 and J_3 :
  arguments of the arc tangent during the reduction
- AJ1 : coefficient J_1
- AJ2 : coefficient J_2
- AJ3 : coefficient J_3 multiplied by Z_c
- BJ3 : arc tangent part of J_3
- WYP, WZP : components of the velocity induced by the element at M
  in the local system of coordinates

Segment element

a) Values computed by COORD, used in WXYZ

- XCT, YCT, ZCT : coordinates of the centers of the element
- DSX, DSY, DSZ : half components of the vector $\overrightarrow{M_1M_3}$
- DS2 : half length of the segment
- RMAX : test value (r^2_{max})
- FVA : precomputed value : $(\varepsilon_2)^2$ / DS2

b) Variables used in WXYZ

- DXX, DYY, DZZ : component of the vector M_oM
- RO2 : squared distance MM_o (R_o^2)
- RO3 : R_o^3
- A,ALPHA : intermediate values used for the calculation of I_o (A,α)

- IO : coefficient I_o
- FACT : intermediate value $-\frac{\ell\, I_c\, \Omega_c}{4\pi\, R_o^3}$
- DSMX, DSMY, DSMZ : components of $(\overrightarrow{M_cM} \times \overrightarrow{\Delta S})/\ell$
- DSM2 : value $\left|(\overrightarrow{M_cM} \times \overrightarrow{\Delta S})/\ell\right|^2$

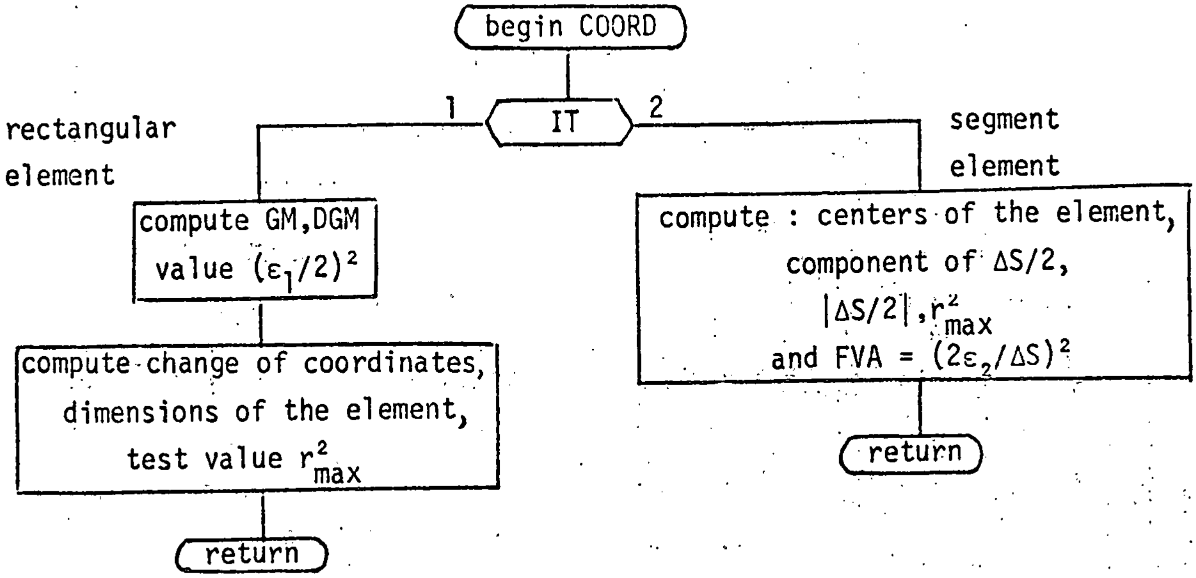
Semi-infinite cylinder element

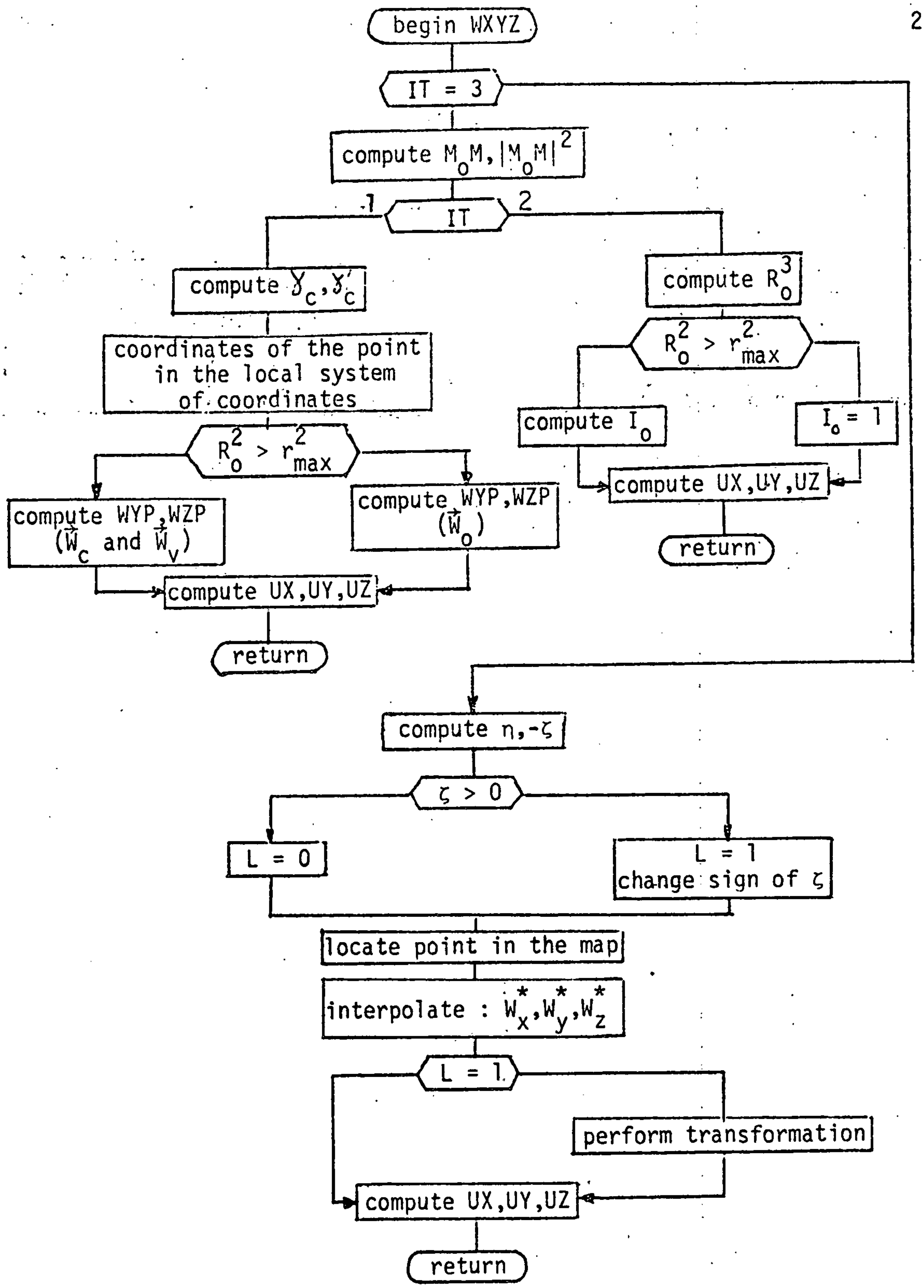
- R : distance from the point to the axis of the cylinder
- ZP : distance from the point to the plane limiting the element
- ETA1, ZETA1 : reduced coordinates of the point ($\eta, -\zeta$)

  (ZETA1 can be + ζ or - ζ)
- L : indicator : set to 1 if ζ is positive.

  Type : integer single precision
- ETA2 : η^2
- ZETA2 : ζ^2
- IETA, IZETA : location of the nearest point in the map of the far wake.

  Type : integer single precision
- E,F,EM1 : interpolation coefficients
- WX_1, WY_1, WZ_1, WX_2, WY_2, WZ_2 : intermediate values, used during the interpolation.
- WX_3, WY_3, WZ_3 : values of W , W , W divided by 2π for the point (η, ζ)


begin COORD
IT
1
2
rectangular element
segment element
compute GM,DGM value $(\varepsilon_1/2)^2$
compute change of coordinates, dimensions of the element, test value r^2_{max}
return
compute : centers of the element, component of $\Delta S/2$, $|\Delta S/2|$, r^2_{max} and FVA = $(2\varepsilon_2/\Delta S)^2$
return



begin WXYZ
IT = 3
compute $M_oM, |M_oM|^2$
IT
1
2
compute γ_c, γ'_c
coordinates of the point in the local system of coordinates
$R_o^2 > r_{max}^2$
compute WYP,WZP ($\vec{W}_c$ and $\vec{W}_v$)
compute WYP,WZP ($\vec{W}_o$)
compute UX,UY,UZ
return
compute R_o^3
$R_o^2 > r_{max}^2$
compute I_o
$I_o = 1$
compute UX,UY,UZ
return
compute $\eta, -\zeta$
$\zeta > 0$
L = 0
L = 1
change sign of ζ
locate point in the map
interpolate : W_x^*, W_y^*, W_z^*
L = 1
perform transformation
compute UX,UY,UZ
return

Subroutine PRTV

Functions : output of the arrays of the common RESUL on the printer
Also : initialization of the arrays

Entries : PRTV,PRTCEN,PRTNOD,PRTVD,CLV,PRTVN,PRTVI.

Arguments : transmitted by commons PARM,GEOM,RESUL

Parameters used : none

Input : none

Local variables ; - I,J : loop variables
- IDEL : indicator, depends on the entry points
- variables beginning by L : format labels

Initialy a debugging tool, the output of PRTV may be used to control the convergence of the program. Conceptionally, by using a single call to the entries of PRTV, a formatted printout of the desired array is executed.
In the final program the calls to PRTV are commanded by some of the parameters therefore the desired output can be obtained without modification of the program.
The name of the entry called is first printed, then the arrays. Each line begins with the name of the array. Values are always printed in increasing radial positions along the line direction, and in increasing azimuthal positions in the vertical direction, one block of lines for each azimutha; position.

Entries : - PRTV : output of the components of the induced velocities at the centers and of the influence coefficients (arrays WXNC,WYNC,WZNC,WXIC,WYIC,WZIC, TWX,TWY,TWZ)

Are also computed and printed the radial components of induced velocities (WR).

- PRTCEN : output of the positions of the centers (arrays XNC, YNC, ZNC, XIC, YIC, ZIC)
- PRTNOD : output of the positions of the nodes (arrays WXNC, WYNC, WZNC)
- PRTVN : output of the components of the velocities induced at the center of the near wake only. (arrays WXIC, WYIC, WZIC)
- CLV : initialisation of the arrays. WXNC, WYNC, WZNC, WXIC, WYIC, WZIC, TWX, TWY, TWZ.

  This entry is normally called at the beginning of the subroutine LOOP2, but is also called when the arrays have been used for a verification routine.
- PRTVD : the action is identical to PRTV plus CLV (execpt that the influence coefficients are not printed). The printed values therefore represent the last call to PRTVD or CLV.

Subroutine LOOP1

Function : - loop on the circulation

- convergence test

Entries : LOOP1

Arguments : transmitted by commons

Input : none

Output : on (IWR)

- intermediate results and test results (ITRCL1=1)

- return code (ITRCL1=1 or ITRACE=1)

- no convergence message

Parameters used : ITRCL1,ITRACE

Return code : ITEST

0 : convergence reached

1 : the convergence of the case is not reached

2 : convergence is not reached with LIM1 iterations on the circulation

The subroutine is composed of two parts :

a) Determination of the distribution of the induced velocities along the blade, and the distribution of the calculation for the current geometry (explicited by the influence coefficients)

b) Convergence test for the case, weighting of convergence is not reached.

Also : calculation of the radial components of the induced velocities (WX)

The convergence return code is set to 2 if part a) does not converge and set to 1 in part b) if convergence is not reached for the case

Details

a) At the beginning of part a) : the arrays WYC,WZC (axial and tangential components of the velocities induced on the blades) and GAMMC (distribution of circulation along the blades) are saved in WYCTT,WZCTT,GTEMP Then an iterative procedure is used to determine the new distribution of induced velocities and of circulation,compatible with the current geometry (position of the wake in space)
The procedure is based on the influence coefficients (TWX(I,J), TWY(I,J),TWZ(I,J)) which give the components of the induced velocity on a blade section located at ETANC(I+1), by a unit circulation at a blade section located at ETANC(J+1)
If this procedure does not converge within LIM1 iterations, the return code (ITEST) is set to 2 and the case is finished.
(This situation may occur if the influence coefficients are unrealistic because of an error)
The convergence test is fixed by the value EPS which is equal to .0005 times the mean axial induced velocity ; both the axial and the tangential components of the induced velocities must pass the test.

A weighting is done for each iteration, based on the average values of the axial induced velocity (old and new distribution of induced velocities for the precedent and current case).

b) Convergence test for the case

Are compared : the distribution of axial and tangential components of the induced velocities along the blades obtained from a) and those distributions coming from the precedent iteration

The test value is EPS=.01 times the mean axial induced velocity

An array (MTEST) may be printed (line labelled TEST) by specifying ITRCL1=1, which shows the results of the test for a given position

A value : 0 : means that the position passes the test

1 : means that the tangential component does not pass the test

2 : means that the axial component does not pass the test

3 : means that both components do not pass the test

Weighting : If convergence is not obtained

- The old and new distributionsof circulation are weighted
- The corresponding distributions of induced velocities along the blade are computed using the influence coefficients

Local variables :

All variables are reals single precision except loop variables, labels and when specified.

- variables beginning by L : format labels
- PI : constant π
- PI2 : constant π^2
- EPS : convergence test value
- TLAM : current value of tangent λ
- FLAMDA : current value of λ
- U : current value of the total velocity relative to the blades
- ALP : attack angle (stall corrected)
- KTEST : internal convergence test result

  KTEST = 1 : convergence obtained

  KTEST = 0 : no convergence

  Type : integer single precision
- FACI,FNI1 : sum of the axial induced velocities on the blade obtained from the current iteration (new distribution)
- FACIE,FVI1 : sum of the axial induced velocities on the blade used by the current iteration (old distribution)
- FNI2 : value of FNI1 of the precedent iteration
- FVI2 : value of FVI1 of the precedent iteration
- XO : intermediate value (calculation of the weighting factors)
- FACV : weighting factor for the old distribution

- WYCT(14) : distribution of the tangential components of induced velocities on the blade obtained from the current iteration (new distribution)
- WZCT(14) : same as WYCT, axial components
- WYCTT(14) : saved distribution of the tangential components of the induced velocities on the blade, (from the last iteration on the geometry)
- WZCTT(14) : same as WYCTT, axial components
- GTEMP(14) : saved distribution of circulation (from the last iteration of the geometry)
- MTEST : results of the final convergence test

Note : All arrays are relative to the inner centers of the blades except GTEMP which follows the convention of GAMMC (2 to NNCR1)

(GAMMC(I) = circulation at η = ETQNC(I))

(WYT(I) = axial induced velocity at η = ETANC(I+1), etc...)

Subroutine WNOFC

Function : determination of the velocities induced at the nodes from the velocities induced at the centers, and weighting of the old and new distribution of the induced velocities.

Also : weighting of influence coefficients

Entries : - INIWN : calculation of the constants used in WNOFC.

- WNOFC

Arguments : transmitted by commons

Input : none

Output : on (IWR)

- by INIWN : interpolation coefficients, if ITRACE $\neq$ 0

- by WNOFC : - if ITRCW $\neq$ 1 : velocities induced at the nodes, influence coefficients

- if ITRCW = 2 : same action as ITRCW, but more detailed (velocities induced at the nodes after each step)

Parameters used : ITRACE,ITRCW,ITRANS

The subroutine is composed of two parts, called from the entries INIWN and WNOFC.

In INIWN, all the interpolation coefficients are computed, they are saved in the common CWNOF, are printed if ITRACE = 1).

Note : the coefficients are computed only once at the beginning of each case, therefore CWNOF must reside in the root segment if the overlay structure is used.

WNOFC by itself is composed of four subsections :

a) Interpolation of the velocities at the nodes from the velocities induced at the centers (in rectangular coordinates).

b) Transformation rectangular/cylindrical coordinates.

c) Weighting of the new induced velocities with those coming from the precedent iteration.

The weighting factors are :

- FRV,FRN for the radial components
- FTV,FTN for the tangential components
- FZV,FZN for the axial components

The coefficients FRV,FTV,FZV are relative to the old distribution of induced velocities ; FRN,FTN,FZN are relative to the new distribution of induced velocities.

d) Weighting of the Y and Z components of the influence coefficients with those coming from the precedent iteration

- FZV : weighting factor for the old set of influence coefficients
- FZN : weighting factor for the new set of influence coefficients

Variables in common CWNOF :

(All variables are reals single precision, execpt when specified)

Note : An index I (or I1) always indicates a radial direction and the index J (or J1) always indicates an azimuthal direction.

- AAN(15) : interpolation coefficients : node I versus the centers I and I+1 of the near wake (radial position). Weighting factor AAN(I) for the center I and 1-AAN(I) for the centerI+1
- AAI(15) : same as AAN, intermediate wake
- ACB(15) : interpolation coefficient, used with the array ICB : node I of the near wake versus the centers I1 and I1+1 of the intermediate wake. I1=ICB(I) (weighting factor ACB(I) for the centers I1 and 1- ACB(I) for the center I1+1)
- ICB(15) : array of address, used with ACB
  Type : integers single precision
- BCB(16) : interpolation coefficients : node J of the transition wake (defined by the near wake) versus the centers J1 and J1+1 of the intermediate wake J1=JCB(J)
  (weighting factor BCB(J) for the centers J1 and 1-BCB(J) for the center J1+1)
- ICB(16) : array of addresses, used with BCB
  Type : integers single precision

- ACC(7) : interpolation coefficients : node I of the intermediate wake, versus the nodesI1 and I1+1 of the near wake, radial direction
  I1=ICC(I)
  (weighting vactor ACC(I) for the node I1 and 1=ACC(I) for the node I1+1)
  Note : used for the transition wake.
- ICC(7) : array of addresses used with ACC
  Type : integers single precision
- BCC(7) : interpolations coefficients : node J of the intermediate wake, versus the nodes J1 and J1+1 of the near wake, azimuthal direction
  J1=JCC(I)
  (weighting factor BCC(J) for the node J1 and 1-BCC(J) for the node J1+1
- JCC(7) : array of addresses used with ICC
  Type : integers single precision

Variables in common WNDATA

This common contains the values of the induced velocities at the nodes, before the weighting with the velocities of the precedent iteration.
The arrays contain the components in rectangular coordinates then in cylindrical coordinates.
The common is used only in WNOFC as a work area ; if the program is used without the overlay structure, WDATA and some commons in other subroutines can be renamed under an identical name, to decrease the size of the program.
All variables are reals single precision

- WXNVT(15,18) : X components of the velocities induced at the nodes of the near wake.
  (WXNVT(I,J) is the velocities induced at the nodes of coordiantes XNV(I,J),YNV(I,J)ZNV(I,J))
- WYNVT(15,18) : same as WXNVT, Y components
- WZNVT(15,18) : same as WXNVT, Z components
- WXIVT(6,50) : same as WXNVT, intermediate wake
- WYIVT(6,50) : same as WYNVT, intermediate wake
- WZIVT(6,50) : same as WZNVT, intermediate wake
- WRNVT(15,18) : radial components of the velocities induced at the nodes of the near wake
  This array is declared in equivalence, element by elemeNt, with WXNVT
- WTNVT(15,18) : same as WRNVT, tangential components, declared in equivalence with WUNVT

- WRIVT(6,50) : same as WRNVT, intermediate wake declared in equivalence with WXIVT
- WTIVT(6,50) : same as WTNVT, intermediate wake declared in equivalence with WYIVT

Local variables :

(All variables are reals single precision, except IDEB and the loop variables)

- variables beginning by L : format labels used for the output
- ET,ETI : used during the calculation of the interpolation coefficients : current radial or azimuthal position, used for the comparisons.
- A : current first radial interpolation coefficient (from AAN,AAI,ACB,ACC)
- A1 : current second radial interpolation coefficient (always equal to 1-A)
- B : current first azimuthal interpolation coefficient (from BCB,BCC)
- B1 : current second azimuthal interpolation coefficient (always equal to 1-B)
- IDEB : index of the beginning of the first azimuthal position of the intermediate wake, transition wake not included. IDEB=5 if the transition wake exist ; 1 otherwise.

  Type : integer single precision
- ET2,ET3 : used during the calculation of F
- F,F1 : interpolation factors, used for the transition wake : for the fouth azimuthal position of the nodes of the intermediate factor for the nodes NNVA of the near wake, and F1 the interpolation factor

wake, and Fl the interpolation factor for the fifth azimuthal position of the centers of the intermediate wake (Fl=1-H)

- AA : current first radial interpolation coefficient (from AAC), used for the calculation of the velocities at the fourth azimuthal position of the nodes of the intermediate wake (apply only in the case of transition wake)
- AA1 : second radial interpolation coefficient, used with AA (AA1=1-AA)
- I,J,I1,J1 : loop variables

Transformation rectangular/cylindrical coordiantes :

For each point :

- WXX : current X component of the induced velocity
- WYY : current Y component of the induced velocity
- XX : current X coordinate of the point
- YY : current Y coordinate of the point
- R : current radial position of the point
- I,J : loop variables

Weighting new/old values of the induced velocities

- FRN,FTN,FZN : weigthing factor for the radial, tangential and axila components of the induced velocities of the current iteration

- FRV,FTV,FZV : same as FRN,FTN,FZN, but for the precedent iteration. (FRV=1-FRN, etc..)

Weighting of the influence coefficients

- FZN : weighting factor for the influence coefficients of the current iteration
- FZV : weighting factor for the influence coefficients of the preceding iteraion
- I,J, : loop variables

Determination of the velocities induced at the nodes, from the velocities induced at the centers (subsection a)

Two cases have to be considered, following the existence or not of a transition wake (parameter ITRANS).

First case : no transition wake.

The velocity at a node (I,J) of the near or intermediate wake is determined by interpolation between the velocities induced at the centers (I,J),(I+1,J),(I,J+1) and (I+1,L+1)

Where I goes from 1 to NNVR(NIVR)

and J goes from 1 to NNVA(NIVA)

The interpolation is performed by the parts noted A and B in the listing

Second case : transition wake

The velocities at the nodes of the near wake, transition wake not included, are determined in the same way as in the first case (part A) (J goes from 1 to NIVA)

For the transition part : the velocities at the nodes NIVA + 1(=NICA) to NNVA are given by the sum of :

1 - The velocities induced at the centers NICA to NNCA of the near wake (part A)

2 - The velocities induced at the centers j to 4 of the intermediate wake (part B)

The velocities at the nodes of the intermediate wake, transition wake not included are determined in the same way as in the first case (part C) (J goes from 5 to NIVA)

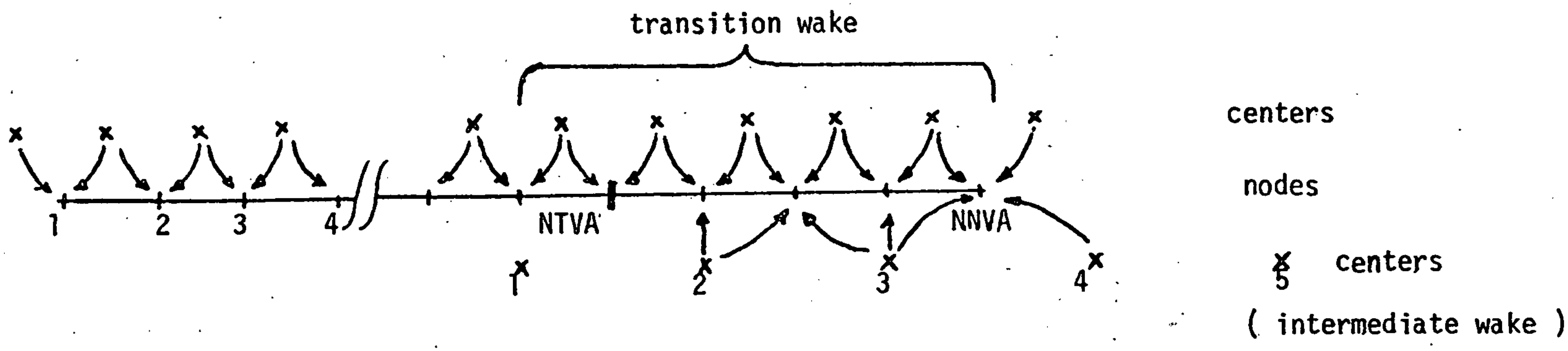
For the azimuthal positions 1,2 and 3 of the transition part of the intermediate wake, the velocities are determined by interpolation of the velocities induced at the nodes of the transition part of the near wake (part D)

The velocities at the azimuthal position 4 of the intermediate wake are determined by interpolation of

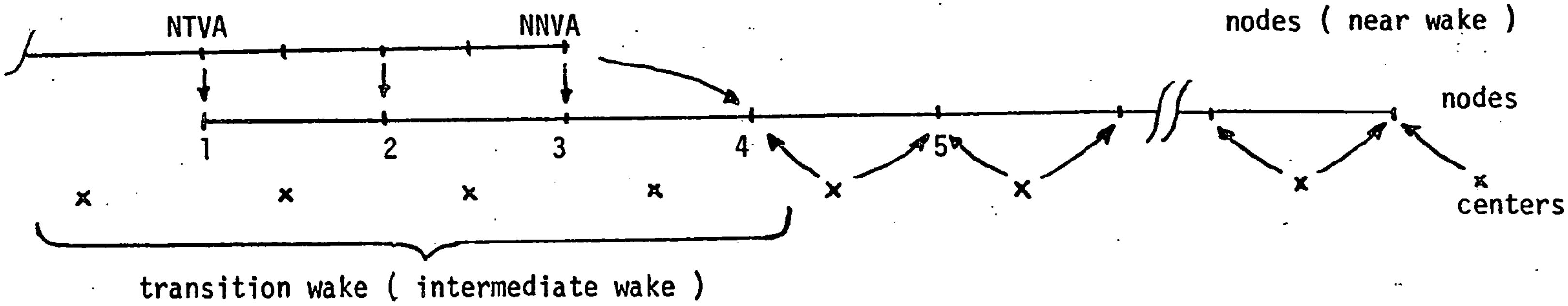
1 - The velocities induced at the center of the azimuthal position 5 of the intermediate wake

2 - The velocities induced at the nodes of the azimuthal position NNVA of the near wake (part E)

Contributions to the nodes of the near wake



Contribution to the nodes of the intermediate wake



Function : integration of the velocities along the streamlines (determination of the new geometry of the wake).

Entry : INTGR

Arguments : transmitted by commons

Parameters used : - ITRTG : trace option
- IVERGR : verification of the integration

Input : none

Output : on (IWR) :
- indices of the reference streamlines
- detailed intermediate results
(if ITRTG = 1 or IVERGR = 1)

Using the velocities induced at the nodes (interpolated by WNOFC, from the velocities at the centers, returned by LOOP2), an updated geometry (positions of the nodes) is computed, by integration.

One must keep in mind, that the velocities have been computed at control points which are slightly different than those which will be returned; the integration must then be performed in cylindrical coordinates, and use the old locations of the nodes to avoid undesirable side effects. Some small corrections, which have no effect once the convergence is obtained, are also used.

Furthermore, while they must be kept on the same streamline for a given radial position, the nodes do not need to have the same azimuth, or to have the same age (here: angle of which the rotor has turned since the points have left the blade). This permits making systematic corrections, to limit the distorsion of the rectangular elements.

The nodes located on a reference streamline will keep a constant time increment between them, and for a given azimuth position, the other nodes will be constrained to be in a plane (P), containing the reference node, its projection on the Z axis, perpendicular to the plane containing the velocity at the reference node and the projection of the reference node on the Z axis.

Local variables

As the subroutine uses many temporary arrays, these have been place in a common (WDATA), if the program is used without the overlay structure, WDATA and some commons in other subroutines can be renamed under an identical name, to decrease the size of the program.

a) Arrays in common (WDATA)

- XT(16),YT(16),ZT(16) : current rectangular coordinates of the nodes of the J^{th} azimuthal position, before and after the correction to limit the distortion of the elements.
- R1N(16) : current radial coordinates of the new nodes of the $(J-1)^{th}$ azimuthal position
- VR1N(16) : current radial velocities at the new nodes of the $(J-1)^{th}$ azimuthal position
- VT1N(16) : current tangential velocities at the node of the $(J-1)^{th}$ azimuthal position
- VZ1(16) : current axial velocities at the nodes of the $(J-1)^{th}$ azimuthal position
- VX2(16),VY2(16),VZ2(16) : current velocities at the nodes of the J^{th} azimuthal position (rectangular coordinates)

b) Variables not in common WDATA

- ETAM : test value : mean radial position, used to choose the reference streamline
- ISAV : saved value of ITRTG, as ITRTG may be temporally modified in INTGR

  Type : integer single precision

- KW : loop variable

  KW = 1 : near wake

  KW = 2 : intermediate wake

- IK : index of the reference streamline

  Type : integer single precision

- DPSI : nominal azimuthal increment

  (DPSIN (IK = 1) or DPSII (IK = 2)),

  divided by two

- NVA : number of azimuthal position of the current wake

  (NNVA or NIVA)

  Type : integer single precision

- NVR : number of radial position of the current wake

  (NNVR or NNVA)

- ET : temporary value, used for the interpolation of the nodes of the first azimuthal position of the intermediate wake

- A,A1 : interpolation coefficients (initialization of the intermediate wake integration)

- R1O : current radial position of the old node of the first azimuthal position of the intermediate wake

  (initialization of the intermediate wake integration)

- AT : radial positions of a node after and before integration)

- R2O : current radial position of a node at azimuthal position J, before integration ("old" node)

- VR2O : current radial component of the velocity at a node

  (azimuthal position J), before radial correction

- VT2O : same as VR2O, tangential component
- R2N : current radial position of a node at azimuthal position J, after integration, but before correction for the distortion of the elements
- VR2N : current radial component of the velocity at a node (azimuthal position J), after radial correction
- VT2N : same as VR2N, tangential component
- TH2 : azimuth of the current node (azimuthal position J); before correction for the distortion of the elements
- CC,SS : cosine and sine of TH2
- I,I1,I2,J : loop variables

Correction for the distortion of the elements

- XO,YO,ZO : rectangular coordinates of the reference node
- VX,VY,VZ : components of the velocity at the reference node (rectangular coordinates)
- ALPHA : correction factor for VX,VY,VZ
- BETA : amplitude of the correction for the current node

Details of the integration

The velocities at a given point, relative to the general system of coordinates (fixed with respect to the rotor) is the sum of the free stream velocity and of the induced velocity.

The component of the free stream velocity at a point M (ρ,ψ,Z)is ($\rho\Omega, 0, V_Z$),

where: ρ is the radial position of the point M

Ω is the angular velocity of the rotor

V_z is the wind velocity

As non dimensional coordinates are used, we have:

$\Omega = 1$

$V_z = \mu$ (advance ratio)

The velocity components at a point M are therefore

$$\left| \begin{array}{l} W_R \\ \rho + W_T \\ \mu + W_Z \end{array} \right.$$

(W_R, W_T, W_Z the radial, tangential and axial components of the velocity induced at M)

For a time increment Δt, during which the rotor has turned-off $\Delta\psi = \Omega\Delta t$ the point has moved from the position (1) (azimuthal position J-1, where $W_R = W_{R1}$, $W_T = W_{T1}$, $W_Z = W_{Z1}$) to the position (2) (azimuthal position J, where $W_R = W_{R2}$, $W_T = W_{T2}$, $W_Z = W_{Z2}$)

During the rotation the mean velocities are

$V_{Rm} = 1/2\ (W_{R1} + W_{R2})$

$V_{Tm} = 1/2\ (W_{T1} + W_{T2})$

$V_{Zm} = 1/2\ (W_{Z1} + W_{Z2})$

The coordinates of the point M_2 (R_2, ψ_2, Z_2) are determined from those of the point M_1 (R_1, ψ_1, Z_1)

by :

$$R_2 = R_1 + V_{Rm}\,\Delta t = R_1 + V_{Rm}\,\Delta\psi$$

$$\psi_2 = \psi_1 + \frac{V_{Tm}}{R_m}\,\Delta t = \psi_1 + \frac{V_{Tm}}{R_m}\,\Delta\psi$$

$$Z_2 = Z_1 + V_{Zm}\,\Delta t = Z_1 + V_{Zm}\,\Delta\psi$$

(R_m = mean radius = ($R_1 + R_2$) / 2)

To avoid problems during the first iterations (where the radial displacement of a point may be large). The radial and tangential velocities are mulitiplied by the ratio of the new and old radius. Using R_{20} the old radial position of the point, the radial and tangential velocities will be given by :

$$V'_{Rm} = V_{Rm}\; R_2/R_{20}$$

$$V'_{Tm} = V_{Tm}\; R_2/R_{20}$$

The new radial position (R_2) will be given by :

$$R_2 = R_1 + \frac{\Delta\psi}{2}\left(V_{R1} + V_{R2}\,\frac{R_2}{R_{20}} \right)$$

$$R_2 = \left(R_1 + \frac{\Delta\psi}{2} V_{R1} \right) \Big/ \left(1 - \frac{\Delta\psi}{2}\,\frac{V_{R2}}{R_{20}} \right)$$

Correction for the distortion of the elements

Because of the cumulative effect of the tangential induced velocities, the rectangular element would be deformed, especially for the intermediate wake

As the control points (nodes) of a given radial position need only to be on the same streamline, the nodes can be moved along the streamline .

Therefore, allowing the nodes to move along the line determined by the velocity at that point (VX2,VY2,VZ2), they are constrained to be in the same plane (P).

The plane contains the node of a reference streamline (position vector $\vec{X}_o$ (X_o, Y_o, Z_o)), its projection on the Z-axis (position vector $\vec{X}_o'$ ($0, 0, Z_o$)) and is perpandicular to the plane (P') containing the point ($0, 0, Z_o$) and the velocity vector at the reference node $\vec{V}$ (V_x, V_y, V_z) the plane (P) is therefore perpandicular to the vector $\vec{V}'$ (V_x', V_y', V_z')

$$\vec{V}' = \vec{V} + \alpha (\vec{X}_o - \vec{X}_o') \qquad V' (\vec{X}_o - \vec{X}_o') = 0$$

$$\alpha = - \frac{\vec{V} \cdot (\vec{X}_o - \vec{X}_o')}{|\vec{X}_o - \vec{X}_o'|^2}$$

Correction of the point :

The position vector of the corrected point ($\vec{X}$) is given by :

$$\vec{X} = \vec{X_t} + \beta \vec{V_2}$$

where - $\vec{X}_t$ is the position vector of the uncorrected point,

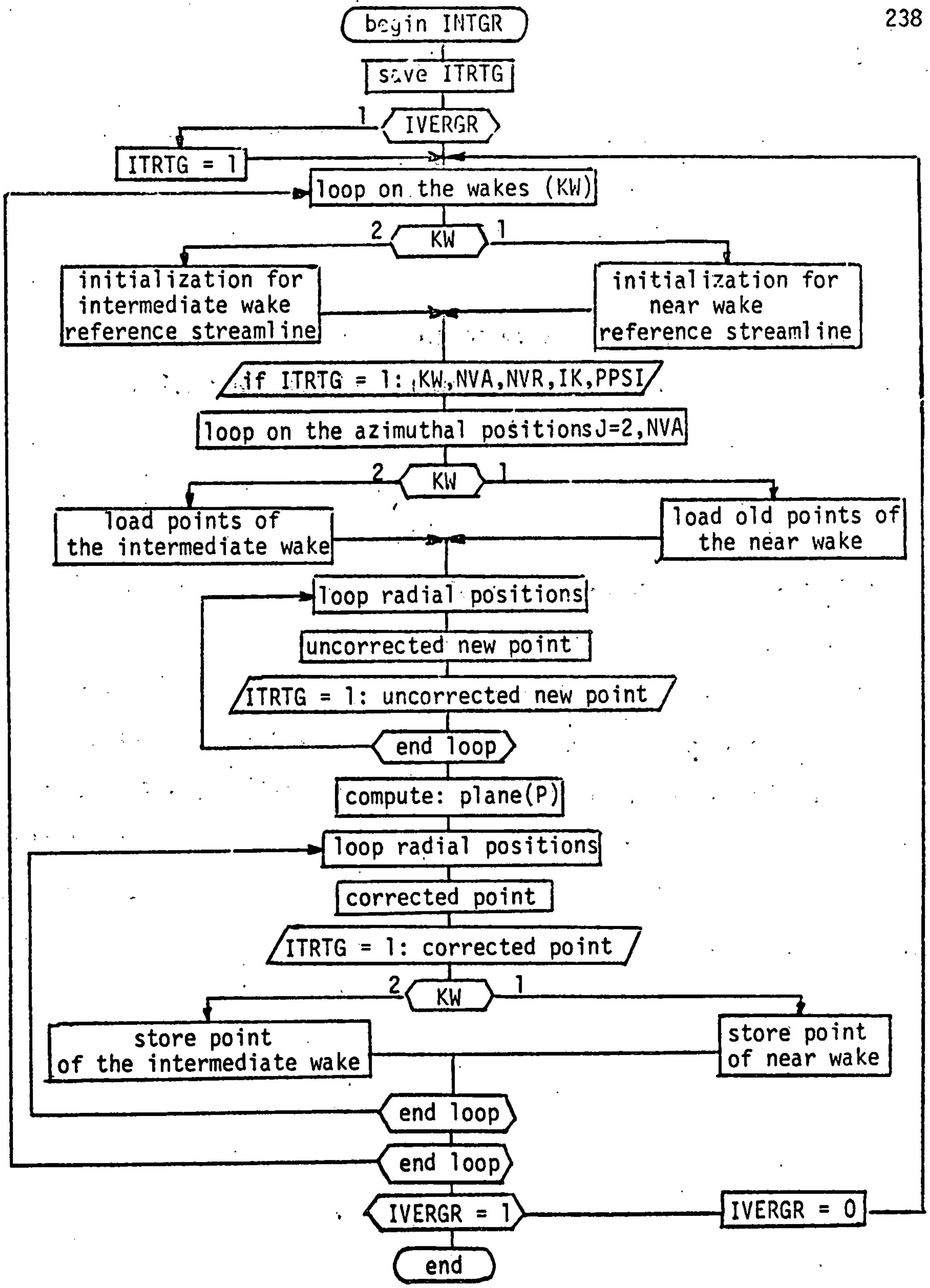
- $\vec{V}_2$ is the velocity at the point

$\vec{X}$ satisfies :

$$(\vec{X} - \vec{X_o}) . \vec{V} = 0$$

therefore :

$$\beta = - \frac{(\vec{X_t} - \vec{X_o}) . \vec{V}'}{\vec{V}' . \vec{V_2}}$$


begin INTGR
save ITRTG
IVERGR
1
ITRTG = 1
loop on the wakes (KW)
KW
2
1
initialization for intermediate wake reference streamline
initialization for near wake reference streamline
if ITRTG = 1: KW,NVA,NVR,IK,PPSI
loop on the azimuthal positions J=2,NVA
KW
2
1
load points of the intermediate wake
load old points of the near wake
loop radial positions
uncorrected new point
ITRTG = 1: uncorrected new point
end loop
compute: plane(P)
loop radial positions
corrected point
ITRTG = 1: corrected point
KW
2
1
store point of the intermediate wake
store point of near wake
end loop
end loop
IVERGR = 1
IVERGR = 0
end

Subroutine OUT

Function : - calculation of the force components and performances
- output of the results

Entries : OUT,OUTINT

Arguments : transmitted by commons

Input : on (IWR) : formated output of the results

Parameters used : none

The subroutine composes the output section of the program.

The first part includes all the quantities relative to the blade and the performance coefficients.

The second part includes the positions of the nodes and the velocities induced at these points (rectangular and cylindrical coordinates).

By calling OUTINT, only the first part is printed ; OUTINT is used only to print intermediate results, after each iteration on the geometry. One can refer to subsection " Description of the user's output " for the format of the output.

Local variables :

As the subroutine uses many temporary arrays, these have been placed in a common (OTDATA) ; if the program is used without the overlay structure, OTDATA and some commons in other subroutines can be renamed under an identical name, to decrease the size of the program.

All variables are reals single precision except loop variable and when specified.

a) Arrays in common OTDATA :

- PSI(15) : azimuths of the nodes of the current azimuthal position (in degrees)
- R(15) : radial positions of the nodes of the current azimuthal position
- GMC(14) : distribution of circulation
- WXY(15) : X components of the velocities induced at the nodes of the current azimuthal position
- WYT(15) : Y components of the velocities induced at the nodes of the current azimuthal position
- FLAMDA(14) : distribution of inflow angle (λ)
- U(14) : distribution of the total velocity relative to the blade
- ALPHA(14) : distribution of attack angle (α)
- LIP(14) : in-plane components of the lift
- TLIP(14) : thrust component of the lift
- DP(14) : drag distribution
- FDP(14) : in-plane components of the drag

- TP(14) : in-plane force distribution
- FP(14) : thrust distribution
- XT(15),YT(15) : X and Y coordinates of the nodes of the current azimuthal position
- WT(15),WR(15) : tangential and radial components of the induced velocities of the current azimuthal position

b) Variables not in common OTDATA :

- All variables beginning by L : format labels used for output
- PI2 : constant : π^2
- PI : constant : π
- CONV : conversion factor radian into degrees ($\pi/180$)
- IOUT : entry point indicator

  IOUT = 1 : entry by OUT

  IOUT = 2 : entry by OUTINT

  Type : integer single precision
- TLAM : current value of tangent λ
- ALPHA1 : equivalent attack angle (stall effect included)
- CD : current drag coefficient
- LT : thrust coefficient $L_t = T/(\rho\pi\Omega^2R^4)$
- LP : power coefficient $L_p = P/(\rho\pi\Omega^3R^5)$
- CT : thrust coefficient $C_t = T/(\rho\pi R^2V^2)$
- CP : power coefficient $C_p = P/(\rho\pi R^2V^3)$
- CTCP : ratio C_t/C_p

- IW : loop in the wakes variables

  IW = 1 : near wake

  IW = 2 : intermediate wake

  Type : integer single precision

- NVR : number of nodes (radial direction)

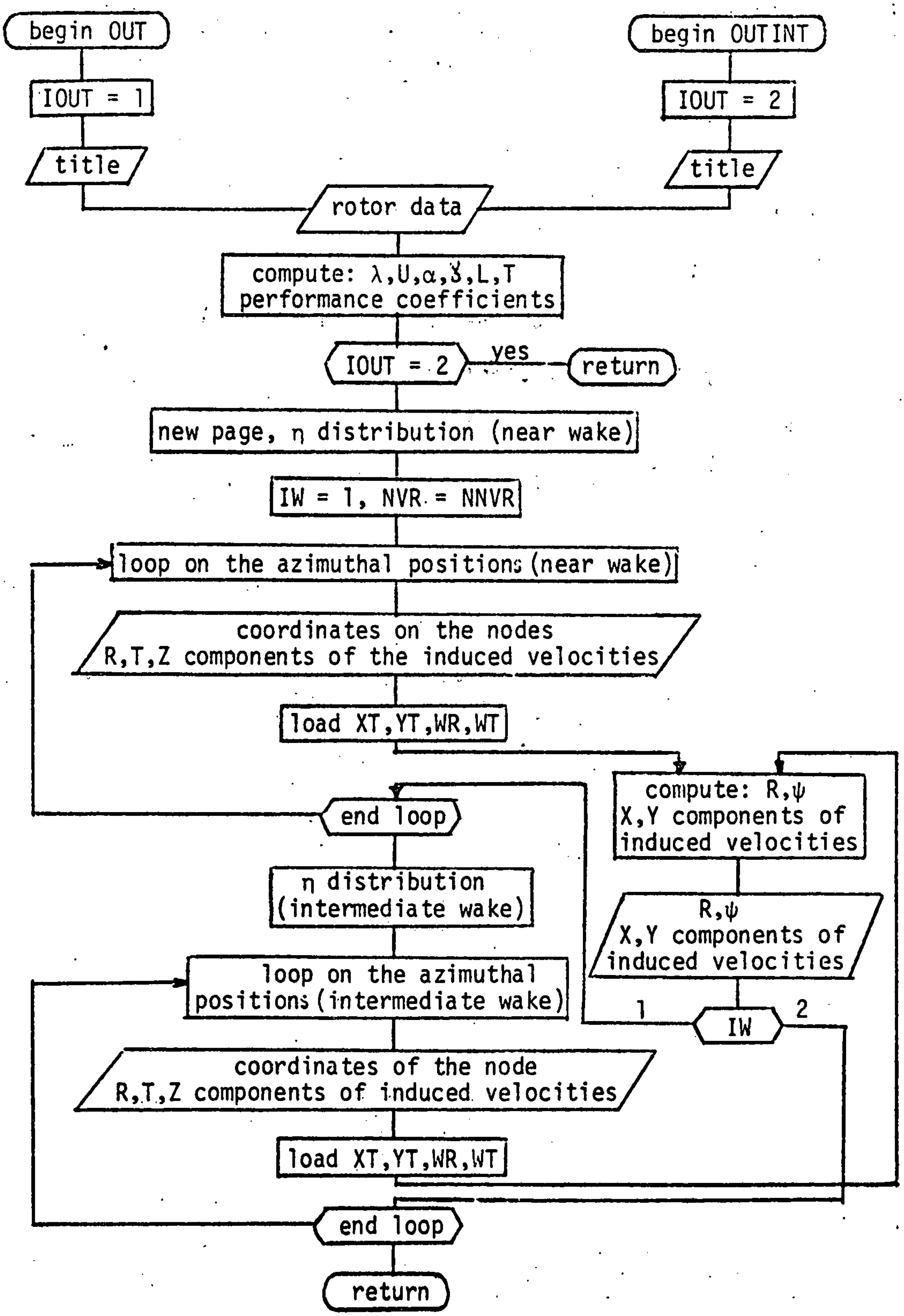
  (NVR = NNVR for the near wake

  NVR = NIVR for the intermediate wake)

  Type : integer single precision

- CC,SS : cosine and sine of the azimuth of the current node

- I,J : loop variables



begin OUT
IOUT = 1
title
begin OUTINT
IOUT = 2
title
rotor data
compute: λ,U,α,ɣ,L,T
performance coefficients
IOUT = 2
yes
return
new page, η distribution (near wake)
IW = 1, NVR = NNVR
loop on the azimuthal positions (near wake)
coordinates on the nodes
R,T,Z components of the induced velocities
load XT,YT,WR,WT
compute: R,ψ
X,Y components of
induced velocities
end loop
R,ψ
X,Y components of
induced velocities
η distribution
(intermediate wake)
loop on the azimuthal
positions (intermediate wake)
1
IW
2
coordinates of the node
R,T,Z components of induced velocities
load XT,YT,WR,WT
end loop
return

Subroutine KTRC

Function : plotting on the printer

Entries : - TRCPT : plotting of the positions of the nodes and the centers (radial and axial coordinates)

- TRN : plotting of the positions of the nodes (radial and axial coordinates)

- TRCVEL : plotting of the velocities

- TRCW : plotting of a side view of the wake

- TRCT : plotting of the tip vortex (radial and axial position versus the azimuth)

- TRCG : plotting of the distribution of circulation along the blade and of the strength of the elements

Arguments : transmitted by commons

and : TRCW(ISCA)

ISCA : size of the plotting

1 : single page width

2 : double page width

Input/Output : none

Parameters used : ITRANS

Subroutines called : TRACE(TRCINI,TRCPTS,TRCSEG,TRCCUB,TRCLIN,TRCEXE)

This subroutine is the interface between the program FWC and the general purpose subroutine TRACE.

For a given entry point, KTRC generates the data in a suitable form for TRACE and calls the corresponding entries.

Local variables :

This subroutine uses large temporary arrays, these have been placed in a common TRCDA, if the program is used without the overlay structure, TRCDA and some commons in the other subroutines, can be renamed under an identical name, to decrease the size of the program.

a) Arrays in TRCDA

- X(300),Y(300),XX1(300),YY1(300),XX2(300) : contain the X and Y coordinates of the points to be plotted.
  (X : horizontal direction)
  (Y : vertical direction)
  Type : reals single precision

b) Variables not in common TRCDA

- ITIT1 to ITIT6 : six arrays containing the titles of the different plottings.
  Type : arrays of integers single precision, dimension (30)
- IPLUS : character " + "
- ICOM : character " ' "
- ISTAR : character " * "
- CHX : character " X "
- CHY : character " Y "

- CHZ : character " Z "
- CHR : character " R "
- CB : character " B "
- IC(15) : characters " 1,2,3,4,5,6,7,8,9,0,A,B,C,D,E "

  (used to number the streamlines)
- CONV : convention factor radians into degrees ($\pi/180$)
- IZ : entry indicator
  - IZ = 1 : entry by TRCPT
  - IZ = 2 : entry by TRCN

  Type : integer single precision
- IP : current number of points

  Type : integer single precision
- FACTX : X scale factor
- FACTY : Y scale factor
- FXX,FXY,FXZ,FYX,FYY,FYZ : projection factors
- XMINS,XMAXS : X definition of the window (in TRCW)
- CC,SS : position of the current point of the tip path
- IDEB : first azimuthal position of the nodes of the intermediate wake, transition wake not included

  1 if ITRANS = 0

  3 if ITRANS = 1
- NBLDS2 : number of blades plotted (in TRCT)

Interpretation of the plottings

- TRCPT : plotting of the nodes and of the centers (radial and axial position).

  The nodes are plotted with a character corresponding to the axial position. The centers with the character " X ".

  The horizontal direction of the page corresponds to the axial position.

- TRCN : identical to TRCPT, only the nodes are plotted.

- TRCVEL : plotting of the velocities induced at the nodes. The nodes are plotted with the character " * ". The velocities are represented by segments attached to the corresponding nodes.

  The grid is the same as the one used in TRCPT.

  Scale : in both directions (radial and axial) a square side width (or heigth) corresponds to a velocity component equal to μ (the free stream velocity).

  The tangential components are plotted with an angle of 45° (character " ' ").

- TRCW : plotting of a side view of the wake.

  Because of the lack of definition, the user can have the plotting on a double width of paper (ISCA = 2), in that case, the plotting is done twice with a different window position (defined by XMINS and XMAXS).

  The view angle can be modified by setting the projection factors (FXX to FYZ) to other values.

A point in space located at (X,Y,Z) will be projected into a point (x,y) with

$$x = X * FXX + Y * FXY + Z * FXZ$$

$$y = X * FYX + Y * FYY + Z * FYZ$$

The total window used is defined by (-1,2) in x direction, and (-1.5, +1.5) in y direction.

Are plotted :

- the blade tip path (character " * ")
- the three axis (character " X ", " Y ", and " Z ")
- the streamlines of the near wake and of the intermediate wake (the character is the streamline number)
- the lines joining the nodes in radial direction (character " ' ")
- the projections of the nodes on the rotor plane (the character is the streamline number)

Note : The transition wake is defined only by the near wake.

- TRCG : plotting of the distribution of circulation along the blade and of the distribution of strength of the elements (derivative of the circulation).

  Note : The value given by the scale must be multiplied by 10 to obtain the value of the circulation.

- TRCT : plotting of the tip vortex.
  - character " Z " : axial position of the tip vortex versus the azimuth.
  - character " R " : radial position of the tip vortex versus the azimuth.

Notes : - the azimuth scale is in vertical direction and in degrees.

- the radial position is multiplied by 2 and the value .6 on the X scale corresponds to a radial position of 1.

Subroutine TRACE

Function : plotting on the printer

Entries : TRCINI : intitialisation on the plotting

TRCPIS : definition of a curve composed only of points

TRCLIN : definition of a continuous cruve passing through a set of points, by segments

TRCSEG : definition of a curve : segment of length DELX, DELY, attached at points X,Y.

TRCCUB : definition of a continuous curve passing through a set of points, by segments of cubics.

TRCEXE : execution of the plotting

Input : on the unit specified by (JOUT), which specifies a printer

- error messages
- plottings

Arguments : for TRCINI :

- XI,XA : limits of the plotting in horizontal direction. (X coordinates of the corners of the plotting).

  Type : reals single precision

  Limitation : XA > XI

- YI,YA : limits of the plotting in vertical direction (Y coordinates of the corners of the plotting).

  Type : reals single precision

  Limitation : YA > YI

- NCX,NCY : size of the plotting on the page. The number of characters will be 20 * NCX + 1, in X direction and 10 * NCY + 1, in Y direction.

Type : integers single precision

Limitations : 1 NCX 6

1 NCY 50

- IXY : work area supplied by the calling routine.

  Type : array of integers half precision.

  (IBM : 2 bytes or half word).

- IDIM : size of IXY

  Type : integer single precision.

- ITIT1 : title of the plotting : 120 characters in 30 array elements.

  Type : array of integers single precision.

- JGRIG : indicator : optional plotting of the grid.

  Range : 0 or 1

  0 : the grid is not plotted

  1 : the grid is plotted.

- JOUT : output dataset reference number (must specify a printer).

  Type : integer single precision

  For TRCPTS : ICAR, X, Y, N.

  For TRCLIN : ICAR, X, Y, N.

  For TRCSEG : ICAR, X, Y, DELX, DELY, N.

  For TRCCUB : ICAR, X, Y, N, IDIR.

- ICAR : character used for the current curve.

  Type : integer single precision.

- X,Y : X and Y coordinates of the points defining the curve, N points must be defined.

  Type : arrays of reals single precision, dimension N .

- DELX,DELY : components of the segments attached to the points, N points must be defined

  Type : arrays of reals single precision

- N : number of points in the arrays X,Y,DELX,DELY.

  Type : real single precision

  Limitations: for TRCCUB $N \geqslant 4$

  for TRCLIN $N \geqslant 2$

  for TRCSEG $N \geqslant 1$

  for TRCPTS $N \geqslant 1$

- IDIR : parameters of TRCCUB : direction of the curve

  Type : integer single precision

  Range : 1 or 2

  1 : a cubic interpolation of the form $Y = aX^3 + bX^2 + cX + d$ is used. The elements of the array X must be in increasing order.

  2 : a form $Y = aX^3 + bX^2 + cX + d$ is used. The element of the array Y must be in increasing order.

TRCEXE has no arguments.

Limitations : - 30 curves on the same plotting

- IDIM-1 points on the same plotting
- 10 superpositions on a single line

Notes : - the validity of the arguments is verified.

- all points outside the window are ignored.
- when an error is detected, the control is immediately given back to the calling routine, up to the next call to TRCINI.
- if one of the limitations (number of points or number of curves) is reached, the plotting is immediately executed, all successive calls are ignored up to the next call to TRCINI.
- TRACE cannot be overlaid between the call to TRCINI and the call to TRCEXE.

This general purpose subroutine have been developed for the program FWC, but is independent from its data structure. Using this routine, the user can plot the locations of the control points, vectors representing the induced velocities, the distribution of circulation, etc...

Although the definition is not excellent (limited by the space between two characters), the results can be analysed very rapidly.

Utilisations : for each plotting, a call to TRCINI initializes the routine (definition of the window, title, size of the plotting on the page, etc...), then by a serie of calls to the entries TRCPTS, TRCLIN, TRCSEG, TRCCUB, the user defines the curves to be plotted (one curve correspond to a call, except when two curves are plotted with the same character, in that case, they are merged internally), the integer coordinates are computed and saved under a compressed form.

''+h a final call to TRCEXE, the plotting is executed.

Internal variables : all variables are integer single precision, except when specified.

- L : indicator :

  Equal to 0 : TRCINI has not been called or an error has been detected.

  Equal to 1 : TRCINI has been called, no error has been detected.

- NX : number of characters in X directions (NCX*20+1)
- NY : number of characters in Y directions (NCY*10+1)
- DX,DY : width and heighth of the window

  DX = XA - XI, DY = YA - YI

  Type : reals single precision
- NCURB : current number of curves.
- NP : current number of points in IXY.
- IX,IY : integer coordinates of the current point in the local system of coordinates.
- N1 : total number of segments (TRCLIN and TRCSEG).
- ISEG : entry indicator : 0 : entry by TRCLIN

  1 : entry by TRCSEG

  Loop variable in TRCCUB.
- IX1,IY1,IX2,IY2 : integer coordinates of the current point of a segment in the local system of coordinates.
- NPZ : number of characters for the current segment.
- XXX,YYY : increment in X and Y direction (TRCLIN and TRCSEG) coordinates of the current

point in the user's system of coordinates (TRCCUB).

Type : reals single precision.

- ND : first point of the segment (1 if the segment is only 1 character long, or if it is the first segment defined by TRCLIN ; 2 otherwise).
- LL : indicator : if set to 1 : indicates that the current curve contains at least one point to be plotted.
- LV : indicator : if set to 1 : indicates that the limit of IDIM-1 points is exceeded.
- IB,IE : array of pointers to the beginning and the end of each curve in IXY.
- JCAR : array of character the Ith curve is plotted with the character JCAR(I).
- I,J,ISEG,K : loop variables.
- IRET,KRET : assigned GOTO variables.

For TRCCUB :

- II : pointer, first of the four points needed for the cubic interpolation.
- A,B,C,D : a,b,c,d of the equation

  $$Y = aX^3 + bX^2 + cX + d.$$

  Type : reals single precision.
- DET,S,P,FC,FC1 : intermediate variables for the calculation of a,b,c,d.

  Type : reals single precision.

- XX,YY : arrays containing the coordinates of the four points used in the cubic interpolation.
  Type : arrays of reals single precision.

Shell's sorting :

- JB : pointer to the first value to be sorted.
- NN : pointer to the last value to be sorted.
- ID : distance between the two values to be sorted.
- IT : indicator : set to 1, an exchange has been done.
- IS : tempory variable used during the exchange.

For TRACE :

- IP : character "." (plotting of the grid)
- IBL : space character.
- NCX2 : value NCX+1.
- AX : array used for the plotting of the grid (X direction).
  Type : array of reals single precision.
- AY : value used for the plotting of the grid (Y direction).
  Type : real single precision.
- ITIT : title of the plotting (copied from ITIT1).
- ITAB : array of characters : contains the current line.
- NY1 : value of NY+1.
- NSUP : current number of superpositions.

- NOVER : indicator : if set to 1 : a superposition is needed.
- JB,JE : pointer to the beginning and the end of the curve in the array IXY.
- IT : pointer to the first non processed point of the current curve IXY.

Internal functions :

KX : transformation : user X coordinates to local integer X coordinate.

KY : transformation : user Y coordiantes to local integer Y coordinate.

<u>Condensed form of the integer coordinates of a point.</u>

To limit the size of the array IXY, a condensed form was needed, with it a definition of one point takes only two bytes.

An integer half precision variable (2 bytes) range from - 32768 to + 32767. The integer X coordinate (IX) of a point can range from 1 to 121 (limit of 6 for NCX) and the integer (IY) can range from 1 to 501 (limit of 50 for NCY), therefore there are only 60621 possible coordinates for a point using a condensed coordinate

IXY = IX * IY - 32000 (which can range from - 31998 to 28742) , IX and IY can be found by IY = (IXY + 31998) / NX and IX = IXY + 32000 - NX * IY

Notes : In TRCEXE the value - 32000 is placed in the array element of indicate that the point is already processed.

- The value 32700 is placed in IXY(1) and tested (in case the array IXY has accidentaly overlaid).

Cubic interpolation

A unique cubic of the form $Y = aX^3 + bX^2 + cX + d$ can be determined by four points (x_i, y_i) for i = 1 to 4.

a,b,c and d can be found by solving a linear system of four equations with four unknown ; calling D_i the i^{th} determinant.

$D_i = (x_i - x_j) (x_i - x_k) (x_i - x_l)$

(i, j, k, l all different)

We have :

$$a = \sum_{i=1}^{4} y_i / D_i$$

$$b = -\sum_{i=1}^{4} y_i (x_j + x_k + x_l) / D_i$$

$$c = +\sum_{i=1}^{4} y_i (x_l x_k + x_j x_k + x_l x_j) / D_i$$

$$d = -\sum_{i=1}^{4} y_i (x_j x_k x_l) / D_i$$

An equivalent formulation, which have been used is :

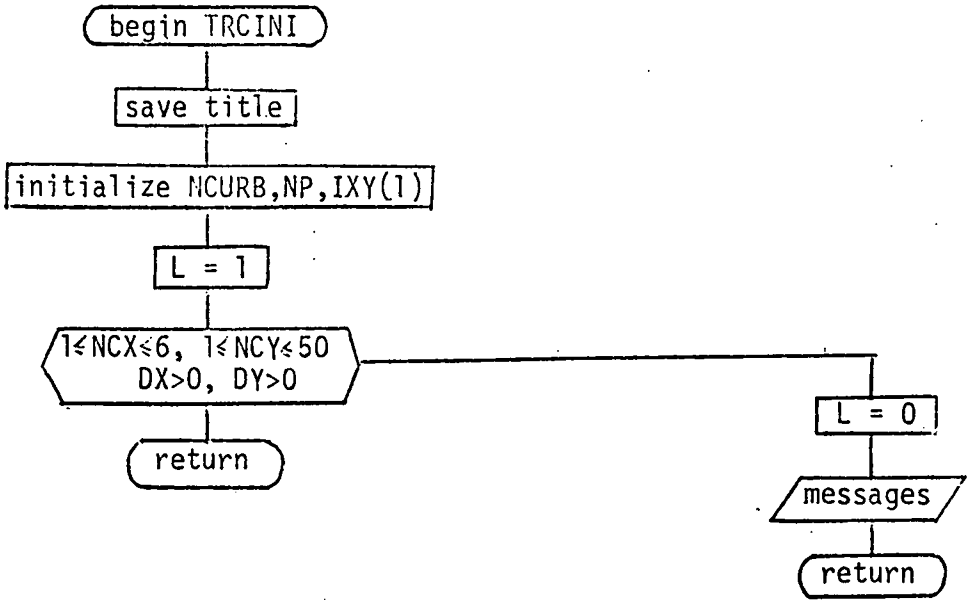
$$D'_i = y_i / \left[\prod_{j \neq i} (x_i - x_j) \right]$$

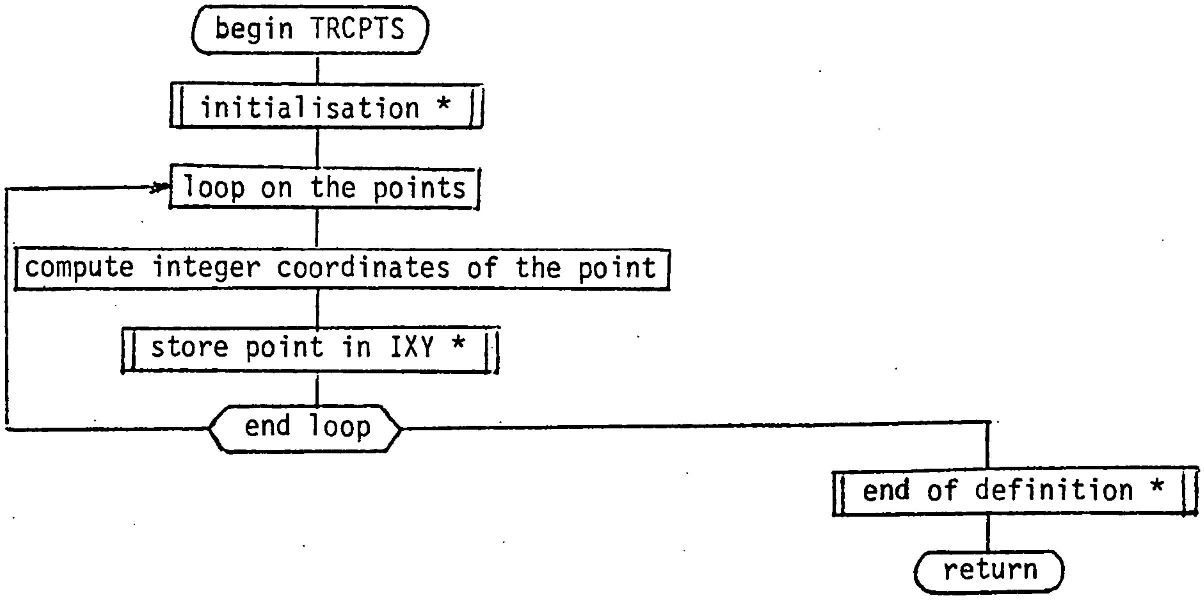
$$a = \sum_i D'_i$$

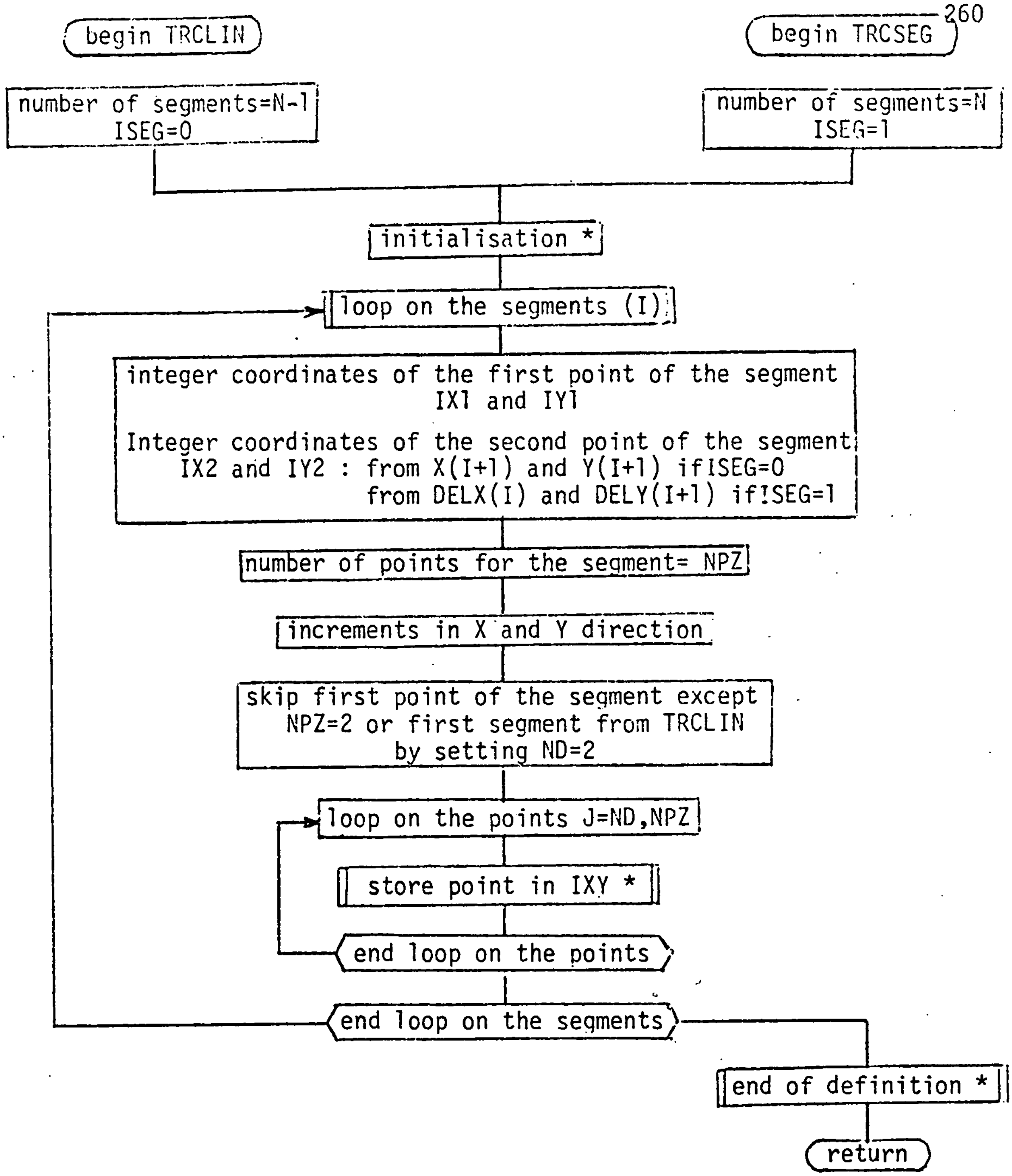
$$b = \sum_i D'_i \left(\sum_{j \neq i} x_j \right)$$

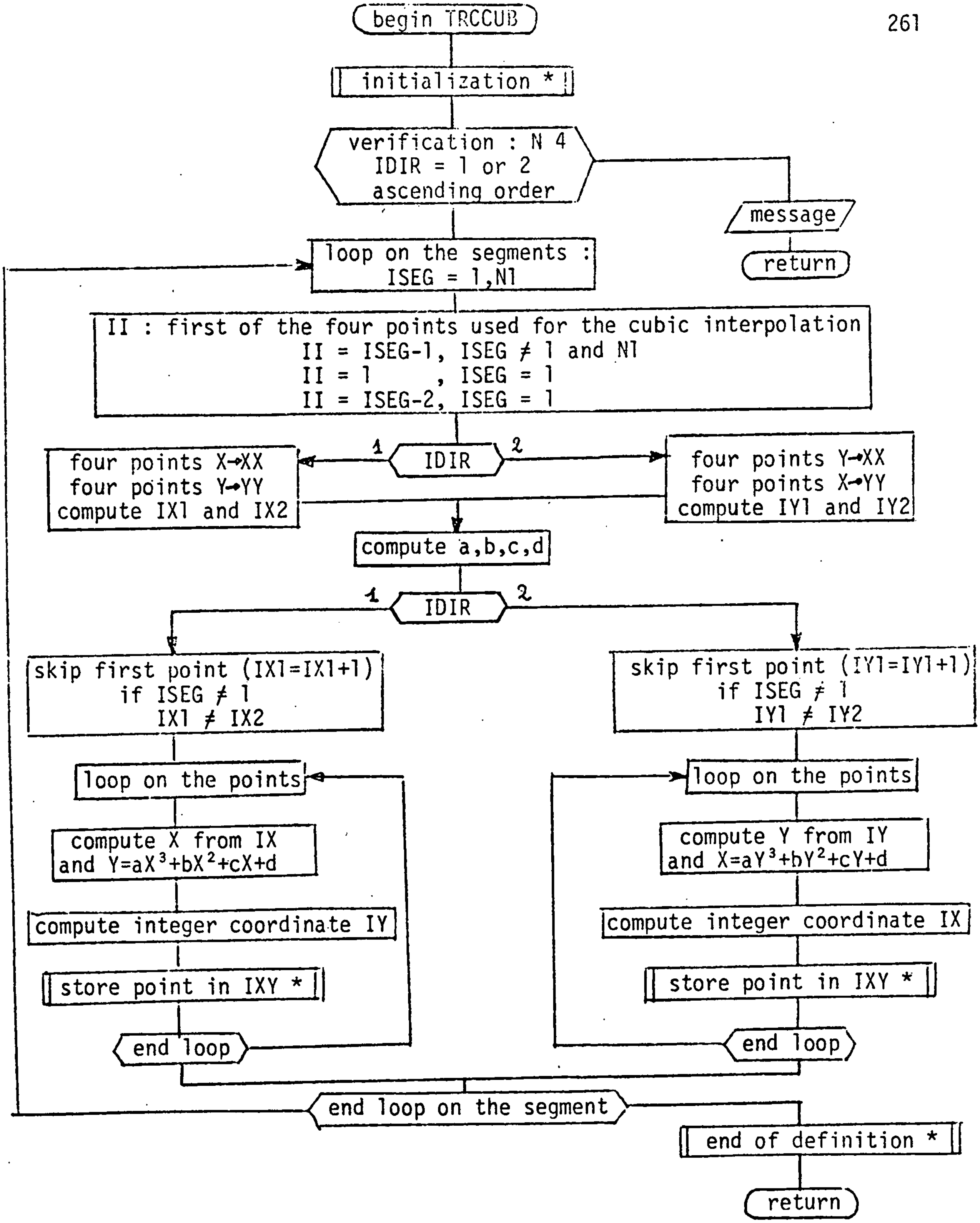
$$c = \sum_i D'_i \left[\sum_{j \neq i} \left(\prod_{\substack{k \neq i \\ k \neq j}} x_k \right) \right]$$

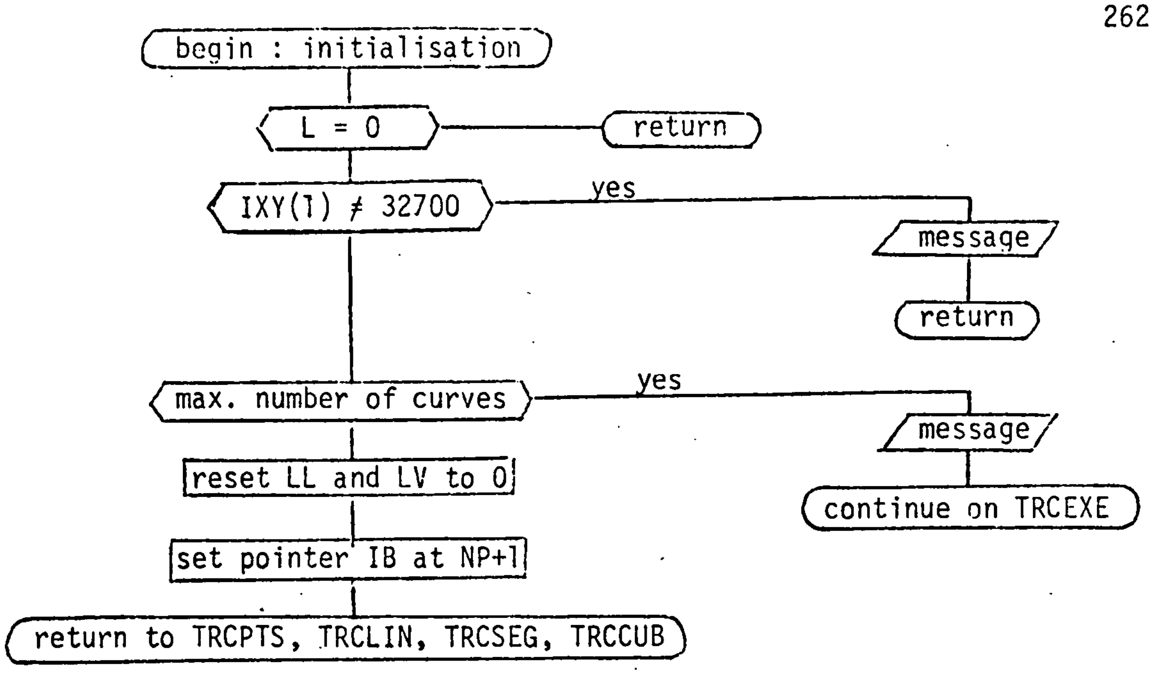
$$d = \sum_i D'_i \left[\prod_{j \neq i} x_j \right]$$

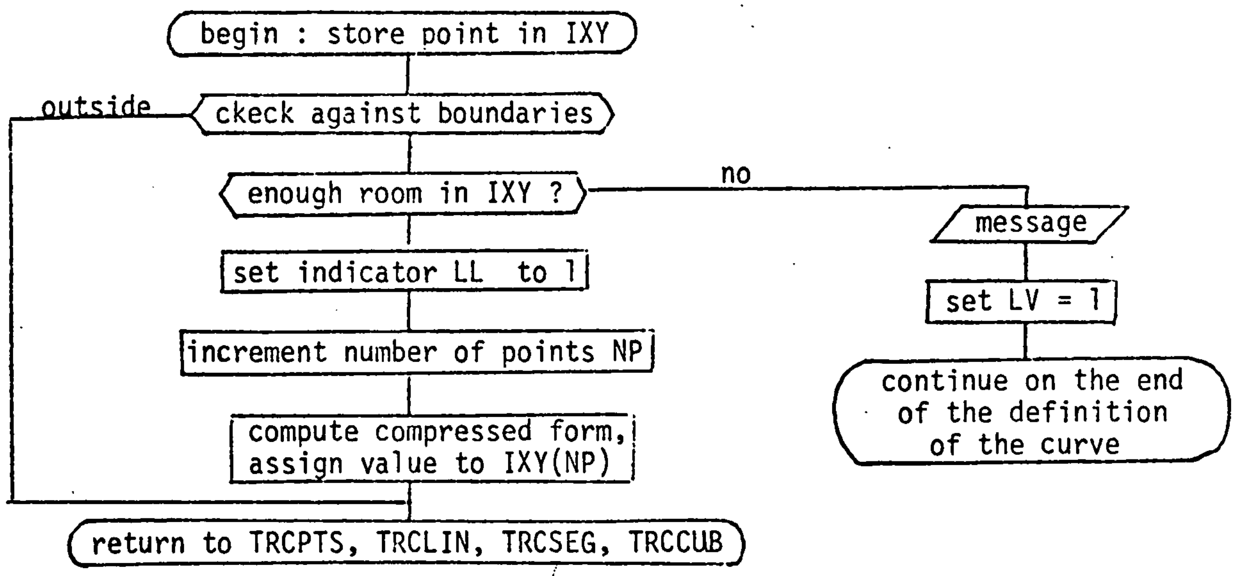

begin TRCINI
save title
initialize NCURB,NP,IXY(1)
L = 1
1≤NCX≤6, 1≤NCY≤50
DX>0, DY>0
return
L = 0
messages
return

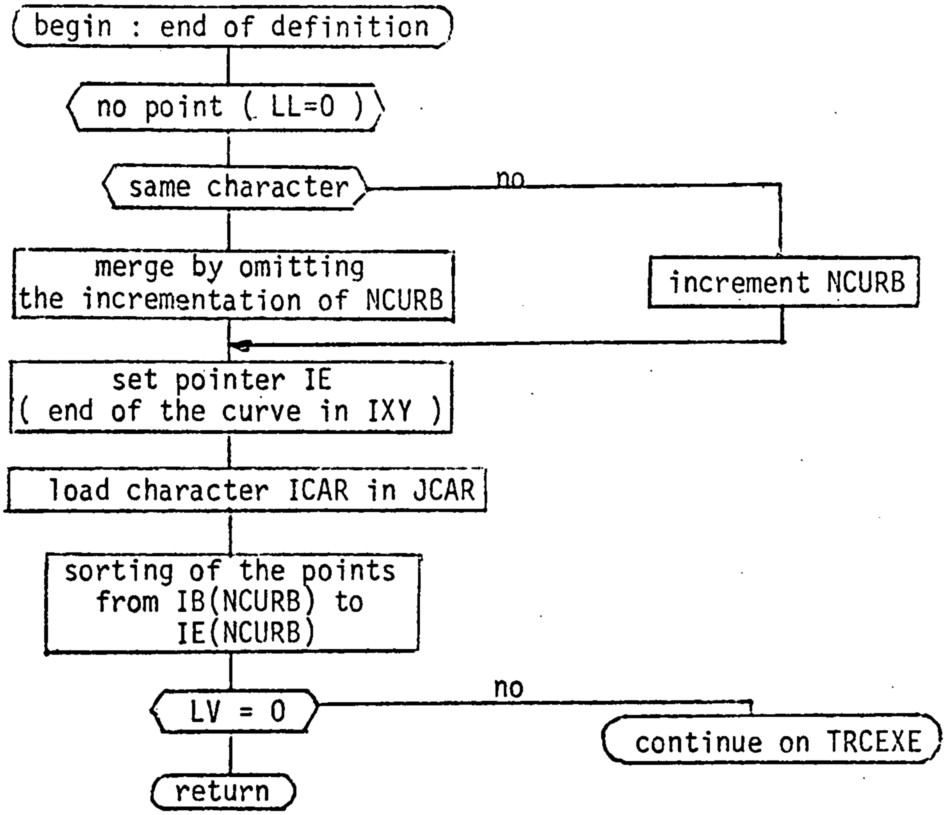

begin TRCPTS
initialisation *
loop on the points
compute integer coordinates of the point
store point in IXY *
end loop
end of definition *
return

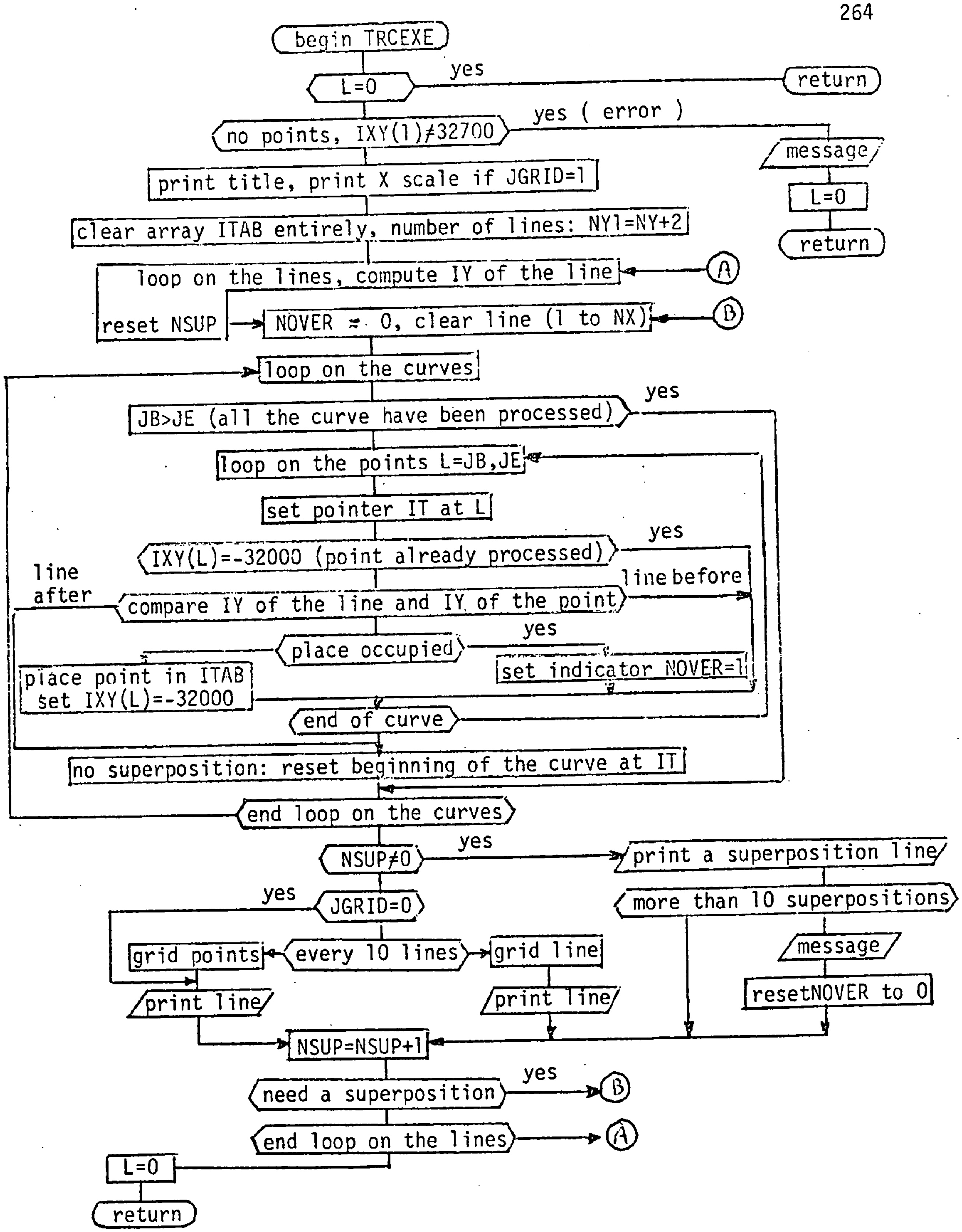

begin TRCLIN
begin TRCSEG
number of segments=N-1
ISEG=0
number of segments=N
ISEG=1
initialisation *
loop on the segments (I)
integer coordinates of the first point of the segment
IX1 and IY1
Integer coordinates of the second point of the segment
IX2 and IY2 : from X(I+1) and Y(I+1) if ISEG=0
from DELX(I) and DELY(I+1) if ISEG=1
number of points for the segment= NPZ
increments in X and Y direction
skip first point of the segment except
NPZ=2 or first segment from TRCLIN
by setting ND=2
loop on the points J=ND,NPZ
store point in IXY *
end loop on the points
end loop on the segments
end of definition *
return


begin TRCCUB
initialization *
verification : N 4
IDIR = 1 or 2
ascending order
message
return
loop on the segments :
ISEG = 1,N1
II : first of the four points used for the cubic interpolation
II = ISEG-1, ISEG ≠ 1 and N1
II = 1 , ISEG = 1
II = ISEG-2, ISEG = 1
IDIR
1
2
four points X→XX
four points Y→YY
compute IX1 and IX2
four points Y→XX
four points X→YY
compute IY1 and IY2
compute a,b,c,d
IDIR
1
2
skip first point (IX1=IX1+1)
if ISEG ≠ 1
IX1 ≠ IX2
skip first point (IY1=IY1+1)
if ISEG ≠ 1
IY1 ≠ IY2
loop on the points
compute X from IX
and Y=aX³+bX²+cX+d
compute integer coordinate IY
store point in IXY *
end loop
loop on the points
compute Y from IY
and X=aY³+bY²+cY+d
compute integer coordinate IX
store point in IXY *
end loop
end loop on the segment
end of definition *
return


begin : initialisation
L = 0
return
IXY(1) ≠ 32700
yes
message
return
max. number of curves
yes
message
continue on TRCEXE
reset LL and LV to 0
set pointer IB at NP+1
return to TRCPTS, TRCLIN, TRCSEG, TRCCUB


begin : store point in IXY
outside
ckeck against boundaries
enough room in IXY ?
no
message
set LV = 1
continue on the end of the definition of the curve
set indicator LL to 1
increment number of points NP
compute compressed form, assign value to IXY(NP)
return to TRCPTS, TRCLIN, TRCSEG, TRCCUB


begin : end of definition
no point (LL=0)
same character
no
merge by omitting the incrementation of NCURB
increment NCURB
set pointer IE (end of the curve in IXY)
load character ICAR in JCAR
sorting of the points from IB(NCURB) to IE(NCURB)
LV = 0
no
continue on TRCEXE
return


begin TRCEXE
L=0
yes
return
no points, IXY(1)≠32700
yes (error)
message
L=0
return
print title, print X scale if JGRID=1
clear array ITAB entirely, number of lines: NY1=NY+2
loop on the lines, compute IY of the line
A
reset NSUP
NOVER = 0, clear line (1 to NX)
B
loop on the curves
JB>JE (all the curve have been processed)
yes
loop on the points L=JB,JE
set pointer IT at L
IXY(L)=-32000 (point already processed)
yes
line after
compare IY of the line and IY of the point
line before
place occupied
yes
place point in ITAB
set IXY(L)=-32000
set indicator NOVER=1
end of curve
no superposition: reset beginning of the curve at IT
end loop on the curves
NSUP≠0
yes
print a superposition line
JGRID=0
yes
more than 10 superpositions
grid points
every 10 lines
grid line
message
print line
print line
resetNOVER to 0
NSUP=NSUP+1
need a superposition
yes
B
end loop on the lines
A
L=0
return

Program GENERF

Function : generation of the map of the far wake

Entry : GENERF

Argument : none

Parameter used : none

Input : none

Output : on unit (6), which specifies a printer :

- array ETA of the locations of the control points.
- arrays WX, WY, WZ : map of the far wake.

On unit (8) : (sequential file) map of the far wake.

Although compiled as a subroutine, GENERF is a complitely independent program. This program first computes a set of control points, (array ETA) and then, the velocities induced by a unit semi-infinite cylinder at these points. All the values are printed (for control) and written on a sequential file.

All these values compose the map of the far wake (semi-infinite cylinder). Using this map, the velocities induced at any point in space by a semi-infinite cylinder can be obtained rapidly by interpolation in the arrays WX, WY, WZ.

One can refer to the appendix B for the notations and the determination of the formulas.

The array element WX(I,J) contains the value $\dfrac{W_x^*(\eta,\zeta)}{2\pi}$

where η = ETA(I) and ζ = - ETA(J)

WY(I,J) the value $\dfrac{W_Y^*(\eta,\zeta)}{2\pi}$

WZ(I,J) the value $\dfrac{W_Z^*(\eta,\zeta)}{2\pi}$

All the values are computed with a relative precision of 10^{-4}. The division by 2π is done in GENERF to avoid it in INDVEL.

Choice of the control points

The map extends from 0. to approximately 20 in η direction, and from 0 to 200 in ζ direction.

To limit the size of the map, while keeping enough accuracy, a variable mesh size was needed (small mesh size for the small values of η and ζ, where the velocities are large and vary rapidly, large mesh size for the large values of η and ζ), and the number of points in ζ direction limited to 40. To locate easily the points in the map during its use, a rational fraction was choosen.

A velocity component at a point of coordinates (η,ζ) can be interpolated between the four arrays elements (I,J) (I+1,J), (I,J+1) and (I+1,J+1) where I : integer part of $\left[\dfrac{a\eta^2+b\eta+1}{c\eta^2+b\eta+1}\right]$ (an identical formulation has been used for the η and ζ directions)

The formulation satisfies : I = 1 for $\eta = 0$

To evaluate the coefficients a,b,c,d (constants (AA, BB, CC, DD in the program), four equations are needed.

They are : I = -2 for η = .1 (mesh size of .1 around η = o)

I = 20 for η = 15 (the middle of the map corresponds approximatly to 200).

I = 39 for η = 190 (mesh size of 10. around η = 200)

I = 40 for η = 200 (maximum value of η).

This represent a system of 4 linear equations with 4 unknowns.

We arrive at $a = 5.12160\ 10^{-2}$

$b = 11.0501$

$c = 6.76976\ 10^{-7}$

$d = .527592$

The formulation can be inverted and the position of the control point ETA(I) is given by :

$$\eta(i) = \left[\frac{-(i * d - b) - \sqrt{(i * d - b)^2 - 4(i * c - a)(i-1)}}{2(i * c - a)} \right]$$

As the range in η is smaller than the range in ζ the map is limited to 22 elements (ETA(22) ≈ 20) in this direction.

SEMI-RIGID WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETAO 0.035 RAD
MU= 0.105
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WX	-0.00690739	-0.00354359	0.00342347	0.01243281	0.01912123	0.02389324	0.02721006	0.02813129
WY	0.02006721	0.01347888	0.00846978	0.00532218	0.00405711	0.00347845	0.00235051	0.00293975
WZ	-0.06189186	-0.05353374	-0.04729566	-0.04426629	-0.04242026	-0.04172742	-0.04277101	-0.04695357
U	0.24844587	0.29294074	0.41253263	0.60833764	0.75663126	0.85579538	0.92980069	0.97960007
LAM	0.17429924	0.17661262	0.14033478	0.09999341	0.08280218	0.07400113	0.06699568	0.05928959
THET	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658
ALPH	0.13939261	0.14170599	0.10542816	0.06508678	0.04789560	0.03909455	0.03208910	0.02438301
GAMM	0.01813244	0.02173465	0.02277191	0.02073109	0.01897424	0.01751745	0.01562183	0.01250606
LIP	0.00450493	0.00636696	0.00939415	0.01261150	0.01435651	0.01499135	0.01452519	0.01225094
TLIP	0.00443667	0.00626792	0.00930180	0.01254850	0.01430732	0.01495032	0.01449260	0.01222941
FLIP	-0.00078124	-0.00111865	-0.00131400	-0.00125897	-0.00118739	-0.00110836	-0.00097240	-0.00072593
DP	0.00010141	0.00014331	0.00022063	0.00037371	0.00053178	0.00065694	0.00075751	0.00082343
TDP	0.00001759	0.00002518	0.00003086	0.00003731	0.00004398	0.00004857	0.00005071	0.00004879
FDP	0.00009987	0.00014108	0.00021846	0.00037184	0.00052996	0.00065514	0.00075581	0.00082198
TP	0.00445425	0.00629309	0.00933266	0.01258581	0.01435130	0.01499889	0.01454331	0.01227820
FP	-0.00068136	-0.00097757	-0.00109554	-0.00088713	-0.00065743	-0.00045322	-0.00021659	0.00009605

CT = 0.53107E 00
CP =-0.16958E 00
LT = 0.98551E-02
LP =-0.19632E-03

CT/CP=-0.31316E 01

Table 1: Case number 1

FREE WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETAC 0.035 RAD
MU= 0.105
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WXC	-0.00443090	-0.00219909	0.00483261	0.01381350	0.01965054	0.02318192	0.02581358	0.02651423
WYC	0.01874966	0.01455489	0.00916151	0.00590643	0.00452592	0.00394162	0.00394605	0.00413144
WZC	-0.05799785	-0.05308780	-0.04755423	-0.04424042	-0.04200734	-0.04106887	-0.04255347	-0.04509711
U	0.24823993	0.29417145	0.41317439	0.60894507	0.75715029	0.85633123	0.93104219	0.98096198
GAC	0.02022196	0.02194693	0.02262392	0.02073622	0.01918189	0.01785360	0.01570423	0.01345506
LAM	0.19049138	0.17739803	0.13948691	0.09994459	0.08329314	0.07472640	0.06712192	0.06110344
ALP	0.15558475	0.14249140	0.10458026	0.06503797	0.04838656	0.03981982	0.03221534	0.02619686
THE	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658
LIP	0.00501990	0.00645616	0.00934762	0.01262722	0.01452357	0.01528859	0.01462130	0.01319890
TLIP	0.00492909	0.00635483	0.00925663	0.01256420	0.01447322	0.01524592	0.01458837	0.01317427
FLIP	-0.00095047	-0.00113931	-0.00129965	-0.00125992	-0.00120831	-0.00114140	-0.00098067	-0.00080600
DP	0.00011350	0.00014532	0.00022005	0.00037435	0.00053364	0.00065951	0.00075983	0.00082939
TDP	0.00002149	0.00002564	0.00003059	0.00003735	0.00004440	0.00004924	0.00005096	0.00005065
FDP	0.00011145	0.00014304	0.00021791	0.00037249	0.00053179	0.00065767	0.00075812	0.00082785
TP	0.00495058	0.00638048	0.00928742	0.01260155	0.01451762	0.01529516	0.01463934	0.01322491
FP	-0.00083902	-0.00099627	-0.00108173	-0.00088744	-0.00067653	-0.00048373	-0.00022256	0.00002185

CT = 0.53810E 00
CP =-0.17447E 00
LT = 0.50326E-02
LP =-0.20197E-03

CT/CP=-0.30842E 01

Table 2: Case number 1

SEMI-RIGID WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETAO 0.0 RAD
MU= 0.154
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WX	-0.00574289	0.00002306	0.00762311	0.01891254	0.02755733	0.03296589	0.03586733	0.03685402
WY	0.03258815	0.01857160	0.01547890	0.01009170	0.00807066	0.00727961	0.00739536	0.01081716
WZ	-0.08429509	-0.04458468	-0.07105494	-0.06906837	-0.07017535	-0.07331032	-0.08100039	-0.09549427
U	0.26684648	0.31329465	0.42367351	0.61597538	0.76259786	0.86106586	0.93453950	0.98749435
LAM	0.26425719	0.35674334	0.19700360	0.13834298	0.11012059	0.09377313	0.07812935	0.05941479
THET	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
ALPH	0.26425719	0.35674334	0.19700360	0.13834298	0.11012059	0.09377313	0.07812935	0.05941479
GAMM	0.02794320	0.03280705	0.04370087	0.04461747	0.04396921	0.04227653	0.03822932	0.03071947
LIP	0.00745654	0.01027827	0.01851490	0.02748326	0.03353083	0.03640288	0.03572680	0.03033530
TLIP	0.00719770	0.00963114	0.01815677	0.02722068	0.03332773	0.03624294	0.03561782	0.03028177
FLIP	-0.00194759	-0.00358942	-0.00362395	-0.00379000	-0.00368497	-0.00340861	-0.00278847	-0.00180130
DP	0.00032586	0.00068405	0.00043984	0.00061874	0.00077845	0.00088949	0.00094991	0.00095602
TDP	0.00008511	0.00023889	0.00008609	0.00008533	0.00008555	0.00008329	0.00007414	0.00005677
FDP	0.00031454	0.00064098	0.00043133	0.00061283	0.00077373	0.00088558	0.00094701	0.00095434
TP	0.00728281	0.00987003	0.01824286	0.02730601	0.03341328	0.03632623	0.03569195	0.03033853
FP	-0.00163305	-0.00294844	-0.00319262	-0.00317717	-0.00291124	-0.00252303	-0.00184146	-0.00084697

CT = 0.54339E 00
CP =-0.21873E 00
LT = 0.12887E-01
LP =-0.79886E-03

CT/CP=-0.24843E 01

Table 3: Case number 2

FREE WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETAO 0.0 RAD
MU= 0.154
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WXC	-0.00160378	-0.00076856	0.00449574	0.01472700	0.02288925	0.02830830	0.03429672	0.03532047
WYC	0.03319027	0.02572441	0.01710676	0.01243965	0.01033473	0.00968996	0.01102869	0.01260881
WZC	-0.06412143	-0.05814109	-0.06563580	-0.06680399	-0.06888652	-0.07252163	-0.08130354	-0.09130532
U	0.27338678	0.31563276	0.42636369	0.61861557	0.76508319	0.86354226	0.93884701	0.98959661
UMC	0.02862807	0.03305191	0.04464728	0.04580674	0.04465628	0.04272413	0.03810063	0.03284776
LAM	0.33498996	0.30857754	0.20876366	0.14142430	0.11147803	0.09449416	0.07750911	0.06339622
ALP	0.33498996	0.30857754	0.20876366	0.14142430	0.11147803	0.09449416	0.07750911	0.06339622
THE	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
LIP	0.00782653	0.01043226	0.01903598	0.02833676	0.03416577	0.03689409	0.03577066	0.03250603
TLIP	0.00739146	0.00993951	0.01862267	0.02805385	0.03395369	0.03672950	0.03566327	0.03244073
FLIP	-0.00257305	-0.00316832	-0.00394522	-0.00399416	-0.00380085	-0.00348109	-0.00276978	-0.00205938
DP	0.00047402	0.00056128	0.00063307	0.00063789	0.00079087	0.00089883	0.00095514	0.00098005
TDP	0.00015584	0.00017046	0.00013120	0.00008990	0.00008798	0.00008481	0.00007396	0.00006209
FDP	0.00044767	0.00053477	0.00061932	0.00063143	0.00078596	0.00089482	0.00095227	0.00097808
TP	0.00754732	0.01010997	0.01875387	0.02814375	0.03404167	0.03681430	0.03573722	0.03250282
FP	-0.00212536	-0.00263395	-0.00332589	-0.00336273	-0.00301489	-0.00258627	-0.00181751	-0.00108130

CT = 0.55727E 00
CP =-0.22877E 00
LT = 0.13216E-01
LP =-0.83553E-03

CT/CP=-0.24359E 01

Table 4: Case number 2

FREE WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETA0 0.035 RAD
MU= 0.154
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WXC	-0.00128327	-0.00048063	0.00464309	0.01412700	0.02085144	0.02470255	0.02846504	0.02873173
WYC	0.02295487	0.02530602	0.01597787	0.01078694	0.00863140	0.00795782	0.00884949	0.00997379
WZC	-0.06229305	-0.05711997	-0.06019039	-0.05728259	-0.05651570	-0.05767658	-0.06321973	-0.07018828
U	0.27377146	0.31554615	0.42642438	0.61839688	0.76486850	0.86334789	0.93825114	0.98853296
GMC	0.02866835	0.03304285	0.04172850	0.03954607	0.03720105	0.03475937	0.03045745	0.02586811
LAM	0.34158003	0.31206346	0.22180527	0.15704483	0.12779981	0.11180228	0.09690630	0.08488578
ALP	0.30667341	0.27715683	0.18689865	0.12213820	0.09289318	0.07689565	0.06199972	0.04997919
THE	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658
LIP	0.00784858	0.01042654	0.01779405	0.02445517	0.02845391	0.03000943	0.02857674	0.02557148
TLIP	0.00739513	0.00992296	0.01735812	0.02415422	0.02822186	0.02982207	0.02844266	0.02547941
FLIP	-0.00262909	-0.00320119	-0.00391453	-0.00382479	-0.00362651	-0.00334814	-0.00276493	-0.00216805
CP	0.00041861	0.00048462	0.00041618	0.00055636	0.00069784	0.00080475	0.00087456	0.00091601
TDP	0.00014023	0.00014879	0.00009155	0.00008701	0.00008894	0.00008979	0.00008462	0.00007766
FCP	0.00039443	0.00046121	0.00040598	0.00054951	0.00069215	0.00079973	0.00087046	0.00091271
TP	0.00753536	0.01007175	0.01744968	0.02424123	0.02831680	0.02991185	0.02852727	0.02555707
FP	-0.00223466	-0.00273997	-0.00350855	-0.00327528	-0.00293436	-0.00254841	-0.00189447	-0.00125534

CT = 0.47634E 00
CP =-0.23044E 00
LT = 0.11297E-01
LP =-0.84164E-03

CT/CP=-0.20671E 01

Table 5: Case number 3

SEMI-RIGID WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETAO 0.035 RAD
MU= 0.154
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WX	-0.00620236	-0.00060898	0.00584307	0.01521456	0.02207758	0.02611104	0.02818136	0.02885750
WY	0.03315381	0.01851357	0.01498725	0.00931589	0.00722514	0.00643869	0.00646924	0.00882528
WZ	-0.08058119	-0.04544507	-0.06159778	-0.05732181	-0.05632180	-0.05768217	-0.06260788	-0.07313573
U	0.26840576	0.31294626	0.42514920	0.61692816	0.76349759	0.86183745	0.93573666	0.98721164
LAM	0.27705967	0.35426110	0.21908450	0.15734798	0.12829059	0.11199301	0.09779662	0.08202237
THET	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658	0.03490658
ALPH	0.24215305	0.31935447	0.16417788	0.12244135	0.09338397	0.07708639	0.06288999	0.04711579
GAMM	0.02810642	0.03277057	0.04099807	0.03955007	0.03733057	0.03478464	0.03081200	0.02435348
LIP	0.00754391	0.01025543	0.01743029	0.02439955	0.02850180	0.02997871	0.02883191	0.02404204
TLIP	0.00725621	0.00961860	0.01701365	0.02409812	0.02826757	0.02979090	0.02869414	0.02396121
FLIP	-0.00206347	-0.00355758	-0.00378823	-0.00382339	-0.00364649	-0.00335039	-0.00281517	-0.00196977
UP	0.00029607	0.00057938	0.00040609	0.00055490	0.00069756	0.00080285	0.00087394	0.00090227
TUP	0.00008298	0.00020099	0.00008626	0.00006695	0.00008925	0.00008973	0.00008533	0.00007392
FUP	0.00028473	0.00054340	0.00039638	0.00054804	0.00069183	0.00079782	0.00086976	0.00089924
TP	0.00733720	0.00981958	0.01710191	0.02418507	0.02835681	0.02988062	0.02877947	0.02403513
FP	-0.00177869	-0.00301418	-0.00339185	-0.00327535	-0.00295466	-0.00255257	-0.00194541	-0.00107053

CT = 0.47190E 00
CP =-0.22775E 00
LT = 0.11192E-01
LP =-0.83179E-03

CT/CP=-0.20720E 01

Table 6: Case number 3

SEMI-RIGID WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETA0 0.070 RAD
MU= 0.154
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WX	-0.00676415	-0.00116885	0.00437855	0.01225680	0.01776307	0.02082472	0.02233298	0.02279632
WY	0.03420300	0.01980668	0.01423713	0.00835881	0.00612866	0.00530534	0.00508729	0.00647864
WZ	-0.07939029	-0.05058693	-0.05363949	-0.04730738	-0.04393880	-0.04355464	-0.04506795	-0.05190121
U	0.26967329	0.31241578	0.42631286	0.61784506	0.76416659	0.86241955	0.93604624	0.98702145
LAM	0.28036046	0.33732653	0.23783046	0.17392457	0.14454120	0.12883788	0.11644536	0.10401100
THET	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313
ALPH	0.21054733	0.26751339	0.16801733	0.10411143	0.07472807	0.05902475	0.04663223	0.03419787
GAMM	0.02823921	0.03271502	0.03750309	0.03367925	0.02989899	0.02665247	0.02285431	0.01767300
LIP	0.00761536	0.01022069	0.01598805	0.02080856	0.02284781	0.02298561	0.02139269	0.01744362
TLIP	0.00731802	0.00964468	0.01553801	0.02049462	0.02260955	0.02279510	0.02124782	0.01734935
FLIP	-0.00210718	-0.00338269	-0.00376670	-0.00360090	-0.00329096	-0.00295323	-0.00248545	-0.00181106
DP	0.00025552	0.00045369	0.00036521	0.00049050	0.00062248	0.00072775	0.00080951	0.00085929
TDP	0.00007070	0.00015016	0.00008604	0.00008488	0.00008966	0.00009350	0.00009405	0.00008921
FDP	0.00024555	0.00042812	0.00035493	0.00048310	0.00061599	0.00072172	0.00080403	0.00085464
TP	0.00738872	0.00979483	0.01562405	0.02057950	0.02269921	0.02288860	0.02134186	0.01743856
FP	-0.00186164	-0.00295457	-0.00341177	-0.00311780	-0.00267497	-0.00223151	-0.00168142	-0.00095642

CT = 0.39185E 00
CP =-0.21324E 00
LT = 0.92932E-02
LP =-0.77880E-03

CT/CP=-0.18376E 01

Table 7: <u>Case number 4</u>

FREE WAKE ANALYSIS OF THE WIND TURBINE

ROTOR DATA

2BLADES
SIGMA=0.106
THETAO 0.070 RAD
MU= 0.154
CD= 0.0100+ 0.50*ALPHA**2
STALL ANGLE= 0.200

ETA	0.2250	0.2750	0.4000	0.6000	0.7500	0.8500	0.9250	0.9750
WXC	-0.00006498	0.00014346	0.00502484	0.01315366	0.01809514	0.02046398	0.02237735	0.02213269
WYC	0.03275122	0.02505873	0.01498579	0.00945139	0.00719507	0.00640060	0.00681360	0.00747167
WZC	-0.06062273	-0.05628279	-0.05274114	-0.04671310	-0.04330312	-0.04218379	-0.04473476	-0.04876042
U	0.27407596	0.31556904	0.42716092	0.61882240	0.76524329	0.86366928	0.93819761	0.98809183
GMC	0.02870024	0.03304523	0.03791282	0.03383899	0.03019106	0.02714008	0.02304554	0.01908855
LAM	0.34638365	0.31482887	0.23932892	0.17425311	0.14516503	0.12983078	0.11672771	0.10671014
ALP	0.27707052	0.24501574	0.16951579	0.10443997	0.07535189	0.06001765	0.04691458	0.03689700
THE	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313	0.06981313
LIP	0.00786604	0.01042805	0.01619487	0.02094032	0.02310351	0.02344005	0.02162126	0.01886124
TLIP	0.00739751	0.00991559	0.01573327	0.02062321	0.02286050	0.02324277	0.02147413	0.01875395
FLIP	-0.00267421	-0.00322908	-0.00383901	-0.00363048	-0.00334205	-0.00303470	-0.00251807	-0.00200887
UP	0.00036546	0.00041505	0.00037051	0.00049314	0.00062652	0.00073353	0.00081421	0.00086896
TUP	0.00012423	0.00012852	0.00008783	0.00008550	0.00009063	0.00009497	0.00009482	0.00009255
FUP	0.00034363	0.00039465	0.00035995	0.00040568	0.00061993	0.00072776	0.00080667	0.00086402
TP	0.00752176	0.01504402	0.01582110	0.02070870	0.02295113	0.02333774	0.02156896	0.01884650
FP	-0.00233052	-0.00233443	-0.00347905	-0.00314480	-0.00272213	-0.00230733	-0.00170940	-0.00114485

CT = 0.30819E 00
CP =-0.21894E 00
LT = 0.944365E-02
LP =-0.70962E-03

CT/CP=-0.14187E 01

Table 8: <u>Case number 4</u>

TABLE 9

INPUT FOR THE CASES 1 to 4

- Number of blades (NBLD1) = 2
- Rotor solidity (SIGMA) = .1061
- Untwisted blades : LTWIST = 0
- Stall angle (ALPHAS) = .2
- Drag coefficients (CDO) = .01

  (CDK) = .5
- Root and tip vortices factor (COEFF) = .5
- Rectangular elements thickness (EPS1) = .03
- segment elements core size (EPS2) = .01
- Distribution of the nodes :

  a) Near wake:

  n_{nvr} : KNNVR = 9

  n_{nva} : NTVA = 5, resulting value of NNVA : 9

  $\eta_n(i)$: ETAN = .2, .25, .3, .5, .7, .8, .9, .95, 1.

  b) Intermediate wake:

  n_{ivr} : KNIVR = 6

  n_{iva} : NIVA = 20

  $\eta_i(i)$: ETAI = .2, .275, .6, .85, .95, 1.

- Pitch angle (Θ)

  Case # 1 : THETOD = 2.

  Case # 2 : THETOD = 0.

  Case # 3 : THETOD = 2.

  Case # 4 : THETOD = 4.

- Advance ratio (μ)

  Case # 1 : FMU = .105

  Case # 2 to 4 : FMU = .154

TABLE 10

EXPANSION OF THE WAKE

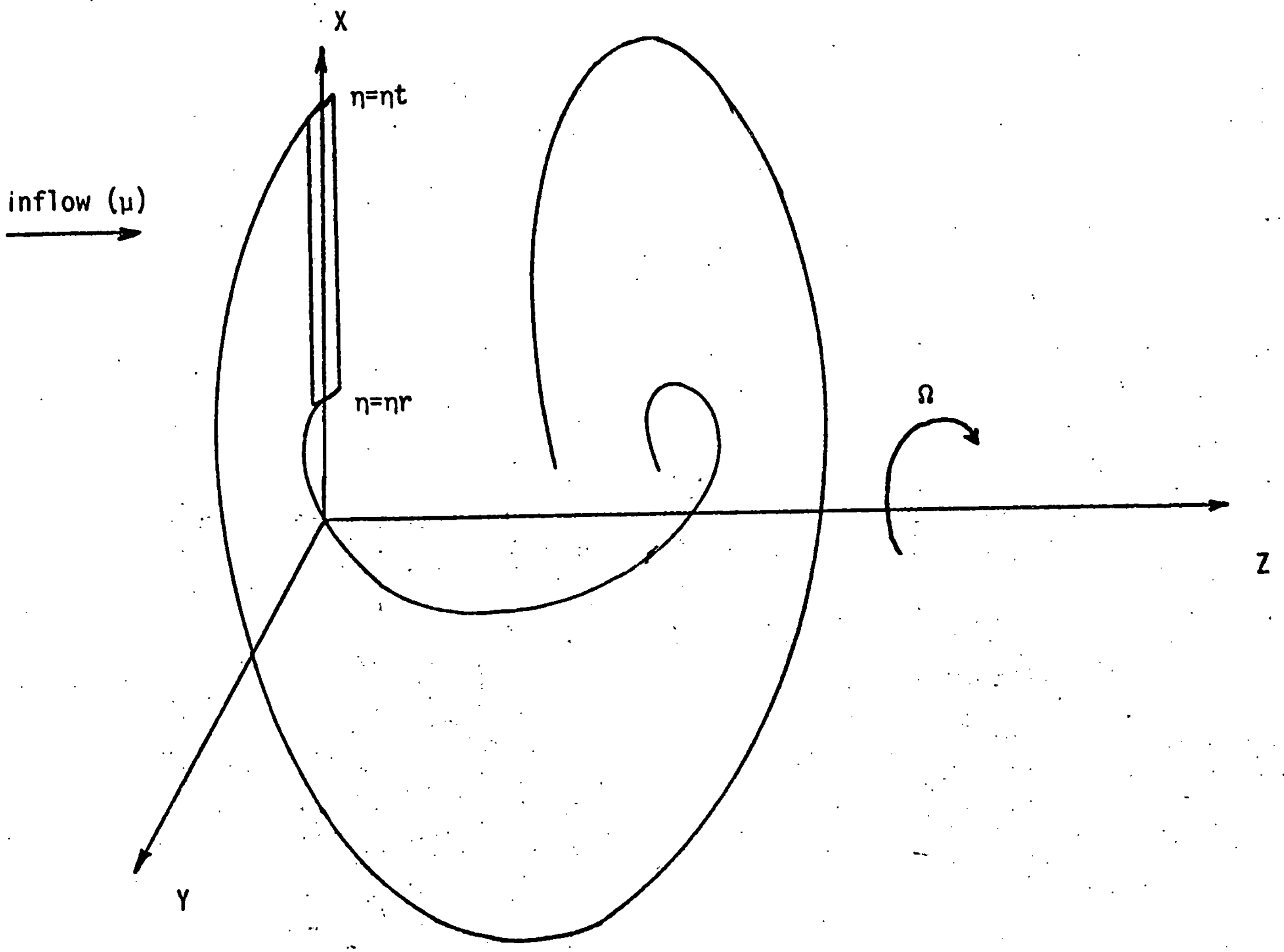
Case number		1	2	3	4
Last position of the tip vortex:	R =	1.092	1.097	1.062	1.039
Last position of the outermost vortex line:	R =	1.206	1.129	1.193	1.189

The outermost vortex line originates from $\eta(5) = .95$

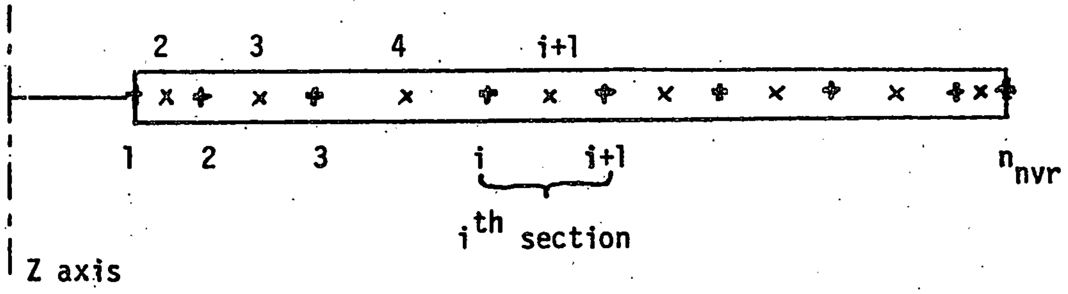
ROLL-UP EFFECT

Azimuth for which the outermost vortex line has a larger radial position than that of the tip vortex (ψ)

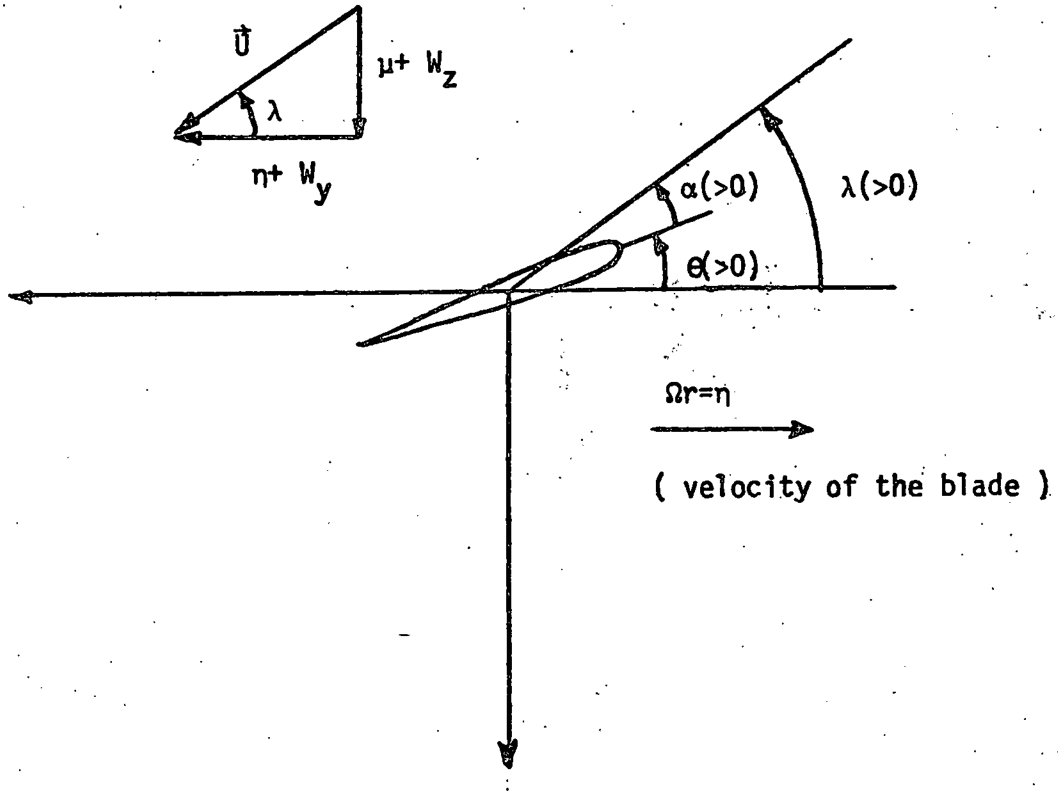
Case number	1	2	3	4
ψ (degrees)	110	45	60	75



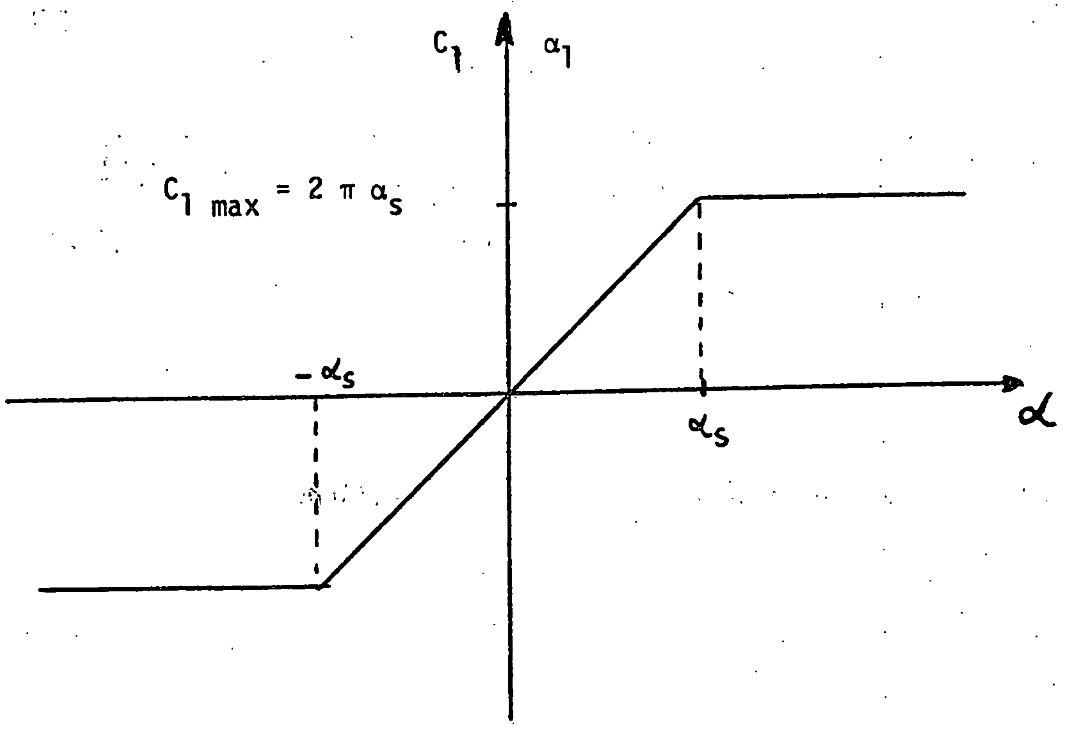
(figure 1) system of coordinates



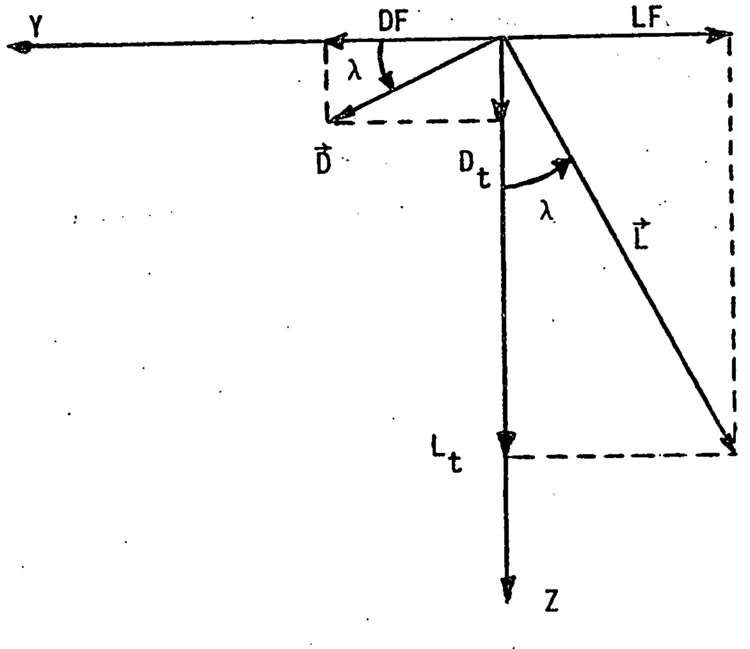
(figure 2) decomposition of the blade into sections



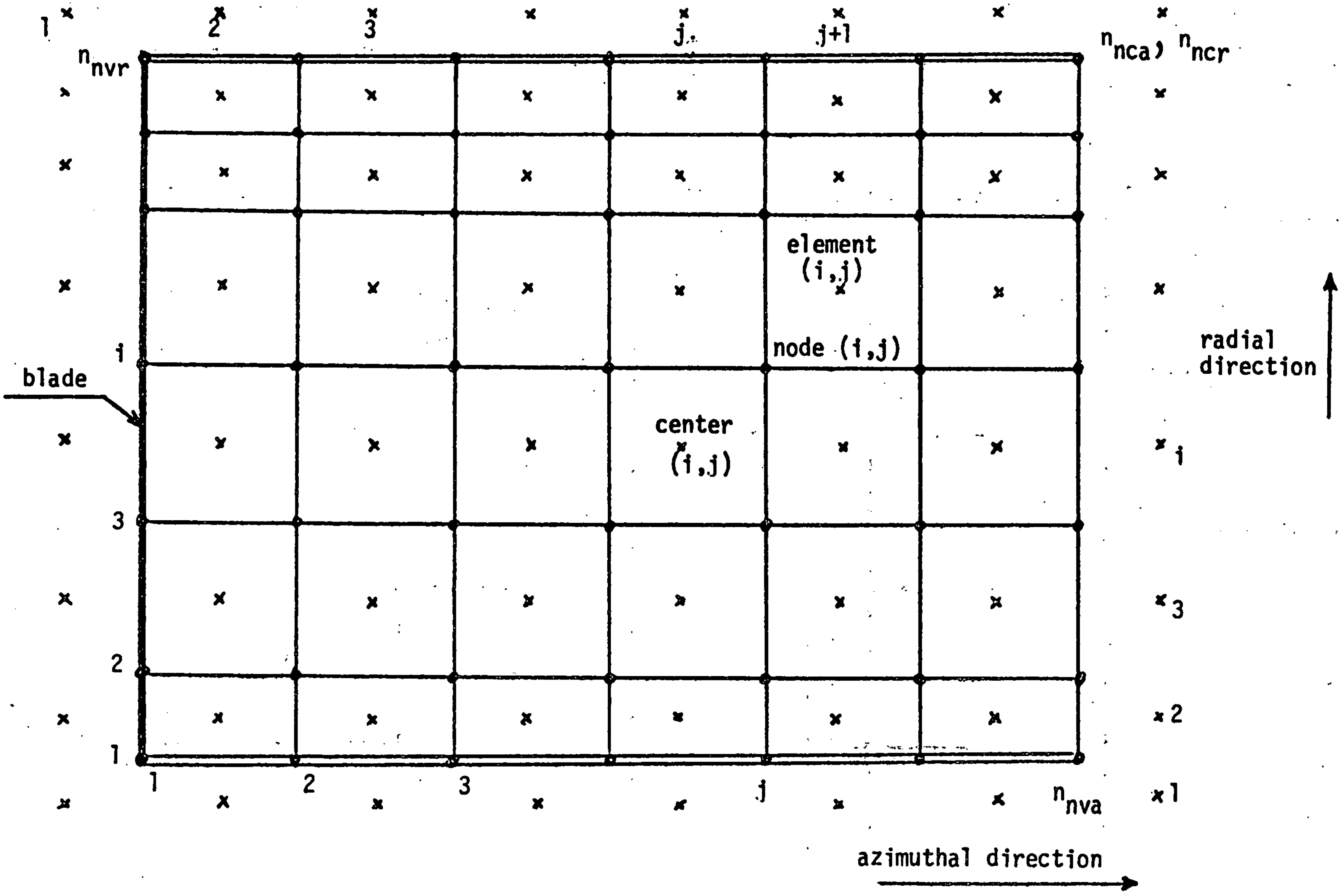
(figure 3) sign conventions for the angles



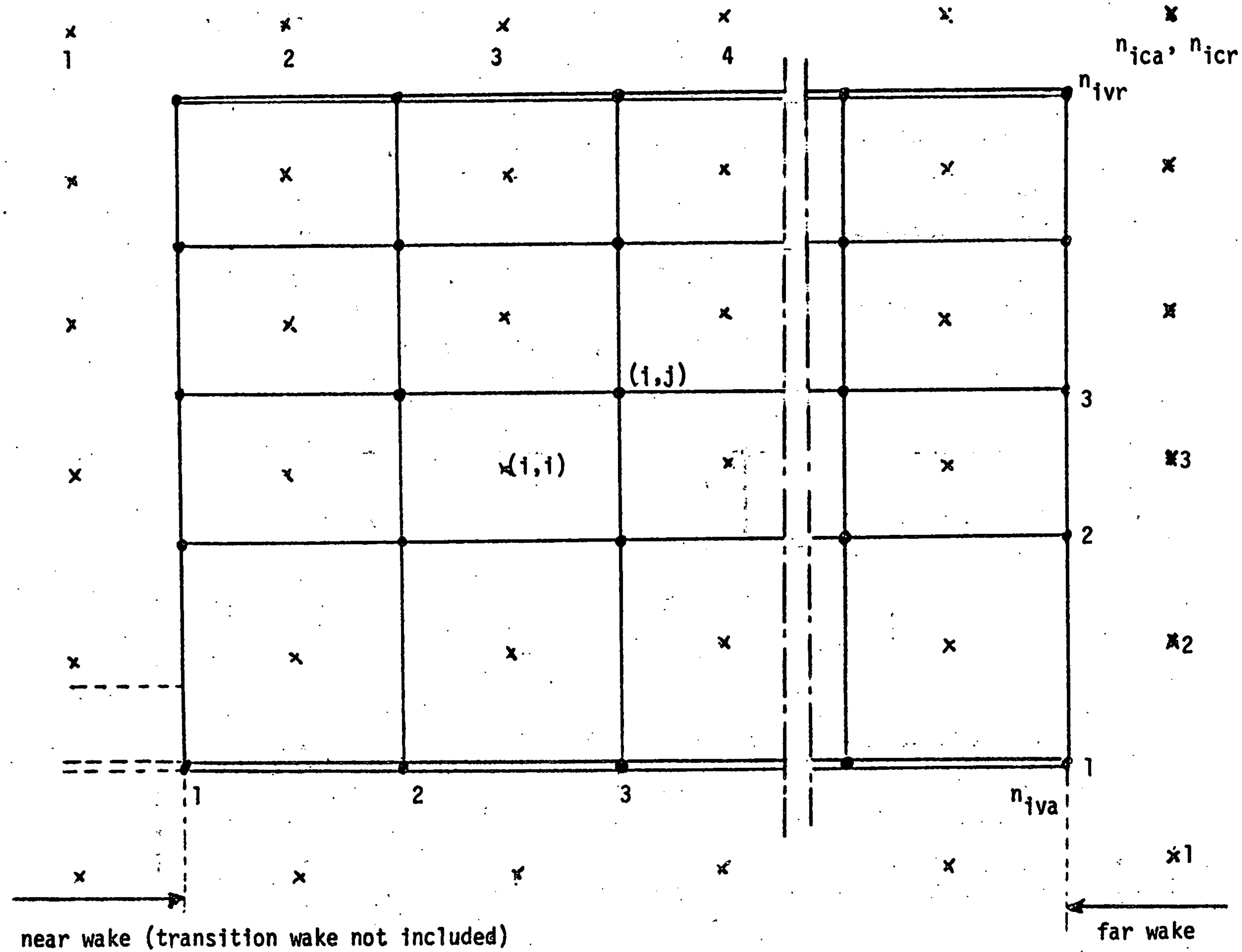
(figure 4) lift coefficient representation



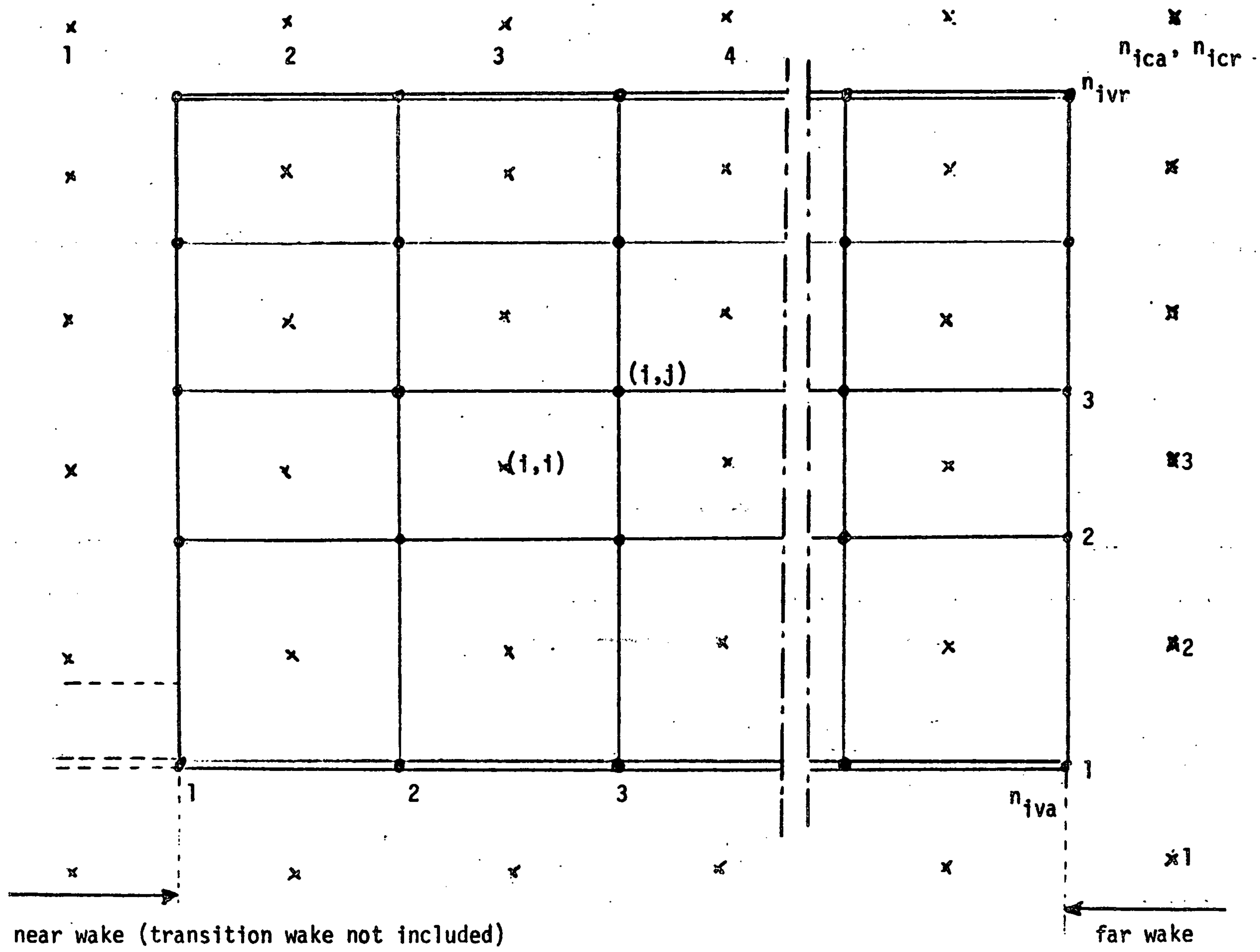
(figure 5) lift and drag components



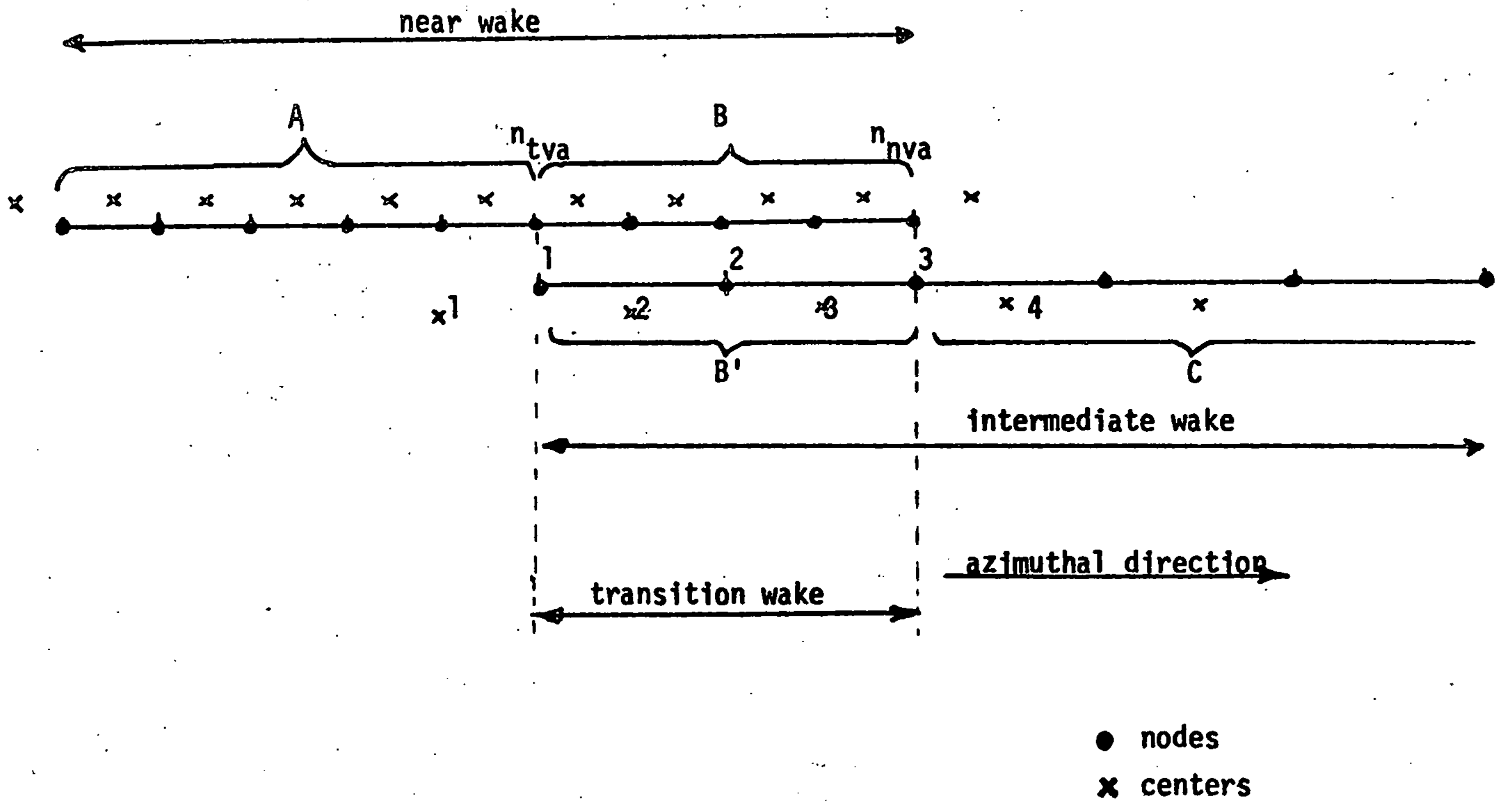
(figure 6) near wake definition



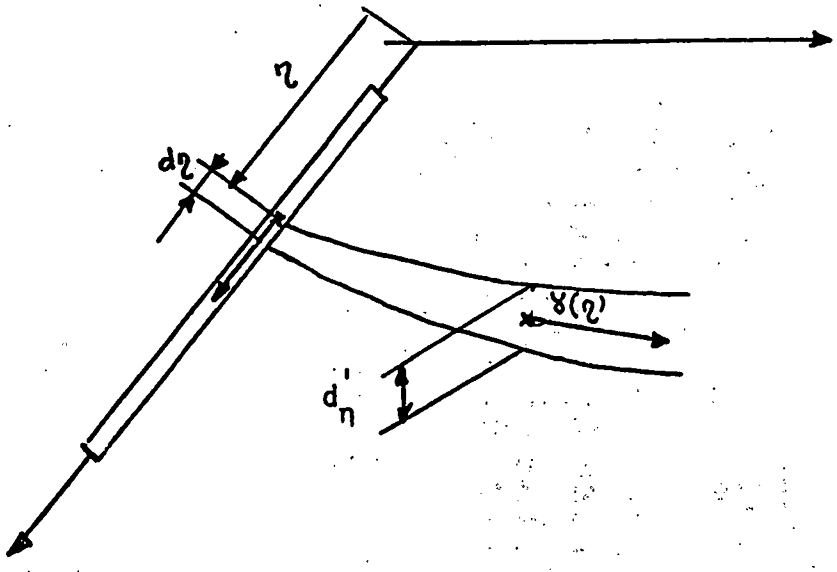
(figure 7) intermediate wake definition



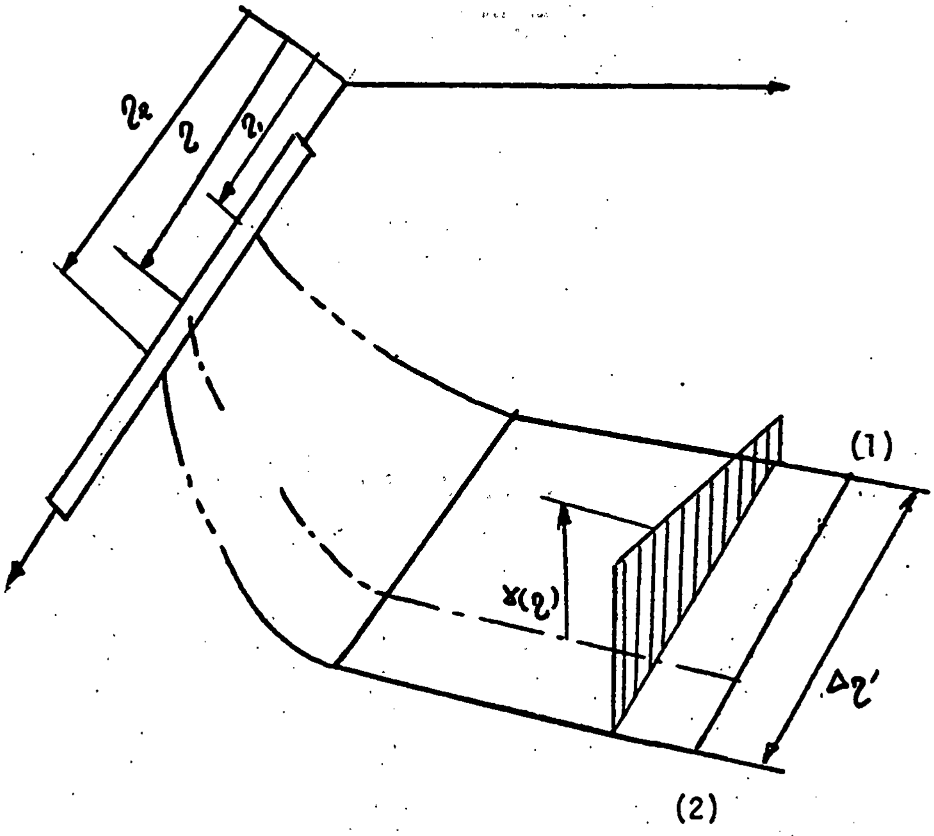
(figure 7) intermediate wake definition



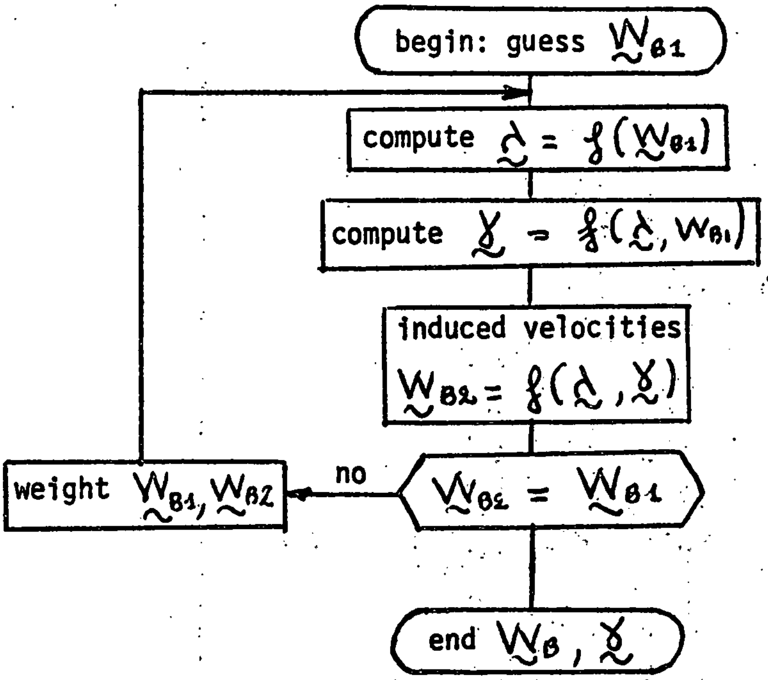
(figure 8) implementation of the transition wake



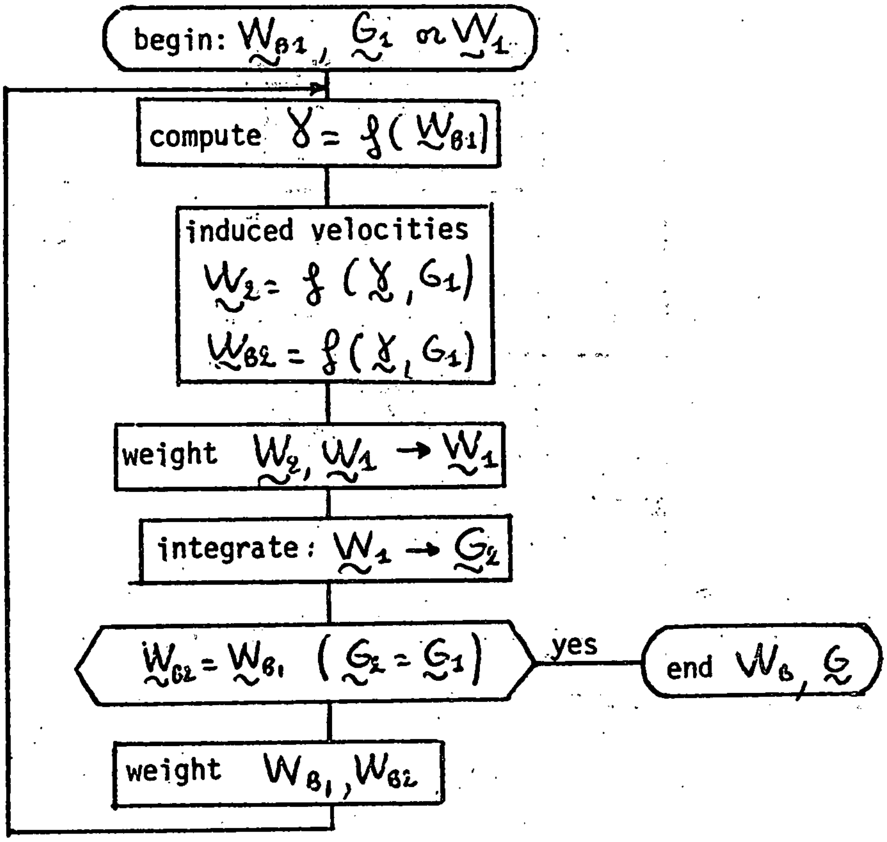
(figure 9) strength of the sheet element of width d_{η}'



(figure 10) strength of a rectangular element

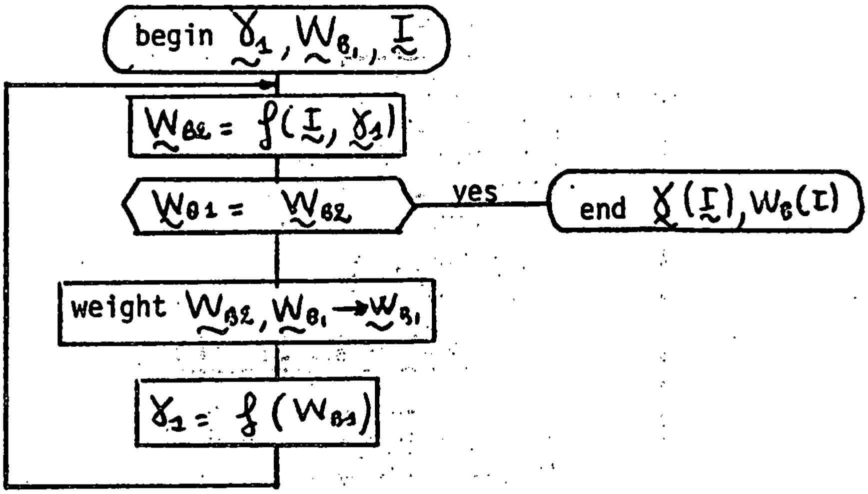


(figure 11) iterative procedure, case of the semi-rigid wake model



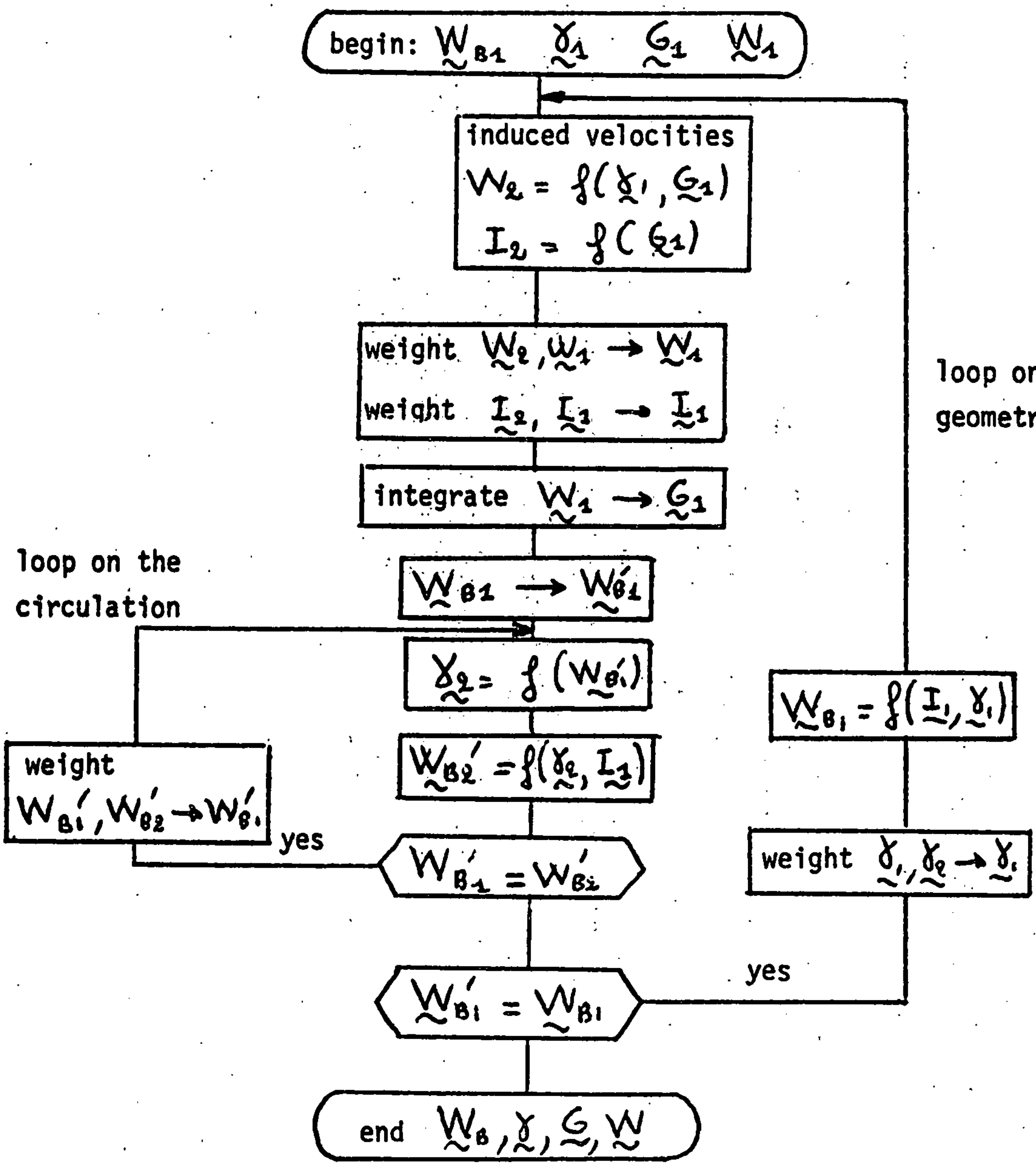
(figure 12) iterative procedure, case of the free wake model

(direct method)



$\underset{\sim}{\gamma}$ and $\underset{\sim}{W}_B$ are then compatible with the geometry, but the geometry is not then compatible with $\underset{\sim}{\gamma}$

(figure 13) loop on the circulation



(figure 14) complete iteration procedure

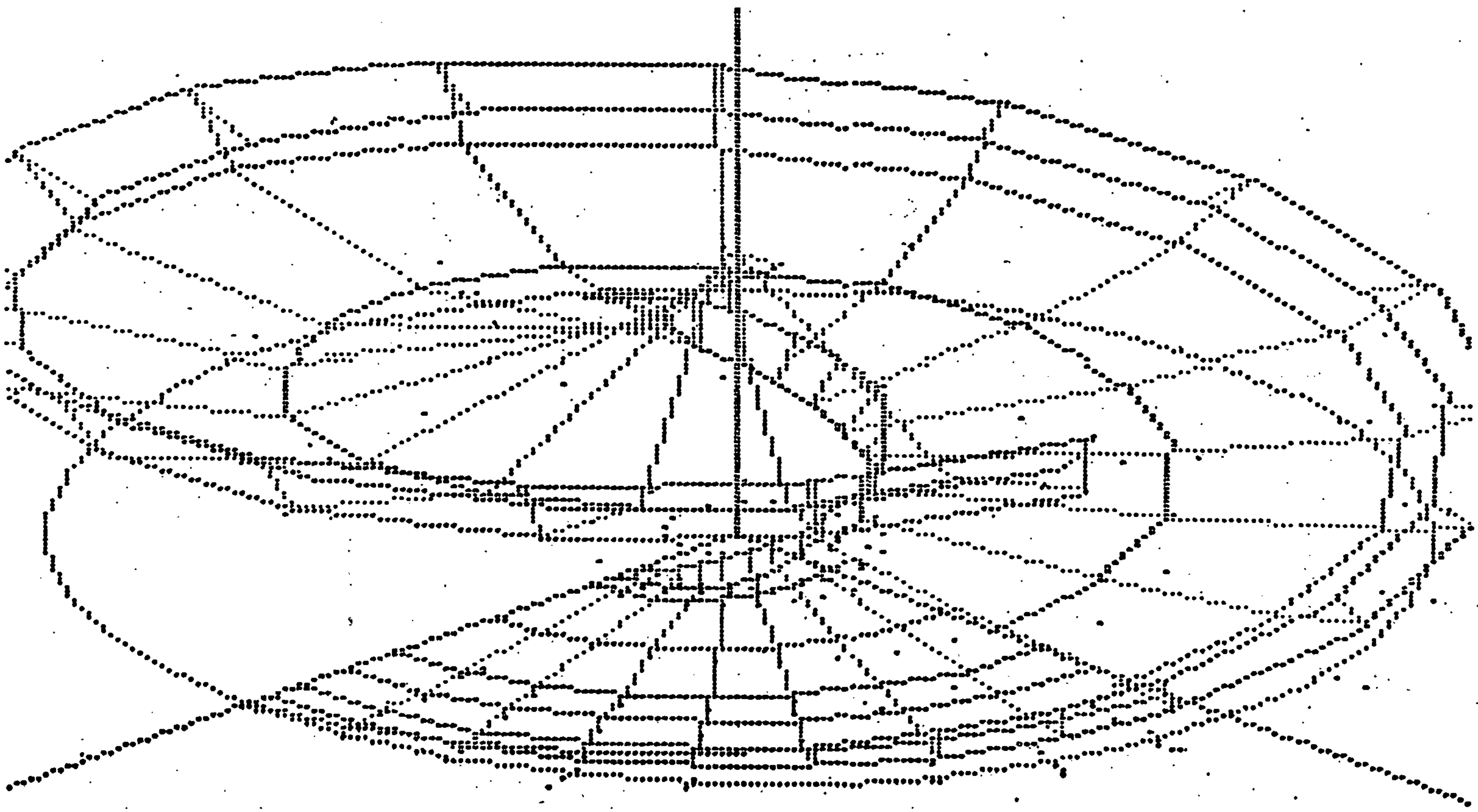
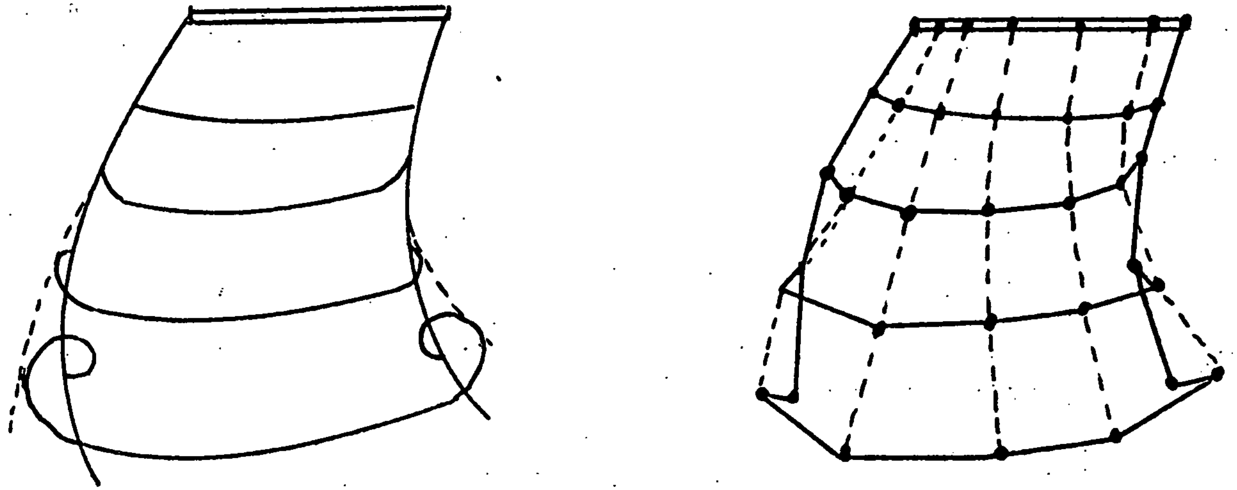


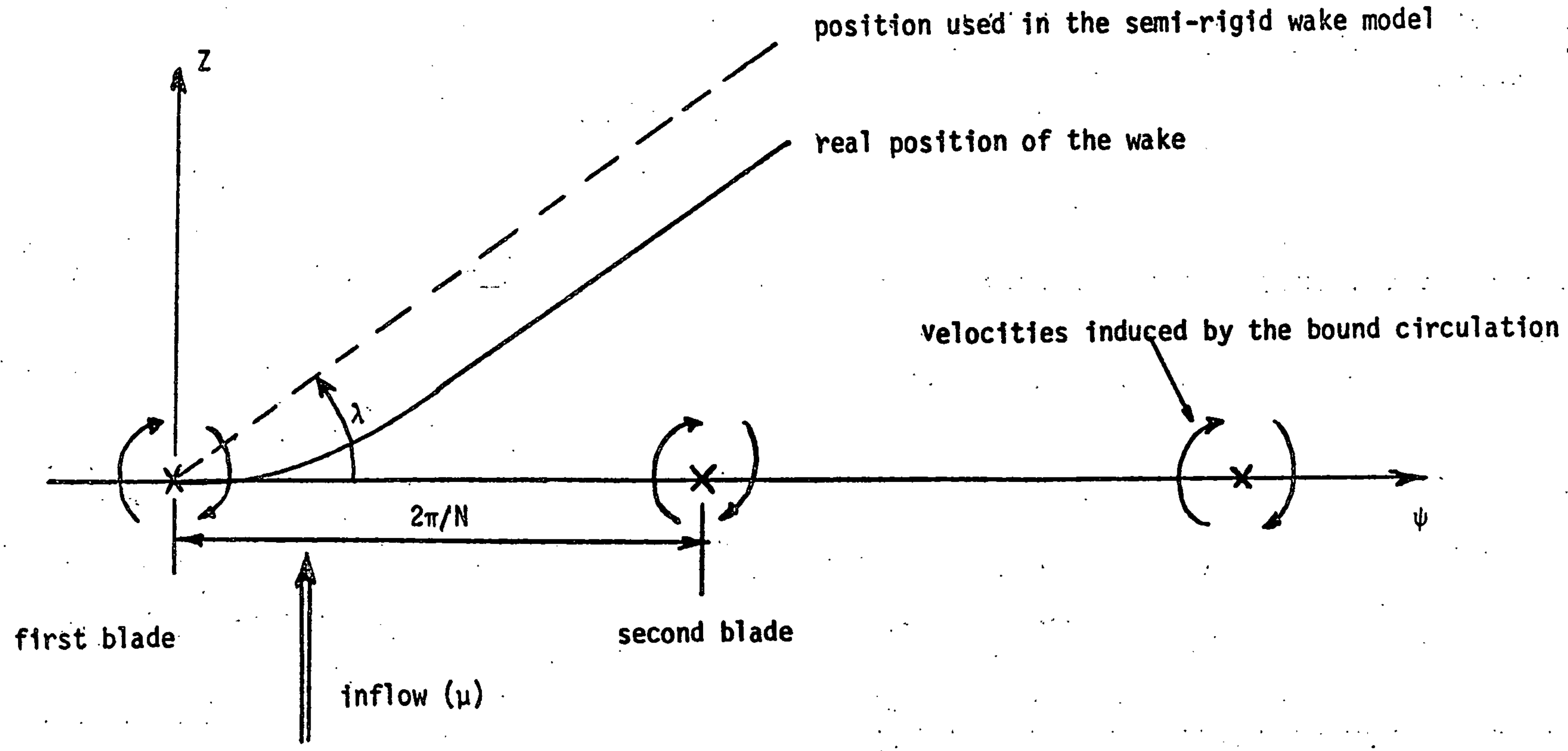
figure 15 Wake geometry for the case 3 (μ = .154, θ = 2^0)



a) physical roll-up

b) folding of the last section as computed

(figure 16) roll-up of the wake



(figure 17) deflection of the wake

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