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Geometric Considerations for a Waypoint Guidance Method

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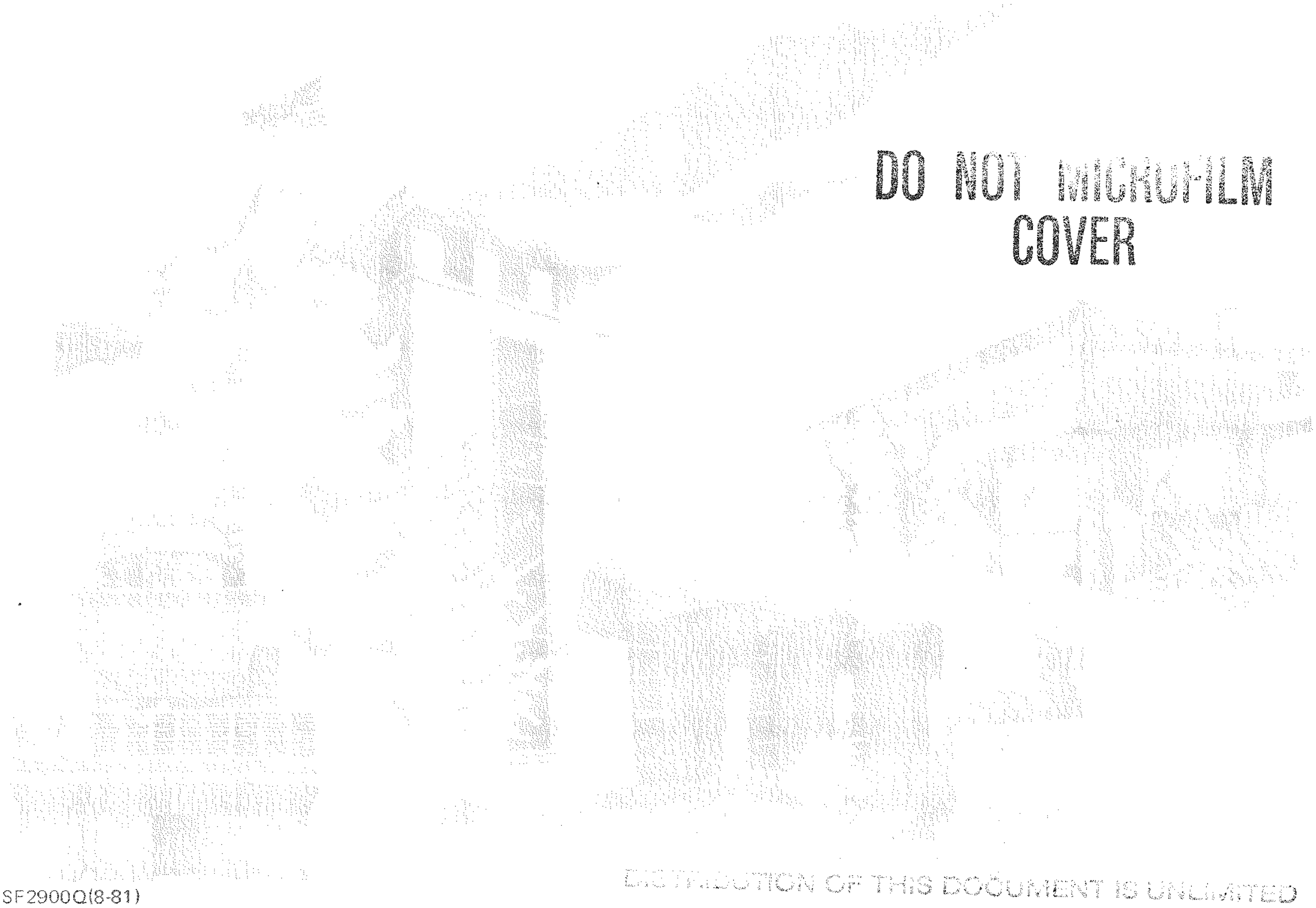

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Geometric Considerations for a Waypoint Guidance Method

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Abstract

Equations are developed for application to waypoint guidance. They provide a measure of deviation from a great circle-like course between waypoints. Ellipsoidal earth geometry is included. The approach utilizes planes that are described in analytic geometry terms. Position information input is in latitude, longitude, and altitude format.

Geometric Considerations for a Waypoint Guidance Method

Introduction

Waypoint guidance is a technique used to steer an autonomous vehicle along a desired course. Waypoints are designated at intervals along that course. The guidance algorithms issue steering commands that, if followed, result in the vehicle's travelling from waypoint to waypoint. Thus, the vehicle remains on or very close to the desired trajectory.

Waypoints can be specified by latitude, longitude, and altitude in geographic coordinates. The vehicle's current location can be described similarly. This is enough information to determine the direction of travel that would take the vehicle to the waypoint.

The shortest route from one point on a spherical surface to another on the same surface is a great circle. A great circle lies in a plane that contains the center of the sphere. The plane also is normal to the surface of the sphere along the great circle (Figure 1). The earth actually is distorted slightly from spherical; it is ellipsoidal. An ellipsoidal earth model will be used in this report. Features of the great circle will be applied to effect good approximations of shortest routes to waypoints. Therefore, only the endpoints of long, straight course segments need be specified.

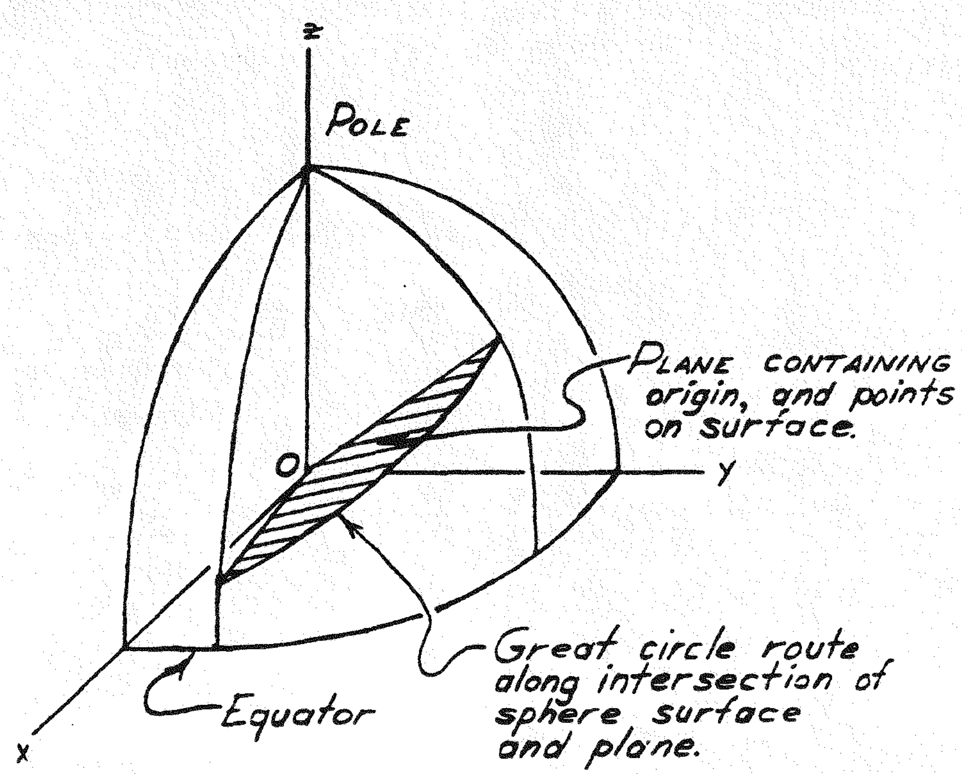

Figure 1. Great Circle Route on Spherical Earth

Analytic Geometry Principles

Solid analytic geometry will be used here to define two planes in the ellipsoidal earth (Figure 2).

- Plane A will contain the earth center and a point v that can be on, above, or below the earth ellipsoid. It will be perpendicular to the equatorial plane.
- Plane B will contain both a destination waypoint w and the point v. It also will include the line normal to the geographic surface at point v. Typically, it will not be perpendicular to the equatorial plane, nor will it include the earth center.

The intersection of plane B with an ellipsoid surface will be used to define an approximation to the shortest route from point v to the waypoint w. The ellipsoid surface could be either at sea level or at altitude h. The course angle to the waypoint will be defined by the angle between plane A and plane B, clockwise from north. Both planes will be perpendicular to the ellipsoid surface at their intersection. Therefore, the angle obtained from analytic geometry will lie on the tangent to the ellipsoidal surface. The computed angle will be the acute angle between planes A and B. A simple logic scheme will determine the quadrant in which the course will lie.

Altitude differences between point v and the waypoint will be handled separately. Altitude changes could be made at specified rates of climb or descent.

Equations of planes

The equation of a plane can be written in the form

$$Ax + By + Cz + D = 0 \,. \quad (1)$$

This equation has the following form when the plane is perpendicular to one of the three coordinate planes.

$$Ax + By + D = 0, \text{ perpendicular to xy-plane} \quad (2)$$

$$By + Cz + D = 0, \text{ perpendicular to yz-plane} \quad (3)$$

$$Ax + Cz + D = 0, \text{ perpendicular to xz-plane} \,. \quad (4)$$

The equation has the following form when the plane is perpendicular to one of the coordinate axes.

$$Ax + D = 0, \text{ perpendicular to x-axis}$$

$$By + D = 0, \text{ perpendicular to y-axis} \quad (5)$$

$$Cz + D = 0, \text{ perpendicular to z-axis} \,.$$

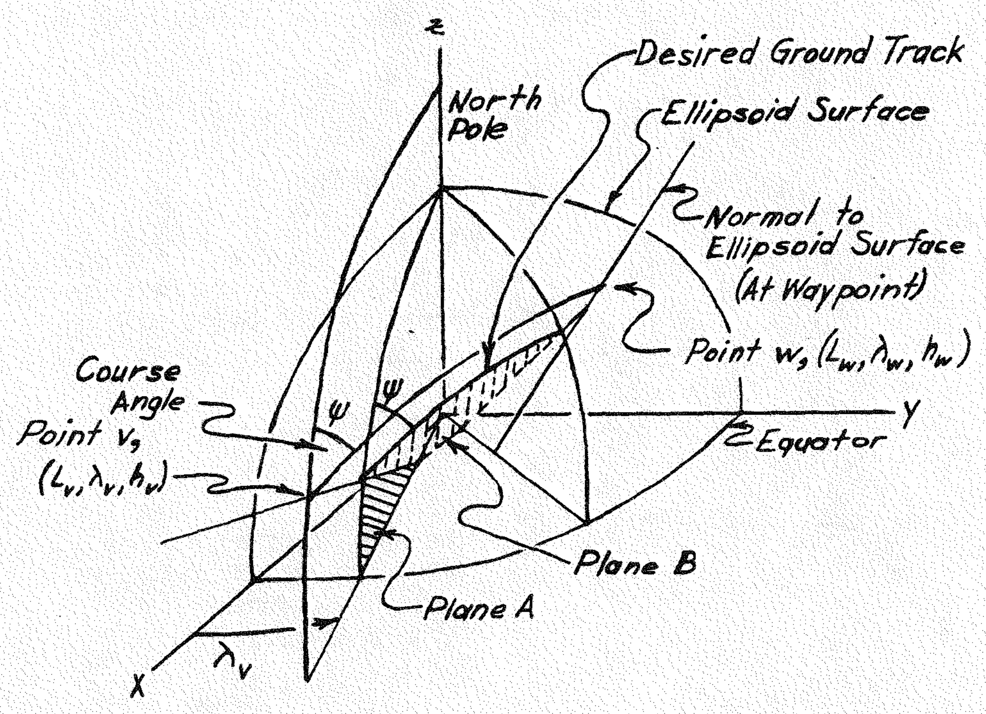

Figure 2. Planes Defined

Derivation of equation of a plane

This method will yield the equation of a plane that has the form of equation (1).

$$Ax + By + Cz + D = 0 \ .$$

The forms of Eqs. (2) through (5) are just special cases of Eq. (1). Therefore, the method will work for them, also.

Form the determinant

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \qquad (6)$$

where x,y,z are variables. The quantities

$$x_i,y_i,z_i; \ i = 1,2,3$$

are the coordinates of three points in the plane.

Expansion of the determinant yields

$$\begin{aligned} &x[y_1(z_2 - z_3) - z_1(y_2 - y_3) + (y_2z_3 - y_3z_2)] \\ &-y[x_1(z_2 - z_3) - z_1(x_2 - x_3) + (x_2z_3 - x_3z_2)] \\ &+z[x_1(y_2 - y_3) - y_1(x_2 - x_3) + (x_2y_3 - x_3y_2)] \qquad (7) \\ &-[x_1(y_2z_3 - y_3z_2) - y_1(x_2z_3 - x_3z_2) \\ &\quad + z_1(x_2y_3 - x_3y_2)] = 0 \ . \end{aligned}$$

The coefficients of x,y,z will be A,B,C respectively. The constant term will be D.

Acute angle between planes

The acute angle θ between two planes that intersect can be calculated from the following equation.

$$\cos\theta = \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}} \right| \ . \qquad (8)$$

A_1,B_1,C_1 are the coefficients of the equation of one plane. A_2,B_2,C_2 are the coefficients of the other plane.

Application of Analytic Geometry

Plane A in this application will always be perpendicular to the xy-plane and will include the z-axis and origin. It will have the form given by Eq. (2). The constant term D for plane A will be zero since the plane includes the origin, i.e., the earth center.

A,B,C and D are determined by the coordinates of points in the plane. Earth geometry is required for locations of a few points. The normal to earth surface through a point such as v or w is shown generically in Figure 3. The point at which a normal to the surface intersects the equatorial plane is denoted as E. The point at which that normal intersects the earth surface (the ellipsoid) is denoted as S. Other geometric parameters are given in Table 1.

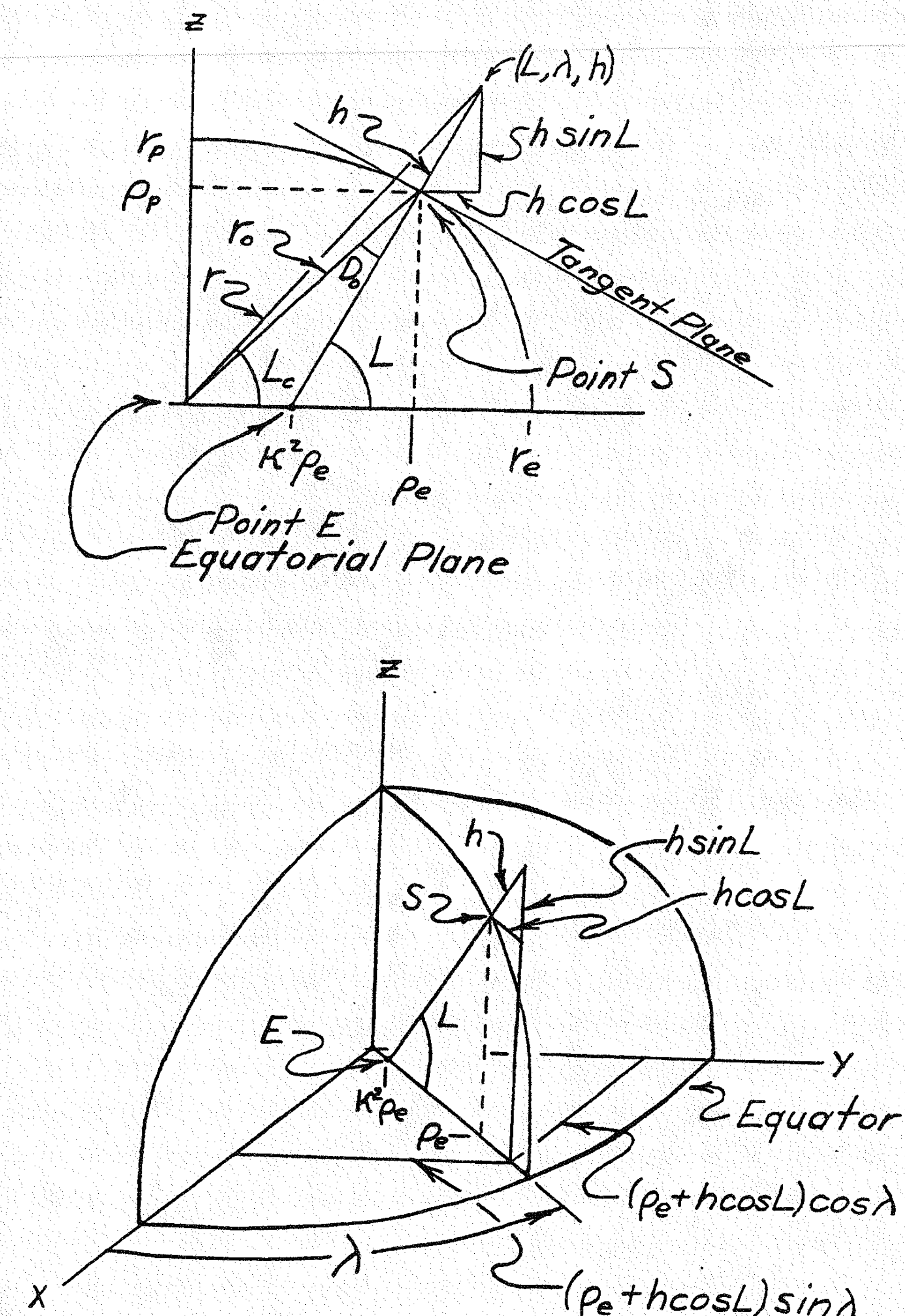

Figure 3. Coordinates of Points

Table 1: Earth Geometry Parameters

$D_o = e\sin 2L + \epsilon; \epsilon \approx 0$

e = ellipticity of earth ellipsoid

$= (r_e - r_p)/r_e$

$= 1/298.25$

$= 0.0033529$

$r_e = 6378160.$ meters

$r_o = r_e(1 - e\sin^2 L)$

$r_p = r_e(1 - e)$

$\kappa^2 = 2e(1 - e/2)$

$= 0.006694558$

$\rho_e = r_o(\cos L\cos D_o + \sin L\sin D_o)$

$\rho_p = r_o(\sin L\cos D_o - \cos L\sin D_o)$

The equation for Plane B will have the form of Eq. (1). It will include the points v and w. Plane B also will include the point E_v where the normal to the surface at point v intersects the equatorial plane.

The point v has the coordinates

$$[(\rho_{ev} + h_v\cos L_v)(i\cos\lambda_v + j\sin\lambda_v)], k(\rho_{pv} + h_v\sin L_v) \quad (9)$$

where i,j,k are unit vectors on the x,y,z axes, respectively. Latitude is L, longitude is λ, and altitude above the earth ellipsoid is h. The point w at the waypoint has the coordinates

$$[(\rho_{ew} + h_w\cos L_w)(i\cos\lambda_w + j\sin\lambda_w)], k(\rho_{pw} + h_w\sin L_w) \ . \quad (10)$$

The coordinates of the point E_v, which is below point v and on the equatorial plane, are

$$(i\kappa^2\rho_{ev}\cos\lambda_v),(j\kappa^2\rho_{ev}\sin\lambda_v) \ . \quad (11)$$

The points defined by Eqs. (9), (10), and (11) are sufficient to define and derive equations for planes A and B. Once that is done, then the course angle to the waypoint can be computed using Eq. (8).

The quadrant that contains the course to the waypoint can be determined by the logic diagram given in Figure 4.

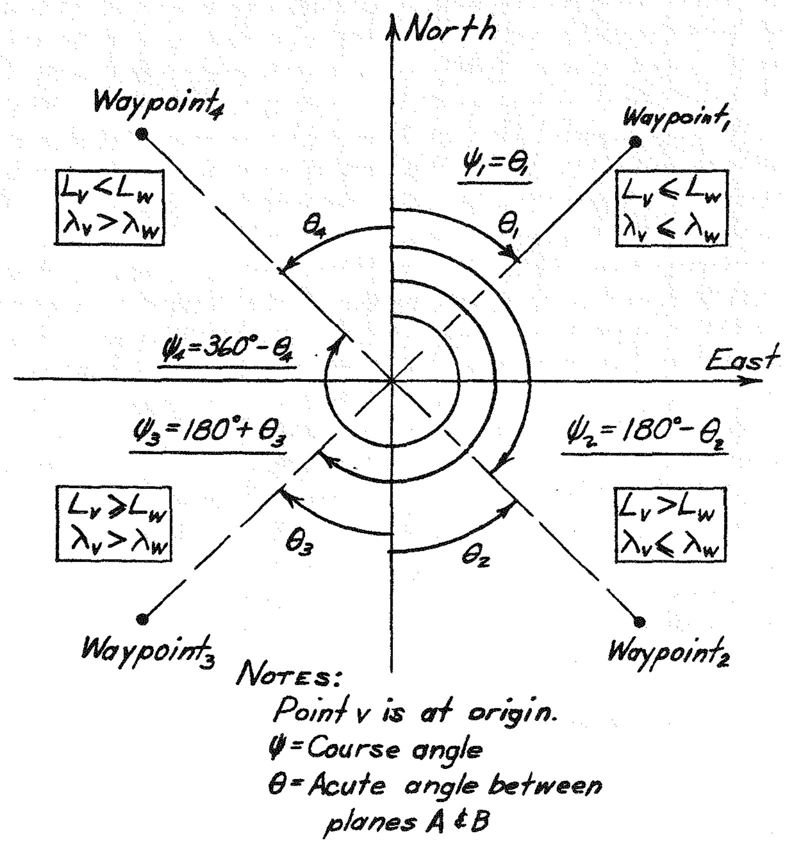

Figure 4. Logic to Determine Quadrant of Course Angle

Distance From Desired Path

The foregoing geometric considerations are suitable for partitioning the guidance problem into two planes of control:

- Altitude (vertical)
- Course (horizontal)

Altitude position errors can be determined by reference to an altimeter or the computed altitude of an inertial navigation system (INS).

Horizontal course errors can be determined from geometry and the latitude, longitude, and altitude (L,λ, and h) outputs of an INS.

The distance from a point x_a,y_a,z_a to a plane that is represented by the equation

$$Ax + By + Cz + D = 0$$

is

$$d = \left| \frac{Ax_a + By_a + Cz_a + D}{\sqrt{A^2 + B^2 + C^2}} \right| . \quad (12)$$

Whether course error is to the left or right of the desired course can be defined by the following equations. Refer also to Figure 5.

$$\psi_e = \psi - \psi_a < 0; (x_a,y_a,z_a) \text{ is right of course} \quad (13)$$

and

$$\psi_e = \psi - \psi_a > 0; (x_a,y_a,z_a) \text{ is left of course} \quad (14)$$

where

ψ = desired course angle on the desired track
ψ_a = bearing angle from point v to the aircraft position
$\psi_e = \psi - \psi_a$.

The logic described in Figure 4 can be used to assist in the determination of ψ_e.

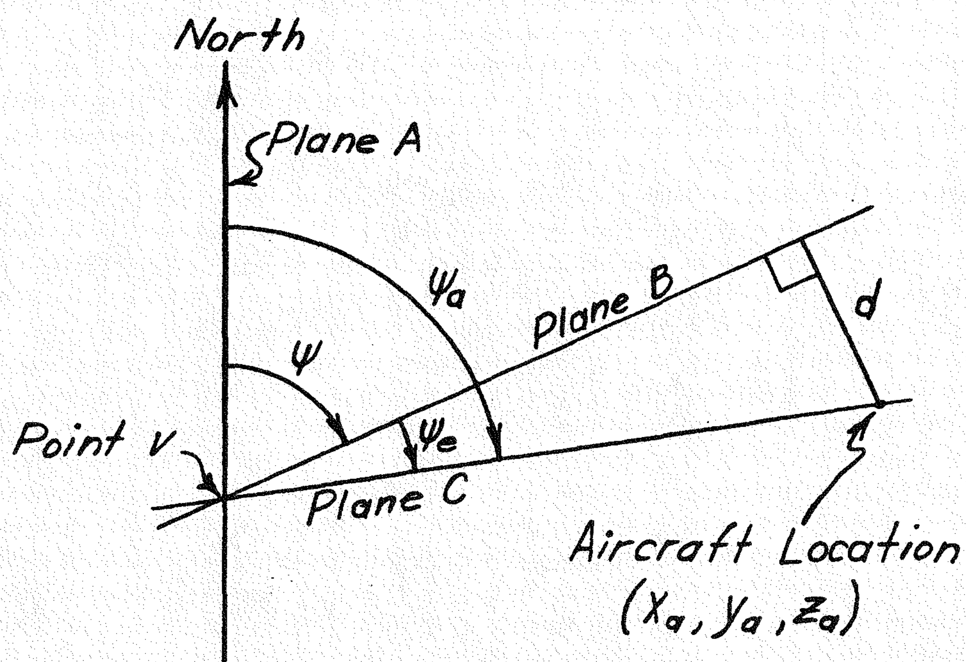

Figure 5. Determination of Error Polarity

Implementation

Consider a desired flight path in which a previous waypoint is at point v (see Figure 2) and the next waypoint is at point w.

Compute x_v, y_v, z_v from L_v, λ_v, h_v.

The coordinates of point v, referenced to a rectangular, earth centered system, are

$$x_v, y_v, z_v = (\rho_{ev} + h_v \cos L_v)\cos\lambda_v, \quad (\rho_{ev} + h_v \cos L_v)\sin\lambda_v, \rho_{pv} + h_v \sin L_v \qquad (15)$$

where

$$\rho_{ev} = r_{ov}(\cos L_v \cos D_{ov} + \sin L_v \sin D_{ov}) \qquad (16)$$

$$\rho_{pv} = r_{ov}(\sin L_v \cos D_{ov} + \cos L_v \sin D_{ov}) \qquad (17)$$

and from Table 1

$$\left.\begin{aligned} r_{ov} &= r_e(1 - e\sin^2 L_v) \\ D_{ov} &= e\sin 2L_v \\ r_e &= 6378160.\text{m},\ e = .0033529 \end{aligned}\right\} \cdot \qquad (18)$$

Compute x_w, y_w, z_w from L_w, λ_w, h_w.

The waypoint coordinates at point w, in the earth-centered rectangular axes are

$$x_w, y_w, z_w = (\rho_{ew} + h_w \cos L_w)\cos\lambda_w, \quad (\rho_{ew} + h_w \cos L_w)\sin\lambda_w, \rho_{pw} + h_w \sin L_w \qquad (19)$$

where

$$\rho_{ew} = r_{ow}(\cos L_w \cos D_{ow} + \sin L_w \sin D_{ow}) \qquad (20)$$

$$\rho_{pw} = r_{ow}(\sin L_w \cos D_{ow} + \cos L_w \sin D_{ow}) \qquad (21)$$

and

$$r_{ow} = r_e(1 - e\sin^2 L_w)$$

$$D_{ow} = e\sin 2L_w \, .$$

Compute x_{Ev}, y_{Ev}, z_{Ev}.

The intersection of the equatorial plane and the normal to the ellipsoid surface through point v has the following coordinates.

$$\left.\begin{aligned} x_{Ev} &= \kappa^2 \rho_{ev} \cos\lambda_v \\ y_{Ev} &= \kappa^2 \rho_{ev} \sin\lambda_v \\ z_{Ev} &= 0 \text{ by definition of point E} \end{aligned}\right\} \qquad (22)$$

where

$$\kappa^2 = 0.006694558$$

Compute Coefficients A, B, C, and D of plane A.

Three points that lie in plane A are (refer to Figure 3)

x_v, y_v, z_v given in Eq. (15)

$x_v, y_v, 0$

$0, 0, \rho_{pv}$; ρ_{pv} is given in Eq. (17) .

The coefficients of the equation for plane A can be computed by referring to Eq. (7).

$$A_A = y_v(0 - \rho_{pv}) - z_v(y_v - 0) + (y_v \rho_{pv} - 0)$$

$$B_A = -[x_v(0 - \rho_{pv}) - z_v(x_v - 0) + (x_v \rho_{pv} - 0)]$$

$C_A = 0$ since plane A is perpendicular to the xy-plane

$D_A = 0$

since plane A contains the origin of the xyz-axes.

Simplification yields

$$\left.\begin{aligned} A_A &= -y_v z_v \\ B_A &= x_v z_v \end{aligned}\right. \qquad (23)$$

$C_A = 0$

$D_A = 0\,.$

Therefore plane A has the equation

$$-(y_v z_v)x + (x_v z_v)y = 0\ . \tag{24}$$

Compute Coefficients A, B, C, and D of plane B.

Three points in plane B are

$x_v, y_v z_v$, given in Eq (15)

x_w, y_w, z_w, given in Eq (19)

x_{Ev}, y_{Ev}, 0, given in Eq (22) .

The coefficients in the equation for plane B are

$$A_B = y_v(z_w - 0) - z_v(y_w - y_{Ev}) + (y_w 0 - y_{Ev} z_w)$$

$$B_B = -[x_v(z_w - 0) - z_v(x_w - x_{Ev}) + (x_w 0 - x_{Ev} z_w)]$$

$$C_B = x_v(y_w - y_{Ev}) - y_v(x_w - x_{Ev}) + (x_w y_{Ev} - x_{Ev} y_w)$$

$$D_B = -\ [x_v(y_w 0 - y_{Ev} z_w) - y_v\,(x_w 0 - x_{Ev} z_w) + z_v(x_w y_{Ev} - x_{Ev} y_w)]\ .$$

Simplification yields

$$\left.\begin{aligned} A_B &= y_v z_w - y_w z_v + y_{Ev} z_v - y_{Ev} z_w \\ B_B &= -x_v z_w + x_w z_v + x_{Ev}\,(z_w - z_v) \\ C_B &= x_v(y_w - y_{Ev}) - y_v(x_w - x_{Ev}) \\ &\quad + (x_w y_{Ev} - x_{Ev} y_w) \\ D_B &= x_v y_{Ev} z_w - x_{Ev} y_v z_w \\ &\quad - z_v(x_w y_{Ev} - x_{Ev} y_w) \end{aligned}\right\}\ . \tag{25}$$

Therefore, the equation for plane B is

$$\begin{aligned} &x[y_v z_w - y_w z_v + y_{Ev}(z_v - z_w)] \\ &+y[-x_v z_w + x_w z_v + x_{Ev}(z_w - z_v)] \\ &+z[x_v(y_w - y_{Ev}) - y_v(x_w - x_{Ev}) + (x_w y_{Ev} - x_{Ev} y_w)] \\ &+(x_v y_{Ev} - x_{Ev} y_v)z_w - (x_w y_{Ev} - x_{Ev} y_w)z_v = 0\ . \end{aligned} \tag{26}$$

Compute course angle of desired track.

The acute angle between plane A and plane B is

$$\theta = \cos^{-1}\left|\frac{A_A A_B + B_A B_B + C_A C_B}{\sqrt{A_A^2 + B_A^2 + C_A^2}\sqrt{A_B^2 + B_B^2 + C_B^2}}\right|\ . \tag{27}$$

Now, the course angle can be computed by determining the quadrant in which theta applies. Refer to Figure 4.

$$\left.\begin{aligned} L_v &: L_w \\ \lambda_v &: \lambda_w \end{aligned}\right\} \Rightarrow \psi\ . \tag{28}$$

Determine distance of aircraft from desired track.

The desired track of travel is the line of intersection of plane B and the ellipsoidal surface at the desired altitude, h. The horizontal distance of the aircraft (at a point x_a, y_a, z_a) from the desired track is the distance between the aircraft and plane B. That distance is

$$d = \left|\frac{A_B x_a + B_B y_a + C_B z_a + D_B}{\sqrt{A_B^2 + B_B^2 + C_B^2}}\right|\ . \tag{29}$$

The vertical distance of the aircraft from the desired altitude is

$$h_{error} = h_{desired} - h_a\ . \tag{30}$$

Determine position of aircraft relative to plane B.

Course angles are computed as positive, clockwise from north. Recall the notation in Figure 5. The expression

$$\mathrm{Sign}[\psi_e = \psi - \psi_a] \tag{31}$$

must be evaluated; therefore, ψ_a must be determined. This will require that

- the equation of a plane (plane C) similar to plane B but containing the aircraft position and the waypoint v be derived
- the acute angle θ between plane A and plane C be computed
- ψ_a be determined
- ψ_e be determined, using Eqs. (13) and (14), and Figure (5)
- the following logic be resolved

  $\text{Sign}[\psi_e] < 0 \Rightarrow$ Aircraft to right of course line
  $\text{Sign}[\psi_e] > 0 \Rightarrow$ Aircraft to left of course line .

The equation for plane C will have the same form as that for plane B. It will be

$$
\begin{aligned}
&x[y_a z_v - y_v z_a + y_{Ea}(z_a - z_v)] \\
&+y[-x_a z_v + x_v z_a + x_{Ea}(z_v - z_a)] \\
&+z[x_a(y_v - y_{Ea}) - y_a(x_v - x_{Ea}) \\
&\quad +(x_v y_{Ea} - x_{Ea} y_v)] \\
&+x_a y_{Ea} z_v - x_{Ea} y_a z_v - z_a(x_v y_{Ea} - x_{Ea} y_v) = 0
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
x_a &= (\rho_{ea} + h_a \cos L_a)\cos\lambda_a \\
y_a &= (\rho_{ea} + h_a \cos L_a)\sin\lambda_a \\
z_a &= \rho_{pa} + h_a \sin L_a \,.
\end{aligned}
$$

$$
\begin{aligned}
\rho_{ea} &= r_{oa}(\cos L_a \cos D_{oa} + \sin L_a \sin D_{oa}) \\
\rho_{pa} &= r_{oa}(\sin L_a \cos D_{oa} + \cos L_a \sin D_{oa})
\end{aligned}
\tag{33}
$$

$$r_{oa} = r_e\,(1 - e\sin^2 L_a)$$

$$D_{oa} = e\sin^2 L_a \,.$$

Concluding Remarks

The information about d, ψ, ψ_e and altitude error is sufficient for a waypoint guidance scheme that does not require the time rate of change of ψ or ψ_e. The rate of change of altitude is available from inertial navigation systems.

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