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TITLE: DIMENSIONS, INDICES, AND CONGRUENCE CLASSES OF REPRESENTATIONS OF AFFINE KAC-MOODY ALGEBRAS (with examples for affine E_8)

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DIMENSIONS, INDICES AND CONGRUENCE CLASSES OF REPRESENTATIONS OF AFFINE KAC-MOODY ALGEBRAS (with examples for affine E_8)

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INTRODUCTION

The purpose of this lecture is to introduce, describe and illustrate affine generalizations of some familiar notions from the representation theory of semisimple Lie algebras/groups. We touch upon the multiplicity of a weight and the dimension, congruence class, and indices of a representation. Our examples of the highest weight representations of affine E_8 can be considered as a preview of far more extensive results of this type to appear (Kass et al., 1987).

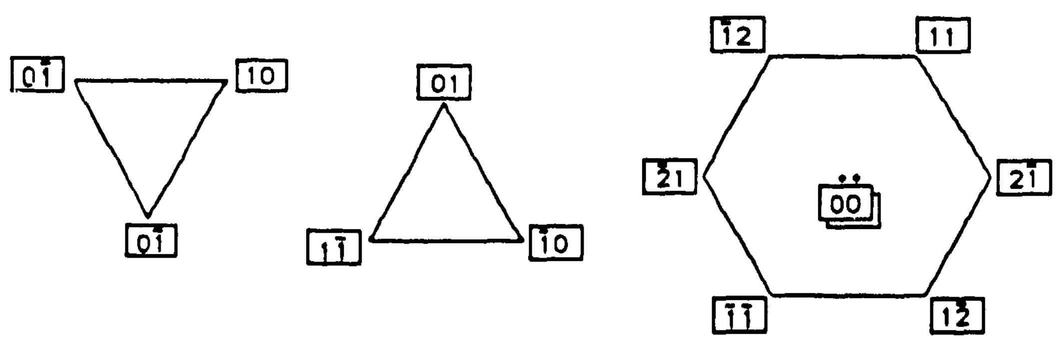
THE WEIGHT SYSTEM OF A REPRESENTATION

First we recall some familiar facts. Finite-dimensional irreducible representations of $SU(2)$ can be specified by the "angular momenta" J, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ of the representation, and the basis vectors or "angular momentum states" within the representation J can be labelled by the "projection" M of J, where M takes on each value $J, J-1, \ldots, -J$. Convenience and consistency with higher rank algebras leads us to change these conventions and to take J as twice the angular momentum and M as twice its projection, giving

$$J \in 0,1,2,\ldots\ , \qquad M \in (J, J-2, \ldots, -J) = \Omega(J),$$

where $\Omega(J)$ is called the *weight system* of the $SU(2)$ representation J. We will sometimes denote the representation simply by J and sometimes by $L(J)$.

A finite-dimensional irreducible $SU(3)$ representation is determined by its *highest weight* $\Lambda = (p,q)$, with integer $p,q \geq 0$. The inherent geometry of the algebra allows us to draw the weight system $\Omega(\Lambda)$ as a triangle (for highest weight $(p,0)$ or $(0,q)$), a hexagon (if $pq > 0$), or a point (for the one-dimensional representation with highest weight $(0,0)$). For example, one has the following weight systems $\Omega(\Lambda)$ for the representations $L(\Lambda)$ with highest weights $\Lambda = (1,0)$, $(0,1)$ and $(1,1)$:



In the last example the symbolism represents the fact that there are two vectors of weight $(0,0)$. In general we suppress the commas between the coordinates of the weight and economize space by using an overbar instead of a minus sign.

Briefly, the weights of a finite-dimensional simple complex Lie algebra g "live" in a Euclidean lattice, and their coordinates are given in terms of the basis of *fundamental weights* $\lambda_1, \lambda_2, \ldots \lambda_\ell$ of that lattice. This basis and its dual basis of *simple roots* $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are related by the matrix CM, the *Cartan matrix*, which is the change of basis matrix between the two bases, and which completely describes the algebra. The same information contained in CM can be given by a graph, the *Dynkin diagram* of g (DD). The Cartan matrices and Dynkin diagrams for each of the finite-dimensional simple complex Lie algebras can be found in many places (see Bremner et al., 1985). A thorough development of the representation theory of these algebras can be found in (Humphreys, 1972).

For an affine Kac-Moody algebra g an irreducible representation is again given by the tuple Λ, which now has $\ell + 1$ coordinates $\lambda_0, \lambda_1, \ldots \lambda_\ell$. Although these coordinates completely describe a module, they do not completely describe Λ within the weight lattice, as we shall soon see, and we generally call this $\ell + 1$-tuple the *weight label* of Λ.

The distinct weights of $\Omega(\Lambda)$ can be calculated recursively from the highest weight $\Lambda = (p_0, \ldots, p_\ell)$ (ℓ is the *rank* of g, and of CM) with non-negative integer coordinates p_k (if g is finite, there is no p_0) using the following algorithm:

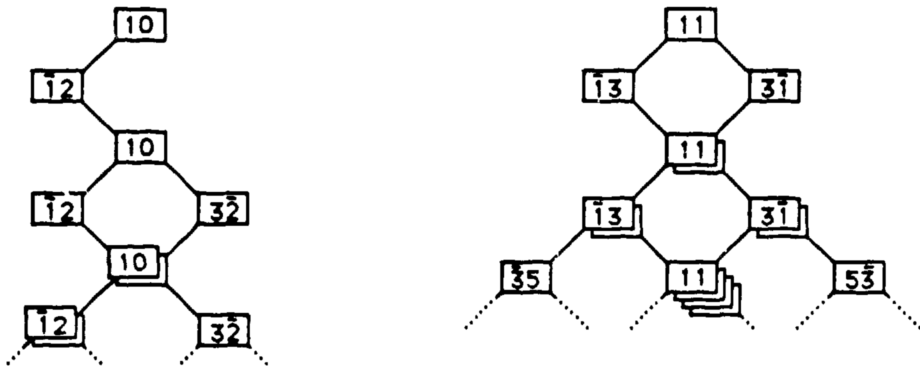
(a) Put Λ into $\Omega(\Lambda)$ and let $\mu = \Lambda$.

(b) For $\mu \in \Omega(\Lambda)$, for any positive coodinate p_k of μ, add the p_k weights $\Lambda - \alpha_k, \Lambda - 2\alpha_k, \ldots, \Lambda - p_k\alpha_k$ to $\Omega(\Lambda)$. In the basis of fundamental weights, the vector α_k is the k-th row of CM for g (in some conventions α_k is the k-th column of CM).

(c) Repeat (b), replacing μ by each of the weights found in (b).
The procedure terminates for finite-dimensional Lie algebras and continues forever in the affine case.

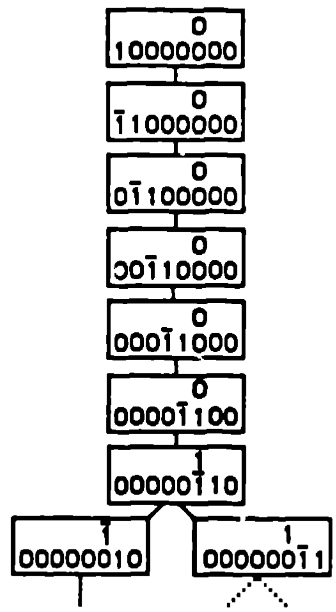
The above algorithm computes only the list of distinct weights of a module. The number of vectors of weight μ in $L(\Lambda)$ (the *multiplicity* of μ in $L(\Lambda)$) is difficult to compute in general, and can be found in published tables (Bremner et al., 1985 (finite case); Kass et al., 1987 (affine case)).

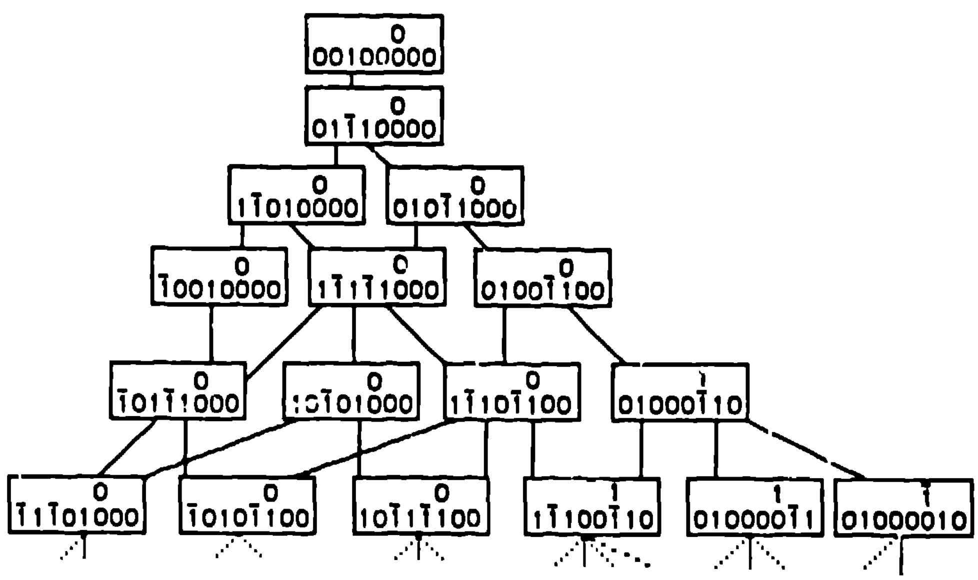
We now consider a few examples in the affine case. Each of the finite-dimensional simple Lie algebras has an "affinization." For $g = A_1$, the Lie algebra of $SU(2)$, the affinization, $A_1^{(1)}$ has $CM = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Consider its representations with highest weights $\Lambda = (10)$ and (11). Arranging $\Omega(\Lambda)$ into horizontal slices according to the number of simple roots which have been subtracted from Λ (we call this the *principal slicing* of the representation), we obtain the following diagrams:



Lines indicate subtraction of some simple root α_i. Note in particular that the weight labels reappear, shifted by units of $\alpha_0 + \alpha_1$. In the affine case, the simple roots are linearly dependent when projected onto the space generated by the fundamental weights, and an additional label would be needed to separate the repeated occurences of each weight label. Already we can see one of the generalizations of a notion in the finite-dimensional case. While we could find the multiplicity of each weight and treat these as separate identities as is generally done in the finite case, it is instructive to write a generating function for the multiplicities of each weight label, using powers of q to separate the occurences. Thus in the first example above, where the weight (10) occurs with multiplicities $1, 1, 2, \ldots$, we could say that its multiplicity is $1 + q + 2q^2 + \ldots$. Remarkably, this particular power series is exactly the generating function of the classical partition function, where the coefficient of q^n is the number of partitions of the integer n. In general, the generalized multiplicity (with an appropriate power of q on the outside) is a modular form, and many interesting identities and properties are known for these series (see Kac and Peterson, 1984). As the next examples we take two representations of the algebra $E_8^{(1)}$, the affinization of the 248-dimensional algebra E_8. To remain independent of numbering conventions, we write the coordinates of the simple roots as well as the weight coordinates in Dynkin diagram form. One has (in the principal slicing):

$$\alpha_0 = 2\bar{1}000\overset{0}{0}00$$
$$\alpha_1 = \bar{1}2\bar{1}00\overset{0}{0}00$$
$$\alpha_2 = 0\bar{1}2\bar{1}0\overset{0}{0}00$$
$$\alpha_3 = 00\bar{1}2\bar{1}\overset{0}{0}00$$
$$\alpha_4 = 000\bar{1}2\overset{0}{\bar{1}}00$$
$$\alpha_5 = 0000\bar{1}\overset{\bar{1}}{2}\bar{1}0$$
$$\alpha_6 = 00000\overset{0}{\bar{1}}2\bar{1}$$
$$\alpha_7 = 00000\overset{0}{0}\bar{1}2$$
$$\alpha_8 = 00000\overset{2}{\bar{1}}00$$





All weights shown here have multiplicity 1. Higher multiplicities occur further down when other weights with non-negative coordinates (dominant weights) appear.

INVARIANT CHARACTERISTICS OF REPRESENTATIONS

There exist several quantities which are easy to determine for a representation Λ of g of any type, and which are often very useful in calculations. Suppose that one has been given g and a representation $L(\Lambda)$ of g with highest weight $\Lambda = (p_0, p_1, \ldots, p_\ell)$. (If g is finite-dimensional, simply let $p_0 = 0$.) Consider the two integers:

Congruence number of Λ	$C(\Lambda) = \Sigma_{i=0}^{\ell} b_i p_i \mod(\det CM)$.
Level of Λ	$L(\Lambda) = \Sigma_{i=0}^{\ell} c_i p_i$.

Here $\det CM$ is the determinant of the Cartan matrix. The coefficients b and c depend on the algebra only. They are found, for example, in (Bremner et al., 1984) and (Kass et al., 1987). For more details and examples of $C(\Lambda)$ see (Lemire and Patera, 1982). In the case of $SU(2)$, $C(J) = \pm 1$ for odd (even) dimensional representations; for $SU(3)$, $C(\Lambda)$ is the familiar triality number.

Two representations Λ and Λ' of g belong to the same *congruence class* provided

$C(\Lambda) = C(\Lambda')$	for finite g;
$C(\Lambda) = C(\Lambda')$ and $L(\Lambda) = L(\Lambda')$	for affine g.

It is always true that $C(\alpha_i) = 0$ and $L(\alpha_i) = 0$ for a simple root α_i, hence $C(\Lambda) = C(\mu)$ and $L(\Lambda) = L(\mu)$ for any $\mu \in \Omega(\Lambda)$, since μ is obtained from Λ by the subtraction of simple roots. It can also be shown that for g finite-dimensional, there are a finite number $(\det CM)$ of congruence classes each containing infinitely many representations. In contrast, an affine g has infinitely many congruence classes each containing a finite number of irreducible representations.

The most common use of $C(\Lambda)$ is in computing tensor products of representations or tensor powers of a single representation (with or without a particular permutational symmetry). Thus if

$$\Lambda\otimes\Lambda' = \Lambda_1 \oplus \ldots \oplus \Lambda_1 \quad \text{and} \quad \Lambda\otimes\ldots\otimes\Lambda_{YT} = \Lambda_1\oplus\ldots\oplus\Lambda_n,$$

where the subscript YT denotes a (Young tableau) permutation symmmetry of the tensor power of Λ, one has

$$C(\Lambda\otimes\Lambda') = C(\Lambda) + C(\Lambda') = C(\Lambda_1) = .. = C(\Lambda_l) \bmod(\det CM),$$

$$C(\Lambda\otimes\ldots\otimes\Lambda)_{YT} = C(\Lambda) + \ldots + C(\Lambda) = C(\Lambda_1) = .. = C(\Lambda_n) \bmod(\det CM).$$

The *dimension* $d(\Lambda)$ of a representation Λ of an affine algebra g is of course infinite, but we can proceed as we did with the multiplicity of a weight and write the dimension as a power series in q according to some particular slicing of g, letting the coefficient of q^k be the dimension of the $k-1$-st slice. Such a series will have many of the useful properties of the dimension in the finite case. The dimensions (in the principal slicing) of the above representations of affine A_1 and E_8 are

$$d(10) = 1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \ldots,$$

$$d(11) = 1 + 2q + 2q^2 + 4q^3 + 6q^4 + \ldots,$$

$$d\left(10000\overset{0}{0}00\right) = 1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + 2q^7 + \ldots,$$

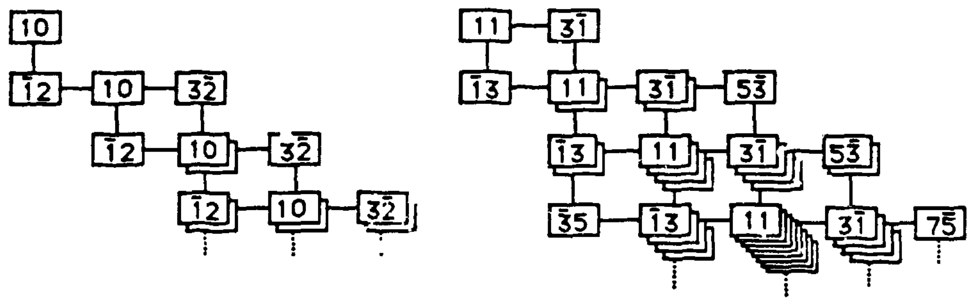
$$d\left(00100\overset{0}{0}00\right) = 1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + \ldots .$$

Another type of slicing giving useful information is a slicing into representations of a finite-dimensional subalgebra of g. If one removes the k-th node from the Dynkin diagram of g (equivalently, the k-th row and column from CM), one obtains DD or CM for a maximal finite-dimensional subalgebra of g, which we denote $g(k)$. (The algebras so obtained are essentially all the maximal finite-dimensional semisimple subalgebras of g. (R. V. Moody, private communication, 1986)) We can then define the $g(k)$-slicing of $L(\Lambda)$ so that the weights of the p-th slice are those weights of the form

$$\mu = \Lambda - (p-1)\alpha_k - (\text{other } \alpha_i\text{'s}) \in \Omega(\Lambda).$$

For the affine A_1 modules above, we can arrange the weights to present the

$g(0)$ slices as horizontal slices and the $g(1)$ slices as vertical slices. In this case both $g(0)$ and $g(1)$ are isomorphic to $SU(2)$, hence the slices are (generally reducible) $SU(2)$ modules.



The first few slices of $L(10)$ and $L(11)$ in the $g(0)$ slicing are

$\Gamma_1=(0),\ \Gamma_2=(2),\ \Gamma_3=(2)\oplus(0),\ \Gamma_4=(2)\oplus(2),\ldots;$

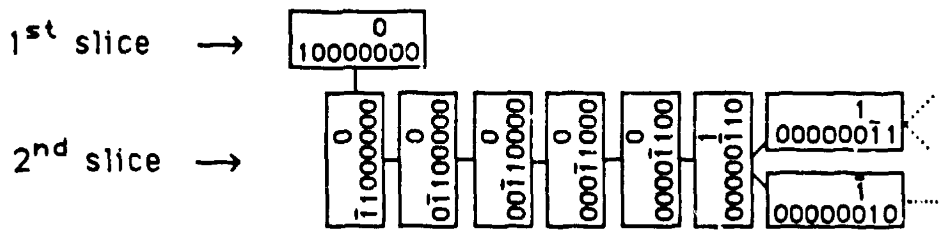
$\Gamma_1=(1),\ \Gamma_2=(3)\oplus(1),\ \Gamma_3=(3)\oplus(3)\oplus(1)\oplus(1),\ldots,$

with corresponding dimension series $\Sigma_i d(\Gamma_i)q^{i-1}$, here giving

$1 + 3q + 4q^2 + 6q^3 + \ldots;$

$2 + 6q + 12q^2 + 18q^3 + \ldots.$

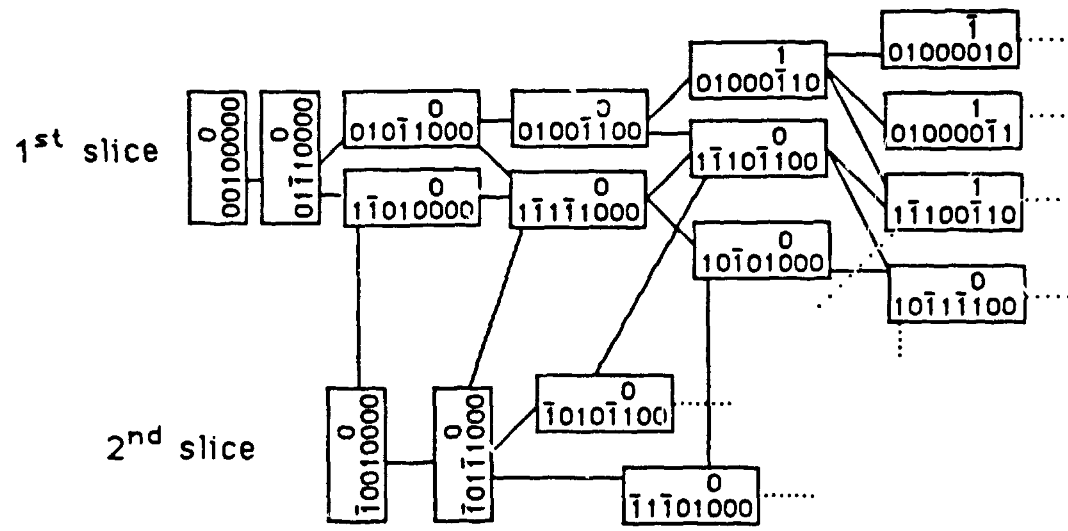
More interesting are the E_8 examples. Consider $\Lambda = \left(\begin{smallmatrix} & & & & & 0 & & \\ 1&0&0&0&0&0&0&0\end{smallmatrix}\right)$ of $g(0) = E_8 \subset$ affine E_8. One has



Viewing the slices by ignoring the p_0 coordinate of each weight, one sees immediately that the top slice here is a one-dimensional E_8 (scalar) representation, while the second slice contains the representation $\Gamma_2 = \left(\begin{smallmatrix} & & & & 0 & & \\ 1&0&0&0&0&0&0\end{smallmatrix}\right)$ of dimension 248 (further computation shows that no other representation appears at this slice). The dimension series for this slicing thus begins with

$$d\left(10000\overset{0}{0}00\right) = 1 + 248q + \ldots .$$

For $\Lambda = \left(00100\overset{0}{0}00\right)$, we have

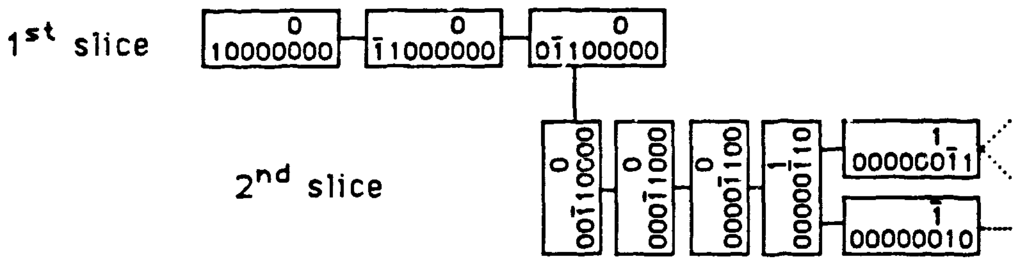


Here the first slice is the E_8 representation $\Gamma_1 = \left(0100\overset{0}{0}00\right)$ with $d(\Gamma_1)$ = 27 000, and the second one contains the representation $\Gamma_2 = \left(0010\overset{0}{0}00\right)$ of dimension 2 450 240.

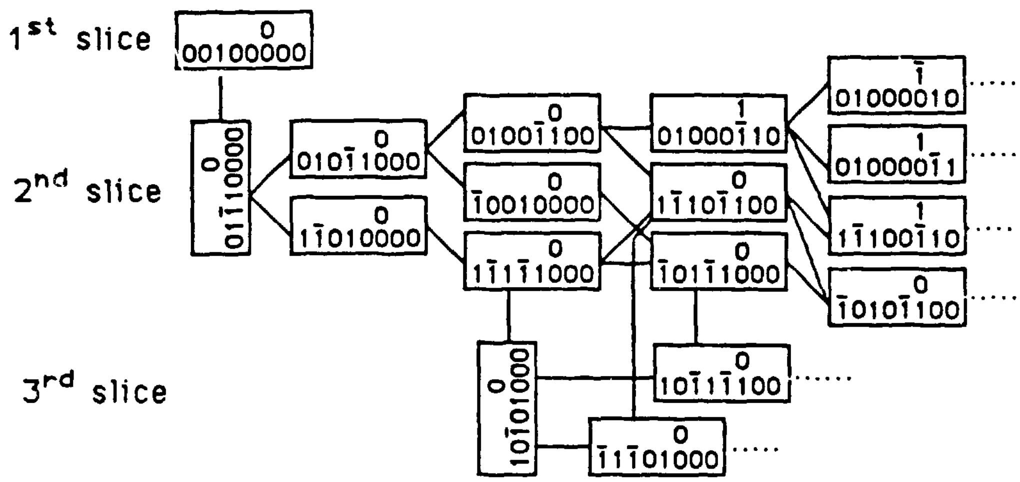
Next let us take $g(2) = SU(3) + E_6$. The slices now are defined by the number of times α_2 has been subtracted from Λ, and affine E_8 weights are read as the subalgebra weights as follows:

$$\left(p_0p_1p_2p_3p_4\overset{p_8}{p_5}p_6p_7\right) = (p_0p_1)\left(p_3p_4\overset{p_3}{p_5}p_6p_7\right)$$

In $g(2)$-slicing the top slice of the basic representation $\Lambda = \left(10000\overset{0}{0}00\right)$ of affine E_8 is an $SU(3)$ triplet with a E_6 singlet, $\Gamma_1 = (10)\left(00\overset{0}{0}00\right)$; the second slice contains $\Gamma_2 = (00)\left(10\overset{0}{0}00\right)$, the singlet of $SU(3)$ and the 27-plet of E_6 :



The other E_8 representation slices with respect to $g(2) = SU(3) + E_6$ as follows:



The top is a singlet of $SU(3) + E_6$, $\Gamma_1 = (00)\left(\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. Next is a triplet of $SU(3)$ and an E_6 representation of dimension 27, $\Gamma_2 = (01)\left(\begin{smallmatrix} & & 0 & & \\ 1 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$; the third slice is $\Gamma_3 = (10)\left(\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 0 & 0 & 0 \end{smallmatrix}\right) \oplus (02)\left(\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right) \oplus (10)\left(\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}\right)$. The dimension is then $d\left(\begin{smallmatrix} & & & & & 0 & & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ $= d(\Gamma_1) + d(\Gamma_2)q + \ldots = 1 + 81q + 1296q^2 + \ldots$.

Finally let us point out a general property of the representations Γ of the subalgebra $g(k)$ in different slices: all irreducible components of a slice belong to the same congruence class, though the congruence classe may vary from slice to slice. It is not difficult to write specific rules for each case.

The *indices of representations* of finite Lie algebras g (Patera et al., 1976) as well as the anomaly numbers (Patera, Sharp, 1981) also generalize into power series (different series in different slicings) retaining all their useful properties. The index of degree k can be defined in the affine case as the power series

$$I^{(k)}(\Lambda) = \sum_{j=1}^{\infty} q^{j-1} I^{(k)}(\Gamma_j)\,.$$

Here $I^{(k)}(\Gamma_j)$ is the index of degree k (Patera et al., 1976; Patera, Sharp, 1981) in the finite-dimensional case. Using its definition, we have, for example

$$I^{(2k)}(\Lambda) = \sum_{j=1}^{\infty} q^{j-1} I^{(2k)}(\Gamma_j) = \sum_{j=1}^{\infty} q^{j-1} \sum_{\mu\in\Omega(\Gamma_j)} (\mu,\mu)^k\,.$$

Let $I^{(k)}(\Lambda)$ be the index and $d(\Lambda)$ the dimension of a representation Λ. Then we have all the properties from (Patera, Sharp, 1981; McKay et al., 1981). As an example,

$$I^{(2)}(\Lambda\otimes\Lambda') = I^{(2)}(\Lambda)d(\Lambda') + d(\Lambda)I^{(2)}(\Lambda'),$$

or, denoting by a Young tableau a permutation symmetry component of a tensor power $\Lambda\otimes\ldots\otimes\Lambda$,

$$I^{(2)}(\underbrace{\square\square\cdots\square}_{\leftarrow k \rightarrow}) = \frac{d(\Lambda)+k)!}{(d(\Lambda)+1)!(k-1)!}\, I^{(2)}(\Lambda)\ ,$$

$$I^{(2)}\left(k\left\{\begin{matrix}\square\\ \vdots \\ \square\end{matrix}\right.\right) = \frac{d(\Lambda)-2)!}{(d(\Lambda)-k-1)!(k-1)!}\, I^{(2)}(\Lambda)\ ,$$

$$I^{(4)}(\square\square) = (d(\Lambda)+8)\, I^{(4)}(\Lambda) + \frac{\ell+2}{\ell}\left(I^{(2)}(\Lambda)\right)^2\ ,$$

$$I^{(4)}\left(\begin{matrix}\square\\ \square\end{matrix}\right) = (d(\Lambda)-8)\, I^{(4)}(\Lambda) + \frac{\ell+2}{\ell}\left(I^{(2)}(\Lambda)\right)^2\ .$$

Using the anomaly-index (Patera, Sharp, 1981) as $I^{(k)}(\Gamma_j)$, one has also

$$I^{(3)}(\square\square) = (d(\Lambda)+4)I^{(3)}(\Lambda)\ ,$$

$$I^{(3)}\left(\begin{matrix}\square\\ \square\end{matrix}\right) = (d(\Lambda)-4)I^{(3)}(\Lambda)\ , \quad \text{etc.}$$

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