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Classical geometrical interpretation of ghost fields and anomalies in Yang-Mills theory and quantum gravity. [1]

Jean THIERRY-MIEG

Lawrence Berkeley Laboratory,
University of California, Berkeley, California 94720,
and CNRS, Observatoire de Meudon, 92190 France

To Yuval Ne'eman, on his 60th birthday.

Abstract

The reinterpretation of the BRS equations of Quantum Field Theory as the Maurer Cartan equation of a classical principal fiber bundle leads to a simple gauge invariant classification of the anomalies in Yang Mills theory and gravity.

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1 Introduction

The idea that classical Yang-Mills theory [1] should be formulated over a principal fiber bundle, locally the product of space-time by a Lie group, was first expressed by Ikeda and Miyachi [2] in 1956 and later by Lubkin [3] in 1963. It was widely accepted only in 1975 when Wu and Yang [4] showed that this geometrical setting was necessary for a proper understanding of the solitons.

Remarkedly, the same classical geometry controls the quantized theory. I have shown in 1978 [5] that the globally defined Darboux-Maurer-Cartan-Ehresmann (DMCE) structural equations of the principal fiber bundle [6] imply in any given gauge, i.e. over a local section, the Becchi Rouet Stora (BRS) equations of the quantum field theory [7] and hereby control its unitarity and renormalisability [7,8]. The recent finding [9] that the BRS equations also control the algebraic classification of the anomalies greatly increases the interest of this identification.

The aim of my talk is to show that one may identify the anomalies with the secondary characteristic classes of the principal fiber bundle, and hereby obtain their complete classification in a gauge invariant geometrical way.

2 The Darboux-Maurer-Cartan-Ehresmann equation

Let $\mathcal{P}$ denote a differentiable fiber bundle of dimension d+n over a base $\mathcal{B}$ of dimension d. Let Π be the projection map. Let us adorn with a $\tilde{}$ the exterior differential $\tilde{d}$ and any exterior form over $\mathcal{P}$. Let x^μ denote a coordinate system over $\mathcal{B}$. Using the cotangent map Π^*, we can pull back on $\mathcal{P}$ the dx^μ :

$$\tilde{d}x^\mu = \Pi^*(dx^\mu) = \tilde{d}(x^\mu \circ \Pi) \ . \tag{1}$$

Let us now introduce over $\mathcal{P}$ a field of one-forms $\tilde{A}$ valued into a Lie algebra $\mathcal{A}$ of dimension n :

$$\tilde{A} \in \Omega^1(\mathcal{P};\mathcal{A}) \ , \quad \tilde{A} = \tilde{A}^a \lambda_a \ , \quad \lambda_a \in \mathcal{A} \ , \tag{2}$$

such that the n $\tilde{A}^a$ together with the d $\tilde{d}x^\mu$ define a moving frame (Cartan's repère mobile) over $\mathcal{P}$.

Note that $\tilde{A}$ is a field of one-forms valued into a finite dimensional vector space $\mathcal{A}$, or [illegible] a collection of vectors of the infinite dimensional cotangent space

($\tilde{P}_t = \bigcup P_{x_0 \in M} P$). As such, $\tilde{A}$ generates an infinite dimensional Grassmann algebra as defined by Berezin [10]. The space of multilocal polynomes in $\tilde{A}$ (Green's functions) is infinite dimensional, whereas the exterior product of $\tilde{A}$ defined over the same point of P (local operators) are of maximal degree d. In other words, $\tilde{A}$ is not a function on P valued into a finite Grassmann algebra Λ, but on the contrary a Grassmann field valued in a finite vector space A . A certain confusion was caused by [BH] where this distinction was overlooked.

Let us now consider the 2-form of curvature $\tilde{F} = \tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A},\tilde{A}]$. We restrict the geometry by imposing that $\tilde{F}$ should be purely horizontal, i.e. that $\tilde{F}$ can be expressed on the dx^μ only

$$\tilde{F} = F = \frac{1}{2} F_{\mu\nu}\, \tilde{d}x^\mu \tilde{d}x^\nu \ . \tag{3}$$

This is the celebrated Darboux-Maurer-Cartan-Ehresmann structure equation of the principal fiber bundle [8].

Consider a system of n vector fields on P dual to $\tilde{A}$:

$$\tilde{D}_a \lrcorner \tilde{A}^b = \delta_a^b \ , \quad \tilde{D}_a \lrcorner \tilde{d}x^\mu = 0 \ . \tag{4}$$

The DMCE equation implies that, under Poisson bracket, the n vector fields $\tilde{D}_a$ generate a finite Lie algebra isomorphic to A :

$$[\tilde{D}_a, \tilde{D}_b]_{P.B.} = \tilde{D}_{[a,b]_A} \ . \tag{5}$$

Therefore the Lie group G, with Lie algebra A , acts as a transformation group on P . Note however that the structure is only local :

a) P may admit no global section over B , as usual,

b) P may admit no global trivialization map over the group.

Indeed, we have not specified the action of the center of the group. In this respect, P has a weaker structure than is usually assumed for a principal fiber bundle. We do not know how to resolve this ambiguity or if it plays a role in quantum field theory, but we have to live with it, because the local DMCE equation is the only one preserved by the renormalization.

3 The Becchi-Rouet-Stora equations.

A section Σ , or gauge choice, is a map from an open subset of the base B into P not tangent to the fibers. Using the tangent map Σ_*, we can transport forward the vectors ∂_μ tangent to the base onto $\tilde{P}_*$:

$$(\tilde{\partial}_\mu)_\Sigma = \Sigma_*(\tilde{A}) \tag{6}$$

and pull back the Ehresmann connection $\tilde{A}$ onto the base :

$$A_\Sigma = \Sigma^*(\tilde{A}) \ . \tag{7}$$

A_Σ is the Yang-Mills one-form in the gauge Σ . However, $\tilde{A}$ contains more information than A_Σ alone. Given the section Σ , we can complete a coordinate coframe on $\tilde{P}$ by choosing some coordinates y^i along the fibers such that y^i is zero on Σ , i.e. $\tilde{d}y^i$ is normal to $\tilde{\partial}_\mu$:

$$(\tilde{\partial}_\mu)_\Sigma \lrcorner (\tilde{d}y^i)_\Sigma = 0 \ , \quad \tilde{\partial}_i \lrcorner \tilde{d}x^\mu = 0 \ . \tag{8}$$

The orientation of the y^i along each fiber is arbitrary, however, if we expand the connection $\tilde{A}$ on this coframe :

$$\tilde{A}^a = (A^a_\mu)_\Sigma \, \tilde{d}x^\mu + A^a_b (\tilde{d}y^b)_\Sigma \tag{9}$$

the matrix A^a_b gives at each point the orientation of the coordinate vectors $\tilde{\partial}_a$ with respect to the Killing vectors $\tilde{D}_a$.

We call Faddeev Popov ghost the section dependent object :

$$c = A^a_b \, (\tilde{d}y^b)_\Sigma \ . \tag{10}$$

In the same coordinate system, we call Becchi-Rouet-Stora operator the differential :

$$s = (\tilde{d}y^a) \, \tilde{\partial}_{y^a} \ . \tag{11}$$

In these notations, the DMCE equation splits into 3 components known as the BRS equations [7] :

$$dA + \frac{1}{2} [A, A] = F(A) \ , \quad sA + dc + [c, A] = 0 \ , \quad sc + \frac{1}{2} [c, c] = 0 \ . \tag{12}$$

or in a more concise form :

$$\tilde{F}(A + c) = F(A) \tag{13}$$

This equation, that I first proposed in 1978 [5] has been recently nicknamed by Stora the Russian (?) formula.

4 Characteristic classes.

In this section, we wish to find the cohomology classes of $\tilde{F}$, i.e. the closed exterior forms $\tilde{\omega}$ of degree p modulo exact forms :

$$\tilde{d}\tilde{\omega}(\tilde{\lambda},\tilde{F}) = 0 \quad , \quad \tilde{\omega} \equiv \tilde{\omega} + \tilde{d}\tilde{K} \ . \tag{14}$$

If $p \leq d$, then the $\tilde{\omega}$ are exterior polynomes in $\tilde{F}$ and represent the primary characteristic classes of the manifold. However, when $p = d + g$, these polynomes vanish since $\tilde{F}$ is horizontal and $\tilde{\omega}$ represents the secondary classes. These forms correspond to the g-cocycles of Zumino [12].

We shall perform the classification intrinsically, without choosing a section, and thus without decomposing $\tilde{\lambda}$ into its gauge and ghost components. In the next section, we shall prove that this geometric problem is equivalent to the classification of the anomalies of Yang Mills theory which may obstruct the gauge invariance, i.e. section independence, of the renormalized action.

We proceed in two steps :

a) we relax the DMCE equation. Then $\tilde{F}$ and $\tilde{\lambda}$ are independent and the cohomology of $\tilde{d}$ on the space of exterior polynomes in $(\tilde{\lambda}\ ,\tilde{F})$ becomes trivial. Then the classification of forms of degree $d + g$ modulo exact forms becomes equivalent to the classification of closed forms I of degree $d + g + 1$.

$$\tilde{d}\tilde{\omega}(\tilde{\lambda},\tilde{F}) = \tilde{I}(\tilde{\lambda},\tilde{F}) \tag{15}$$

b) we impose the DMCE equation $\tilde{F} = F$. A polynome in $\tilde{F}$ of degree $\frac{d}{2}$ or higher vanishes as a consequence of the horizontality of F. Observe now that if $u(\tilde{\lambda},\tilde{F}) = \tilde{d}v(\tilde{\lambda},\tilde{F})$, then u is always of higher degree in $\tilde{F}$ than v. Therefore, if v vanishes because of the DMCE condition, u vanishes a fortiori. Reciprocally, if u does not vanish, nor does v. Thus, when I is annihilated by the DMCE equation but not $\tilde{\omega}$, $\tilde{\omega}$ represents a cohomology class of $\tilde{d}$ on $\tilde{F}$.

The general solution [10] is the product of q Chern Simons forms Q_{m_j} by a Weyl invariant polynome P_r in F of degree r :

$$\tilde{\omega}(\tilde{\lambda},\tilde{F}) = \prod_{j=1}^{q} Q_{m_j}(\tilde{\lambda},\tilde{F})\,P_r(\tilde{F}) \ , \quad \tilde{d}Q_{m_j}(\tilde{\lambda},\tilde{F}) = P_{\frac{m_j+1}{2}}(\tilde{F}) \ . \tag{16}$$

The coefficients are subject to the constraints :

$$\sum_{j=1}^{q} m_j + 2r = d + g \ , \quad g \leq \sum_{j=1}^{q} m_j \leq g + min(m_j) \tag{17}$$

For any value of g such that $d+g$ is odd, there exists a solution with $q=1$. This includes the Deser-Jackiw-Templeton topological mass term in odd dimension and the usual $g=1$ ABJ anomaly in even dimension with its associated $g=2$ Faddeev anomaly in the Hamiltonian formalism. Unusual solutions with $q>1$ occur with $g=1$ only in the presence of two U(1) groups in odd dimension :

$$\tilde{\omega} = \tilde{A} \wedge \tilde{A}' P(F,F') \tag{18}$$

With $q=2$, $g=3$, d odd, we note the SU(2).U(1) anomaly :

$$\tilde{\omega} = Tr_{SU(2)}(\tilde{A}^a, \tilde{A}^b, \tilde{A}^c)\ \tilde{A}^{U(1)}\ P_{\frac{d-1}{2}}(F) \tag{19}$$

5 The Wess-Zumino and the descent equations.

Consider a closed non exact form $\tilde{\omega}$ of degree $d+g$:

$$\tilde{d}\tilde{\omega}(\tilde{A},F) = 0\ ,\quad \tilde{\omega} \neq \tilde{d}\tilde{K}\ . \tag{20}$$

If we choose a section Σ and expand $\tilde{\omega}$ in gauge and ghost components :

$$\tilde{\omega}_{d+g} = \sum_{i=0}^{d} \omega^{g+i}_{d-i}\ ,\qquad \omega^{g+i}_{d-i} = \frac{1}{(g+i)!}\left(c\frac{\partial}{\partial \tilde{A}}\right)^{g+i} \tilde{\omega}(\tilde{A},F)\ \Big|_{\tilde{A}=A}\ . \tag{21}$$

The expansion of $\tilde{\omega}_{d+g}$ starts with ω^g_d since higher horizontal forms vanish identically. If we expand the closure equation, we obtain a set of equations known as *the descent* :

$$s\omega^{g+i}_{d-i} + d\omega^{g+i+1}_{d-i-1} = 0\ . \tag{22}$$

Integrating the $i=0$ equation over the base and discarding the surface terms, we see that ω^g_d satisfies the dual *Cartan* form of the Wess Zumino consistency condition [6] which defines the possible anomalies :

$$\int s\omega^g_d = 0\ . \tag{23}$$

However, in quantum field theory, the anomaly considered as a quantum correction to the BRS variation of 1pi action [7,8] is a priori a function of A_μ, c, the antighost and the source operators) and (23) could have many more solutions. But recently, I have shown [13] that all Yang-Mills anomalies are of the type (20-22). The proof involves 4 steps :

a) using auxiliary fields, the antighosts and source operators are gauged away (Dixon's problem [9]);

b) the Yang Mills field A_μ is shown, by lengthy Taylor expansions, to contribute as an exterior form $A_\mu \tilde{d}x^\mu$ to those anomalies such that $\epsilon\omega \neq 0$;

c) the ghost c and Yang Mills form A are combined into $\tilde{A} = A + c$;

d) ω^0_4 is shown to be the first term of the expansion of some $\tilde{d}$ cohomology class $\tilde{\omega}$;

The quantum field theory problem is therefore reduced to the geometric problem studied in the preceding section and its generic solution is obtained by expanding (16) using the descent equation (20). Steps c) and d) have been established by several authors [14] with similar results (in particular Viallet at this conference).

As we have seen in section 3, the intrinsic geometrical classification formally dispenses of proving b,c,d. Indeed, if the anomaly has an intrinsic meaning, it is not a coordinate artefact and must be globally defined. Therefore it must depend on c and A_μ only through the intrinsic combination $\tilde{A}$ and on d and s only through $\tilde{d}$.

6 Gravitational anomalies.

It is extremely simple to include general relativity in this formalism. One just has to replace the moving frame $(\tilde{A}, \tilde{d}x^\mu)$ of P by a Poincaré valued one-form field $(\tilde{\omega}, \tilde{e})$ over a 10 dimensional manifold M, the Regge Ne'eman group manifold [16]. $\tilde{\omega}$ denotes a connection form for the Lorentz group, which plays the role of the Yang Mills group, and $\tilde{e}$ is the 'quantized' vierbein. By 'quantized', I mean that in an arbitrary system of coordinates $\tilde{e}$ decompose as a classical vierbein plus ghost field:

$$\tilde{e}^a = e^a_\mu \tilde{d}x^\mu + \eta^a . \tag{24}$$

η^a is the ghost field of local translations in the tangent space. Equivalently, one may express η^a in terms of a ghost vector field ξ which is the ghost of local diffeomorphisms (Kibble formalism):

$$\eta^a = \xi \lrcorner \tilde{e}^a \; , \quad \xi = \xi^\mu \partial_\mu \; , \quad \xi^\mu = \eta^a (e^{-1})^\mu_a \; . \tag{25}$$

The generalized DMCE-BRS equations, also known as the rheonomy conditions, state that $\tilde{e}$ and all its Lorentz covariant exterior differentials can be expanded over $\tilde{e}$ with classical coefficients.

$$\tilde{F} = F_b \tilde{e}^b \; , \quad \tilde{T}^a(\tilde{e},\tilde{\omega}) = \frac{1}{2} T^a_{bc}(e,\omega)\tilde{e}^b\tilde{e}^c \; , \quad \tilde{R}^{ab}(\tilde{\omega}) = \frac{1}{2} R^{ab}_{cd}(\omega)\tilde{e}^c\tilde{e}^d \; , \tag{26}$$

where :

$$T^a = De^a = de^a + \omega^a_b e^b \ , \quad R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} \ . \tag{27}$$

The system is closed and consistent since :

$$De = T \ , \quad DT = Re \ , \quad DR = 0 \ . \tag{28}$$

Considering the dual vector fields $(\tilde{D}_{ab}, \tilde{D}_a)$, one may easily verify that the $\tilde{D}_{ab}$ represent the Lorentz algebra [15] :

$$[\tilde{D}_{ab}, \tilde{D}_{cd}] = \tilde{D}_{[ab,cd]} \ . \tag{29}$$

The difference between $\mathcal{M}$ and a principal fiber bundle is that there is no predefined projection map, but the space develops a 'spontaneous fibration' along the $\tilde{D}_{ab}$ directions as a result of the DMCE-BRS equations [16].

The best choice of variables to classify the anomalies is to devellop the $\tilde{\omega}$ themselves on the $\tilde{e}$:

$$\tilde{\omega} = \omega_a \tilde{e}^a + \Omega' \tag{30}$$

and to introduce [17] a 'translation covariant' BRS operator s' such that the 'alibi' active translation, or displacement in the tangent space, parametrized by the ghost η^a is compensated for by and 'alias' transformation, a passive relabelling of the coordinates, or displacement along the curved section, induced by a Lie derivative along the ghost vector field ξ associated to η^a :

$$s' = s - \mathcal{L}_\xi \ , \tag{31}$$

In these variables, $\tilde{e}$ has no ghost ! and the structure equations read [18] :

$$S'\tilde{e}^a = s'\tilde{e}^a + \Omega'^a_b \tilde{e}^b = 0 \ , \quad s'\xi = -\frac{1}{2}[\xi,\xi] \ , \tag{32}$$

$$s'\omega = D\Omega' \ , \quad s'\Omega' = -\frac{1}{2}[\Omega',\Omega'] \ . \tag{33}$$

These beautifully simple equations show that the classification of the anomalies of general relativity is reduced to the classification of the anomalies of a Yang Mills theory of the Lorentz group [12,18] since the local cohomology of s and s' are identical :

$$\int s'\omega^g_d = \int s\omega - d(\xi \lrcorner \omega) - (\xi \lrcorner d\omega) = \int s\omega \tag{34}$$

Indeed, the second term is exact and the third vanishes since $d\omega$ is a horizontal (d+1) form.

I have developped this presentation of the gauge structure of quantum gravity in several steps. First Regge and Ne'eman [15] analysed the classical theory and obtained the structure equations as equations of motion which Ne'eman and I reinterpreted as BRS equations [16]. Later with Ne'eman and Takasugi [17] we introduced the $\mathfrak{s}$ operator and finally with Baulieu [18] we have simplified the equations and classified the anomalies. In this last paper, our proof that all anomalies can be written as exterior forms is incomplete. We were unable to gauge away the BRS closed objects of the form $\phi = \int \xi^\mu \partial_\mu \Delta \sqrt{g}$ where Δ is an arbitrary scalar. This is however possible since Alvarez and Zumino have found that ϕ is s exact : $\phi = -\int s\Delta\sqrt{g}\,Log(\sqrt{g})$.

7 Conclusion.

The geometrical formalism reviewed here leads to a clear understanding and a simple classification of the anomalies of Yang Mills theory and general relativity. A straightforward generalisation leads to the quantisation of antisymmetric tensor gauge fields and it is hoped that the formalism can be extended to supergravity.

It is rewarding to see that the geometrical understanding of the anomalies, the result of a rear guard study by mathematical physicists sometimes considered as futile by the model builders, has finally led to a renewal of unified theories through the discovery of the cancelation of anomalies of d=10, N=1 supergravity when the gauge group is SO(32) (Green and Schwarz [19]) or $E_8 \otimes E_8$ (Thierry-Mieg [20]).

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