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TITLE: A SEMIGROUP ASSOCIATED WITH A QUASI-LINEAR SYSTEM IN WHICH THE COUPLING IS LINEAR

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A SEMIGROUP ASSOCIATED WITH A QUASI-LINEAR SYSTEM IN WHICH THE COUPLING IS LINEAR

by

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1. Introduction

Until now, semigroup solution has been successful mainly for linear and mildly nonlinear initial value problems having but a single partial differential equation (PDE). Attempts to apply standard semigroup generation principles [Refs. 1, pp. 109, 110; 5, Chaps. IX, XIV] to systems of nonlinear PDEs have encountered difficulties, though papers have appeared that treat mildly nonlinear systems [Ref. 2]. The purpose of this work is to demonstrate ideas, involving the Trotter product formula, that may be efficacious in finding semigroups for coupled systems of quasi-linear PDEs. This is a class for which semigroup generation principles involving accretiveness of an operator have generally failed. These Trotter product ideas were originally explained by J. Marsden, using a profound Banach manifold approach [Ref. 4].

We select one of the simplest quasi-linear coupled systems for our study:

$$\begin{aligned} &\rho_t + \phi(\rho)_x + \sigma u_x = 0 \quad , \quad \sigma \geq 0 \\ &u_t + \sigma\rho_x + \psi(u)_x = 0 \quad , \quad \rho(x,0) = \rho_0(x) \quad , \quad u(x,0) = u_0(x) \quad , \end{aligned} \tag{1}$$

but choose to work in the space $(L^1(R))^2 = L^1(R) \times L^1(R)$, where R is the real axis. Thus (1) is a linear perturbation on a separated nonlinear problem (for $\sigma = 0$), that has been successfully treated using semigroup generation methods by M. Crandall [Ref. 1]. Crandall user a more general setting than used here for the separated problem however.

In (1) we take ϕ and ψ to be twice differentiable monotone increasing functions with bounded first derivative, and suppose that $\phi(0) = \psi(0) = 0$.

We can write problem (1) abstractly as follows:

$$\frac{dw}{dt} + A_1 w + A_2 w \ni 0 \quad , \quad w(0) = w_0 = \begin{pmatrix} \rho_0 \\ u_0 \end{pmatrix} \quad , \tag{2}$$

where

$$w = \begin{pmatrix} \rho \\ u \end{pmatrix} , \quad A_1 w \ni \begin{pmatrix} \phi(\rho)_x \\ \psi(u)_x \end{pmatrix} \quad , \text{ and } A_2 w = \sigma \begin{pmatrix} u_x \\ \rho_x \end{pmatrix} \quad .$$

The set-valued operator A_1 is as given in Ref. 1, pp. 111-112. The joint domain D(A) of A_1 and A_2 includes those absolutely continuous vectors in $(L^1(R))^2$ that both map into $(L^1(R))^2$.

2. Two Component Problems

The idea is to solve two separate problems:

(a) $\frac{dw}{dt} + A_1 w \ni 0$

or $\rho_t + \phi(\rho)_x = 0$

$u_t + \psi(u)_x = 0$

with

$\rho(x,0) = \rho_{10}$, $u(x,0) = u_{10}$

(b) $\frac{dw}{dt} + A_2 w = 0$

or $\rho_t + \sigma u_x = 0$

$u_t + \sigma\rho_x = 0$

with

$\rho(x,0) = \rho_{20}$, $u(x,0) = u_{20}$,

and then to put the respective semigroups S_1, S_2 for problems (a) and (b) together, using Trotter's product.

Problem (a) has two uncoupled PDEs that can be solved locally, using the familiar method of characteristics for first order equations. This solution carries the regularity of the initial data up to a finite breakdown time. Problem (a) has been globally solved in the sense of distributions by M. Crandall [Ref. 1], who used the accretiveness properties of the operator A_1 to show the existence of the solution operator $S_1(t)$ having contraction semigroup properties.

If $S_\phi(t)$, $S_\psi(t)$ are the two componentwise Crandall contraction semigroups in problem (a), then

$$w = \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} S_\phi(t) & 0 \\ 0 & S_\psi(t) \end{pmatrix} \begin{pmatrix} \rho_{10} \\ u_{10} \end{pmatrix} = S_1(t) w_{10} \quad , \quad w_{10} \in \overline{D(A)} \quad .$$

Here, $S_\phi(t)\rho_{10}$, $S_\psi(t)u_{10}$ are the unique weak entropy solutions for the uncoupled components of problem (a), produced by semigroup generation principles based on the accretiveness and range conditions satisfied by the respective operators $\phi(\cdot)_x$, $\psi(\cdot)_x$ in $L^1(R)$. S_ϕ and S_ψ are contractions in $L^1(R)$, but they do not preserve regularity since they can map continuous functions into finitely discontinuous functions. Because elementary methods are locally valid however, if the initial w_{10} has a prescribed smoothness, this will persist in the generalized solution, for some small time interval.

Problem (b) is really a linear wave problem. By differentiating and substituting, one obtains $\rho_{tt} = \sigma^2 \rho_{xx}$. We could write the semigroup solution as follows:

$$w = \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sigma} \frac{\partial}{\partial t} \int_{x-\sigma t}^{x+\sigma t} \cdot \, d\xi & -\frac{1}{2} \int_{x-\sigma t}^{x+\sigma t} \frac{\partial}{\partial \xi} \cdot \, d\xi \\ -\frac{1}{2} \int_{x-\sigma t}^{x+\sigma t} \frac{\partial}{\partial \xi} \cdot \, d\xi & \frac{1}{2\sigma} \frac{\partial}{\partial t} \int_{x-\sigma t}^{x+\sigma t} \cdot \, d\xi \end{pmatrix} \begin{pmatrix} \rho_{20}(\xi) \\ u_{20}(\xi) \end{pmatrix} d\xi = S_2(t) w_{20} \quad ,$$

$$w_{20} \in D(A) \quad .$$

It is better to write directly

$$w = \begin{pmatrix} \rho(x,t) \\ u(x,t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \rho_{20}(x+\sigma t) + \rho_{20}(x-\sigma t) - u_{20}(x+\sigma t) + u_{20}(x-\sigma t) \\ -\rho_{20}(x+\sigma t) + \rho_{20}(x-\sigma t) + u_{20}(x+\sigma t) + u_{20}(x-\sigma t) \end{pmatrix} \quad . (3)$$

The semigroup (actually group) character of (3) can be checked, i.e., $S_2(t_1 + t_2) = S_2(t_1)S_2(t_2)$.

It is well known that in the space $(L^2(R))^2$ the linear operator $S_2(t)$ would be an isometry. In $(L^1(R))^2$ this is not the case; $S_2(t)$ has the bound 2 if we use the standard cartesian norm: $|| \cdot || = || \cdot ||_{L^1} + || \cdot ||_{L^1}$, where $|| \cdot ||_{L^1} = \int_R | \cdot | dx$. We have, using (3):

$$\begin{aligned}||\rho(x,t)||_{L^1} + ||u(x,t)||_{L^1} &\le ||\rho_{20}(x + \sigma t)||_{L^1} + ||\rho_{20}(x - \sigma t)||_{L^1} \\ &+ ||u_{20}(x + \sigma t)||_{L^1} + ||u_{20}(x - \sigma t)||_{L^1} \\ &= 2\{||\rho_{20}(x)||_{L^1} + ||u_{20}(x)||_{L^1}\} \quad ,\end{aligned}$$

and this bound, namely 2, is assumed when ρ_{20}, u_{20} are disjoint approximate rectangle functions of unit L^1 norm. For appropriate $t > 0$, $\frac{1}{2}\rho_{20}(x \pm \sigma t)$ and $\frac{1}{2}u_{20}(x \pm \sigma t)$ separate, and their norms add to twice the initial norm. W. Littman [Ref. 3] relates that in $L^p(R)$, $p \neq 2$, standard energy estimates do not hold for the linear wave problem. Here we see that with $p = 1$, the semigroup solution is expansive.

Let us summarize the above background material:

Theorem 1: Problem (a) has a unique weak global entropy solution, given by a contraction semigroup $S_1(t)$. $S_1(t)$ is not regularity-preserving beyond a finite breakdown time. Problem (b) has a unique global solution expressed as a regularity-preserving semigroup $S_2(t)$, with Lipschitz constant 2. Each of these timewise solutions represents a trajectory in $(L^1(R))^2 = L^1(R) \times L^1(R)$.

3. Some Preparation

According to a theorem given by Yosida [Ref. 5, p. 249], the semigroup $S_2(t)$, being generated by the closed linear operator A_2 with $\overline{D(A_2)} = (L^1(R))^2$, has a resolvent $(I + \lambda A_2)^{-1}$ which obeys the condition $||(I + \lambda A_2)^{-m}|| \le 2$, $\lambda > 0$, $m = 1,2,\cdots$. Thus Yosida's exponential representation [Ref. 5, p 248] is valid:

$$S_2(t)w_{20} = \lim_{\lambda \to 0} e^{tA_2(I + \lambda A_2)^{-1}} w_{20} \quad , \quad w_{20} \in (L^1(R))^2 \quad . \tag{4}$$

Expression (4) contains the "Yosida approximation" $A_{2\lambda}$ as it is usually defined for linear generators: $A_{2\lambda} = A_2(I + \lambda A_2)^{-1}$. $A_{2\lambda}$ is Lipschitz continuous in $(L^1(R))^2$ for given fixed $\lambda > 0$. The convergence in expression (4) is uniform on every compact t-set.

We are now in a position to consider the Trotter product, which can purport to represent the overall semigroup for a λ-approximation to our original problem (1):

$$S_\lambda(t)w_0 = \lim_{n \to \infty} [S_{2\lambda}(\tfrac{t}{n})S_1(\tfrac{t}{n})]^n w_0 \quad , \tag{5}$$

where w_0 is an initial vector. $S_{2\lambda}(t)$ in (5) is the semigroup for the λ-approximation to problem (b):

$$S_{2\lambda}(t) = e^{tA_2(I+\lambda A_2)^{-1}} = e^{tA_{2\lambda}} \quad , \tag{6}$$

(cf. expression (4)). (6) is valid because the Yosida approximation $A_{2\lambda}$ is linear and bounded.

Part of the attractiveness of choosing problem (1) for our study is that we can work entirely in the linear Banach manifold $(L^1(R))^2$, a familiar setting. It is not necessary to re-metrize, or to make the transition to manifold theory outlined by Marsden [Ref. 4, pp. 63-65]. This writer believes, however, that if we had considered a problem that was much more complicated than (1), such a transition to manifolds would have been needed. It is conjectured that, in general, semigroups for systems of quasi-linear PDEs require a manifold approach.

We must satisfy the Marsden requirements for the convergence of (5). Because (1) is among the easiest of the problems of this genre, some of the requirements of Marsden's main theorem [Ref. 4, pp. 53-55] are met in a self-evident way. Marsden conditions (iii) and (iv) [Ref. 4, p. 54] present the most difficulty. The satisfaction of condition (iii) required our introduction of the Yosida approximation $A_{2\lambda}$, which we now use.

<u>Theorem 2:</u> The semigroup $S_{2\lambda}(t)$ in (6) is quasi-contractive, i.e., has a pure exponential for a Lipschitz constant.

<u>Proof</u>: $S_{2\lambda}(t)w$, $w \in (L^1(R))^2$, solves a differential equation similar to that in problem (b):

$$\frac{d}{dt} S_{2\lambda}(t)w + A_{2\lambda}S_{2\lambda}(t)w = 0 \quad ,$$

or

$$S_{2\lambda}(t)w = S_{2\lambda}(0)w - \int_0^t A_{2\lambda}S_{2\lambda}(t')w \, dt' \quad .$$

In the standard cartesian norm of $(L^1(R))^2$, this gives

$$||S_{2\lambda}(t)w|| - ||w|| \le \Big| ||S_{2\lambda}(t)w|| - ||w|| \Big| \le \beta_\lambda \int_0^t ||S_{2\lambda}(t')w|| \, dt' \quad ,$$

where β_λ is the operator norm of $A_{2\lambda}$. Note that $\beta_\lambda \to \infty$ as $\lambda \to 0$. Then Gronwall's inequality gives $||S_{2\lambda}(t)w|| \le e^{\beta_\lambda t}||w||$. Thus for w_1, $w_2 \in (L^1(R))^2$,

$$||S_{2\lambda}(t)w_1 - S_{2\lambda}(t)w_2|| \le e^{\beta_\lambda t}||w_1 - w_2|| \quad . \tag{7}$$

The above proof of Theorem 2 follows, but is much simpler than a similar proof by Marsden [Ref. 4, pp. 68, 69].

Although $S_2(t)$ has Lipschitz constant 2, the Yosida approximation so modifies the operator $A_{2\lambda}$ that $S_{2\lambda}(t)$ has the Lipschitz constant $e^{\beta_\lambda t}$ with $\beta_\lambda > 0$, which approaches unity as $t \to 0$. Such a thing is possible because $A_{2\lambda}$ is continuous for fixed $\lambda > 0$. Note that this quasi-contractiveness property is more or less apparent from (6).

Defining Chernoff's operator $K_\lambda(t) = S_{2\lambda}(t)S_1(t)$ [Ref. 4, pp. 52,66] for problem (1), we get from (7),

$$\|K_\lambda(t)w_1 - K_\lambda(t)w_2\| \le e^{\beta_\lambda t}\|w_1 - w_2\| \quad , \quad w_1,w_2 \in (L^1(R))^2 \quad . \tag{8}$$

Thus our $K_\lambda(t)$ satisfies Marsden condition (iii) for fixed $\lambda > 0$.

The next result takes the place of Marsden Lemma 2.2 [Ref. 4, p. 55] and results thereby in some relaxation of the strictures in Ref. 4.

Theorem 3: Let $y \in W$, where $W \subset (L^1(R))^2$ is a closed bounded region. Then for $0 \le t \le T$, there exists a bounded region V^T, with $W \subset V^T$, such that the sequence of iterates $\{[K_\lambda(\frac{t}{n})]^n y\}$ of Chernoff's operator remains in V^T, $n = 1,2,3,\cdots$.

Proof: Our Chernoff operator has the property that $K_\lambda(t)\theta = \theta$, $t > 0$, where θ is the null element of $(L^1(R))^2$. Then using (8) (Marsden condition (iii)),

$$\begin{aligned}\|[K_\lambda(\tfrac{t}{n})]^n y\| &= \|[K_\lambda(\tfrac{t}{n})]^n y - [K_\lambda(\tfrac{t}{n})]^n \theta\| \\ &\le e^{\beta_\lambda t/n}\left\|[K_\lambda(\tfrac{t}{n})]^{n-1} y - [K_\lambda(\tfrac{t}{n})]^{n-1}\theta\right\| \\ &\quad - \\ &\le e^{\beta_\lambda t}\|y - \theta\| = e^{\beta_\lambda t}\|y\| \le e^{\beta_\lambda T}\|y\| \quad .\end{aligned}$$

Thus $\{[K_\lambda(\frac{t}{n})]^n y\}$, $y \in W$, remains in a region V^T.

It is by virtue of Theorem 3 that the composite semigroup for problem (1), which results from (5), will be a global semigroup. Marsden's semigroup [Ref. 4, p. 54] is local partly because general Banach manifolds lack null elements, or other fixed points for the Chernoff operator.

4. An Approximate Semigroup

We devote this section to the tedious task of showing that Marsden condition (iv) is satisfied [Ref. 4, p. 54].

Theorem 4: The Chernoff operator $K_\lambda(t) = S_{2\lambda}(t)S_1(t)$ is an "approximate semigroup," in the sense that there is a constant C_0 (dependent on x) such that

$$||K_\lambda(t+s)x - K_\lambda(t)K_\lambda(s)x|| \le C_0 st \quad , \tag{9}$$

where

$$x \in V_2^T = V^T \cap (W^{2,1}(R))^2 \quad .$$

Note: V^T is the region of Theorem 3, and $(W^{2,1}(R))^2 = W^{2,1}(R) \times W^{2,1}(R)$.

Proof: We replace the operator A_1 by its Lipschitz continuous Yosida approximation: $A_{1\mu} = \frac{1}{\mu}[I - J_\mu(A_1)]$ (see Ref. 5, p. 447, Eq. (3)) and later take $\mu \to 0$. $A_{1\mu}$ generates a semigroup $S_{1\mu}(t)$ (e.g., by Picard iterations). We shall need the fact that, for given $x \in V_2^T$, $S_1(t)x$ is itself regular up to a finite breakdown time. Also, since $A_{1\mu}$ is accretive (see Ref. 5, p. 448; Crandall showed that A_1 is accretive), we have that $\lim_{\mu\to 0} S_{1\mu}(t)x = \lim_{n\to\infty} (I + \frac{t}{n}A_1)^{-n}x = S_1(t)x$, where the convergence is uniform on any compact t-set [Ref. 5, p. 461].

Let us define two functions:

$$f_\mu(t,s) = S_{2\lambda}(t+s)S_{1\mu}(t+s)x$$
$$g_\mu(t,s) = S_{2\lambda}(t)S_{1\mu}(t)S_{2\lambda}(s)S_{1\mu}(s)x$$

for small s,t, letting $x \in V_2^T$. Recalling that $A_{2\lambda}$ and $S_{2\lambda}$ are linear bounded operators,

$$\frac{\partial}{\partial t} f_\mu(t,s) = \frac{\partial}{\partial s} f_\mu(t,s) = -A_{2\lambda}S_{2\lambda}(t+s)S_{1\mu}(t+s) - S_{2\lambda}(t+s)A_{1\mu}S_{1\mu}(t+s)x \ ;$$

and

$$\frac{\partial^2}{\partial t\partial s} f_\mu(t,s) = \frac{\partial^2}{\partial s\partial t} f_\mu(t,s)$$

$$= A_{2\lambda}[A_{2\lambda}S_{2\lambda}(t+s)S_{1\mu}(t+s)x + S_{2\lambda}(t+s)A_{1\mu}S_{1\mu}(t+s)x]$$

$$+ A_{2\lambda}S_{2\lambda}(t+s)A_{1\mu}S_{1\mu}(t+s)x + D[S_{2\lambda}(t+s)A_{1\mu}]A_{1\mu}S_{1\mu}(t+s)x \ ,$$

where the operator D linearizes in a neighborhood of the .gument the continuous operator in the following bracket. Note that we have used the identities $\frac{d}{dt} S_{1\mu}(t)x + A_{1\mu}S_{1\mu}(t)x = 0$ and $\frac{d}{dt} S_{2\lambda}(t)x + A_{2\lambda}S_{2\lambda}(t)x = 0$. These processes remain confined to $(L^1(R))^2$. Again we obtain

$$\frac{\partial}{\partial t} g_\mu(t,s) = -A_{2\lambda}S_{2\lambda}(t)S_{1\mu}(t)S_{2\lambda}(s)S_{1\mu}(s)x - S_{2\lambda}(t)A_{1\mu}S_{1\mu}(t)S_{2\lambda}(s)S_{1\mu}(s)x$$

$$\frac{\partial}{\partial s} g_\mu(t,s) = -D[S_{2\lambda}(t)S_{1\mu}(t)][A_{2\lambda}S_{2\lambda}(s)S_{1\mu}(s)x + S_{2\lambda}(s)A_{1\mu}S_{1\mu}(s)x]$$

$$\frac{\partial^2}{\partial t\partial s} g_\mu(t,s) = \frac{\partial^2}{\partial s\partial t} g_\mu(t,s)$$

$$= D[A_{2\lambda}S_{2\lambda}(t)S_{1\mu}(t)][A_{2\lambda}S_{2\lambda}(s)S_{1\mu}(s)x + S_{2\lambda}(s)A_{1\mu}S_{1\mu}(s)x]$$

$$+ D[S_{2\lambda}(t)A_{1\mu}S_{1\mu}(t)][A_{2\lambda}S_{2\lambda}(s)S_{1\mu}(s)x + S_{2\lambda}(s)A_{1\mu}S_{1\mu}(s)x] \ .$$

Other derivatives are computed in a similar way. These differentiations are valid because of the assumed differentiablity of ϕ, ψ.

It is our purpose to find out whether $\frac{1}{st} ||f(t,s) - g(t,s)||$ is bounded as $st \to 0$, in whatever way that we may have $st \to 0$. We see plainly that

$$||f_\mu(t,s) - g_\mu(t,s)||$$

$$= ||S_{2\lambda}(t + s)S_{1\mu}(t + s)x - S_{2\lambda}(t)S_{1\mu}(t)S_{2\lambda}(s)S_{1\mu}(s)x|| \to 0$$

as $s \to 0$, or as $t \to 0$. Likewise, taking $\mu \to 0$, we have

$$||S_{2\lambda}(t + s)S_1(t + s)x - S_{2\lambda}(t)S_1(t)S_{2\lambda}(s)S_1(s)x|| \to 0$$

as $s \to 0$, or as $t \to 0$, where we have switched limits by uniform convergence. Thus $||f(t,0) - g(t,0)|| = 0$ along the t-axis of a t,s plane, and $||f(0,s) - g(0,s)|| = 0$ along the s-axis. Moreover, using the expressions obtained above,

$$||f_{\mu t}(t,s) - g_{\mu t}(t,s)|| = ||A_{2\lambda}S_{2\lambda}(t + s)S_{1\mu}(t + s)x + S_{2\lambda}(t + s)A_{1\mu}S_{1\mu}(t + s)x$$

$$-A_{2\lambda}S_{2\lambda}(t)S_{1\mu}(t)S_{2\lambda}(s)S_{1\mu}(s)x - S_{2\lambda}(t)A_{1\mu}S_{1\mu}(t)S_{2\lambda}(s)S_{1\mu}(s)x||$$

$$\to ||A_{2\lambda}S_{2\lambda}(s)S_{1\mu}(s)x + S_{2\lambda}(s)A_{1\mu}S_{1\mu}(s)x$$

$$- A_{2\lambda}S_{2\lambda}(s)S_{1\mu}(s)x - A_{1\mu}S_{2\lambda}(s)S_{1\mu}(s)x|| < \infty \text{ as } t \to 0$$

$$\to 0 \text{ as } t,s \to 0,0 \quad .$$

Letting $\mu \to 0$ (switching limits by the uniform convergence), gives

$$||f_t(t,s) - g_t(t,s)|| \to ||S_{2\lambda}(s)A_1^0S_1(s)x - A_1^0S_{2\lambda}(s)S_1(s)x|| < \infty \qquad (10)$$

as $t \to 0$, A_1^0 being the minimal section of the set-valued accretive operator A_1. We note that (10) is well defined for small s because of the finite breakdown time for $x \in V_2^T$. Equation (10) is also bounded as $\lambda \to 0$, since $A_1^0S_2(s)S_1(s)x$ resides in $(W^{1,1}(R))^2 = W^{1,1}(R) \times W^{1,1}(R)$, and tends to A_1^0x as $s \to 0$.

Again,

$$||f_{\mu s}(t,s) - g_{\mu s}(t,s)|| = ||A_{2\lambda}S_{2\lambda}(t+s)S_{1\mu}(t+s)x + S_{2\lambda}(t+s)A_{1\mu}S_{1\mu}(t+s)x$$

$$-D[S_{2\lambda}(t)S_{1\mu}(t)][A_{2\lambda}S_{2\lambda}(s)S_{1\mu}(s)x + S_{2\lambda}(s)A_{1\mu}S_{1\mu}(s)x]||$$

$$\rightarrow ||A_{2\lambda}S_{2\lambda}(t)S_{1\mu}(t)x + S_{2\lambda}(t)A_{1\mu}S_{1\mu}(t)x - D[S_{2\lambda}(t)S_{1\mu}(t)][A_{2\lambda}x + A_{1\mu}x]|| < \infty$$

as $s \rightarrow 0$,

$\rightarrow 0$ as $t,s \rightarrow 0,0$.

Letting $\mu \rightarrow 0$ (with uniform convergence on compact t-sets),

$$||f_s(t,s) - g_s(t,s)|| \rightarrow ||A_{2\lambda}S_{2\lambda}(t)S_1(t)x + S_{2\lambda}(t)A_1^0S_1(t)x$$

$$- D[S_{2\lambda}(t)S_1(t)][A_{2\lambda}x + A_1^0x]|| < \infty \text{ as } s \rightarrow \infty \quad . \quad (11)$$

We note that (11) is well defined for small t since $x \in V_2^T$. Also it is bounded as $\lambda \rightarrow 0$ since $A_2S_2(t)S_1(t)x$ resides in $(W^{1,1}(R))^2$ for small t, and tends to A_2x as $t \rightarrow 0$.

Thus $||f_s(t,0) - g_s(t,0)|| < \infty$ along the t-axis of the t,s plane, and $||f_t(0,s) - g_t(0,s)|| < \infty$ along the s-axis. This means that along the t-axis, $||f(t,0) - g(t,0)|| = 0$ is assumed at least to first order, while along the s-axis, $||f(0,s) - g(0,s)|| = 0$ is also assumed to first order. We have moreover

$$||f_{\mu st}(0,0) - g_{\mu st}(0,0)|| = ||f_{\mu ts}(0,0) - g_{\mu ts}(0,0)||$$

$$= ||A_{2\lambda}(A_{2\lambda}x + A_{1\mu}x) + A_{2\lambda}A_{1\mu}x + D[A_{1\mu}]A_{1\mu}x$$

$$- A_{2\lambda}(A_{2\lambda}x + A_{1\mu}x) - D[A_{1\mu}](A_{2\lambda}x + A_{1\mu}x)||$$

$$= ||A_{2\lambda}A_{1\mu}x - D[A_{1\mu}]A_{2\lambda}x|| < \infty \quad .$$

Taking the uniform limit as $\mu \rightarrow 0$ again, we see that

$$||f_{st}(0,0) - g_{st}(0,0)|| = ||A_{2\lambda}A_1^0x - D[A^0]A_{2\lambda}x|| \quad (12)$$

is well defined. (12) is bounded as $\lambda \to 0$ since $x \in V_2^T$. ($D[A_1^0]$ is simply $\lim_{\mu\to 0} D[A_{1\mu}]$; $(\phi'(\rho_{10}))\rho_x$, $\psi'(u_{10})u_x)$ is a representation in the variables of (1)). Thus

$$\frac{||f(t,s) - g(t,s)||}{ts} < \infty \quad \text{as } s \to 0, \text{ as } t \to 0, \text{ or as } s,t \to (0,0)$$

simultaneously, regardless of manner. Letting C_0 be a constant larger than (10), (11), or (12), we see that $||f(t,s) - g(t,s)|| \leq C_0 st$, where C_0 is independent of $\lambda > 0$. This verifies (9), the Marsden condition (iv), and the theorem.

Theorem 4 holds for each $x \in \bar{V}^T \cap (W^{2,1}(R))^2$, but the constant C_0 depends on x. If we close $V^T \cap (W^{2,1}(R))^2$ in the norm $||\cdot||_2$ of $(W^{2,1}(R))^2$, however, there is a common C_0 for x in this closure, as is readily seen. Thus (9) holds on a dense set $V_2^T \subset V^T$. It was necessary to use V_2^T in the proof. Since $K_\lambda(t)$ is defined on $\bar{V}_2^T = V^T$ however, inequality (9) is valid on V^T itself. This gives:

Corollary 4': $||K_\lambda(t + s)x - K_\lambda(t)K_\lambda(s)x|| \leq C_0 st$, $x \in V^T$, where C_0 is independent of $\lambda > 0$ and x.

5. Convergence of the Trotter Product

The main Marsden requirements have now been satisfied, namely conditions (iii) and (iv), represented here by (8) and (9) (or more to the point, Corollary 4'). These are conditions on the Chernoff operator $K_\lambda(t) = S_{2\lambda}(t)S_1(t)$. We must now go through a sequence of lemmas, proved by Marsden, which show that the Trotter product (5) converges [Ref. 4, pp. 55-59]. We shall state these lemmas below, using the idiom of this present paper. The proofs are so similar to those of Marsden that there is no need to reproduce them in detail here. The only changes would be the use of the Banach space norm for the metric (replacing Marsden's general metrization), and the ability here to use W and V^T without intersecting with $(W^{2,1}(R))^2$. Also, our Theorem 3 substitutes for Marsden Lemma 2.2, and this permits us to do without Marsden Corollaries 2.4 and 2.4'.

Proposition 5 (Marsden Lemma 2.3): Given $x \in V^T$, there is a constant C_1 (dependent on x and λ) such that

$$||K_\lambda(t)x - [K_\lambda(\frac{t}{\ell})]^\ell x|| \le C_1 t^2 \quad , \quad \ell = 1,2,3,\cdots. \tag{13}$$

Marsden's hypotheses for this result are more involved, but this relates to his manifold setting. By using a natural extension of (9), or Corollary 4', to ℓ variables, we could remove the dependence of the constant C_1 in (13) on λ. This λ dependence results from β_λ in (8). (See Corollary 4" below).

Again, because we have Theorem 3 (Marsden necessarily lacks a corresponding result), we dispense with the corollaries to Lemma 2.3.

Proposition 6 (Marsden Lemma 2.5): Given $x \in W$, there is a constant C_2 (dependent on x and λ) such that for $m \ge n$,

$$||[K_\lambda(\frac{t}{n})]^n x - [K_\lambda(\frac{t}{m})]^m x|| \le C_2 \frac{t^2}{n} \tag{14}$$

for all t with $0 \le t \le T$. The set W was defined in Theorem 3.

Here $[0,T]$ is the arbitrary time interval of our Theorem 3. In contrast to Marsden's situation, T need not be small. The proof of Proposition 6 utilizes Marsden's method of initially considering m to be an integral multiple of n. We have $K_\lambda(\frac{t}{n\ell})^{j\ell}x \in V^T$ by our Theorem 3.

Proposition 7 (Marsden Lemma 2.6): The sequence $\{[K_\lambda(\frac{t}{n})]^n x\}$, $n = 1,2,3,\cdots$, $x \in W$, converges uniformly over $0 \le t \le T$ to a continuous limit: $S_\lambda(t)x$. In particular, for fixed $\lambda > 0$, $S_\lambda(t)x$ is jointly continuous in $(x,t) \in W \times [0,T]$.

The proof [Ref. 4, p. 58] uses the Cauchy criterion based on Proposition 6, and the completeness of $(L^1(R))^2$. Marsden, in his proofs, uses a family of metrics, denumerated by subscript k. The corresponding family of metrics in our work would be the successive Sobolev norms. The reason for Marsden's family of metrics was the desire to produce regularizing properties for the resulting semigroup solutions. Here we do not expect regularizing properties, since the semigroups we started with, namely $S_1(t)$ and $S_2(t)$, did not have regularizing properties. Hence we use a single metric.

Proposition 8 (Marsden Lemma 2.7): $S_\lambda(t)$ is a semigroup. Thus for $x \in W$, $0 \leq s + t \leq T$, $s,t \geq 0$, we have $S_\lambda(t + s) = S_\lambda(t)S_\lambda(s)$ where $S_\lambda(t)$ is defined in (5).

Marsden's clever proof of this result is unchanged except for the use of Banach space norms. Where needed in this proof, we can put $V^T = \tilde{W}$ in Theorem 3, and then generate a new $\tilde{V}^{\tilde{T}}$. $\{[K_\lambda(\frac{t}{n})]^n y\}$ then remains in this $\tilde{V}^{\tilde{T}}$, $n = 1,2,3,\cdots$, provided $y \in \tilde{W}$, $0 \leq t \leq \tilde{T}$. This assures convergence in Proposition 7.

6. The Semigroup Associated With Problem (1)

We have seen that $S_\lambda(t)x = \lim_{n\to\infty} [S_{2\lambda}(\frac{t}{n})S_1(\frac{t}{n})]^n x$, $x \in W$, exists and represents a semigroup of operators in the interval $0 \leq t \leq T$, where T is arbitrary. This is for fixed $\lambda > 0$. The question now is whether $S_\lambda(t)x$, $x \in W$, approaches a limit as $\lambda \to 0$, and whether this limit is a semigroup.

In (8), and subsequently, it was convenient to use the Lipschitz constant $e^{\beta_\lambda t}$ for the Chernoff operator $K_\lambda(t) = S_{2\lambda}(t)S_1(t)$. Here, $\beta_\lambda \to \infty$ as $\lambda \to 0$. There may be a more refined λ-continuous Lipschitz constant for $K_\lambda(t)$ however. Indeed, $K(t) = S_2(t)S_1(t) = \lim_{\lambda\to 0} K_\lambda(t)$ has Lipschitz constant 2, (as does the operator $S_2(t)$).

We shall need the following important extension of inequality (9).

Corollary 4": $||K_\lambda(t_1 + \ldots + t_n)\,x - K_\lambda(t_1)\ \ldots\ K_\lambda(t_n)x||$

$$\leq C_0(t_1t_2 + t_1t_3 + \ldots + t_{n-1}t_n) \tag{15}$$

where the right side has $\frac{n(n-1)}{2}$ summands, and C_0 is the constant in (9), independent of $\lambda > 0$ and $x \in W$, (see Corollary 4').

Proof: Inequality (15) reduces to (9) on all 2-dimensional coordinate planes in n-space. The ratio of the left side of (15) to the right side is bounded by C_0 along any path to the origin involving a 2-dimensional coordinate plane. For more general paths to the origin in $(t_1, t_2, \ldots, t_n)$ space, the same bound is shown to be valid by using a parametrization and ℓ'Hopital's rule.

By putting $t_1 = t_2 = \ldots = t_n = \frac{t}{n}$ in (15), we obtain the inequality

$$||K_\lambda(t)x - [K_\lambda(\tfrac{t}{n})]^n x|| \leq \frac{1}{2}\,C_0 t^2 \tag{16}$$

which is like (13), except that C_0 is independent of λ and x.

Inequality (16) permits us to derive a bound for $S_\lambda(t)x$:

Theorem 9: The semigroup $S_\lambda(t)$ of Proposition 8, represented as a Trotter product in (5), is such that $||S_\lambda(t)x||$ is uniformly bounded as $\lambda \to 0$, $0 \leq t \leq T$, where $T > 0$ is arbitrary.

Proof: Letting $n \to \infty$, and rearranging in (16), we show that $||S_\lambda(t)x|| \leq ||K_\lambda(t)x|| + \frac{1}{2}\,C_0^2 t^2$, x W. We have $K_\lambda(t)x \to K(t)x = S_2(t)S_1(t)x$ as $\lambda \to 0$ however. Since $S_2(t)$ is a linear operator with bound 2 for $t > 0$, and $S_1(t)$ is contracting with $S_1(\Theta) = \Theta$, (where Θ is the null element of $(L'(R))^4$, we see that $||K_\lambda(t)x|| \leq 4\ ||x||$ as $\lambda \to 0$. Hence

$$||S_\lambda(t)x|| \leq C_0^2T^2 + 4\ ||x|| \tag{17}$$

for small $\lambda > 0$. This proves the theorem.

Our method for showing that $S_\lambda(t)$ converges strongly to some semigroup limit $S_0(t)$ as $\lambda \to 0$ depends on proof that $\lim_{\lambda,\lambda'\to 0} \|S_\lambda(t)x - S_{\lambda'}(t)x\| = 0$, and use of the Cauchy criterion and completeness. We estimate $\|S_\lambda(t)x - S_{\lambda'}(t)x\|$ using (16). The success of this method depends evidently on the rapidity of the convergence of $S_{2\lambda}(t)$ to $S_2(t)$, where the λ-approximation $S_{2\lambda}(t)$ of $S_2(t)$ is as given in (6). We have developed a criterion for this, but $S_{2\lambda}$ may not converge fast enough to S_2.

In this connection, it is well to develop information about the Yosida approximation $A_{2\lambda}$ of A_2. By definition, $A_{2\lambda} = \frac{1}{\lambda}[I - J_\lambda(A_2)]$, where the resolvent $J_\lambda(A_2)$ solves the DE system (c.f. problem (b)):

$$\tilde{\rho} + \lambda\, \tilde{u}_x = h_1$$

$$\tilde{u} + \lambda\, \tilde{\rho}_x = h_2,\ h_1,\ h_2 \in L^1(R). \tag{18}$$

The solution of Eqs. (18) in $(L^1(R))^2$ is obtained explicitly:

$$\tilde{\rho}(x) = \frac{1}{2\lambda}\int_{-\infty}^{x} e^{-\frac{x-\tilde{x}}{\lambda}}[h_1(\tilde{x})+h_2(\tilde{x})]d\tilde{x}+\frac{1}{2\lambda}\int_{x}^{\infty} e^{-\frac{\tilde{x}-x}{\lambda}}[h_1(\tilde{x})-h_2(\tilde{x})]d\tilde{x}$$

$$\tilde{u}(x) = \frac{1}{2\lambda}\int_{-\infty}^{x} e^{-\frac{x-\tilde{x}}{\lambda}}[h_1(\tilde{x})+h_2(\tilde{x})]d\tilde{x}-\frac{1}{2\lambda}\int_{x}^{\infty} e^{-\frac{\tilde{x}-x}{\lambda}}[h_1(\tilde{x})-h_2(\tilde{x})]d\tilde{x} \quad .$$

Then, using $A_{2\lambda} = \frac{1}{\lambda}[I - J_\lambda(A_2)]$, and after integration by parts,

$$A_{2\lambda}\begin{pmatrix}\rho\\ u\end{pmatrix} = \frac{1}{2\lambda}\begin{pmatrix} \int_{-\infty}^{x} e^{-\frac{x-\tilde{x}}{\lambda}}[\rho_x(\tilde{x})+u_x(\tilde{x})]d\tilde{x} - \int_{x}^{\infty} e^{-\frac{\tilde{x}-x}{\lambda}}[\rho_x(\tilde{x})-u_x(\tilde{x})]d\tilde{x} \\ \int_{-\infty}^{x} e^{-\frac{x-\tilde{x}}{\lambda}}[\rho_x(\tilde{x})+u_x(\tilde{x})]d\tilde{x} + \int_{x}^{\infty} e^{-\frac{\tilde{x}-x}{\lambda}}[\rho_x(\tilde{x})-u_x(\tilde{x})]d\tilde{x} \end{pmatrix} \quad . \tag{19}$$

As $\lambda \to 0$, (19) tends in the norm of $(L^1(R))^2$ to

$$\begin{pmatrix} \frac{1}{2}(\rho_x+u_x) - \frac{1}{2}(\rho_x-u_x) \\ \\ \frac{1}{2}(\rho_x+u_x) + \frac{1}{2}(\rho_x-u_x) \end{pmatrix} = \begin{pmatrix} u_x \\ \\ \rho_x \end{pmatrix}$$

since $\frac{1}{\lambda}\int_{-\infty}^{x} e^{-\frac{x-\tilde{x}}{\lambda}}\, d\tilde{x} = 1$ and $\frac{1}{\lambda}\int_{x}^{\infty} e^{-\frac{\tilde{x}-x}{\lambda}}\, d\tilde{x} = 1$. Thus expression (19) has δ-tending kernels.

Thus the Yosida approximation of A_2 has components which contain "mollified derivatives." For very small $\lambda \to 0$ furthermore, the components of $A_{2\lambda}$ really are approximate mollified derivatives:

$$A_{2\lambda}\begin{pmatrix}\rho \\ u\end{pmatrix} \simeq \begin{pmatrix} \int_{-\infty}^{\infty} M_\lambda(x,\tilde{x})u_x(\tilde{x})d\tilde{x} \\ \int_{-\infty}^{\infty} M_\lambda(x,\tilde{x})\rho_x(\tilde{x})d\tilde{x} \end{pmatrix}$$

where

$$M_\lambda(x,\hat{x}) = \begin{cases} \frac{1}{2\lambda}\, e^{-\frac{\tilde{x}-x}{\lambda}} , & x < \tilde{x} \\ \frac{1}{2\lambda}\, e^{-\frac{x-\tilde{x}}{\lambda}} , & x > \tilde{x} \end{cases} .$$

We have thus obtained the semigroup $S_\lambda(t)$ corresponding to problem (1), with the derivatives in the coupling terms approximated by mollified derivatives. This semigroup is expressed as a Trotter product, which is uniformly bounded in the strong topology of $(L^1(R))^2$ as $\lambda \to 0$, (Theorem 9). The approximation improves the smaller we take $\lambda > 0$.

We expect to prove actual strong convergence of $S_\lambda(t)$ to a semi-group $S_0(t)$ as $\lambda \to 0$. It may be however that the product formula (5) for $S_\lambda(t)$, itself, does not converge when $\lambda = 0$. If not, it may possibly be because of the Lipschitz constant 2 possessed by $K(t) = S_2(t)S_1(t)$. Product formulas $\lim_{n\to\infty} [K(\frac{t}{n})]^n$ often do not converge when K(t) has Lipschitz constant greater than unity. No general theorem seems to be available on this.

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