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CRACK TIP TOUGHENING BY INCLUSIONS WITH PAIRS OF SHEAR TRANSFORMATIONS†

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ABSTRACT

The deviatoric transformation strain of an inclusion is modeled by applying an equivalent distribution of dislocations along a surface which exhibits a discontinuous change in the transformation strains. This method is applied to qualitatively model the twin structures generated in transformation toughened ceramics. For this case, the transformation shear strain of the inclusion is assumed to consist of a number of symmetrical pairs of (twinning) shears in a rectangular grain. The elastic energy is derived and expressed in terms of elementary functions. With one pair of shears, the inclusion induced toughening effect in the presence of a crack is calculated by applying a recent solution of the crack-dislocation interaction problem. Numerical results show that the toughening due to the inclusion (as compared to that due to dilatation) is not negligible if the inclusion is located within a distance equal to several grain sizes from the crack tip. Moreover, the toughening depends strongly on the orientation of the inclusion relative to the crack.

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1. INTRODUCTION

The martensitic transformation of tetragonal zirconia particles or grains has been shown to increase the toughness of both monolithic zirconia and ceramics containing zirconia inclusions (Becher, Swain and Sōmiya, 1987; Evans and Cannon, 1986; McMeeking and Evans, 1982; Rose, 1987). Analysis of the toughening behavior indicated that the dilatational or volumetric expansion of the transformed particles is a significant contributor to the toughening. Experimental observations of the martensitic transformation of ZrO_2 show that twin structures with different variants are generated (Muddle and Hannink, 1986). The condition required to trigger the transformation seems to remain a subject of investigation, although the transformation mechanism has been shown to be closely related to the local shear (Chen and Morel, 1986). Questions have been raised (McMeeking and Evans, 1982) as to how the orientation of the twins may interact with the crack tip to provide further reduction in the stress intensity.

In the present study, we shall not discuss the criteria for the nucleation of the transformation, but only suggest a method for calculating the toughening contribution as a result of the inclusions which have been subjected to pairs of shear transformations. The method is based on the distribution of dislocations introduced to match the misfit or incompatible deformation along the interfaces between an inclusion or twin and the matrix (Bilby, Bullough and Smith, 1955; Mura, 1987). The twin structure is assumed to be subjected to pairs of shear transformations in a rectangular grain. The energy of the deformation twinning is then calculated and the results show that the elastic twin energy per unit volume decreases as the number of pairs of twins increases. The transformation, therefore, leads to a reduction of the total elastic energy of the system. The stress is found to increase logarithmically as the tip is approached from the matrix side. Moreover, the toughening contribution of the twin structures is analyzed by applying a recent solution of the dislocation-crack interaction problem. The numerical results show that if the twins are located near the crack tip, their contribution to the toughening depends strongly on their orientation and is not negligible as compared to that due to the dilatation toughening.

2. DISLOCATION DENSITY DESCRIPTION FOR INELASTIC INCLUSION

A dislocation density tensor is used to represent the incompatible or misfit deformation across the boundary between an inclusion and the surrounding matrix. It is also used to model the nonuniform inelastic transformation strain within the inclusion. This formulation enables one to solve the inclusion problem with nonuniform transformation strain in terms of the solution of an equivalent dislocation problem.

Let u_k^e and u_k^p denote Cartesian components of the displacements due to elastic and plastic deformations, respectively. The elastic distortion β_{ik}^e and plastic distortion β_{ik}^p are defined, respectively, as the gradients of the displacements,

$$\beta_{ik}^e = \frac{\partial u_k^e}{\partial x_i} \tag{2.1}$$

$$\beta_{ik}^p = \frac{\partial u_k^p}{\partial x_i} \tag{2.2}$$

where x_i ($i = 1, 2, 3$) are the Cartesian coordinates. Since the plastic distortion appears repeatedly in the following text, we shall use the notation β_{ik} without superscript for brevity. The Burgers vector b is defined as a contour integral along a Burgers circuit L with respect to a line segment of a dislocation loop,

$$b_k = -\oint_L \beta_{ik} \, dx_i \tag{2.3}$$

The above line integral can be transformed to a surface integral according to:

$$\oint_L \beta_{ik} \, dx_i = \int_S e_{ilm} \frac{\partial \beta_{mk}}{\partial x_l} dS_i = \int_S e_{ilm} \frac{\partial^2 u_k^p}{\partial x_m \partial x_l} dS_i \tag{2.4}$$

where e_{ilm} is the permutation tensor. If u_k^p were continuous and differentiable, then we should have:

$$\frac{\partial^2 u_k^p}{\partial x_m \partial x_l} = \frac{\partial^2 u_k^p}{\partial x_l \partial x_m} , \tag{2.5}$$

and Eq. (2.4) would be zero. However, Eq. (2.5) is not satisfied everywhere within S due to the disturbance of the u_i^p field induced by the dislocation loop which passes through S. In order to satisfy Eq. (2.3) for a single dislocation line, we must have:

$$e_{ilm} \frac{\partial \beta_{mk}}{\partial x_l} = -b_k \, \nu_i \, \delta(\xi) . \tag{2.6}$$

In the above equation, $\delta(\xi)$ is the two dimensional δ-function and ν_i is the normal direction of the surface S such that:

$$\int_S \nu_i \, \delta(\xi) \, dS_i = 1 . \tag{2.7}$$

In Eq. (2.6), the discrete Burgers vector b_k may be viewed as a distribution of Burgers vectors with the distribution function $b_k \, \delta(\xi)$. Therefore, given more dislocation lines passing through S, we may use

Eq. (2.6) to define a distribution of dislocations. This yields:

$$\alpha_{ik} = \nu_i \; b_k = -e_{ilm} \frac{\partial \beta_{mk}}{\partial x_l} \tag{2.8}$$

where α_{ik} represents the k-th component b_k of the total Burgers vector per unit area that has a unit normal ν_i along the i-th direction. α_{ik} is a tensor quantity because it is composed of two vectors b and ν. A physical view of the distribution of dislocations is to smear out the discrete dislocation lines and to replace them by the equivalent continuous distribution.

It is seen from the above equation, Eq. (2.8), that α_{ik} vanishes if every derivative of the plastic distortion β_{mk} does. Suppose that the derivatives of β_{mk} are nonzero only along certain surfaces in three dimensional space, or lines in two dimensional space. This means that the inelastic strain is uniform everywhere except along these surfaces. Then it is more convenient to represent the inelastic strain field by the discontinuities of β's across these surfaces. This is shown in the following.

In Eq. (2.8), α_{ik} may be interpreted as the divergence of a flux vector $e_{ilm} \, \beta_{mk}$ with $\ell = 1, 2, 3$ as its three components. Suppose that the flux vector is discontinuous across a surface where n_ℓ is the direction cosine n_ℓ and the quantity $[\beta_{mk}]$ is the discontinuity across the surface. We may prescribe a thin layer Δt across the surface and apply the divergence theorem to obtain:

$$\alpha'_{ik} = \lim_{\Delta t \to 0} \alpha_{ik} \, \Delta t = -e_{ilm} \, [\beta_{mk}] \, n_l \; . \tag{2.9}$$

From the above equation, α'_{ik} is defined as the surface dislocation density as a result of the discontinuity $[\beta_{mk}]$. Therefore, an equivalent distribution of surface dislocations can be used to represent the discontinuous change of β_{mk} across the surfaces and the solution of the inclusion problem with nonuniform transformation strain can be reduced to that of an equivalent dislocation problem. We shall formulate the problem more specifically in the next section.

3. TWO DIMENSIONAL INCLUSION

For a two dimensional inclusion, a direct application of Eq. (2.9) leads to:

$$\begin{bmatrix} \alpha'_1 \\ \\ \alpha'_2 \end{bmatrix} = - \begin{bmatrix} [\beta_{11}] & [\beta_{21}] \\ \\ [\beta_{12}] & [\beta_{22}] \end{bmatrix} \begin{bmatrix} s_1 \\ \\ s_2 \end{bmatrix} \quad (3.1)$$

where s is the tangent vector along the interface of discontinuity $[\beta]$. For example, the change in β_{11} component becomes:

$$[\beta_{11}] = \beta^{+}_{11} - \beta^{-}_{11} \quad (3.2)$$

if β^{+}_{11} and β^{-}_{11} are the plastic distortions of the matrix and the inclusion, respectively.

The above representation suggests that we may define a complex surface dislocation density α' by:

$$\alpha' = \alpha'_1 + i\,\alpha'_2 \quad (3.3)$$

and rewrite equation (3.1) as:

$$2\alpha' = -([\beta_{11}] + [\beta_{22}])\frac{dz}{ds} - ([\beta_{11}] - [\beta_{22}])\frac{d\bar{z}}{ds}$$

$$- i\,([\beta_{12}] - [\beta_{21}])\frac{dz}{ds} - i\,([\beta_{12}] + [\beta_{21}])\frac{d\bar{z}}{ds} \tag{3.4}$$

where $z = x_1 + ix_2$. The first two terms in Eq. (3.4) are the effect due to uniform stretching and the last two terms simple shear. By using the complex dislocation density, Eq. (3.4), we have an advantage in calculating some of the contour integrals.

The above description of surface dislocations can be used to calculate the stress distribution as a result of the misfit deformation along a certain surface provided that we know the stress functions due to one single edge dislocation. For an edge dislocation at z_1 in an isotropic and elastic medium with shear modulus μ and Poisson's ratio ν, the stress potentials are :

$$\phi(z) = B\,\alpha\,\frac{1}{z - z_1} \tag{3.5}$$

$$\psi(z) = -B\,\bar{\alpha}\,\frac{1}{z - z_1} + B\,\alpha\,\frac{\bar{z}_1}{(z - z_1)^2} \tag{3.6}$$

where B is a material constant,

$$B = \frac{\mu}{4\pi i(1 - \nu)}\ . \tag{3.7}$$

The solutions of two dimensional inclusion problems can be expressed in terms of the line integrals of equations (3.5) and (3.6) along the inclusion surface with appropriate boundary conditions. Functions $\phi(z)$ and $\psi(z)$ for an inclusion of a general shape in an infinite medium are shown in Appendix A.

These expressions for ϕ and ψ have been applied to circular and elliptic inclusions and the well-known solutions of the inclusion problems (Mura, 1987) were confirmed.

4. TWIN STRUCTURE IDEALIZED AS PAIRS OF SHEAR TRANSFORMATIONS

It is assumed that the twin structure generated by the phase transformation can be represented qualitatively by pairs of inelastic shear deformations in a rectangular grain, as shown schematically in Fig. 1.

Each pair consists of a positive shear S at the right half and a negative shear -S at left,

$$\beta_{12} = \frac{\partial u_2}{\partial x_1} = S \text{ right half} \qquad\qquad (4.1)$$
$$= - S \text{ left half.}$$

The other components of β are zero, including β_{21}. The matrix is assumed to deform elastically. By applying equation (3.4), the misfit strain between the matrix and the inclusion can be used to derive the equivalent distribution of edge dislocations on the top and bottom part of the twin. It is noted that, although there exist changes in plastic strain along the twin boundaries, there is no distribution of dislocations along these boundaries.

Elementary calculation shows that the solution in terms of the complex stress functions for the twinning deformation is:

$$\frac{i}{B\beta_{12}} \phi(z) = 2 \sum_{i=0}^{N} \left[(-1)^i \log \frac{z_{i-} - z}{z_{i+} - z} \right] - \log \frac{z_{o-} - z}{z_{o+} - z} - \log \frac{z_{N-} - z}{z_{N+} - z} \qquad (4.2)$$

$$\frac{i}{B\beta_{12}}\,\psi(z) = 2\sum_{i=0}^{N}\left[(-1)^{i}\left(\frac{\bar{z}_{i-}}{z_{i-}-z}-\frac{\bar{z}_{i+}}{z_{i+}-z}\right)\right]-\left(\frac{\bar{z}_{o-}}{z_{o-}-z}-\frac{\bar{z}_{o+}}{z_{o+}-z}\right)$$

$$-\left(\frac{\bar{z}_{N-}}{z_{N-}-z}-\frac{\bar{z}_{N+}}{z_{N+}-z}\right) . \qquad (4.3)$$

From the potential functions $\phi(z)$ and $\psi(z)$ shown in the above equations, the stress components are:

$$\sigma_1 + \sigma_2 = \frac{2B\beta_{12}}{i} \quad 2 \quad \mathrm{Re}\left[2\sum_{i=0}^{N}(-1)^{i}\log\frac{z_{i-}-z}{z_{i+}-z}\right.$$

$$\left. - \log\frac{(z_{o-}-z)\,(z_{N-}-z)}{(z_{o+}-z)\,(z_{N+}-z)}\right] \qquad (4.4)$$

$$\sigma_2 - \sigma_1 + 2i\sigma_{12} = \frac{2B\beta_{12}}{i}\left[2\sum_{i=0}^{N}(-1)^{i}\left(\frac{\bar{z}_{i-}-\bar{z}}{z_{i-}-z}-\frac{\bar{z}_{i+}-\bar{z}}{z_{i+}-z}\right)\right.$$

$$\left.-\left(\frac{\bar{z}_{o-}-\bar{z}}{z_{o-}-z}-\frac{\bar{z}_{o+}-\bar{z}}{z_{o+}-z}\right)-\left(\frac{\bar{z}_{N-}-\bar{z}}{z_{N-}-z}-\frac{\bar{z}_{N+}-\bar{z}}{z_{N+}-z}\right)\right] \qquad (4.5)$$

5. STRESS INTENSITY

From the stress distribution, Eqs. (4.4) and (4.5), we observe that at the ends of each twinning plane the stress tends to infinity according to:

$$\sigma_1 = \sigma_2 \rightarrow 4Bi\ \beta_{12}\log|z - z_{i+}| \rightarrow -\infty \quad \text{for } z \rightarrow z_{i+} \quad i = \text{odd}$$

$$\rightarrow +\infty \quad \text{for } z \rightarrow z_{i+} \quad i = \text{even} \qquad (5.1)$$

Therefore the stress intensity factor is:

$$K = 4\ Bi\ \beta_{12} = \frac{\mu\ \beta_{12}}{\pi\ (1 - \nu)}\ . \tag{5.2}$$

It is observed that K is linearly proportional to the shear β_{12} deformed within the twin and is independent of the dimension of the transformed inclusion.

6. TOTAL ENERGY

The total elastic energy due to the transformation can be calculated by applying the relation,

$$W = -\frac{1}{2}\int_v \sigma_{ij}\ \varepsilon^*_{ij}\ dv \tag{6.1}$$

where ε^*_{ij} is the inelastic transformation strain and v is the volume of the inclusion prior to the transformation. In the present case it has a rectangular shape, and the transformation consists of several pairs of shears. It has been shown recently (Mura, Jusiuk and Tsuchida, 1985) that the above equation is a valid expression even if the interface between the inclusion and the matrix has a sliding contact boundary condition. The derivation is shown in Appendix B.

After some elementary calculations, we obtain the following expression of energy for the present problem,

$$W = \frac{\mu\beta_{12}^2}{2\pi(1 - \nu)} \sum_{j=1}^{N}\sum_{i=1}^{N} (-1)^{i+j+1} \int_{v_j} \sin 2\theta_i\ dxdy\ . \tag{6.2}$$

where θ_i is defined as the angle between x-axis and the ray from z_{i-} to the point (x,y). As shown in Appendix B, equation (6.2) can be further simplified to logarithmic functions.

The numerical result shows that the energy of the two dimensional is less than that of the three demensional counterpart (Mura, Mori and Kato, 1976). However, the energy of the two dimensional qualitatively described the physical nature exhibited by its three-dimensional counterpart. In addition, the present transformation strain for the case of one pair of shears is obtained via an approximate polynomial transformation (Asaro and Barnett, 1975). Finally, a numerical calculation for the total energy was then carried out and the result is plotted as the bottom curve in Fig. 2. This curve shows a continuous decrease in energy as the number of pairs increases, provided that both the dimension of the inclusion and the magnitude of the shear are fixed.

As discussed earlier, (Evans and Cannon, 1986) the criterion for the nucleation of the twin structure may be related to the minimum total energy of the system. It is possible that an additional energy contribution may come from other sources in which the surface energy along the interface of the discontinuous shear deformation may be a dominant quantity. It is not known how large the surface energy should be. We calculated a series of curves by assigning a range of surface energies. The results are plotted as a series of curves in Fig. 2. Each of these curves has a minimum value. The number of pairs which are required to reach the minimum decreases as the magnitude of the assigned surface energy increases.

7. DISLOCATION AND CRACK INTERACTION

In the preceding sections, we have discussed the use of dislocations to model the misfit created by the inelastic transformation of the inclusion against the matrix. The stress field generated as a result of the misfit, however, depends on the constitutive properties of the inclusion as well as the matrix. At present, we shall assume that both the inclusion and the matrix have the same isotropic elastic constants.

In order to calculate the toughening effect, we need an explicit solution of the dislocation-crack interaction problem. There are several ways to derive the solution (Thomson, 1986). At present, a two-dimensional solution is described in Appendix C. The problem is formulated by invoking a distribution of (virtual) dislocations to model the displacement of the crack surface. The edge dislocation which induces the solution is located near the crack with the complex coordinate β . The problem is schematically shown in Fig. 3 in which a complex coordinate is chosen so that the negative x-axis coincides with the semi-infinite crack surface and the crack tip is located at the origin. Any point in the two-dimensional body is represented by the coordinate $z = x + iy$.

As shown in Appendix C, the complex stress intensity K ($K = K_1 + iK_2$) induced by an edge dislocation which is located at the complex coordinate β with a complex Burgers vector b has the expression:

$$K = -\frac{\sqrt{2\pi}}{\sqrt{\beta}} \left[\bar{a} \left(1 + \frac{\sqrt{\beta}}{\sqrt{\bar{\beta}}}\right) + \frac{a}{2} \left(1 - \frac{\bar{\beta}}{\beta}\right) \right] \qquad (7.1)$$

where K_1 and K_2 are the mode I and mode II stress intensities, respectively, and the complex constant a has been defined in Eq. (3.7).

The distribution function, also complex-valued, is:

$$f(x) = -\frac{1}{A\pi\sqrt{-x}} \left[-\bar{a} \left(\frac{\sqrt{\beta}}{x-\beta} + \frac{\sqrt{\bar{\beta}}}{x-\bar{\beta}} \right) + \frac{a}{2} \left(1 - \frac{\bar{\beta}}{\beta}\right) \frac{(x+\beta)\sqrt{\beta}}{(x-\beta)^2} \right] \qquad (7.2)$$

from which we obtain K by taking the limit:

$$K = \lim_{x \to o} \sqrt{2\pi} \; A\pi \sqrt{-x} \; f(x). \qquad (7.3)$$

The real part of the function f(x) denotes the mode I component of the distribution function and the imaginary part of f(x), the mode II component. The complex displacement for the crack surface is:

$$u = b_o \int_x^o f(x)\, dx \qquad (7.4)$$

where b_o is the magnitude of the Burgers vector for the (virtual) dislocation. The above expressions are valid for b not necessarily parallel to β. For b // β , K reduces to :

$$K = \frac{b\mu}{2\sqrt{2\pi r}\,(1-\nu)} \cos\frac{\theta}{2} \; [3 \sin\theta + i\,(3\cos\theta - 1)] \,. \qquad (7.5)$$

8. INCLUSION INDUCED TOUGHNESS

It has been illustrated earlier that the transformation strain for an inclusion consisting of pairs of shears can be represented by a distribution of dislocations along the inclusion boundary. In the following calculations, the deformation misfit is modeled approximately by a discrete number of edge dislocations.

At present we only use four dislocations to represent the transformation of one pair of shears. The arrangement of the dislocations is schematically shown in Fig. 4. It represents an inclusion which has a rectangular shape of $2\Delta x$ by Δy in dimension and has been subjected to a pair of shears. For the purpose of comparison, a dilatational transformation is approximately represented by a circular arrangement of four dislocations.

The stress intensity K_1 induced by a symmetric pair of inclusions, each consisting of a single pair of shear deformations, is calculated and the result is shown in Fig. 5 where the directions of the shears to the horizontal axis are $\pi/2$ and π. The induced K_1 which has a negative value indicates a toughening effect for the crack tip region.

To estimate the overall K effect on the crack growth, two lines of inclusions are assumed as shown in Fig. 4. The arrangement is similar to that of some known models (Weertman, Lin and Thomson, 1983; L.R.F. Rose, 1987). The double line model shows a change in toughness as the crack grows. It approximately represents the R-curve, or the crack growth resistance curve. The numerical values of these R-curves are plotted in Fig. 6. It is shown that each of the curves varies significantly only near the crack tip region and that the effect is short-ranged. However, since the transformations are induced by the crack tip stress and the density is relatively large near the crack tip region, the local influence on the crack propagation may not be totally discounted. Moreover, results in Fig. 5 also indicate that the toughening effect due to the inclusions consisting of single pair of shears depends strongly on their orientations.

For the purpose of comparison, the numerical results corresponding to dilatational inclusions are plotted in Fig. 7. The dilatational inclusions in most of the locations only result in a toughening effect, except in regions in front of the crack tip where they may cause an increase in crack tip stress intensity and enhance the crack propagation. The R-curve always shows a negative limiting value which contributes to the resistance to the crack growth.

The numerical value of K for an inclusion consisting of single pair of shears has been expressed in terms of a non-dimensional constant:

$$C_1 = \frac{\mu b}{(1-\nu)\sqrt{h}} \left(\frac{\Delta x}{h}\right)^2 \quad . \tag{8.1}$$

In the above equation μ is the shear modulus, ν is Poisson's ratio, b is the Burgers vector, h is a distance and Δx is the distance between the two dislocations (or the grain size). For the R-curve calculation, the non-dimensional constant is $C_1\rho$ where ρ is the number of inclusions per h along the longitudinal direction of the line of inclusions. The K value for the case of pairs of dilatations is plotted with the nondimensional constant:

$$C_2 = \frac{\mu b}{(1-\nu)\sqrt{h}} \left(\frac{\Delta x}{h}\right) \tag{8.2}$$

which depends only on the first power of $\Delta x/h$. The numerical value of K due to either a pair of shears or dilatation has the same order of magnitude, but the non-dimensional constants are different by a factor of $\Delta x/h$. It implies that the influence of a pair of shears is less than the influence of dilatation by the ratio $\Delta x/h$. In other words, the ratio of the stress intensities has the same order of magnitude as that of the grain size vs.the distance from the crack tip. We conclude, therefore, that if a grain with a pair of shear transformations is located near the crack tip within several grain sizes, its influence to the K value should not be neglected as compared to that due to a grain with a dilatation transformation.

9. CONCLUSION AND DISCUSSION

The present investigation shows that it is convenient to use the method of continuous distribution of dislocations to calculate the general inclusion-crack interaction problem. Since the method does not restrict the shape of the inclusion, it can be programmed into an efficient numerical routine such that the more general microstructure-crack interaction problems can be handled.

This paper only suggests a method of calculation. The more difficult and fundamental problem concerning the nucleation of transformation twins remains intact. Important problems such as the determination of the transformation boundary and the density of the transformed grains remain unsolved. Recently, a more-than-additive increase in toughness was observed for some composites if the transformation-toughened ceramic was supplemented with whisker reinforcement (Becher and Tiegs, 1987). This problem seems also related to the criteria of transformation by which the transformation boundary changes substantially as a result of the second toughening mechanism.

APPENDIX A

The potentials for the inclusion problem are:

$$\frac{2}{B}\,\phi(z) = -([\beta_{11}] + [\beta_{22}]) \int \frac{dz_1}{z - z_1} - ([\beta_{11}] - [\beta_{22}]) \int \frac{d\bar{z}_1}{z - z_1}$$

$$- i\,([\beta_{12}] - [\beta_{21}]) \int \frac{dz_1}{z - z_1} - i\,([\beta_{12}]$$

$$+ [\beta_{21}]) \int \frac{d\bar{z}_1}{z - z_1} \qquad (A.1)$$

$$\frac{2}{B}\,\psi(z) = ([\beta_{11}] + [\beta_{22}]) \int \frac{d\bar{z}_1}{z - z_1} + ([\beta_{11}] - [\beta_{22}]) \int \frac{dz_1}{z - z_1}$$

$$- i\,([\beta_{12}] - [\beta_{21}]) \int \frac{d\bar{z}_1}{z - z_1} - i\,([\beta_{12}] + [\beta_{21}]) \int \frac{dz_1}{z - z_1}$$

$$- ([\beta_{11}] + [\beta_{22}]) \int \frac{\bar{z}_1\,dz_1}{(z - z_1)^2} - ([\beta_{11}] - [\beta_{22}]) \int \frac{\bar{z}_1\,d\bar{z}_1}{(z - z_1)^2}$$

$$- i\,([\beta_{12}] - [\beta_{21}]) \int \frac{\bar{z}_1\,dz_1}{(z - z_1)^2} - i\,([\beta_{12}]$$

$$+ [\beta_{21}]) \int \frac{\bar{z}_1\,d\bar{z}_1}{(z - z_1)^2}\,. \qquad (A.2)$$

By straight forward application of Eqs. (A.1) and (A.2), the results for inclusion problems of circular and ellipitc inclusions subjected to uniform eigenstrains were confirmed.

APPENDIX B

The elastic strain energy w of the inclusion and the matrix due to plastic deformation of the inclusion with possible debonding is :

$$W = \frac{1}{2} \int_D \sigma_{ij} \, (u_{i,j} - \varepsilon^*_{ij}) \, dv \; . \tag{B.1}$$

The inclusion domain D is the sum of that for the matrix D-Σ and inclusion Σ. The potential energy for the matrix is the equation when expressed in terms of boundary traction:

$$\int_{D-\Sigma} \sigma_{ij} \, u_{i,j} \, dv = \int_{\partial D} \sigma_{ij} \, n_j \, u_i \, ds - \int_{\partial\Sigma} \sigma_{ij} \, n_j \, u_i \, (out) \, ds \; . \tag{B.2}$$

Also, we have the elastic energy for the inclusion,

$$\int_{\Sigma} \sigma_{ij} \, u_{i,j} \, dv = \int_{\partial\Sigma} \sigma_{ij} \, n_j \, u_i \, (in) \, ds \; . \tag{B.3}$$

The sum of the above two equations gives:

$$\int_D \sigma_{ij} \, u_{i,j} \, dv = - \int_{\partial\Sigma} \sigma_{ij} \, n_j \, [u_i] \, ds \tag{B.4}$$

where $\sigma_{ij} \, n_j = 0$ on ∂D and $[u_i] = u_i \, (out) - u_i \, (in)$.

This discontinuous $[u_i]$ is zero for a bounded interface. For the case of a sliding inclusion, the traction force is continuous and has no shear along $\partial\Sigma$. For this case $\sigma_{ij} \, n_j$ has no tangential shear component. The term $\sigma_{ij} \, n_j \, [u_i]$ vanishes along the interface. Therefore, for both cases Eq. (B.1) becomes :

$$W = - \frac{1}{2} \int_{\Sigma} \sigma_{ij} \, \varepsilon^*_{ij} \, dv \; . \tag{B.5}$$

APPENDIX C

From Eq. (4.5), we can prove that:

$$\sigma_{12} = \mathrm{Bi}\ \beta_{12} \left[2 \sum_{i=0}^{N} (-1)^i (\sin 2\theta_i - \sin 2\theta_{i+}) - (\sin 2\theta_{o-} - \sin 2\theta_{o+}) - (\sin 2\theta_{N-} - \sin 2\theta_{N+}) \right] . \quad (C.1)$$

Substituting the above equation into Eq. (6.1), the elastic energy becomes:

$$W = -\frac{1}{2} \sum_{j=1}^{N} (-1)^j \int_{v_j} \sigma_{12}\ \beta_{12}\ dv$$

$$= -\frac{1}{2} \beta_{12} \sum_{j=1}^{N} (-1)^j \int_{v_j} 4\mathrm{Bi}\beta_{12} \sum_{i=0}^{N} (-1)^i \sin 2\theta_{i-}\ dxdy$$

$$- \int_{v_i} 2\ \mathrm{Bi}\ \beta_{12} (\sin 2\theta_{o-} + \sin 2\theta_{N-})\ dxdy$$

$$= 2\ \mathrm{Bi}\ \beta_{12}^2 \sum_{j=1}^{N} \sum_{i=1}^{N} (-1)^{i+j+1} \int_{v_j} \sin 2\theta_{i-}\ dxdy \quad (C.2)$$

where we have made use of the following relation:

$$\sum_{j=1}^{N} (-1)^j \int_{v_j} \sin 2\theta_{o-}\ dx\ dy = \sum_{j=1}^{N} (-1)^j \int_{v_j} \sin 2\theta_{N-}\ dx\ dy . \quad (C.3)$$

the integral in Eq. (C.2) can be further reduced to:

$$\int_{v_j} \sin 2\theta_{i-} dxdy = \frac{1}{2} (x_j - x_i)^2 \log \left[1 + \frac{(2h)^2}{(x_j - x_i)^2}\right] - \frac{1}{2} (x_{j-1} - x_i)^2 \log \left[1 + \frac{(2h)^2}{(x_{j-1} - x_i)^2}\right] + \frac{1}{2} (2h)^2 \log \left[\frac{(x_j - x_i)^2 + (2h)^2}{(x_{j-1} - x_i)^2 + (2h)^2}\right] . \qquad (C.4)$$

APPENDIX D

The dislocation-crack interaction problem is solved in this appendix. For one edge dislocation located at β as shown in Fig. 3, we formulate the pileup integral equation as:

$$A \int_{-\infty}^{o} \frac{f(x') \, dx'}{x - x'} + \sigma_y + i\sigma_{xy} = 0 \qquad \text{(D.1)}$$

where f(x) is the (virtual) distribution function, A is a constant and

$$\sigma_y + i\sigma_{xy} = \frac{\bar{a}}{x - \beta} + \frac{\bar{a}}{x - \bar{\beta}} + \frac{a\,(-\beta + \bar{\beta})}{(x - \beta)^2} \quad . \qquad \text{(D.2)}$$

The solution to the integral equation is:

$$f(x) = -\frac{1}{A\pi^2\sqrt{-x}} \int_{-\infty}^{o} \frac{\sqrt{-x'}\,[\sigma_y(x') + i\sigma_{xy}(x')]\,dx'}{x' - x}$$

$$= -\frac{1}{A\pi\sqrt{-x}} \left[-\bar{a}\left(\frac{\sqrt{\beta}}{x-\beta} + \frac{\sqrt{\bar{\beta}}}{x-\bar{\beta}}\right) + \frac{a}{2}\left(1 - \frac{\bar{\beta}}{\beta}\right) \frac{(x + \beta)\sqrt{\beta}}{(x - \beta)^2} \right] \qquad \text{(D.3)}$$

The complex stress intensity K $[K \underset{x \to o}{=} \sqrt{2\pi}\, A\pi\, \sqrt{-x}\, f(x)]$ is:

$$K = -\frac{\sqrt{2\pi}}{\sqrt{\beta}} \left[\bar{a}\left(1 + \frac{\sqrt{\beta}}{\sqrt{\bar{\beta}}}\right) + \frac{a}{2}\left(1 - \frac{\bar{\beta}}{\beta}\right) \right] . \qquad \text{(D.4)}$$

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FIGURE CAPTIONS

1. Schematic representation of the deformation twinning for an inclusion consisting of N/2 pairs of shear deformations. Each pair has negative shear -S at left and positive shear +S at right. An equivalent description of the inelastic deformation is represented by a distribution of edge dislocations located along the top and bottom boundaries between the inclusion and the surrounding matrix. Points at the ends of the twin boundaries are labeled by 1+, 1-, ..., N+, N-, with the complex coordinates Z_{1+}, Z_{1-}, ..., Z_{N+}, Z_{N-}, respectively.

2. Total energy per unit volume of grain versus the number of pairs of shear deformations. (a) The curves are plotted with an incremental values of surface energies. The surface energy density increment is 0.0025 along the interface of the shears and the grain is square in shape with volume V. (b) The grain is rectangular in shape.

3. An edge dislocation located at β and Burgers vector b relative to a semi-infinite crack which coincides with the negative real axis.

4. Two lines of inclusions located parallel to the crack surface.

5. Toughening by two symmetrically located inclusions each consisting of one pair of shear transformations located at h and 0.8 h from the crack: a) direction of shear $\phi = \frac{\pi}{2}$ b) direction of shear $\phi = \pi$.

6. R-curves generated by two parallel lines of inclusions. Each inclusion has one pair of shear transformations:
   a) direction of shear $\phi = \frac{\pi}{2}$ b) direction of shear $\phi = \pi$

7. a) Toughening by two symmetrically located inclusions of dilatation, located at h and 0.5 h from the crack
   b) The corresponding R-curves.

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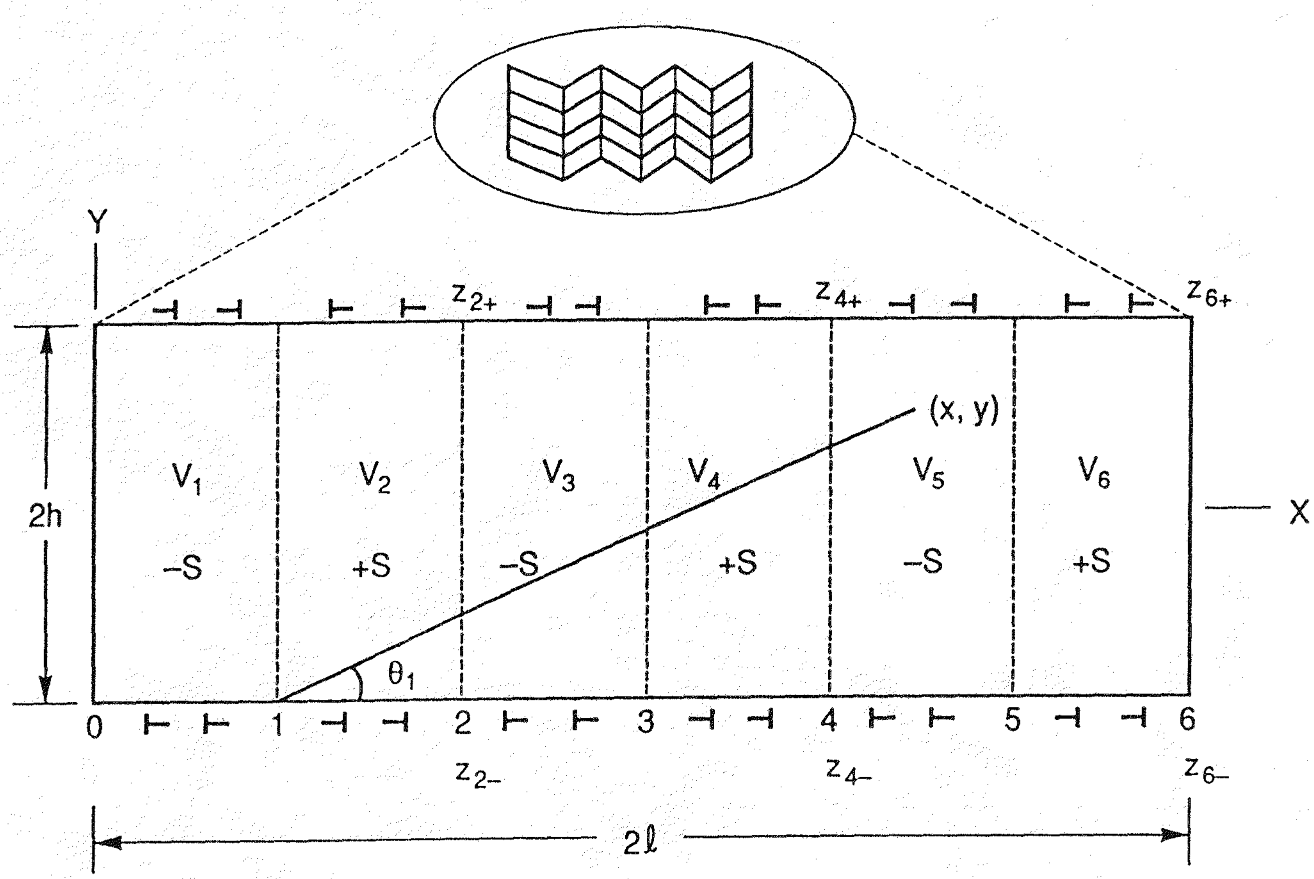

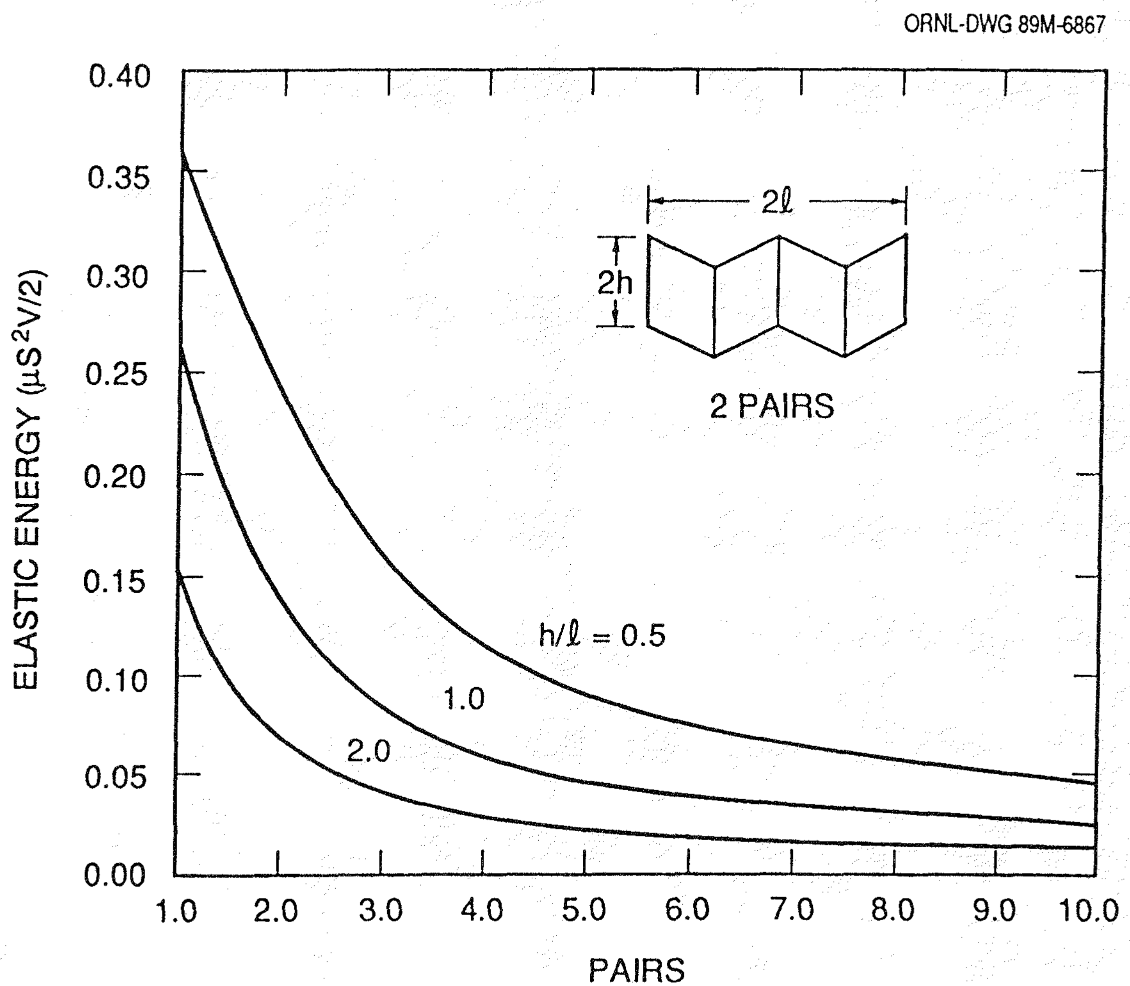
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ELASTIC ENERGY ($\mu S^2V/2$)
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
PAIRS
2ℓ
2h
2 PAIRS
h/ℓ = 0.5
1.0
2.0

2a

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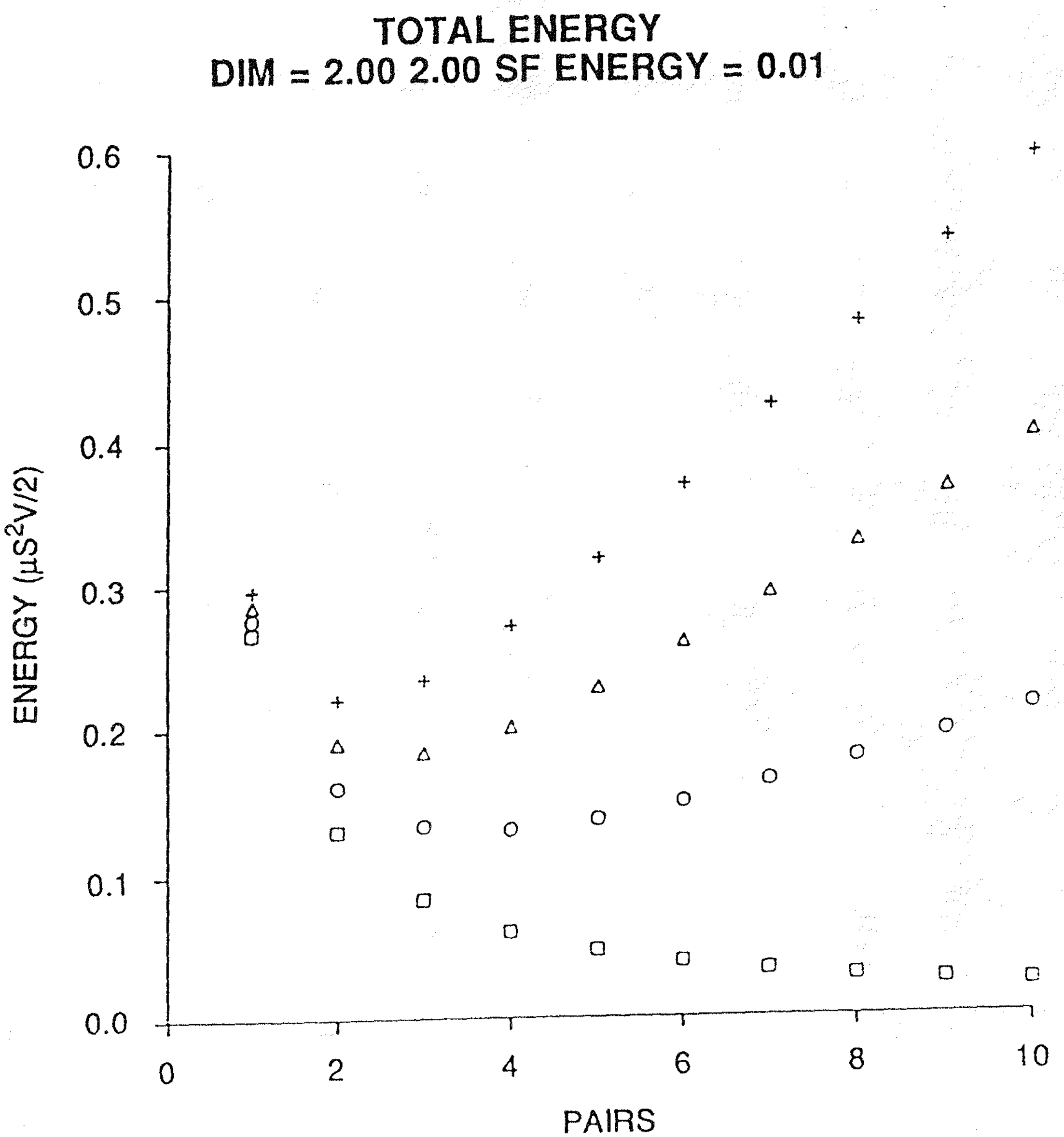
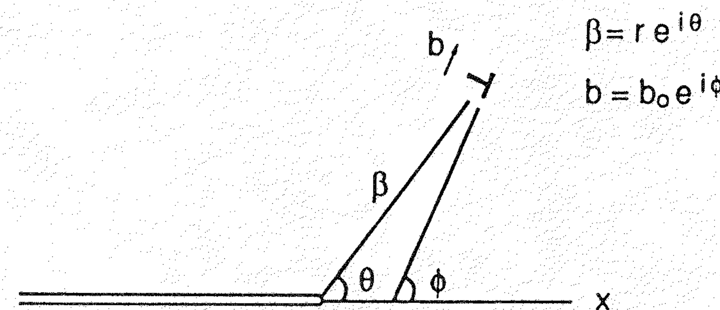


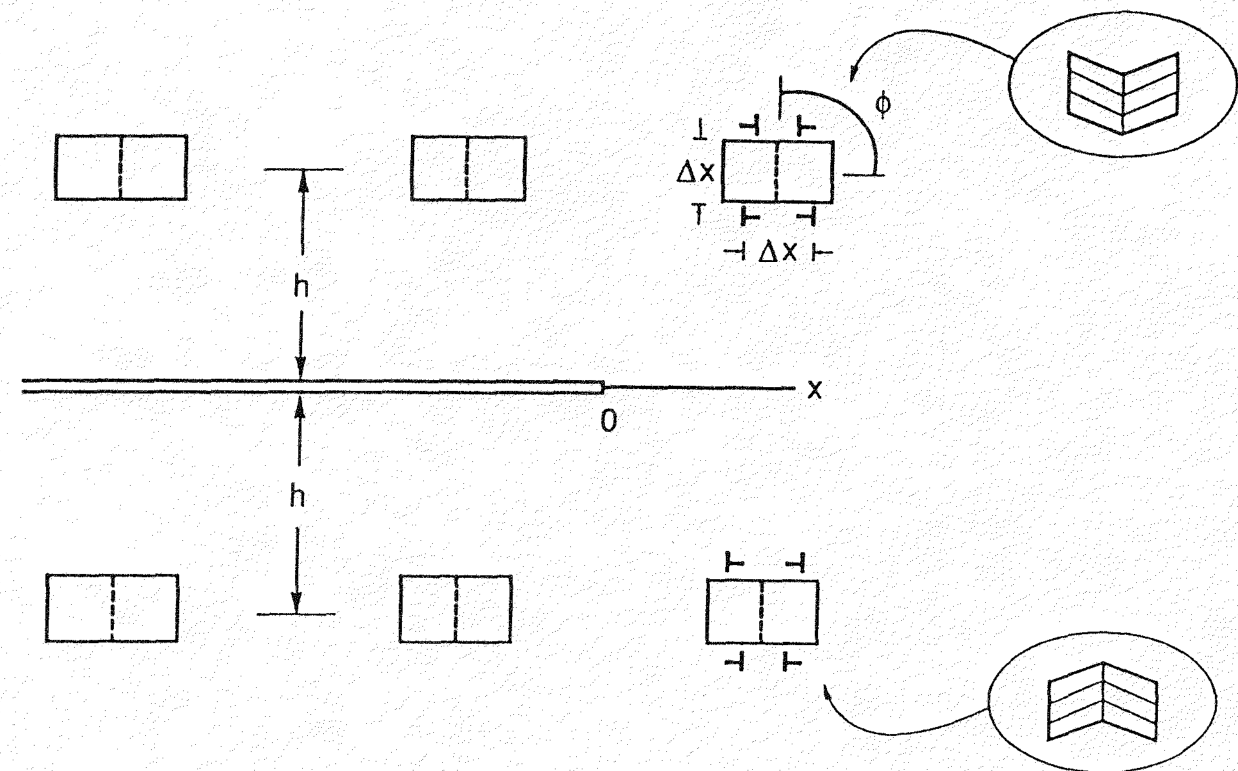
2b

ORNL-DWG 89M-6869



$$K = -\frac{\mu i}{(1-\nu)\sqrt{2\pi r}}\left[e^{-i\phi} - \frac{1}{2} e^{i(\phi-\theta)}\left(1 - e^{-i\theta}\right)\right]\cos\frac{\theta}{2}$$

ORNL-DWG 88M-6868



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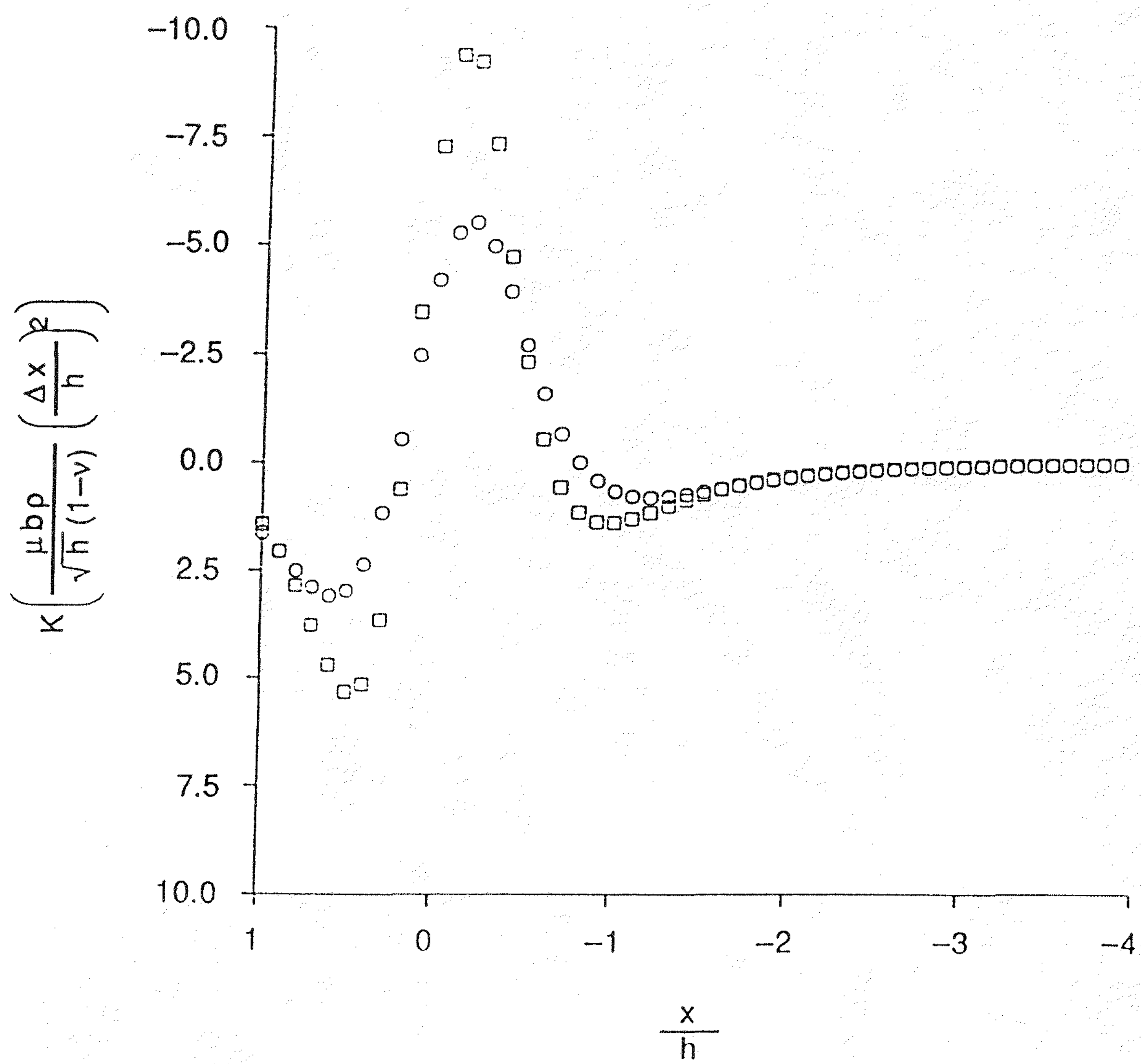
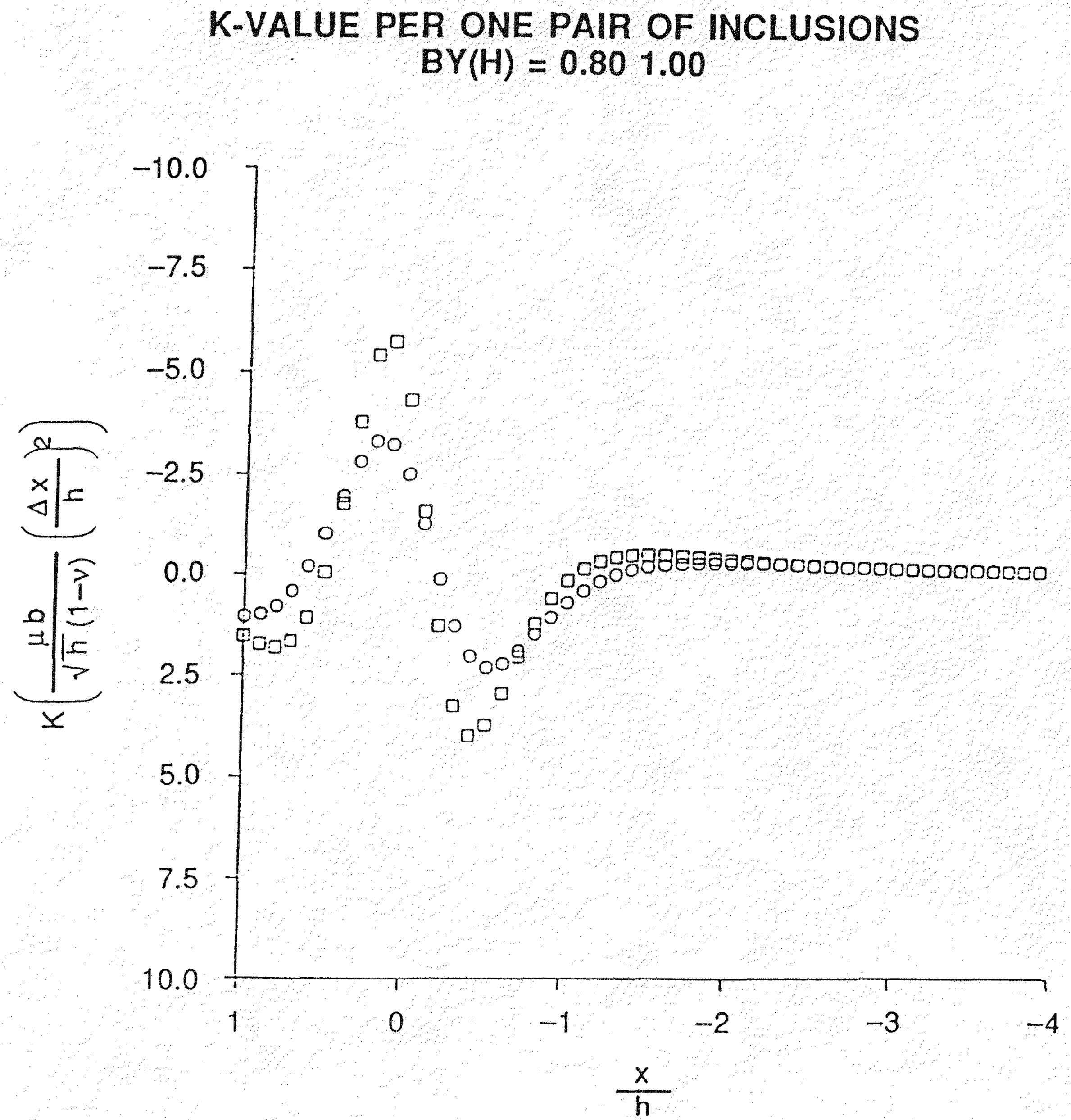
K-VALUE PER ONE PAIR OF INCLUSIONS
BY(H) = 0.50 1.00



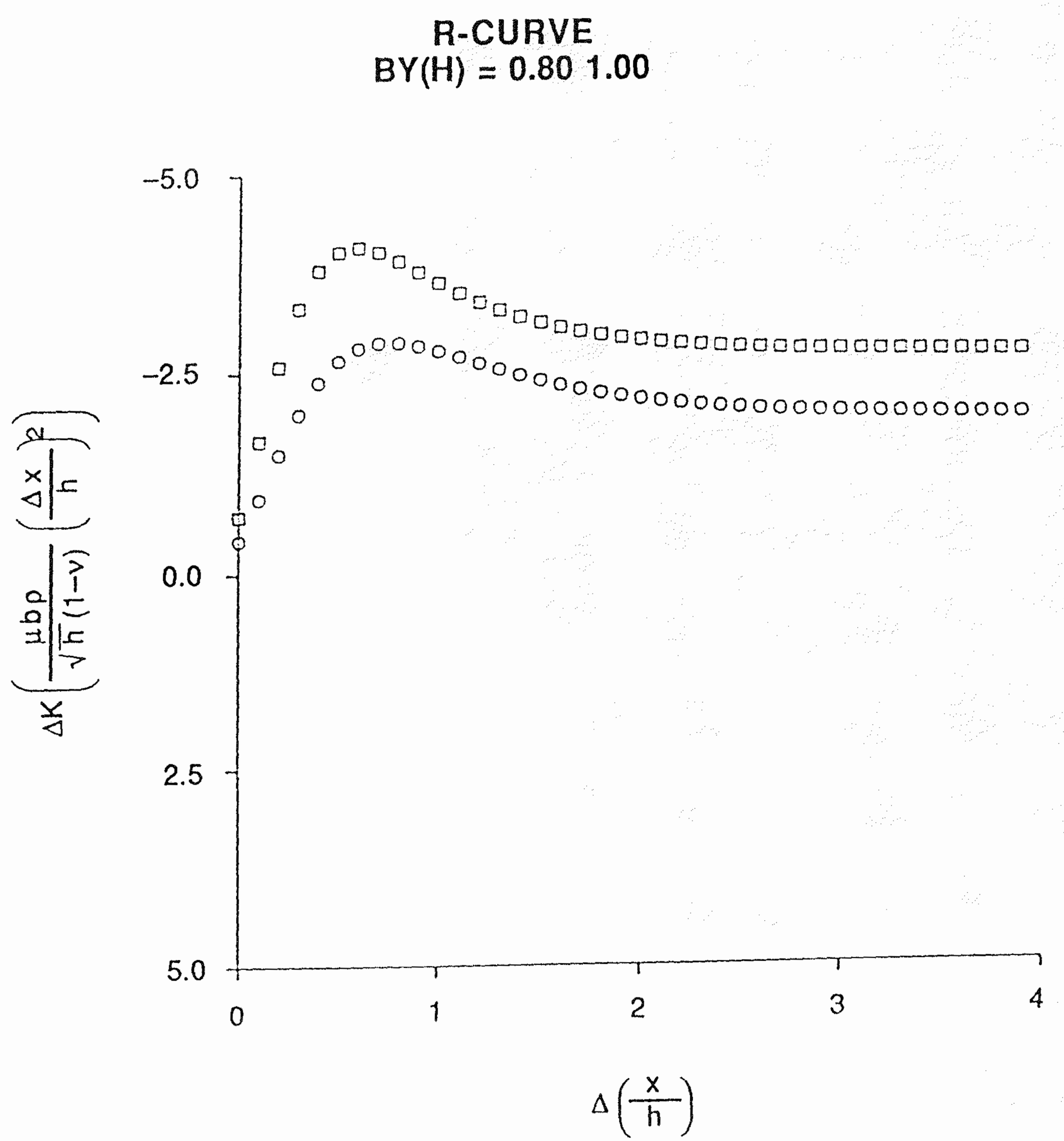
5a

ORNL-DWG 89M-7053



5b

ORNL-DWG 89M-7056



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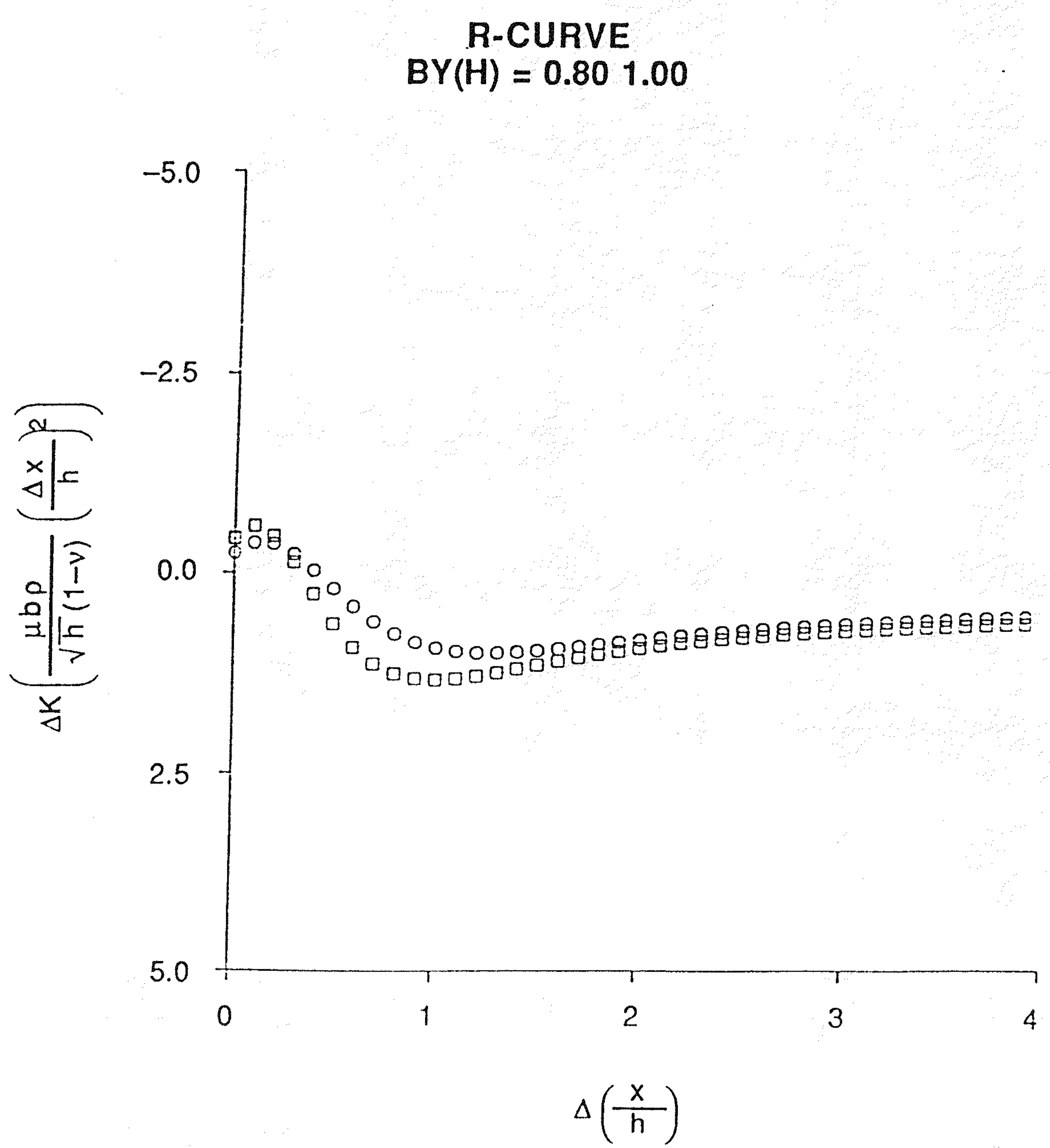

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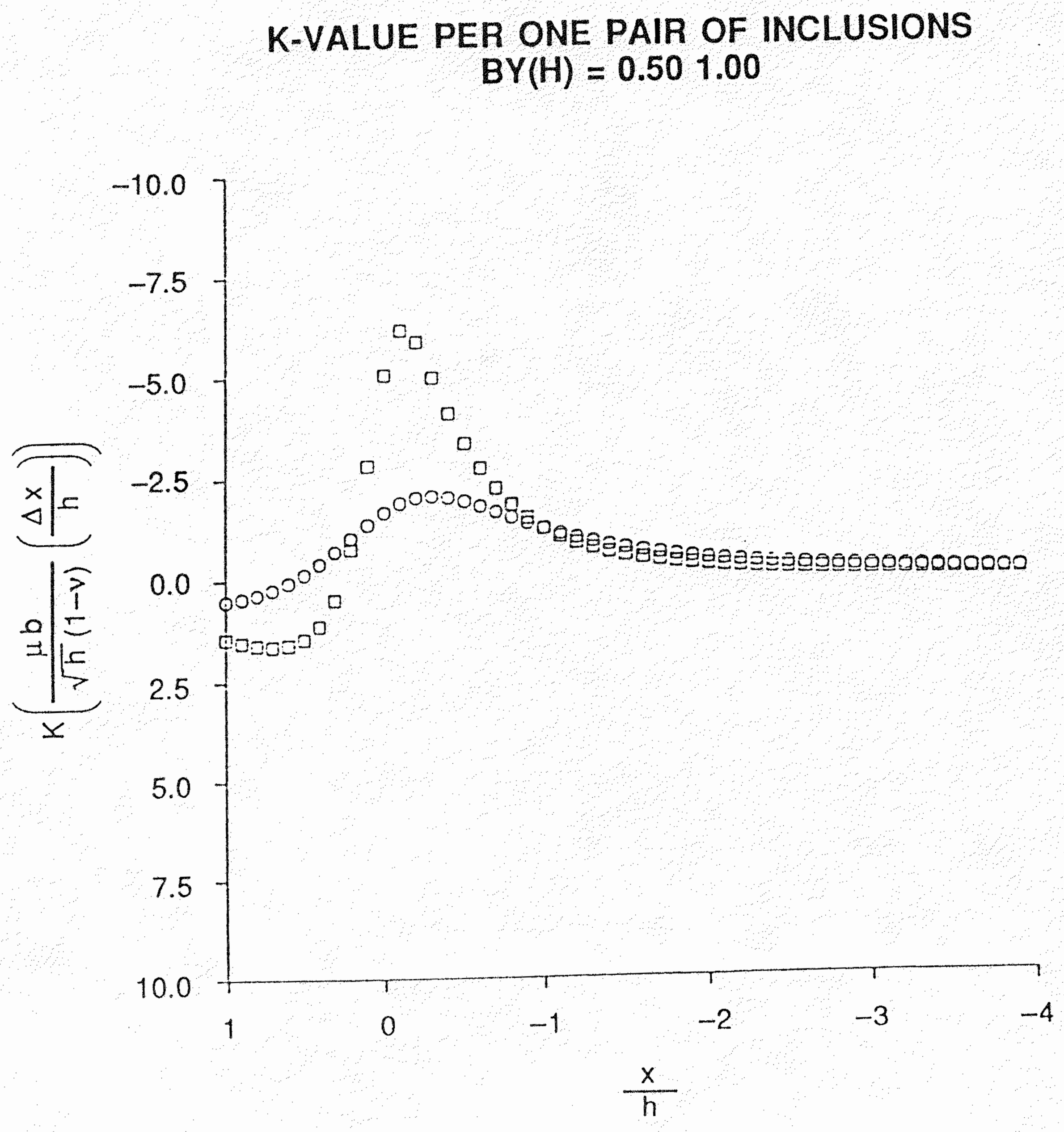

ORNL-DWG 89M-7051

R-CURVE
BY(H) = 0.50 1.00