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THE ELASTIC PROPERTIES OF WOVEN POLYMERIC FABRIC*

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ABSTRACT

The in-plane linear elastic constants of woven fabric are determined in terms of the specific fabric microstructure. The fabric is assumed to be a spatially periodic interlaced network of orthogonal yarns and the individual yarns are modeled as extensible elastica. These results indicate that a significant coupling of bending and stretching effects occurs during deformation. Results of this theoretical analysis compare favorably with measured in-plane elastic constants for Vincel yarn fabrics.

INTRODUCTION

Woven fabrics represent a class of polymeric materials which provide a number of beneficial mechanical properties, the most important of which include strength, flexibility, and relatively low density. Early weaving processes exploiting these properties developed empirically using yarns spun from natural fibers. The development of synthetic polymeric fibers with their wide

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diversity of mechanical, physical and chemical properties offered a corresponding wide diversity of woven fabric properties together with the possibility of engineering fabrics for specific applications. Woven polymeric fabrics, for example, are often used as the filler in building up individual laminates of a composite material. The desire to engineer or design fabrics for specific applications has stimulated considerable interest in theoretical analysis relating their effective mechanical properties to specific aspects of the woven fabric morphology and microstructure.

The first comprehensive investigation of the relationships between various parameters of a woven fabric was apparently the work of Peirce [1]. The Peirce model is strictly a geometrical or kinematical model describing the geometry of woven fabric and no consideration of forces or equilibrium is given. A more physical model utilizing an analysis of the inextensible elastica was developed by Olofsson [2] and Grosberg and Kedia [3]. An excellent summary of the analysis of the mechanical properties of woven fabrics prior to 1969 is included in the monograph by Hearle et al.[4]. More recent summaries have been presented by Ellis [5] and Treloar [6].

The analysis of fabric structure and properties has followed primarily one of two paths: the descriptive geometrical based principally on the work of Peirce [1], or the mechanistic [7]. A number of the mechanistic approaches are based on energy methods [7, 8, 9] in which the strain energy due to yarn stretching is uncoupled from the strain energy due to yarn bending (crimp interchange effect). The bending energy is usually obtained from an analysis of the inextensible elastica and represented in terms of elliptic integrals which are cumbersome to interpret because of the reversal of independent and dependent variables inherent in this approach. In an effort to circumvent these complications, Leaf and Kandil [10] consider a saw-tooth model of plain-woven fabric, which also uncouples the yarn stretching energy from bending energy, and they obtain simple expressions for the effective linear elastic constants in the principle yarn directions in terms of the fabric microstructure. While these results are easily interpreted, they provide estimates of the elastic constants considerably higher than they measure experimentally. An energy method incorporating the Peirce geometry has been used by Hamed and Sadek [11] who minimize the energy using a pattern search computational program and obtain numerical results for uniaxial loading.

In this work we assume the woven fabric consists of a regular network of orthogonal interlaced yarns. We model the individual yarns as extensible elastica and thus couple stretching and bending effects at the outset. Consideration is restricted to biaxial loading in the principal yarn directions in the plane of the fabric; the case of more general loading in the plane, as suggested by the trellis model of Kilby [12], will be the subject of future analysis. The initial unloaded yarn geometry is assumed to be a sequence of alternating circular arcs of constant radius R as considered by Olofsson [13].

We first obtain the solution to the differential equation describing the non-linear deformation of an extensible elastica under the action of a normal contact force $2V$ and a mid-point force T_o. Our interest here is on the effective linear elastic constants, and for small T_o we obtain a linear displacement-force relation for each yarn which depends upon the elastic properties and initial weave geometry of that yarn. The mechanical response of the woven fabric is obtained from the interaction of two of these solutions corresponding to each yarn by enforcing equilibrium and compatibility of displacements. This provides the in-plane linear strain-stress relations for the woven fabric in the two principal yarn directions. These results indicate that a significant coupling of bending and stretching effects occurs during deformation.

The theoretical analysis has been used to estimate the in-plane Young's moduli for one group of fabrics woven from a polymeric Vincel yarn as described by Leaf and Kandil [10]. These theoretical estimates compare favorably with the experimentally measured moduli of [10].

THE EXTENSIBLE ELASTICA

The first step in obtaining expressions for the effective elastic constants for the woven fabric requires an analysis of the force-displacement relation of the individual yarns. Each yarn will be modeled as an extensible elastica, and the geometry of this model is shown in Figure 1. The extensible elastica has been considered in detail by Antman [14] and Tadjbakhsh [15], and in this analysis we make use of the intrinsic coordinates as presented for the inextensible case by Mitchell [16]. The elastica is assumed to deform in the (x, z) plane as shown, and the initial shape is taken to be an arc of a circle of radius R. The undeformed shape of the elastica is defined by the slope

$$\phi = \phi(S_o) = \frac{S_o}{R} , \quad 0 \leq S_o \leq L_o \tag{1}$$

where S_o is arc length along the undeformed curve of total length L_o, and the deformed shape is defined by the slope

$$\psi = \psi(S) , \quad 0 \leq S \leq L \tag{2}$$

with S the arc length along the deformed curve having total length L. The elastica is assumed to stretch linearly under the effect of the axial force $T(S)$ acting through the centroid of the cross-section of area A which provides the relation

$$\frac{dS}{dS_o} = 1 + \frac{T(S)}{EA} \tag{3}$$

where E is the Young's modulus of the elastica. The shear force Q(S) and axial force T(S) as shown in Figure 1c are given by

$$\begin{aligned} Q(S) &= V \text{ Cos } \psi - F \text{ Sin } \psi \\ T(S) &= F \text{ Cos } \psi + V \text{ Sin } \psi \end{aligned} \tag{4}$$

where V and F are the forces at the symmetry point S = 0. In terms of the force T_o applied at the end S = L we find

$$V = T_o \text{ Sin } \alpha \text{ , } F = T_o \text{ Cos } \alpha \tag{5}$$

which, with Equation (4_2), provides the important relation

$$T(S) = T_o \text{ Cos } (\psi - \alpha). \tag{6}$$

The end of the elastica at S = L is assumed to be an inflection point of the yarn, that is, a point of anti-symmetry of the yarn, and at this point the moment must vanish so we have M(L) = 0. The differential equation which describes the non-linear bending of the extensible elastica under the conditions just described is

$$EI \frac{d}{dS} \left\{ (1 + \frac{T}{EA}) \frac{d\psi}{dS} - \frac{d\phi}{dS_o} \right\} = - \frac{dT}{d\psi} \tag{7}$$

and this equation is to be solved subject to the boundary conditions

$$\psi(0) = 0$$

$$\left[\left(1 + \frac{T}{EA}\right) \frac{d\psi}{dS} \right]_{S=L} = \left. \frac{d\phi}{dS_o} \right]_{S_o=L_o} \tag{8}$$

The second condition insures that the moment at the end $S = L$ vanishes. Since the initial shape is taken to be an arc of a circle of radius R, we have

$$\frac{d\phi}{dS_o} = \frac{1}{R}, \quad 0 \leq S_o \leq L_o. \tag{9}$$

A first integral of Equation (7) may be obtained, and with the boundary condition (8_2) leads to

$$R \left[1 + \kappa R^2 \gamma \operatorname{Cos} (\psi - \alpha) \right] \frac{d\psi}{dS} = \Big\{ 1 + 2\kappa R^2 \left[\operatorname{Cos} (\psi_o - \alpha) - \operatorname{Cos} (\psi - \alpha) \right] + \kappa^2 R^4 \gamma \left[\operatorname{Cos}^2 (\psi_o - \alpha) - \operatorname{Cos}^2 (\psi - \alpha) \right] \Big\}^{1/2} \tag{10}$$

where we have made use of Equation (6) and the two constants

$$\kappa = \frac{T_o}{EI}, \quad \gamma = \frac{I}{AR^2}. \tag{11}$$

The constant γ represents a measure of the relative effects of bending and stretching in the deformation of the elastica, and the case $\gamma = 0$ represents the inextensible elastica. Equation (3) provides the relation between dS and dS_o of

$$dS = \left[1 + \kappa R^2 \gamma \operatorname{Cos} (\psi - \alpha) \right] dS_o \tag{12}$$

which is important in evaluating the deformed slope ψ_o at the end. The arc length S may be obtained from Equation (10) as an elliptic integral of Weirstrass' form [14] but this result is not particularly useful for the present application.

We now restrict interest to deformations due to small forces and assume $\kappa R^2 << 1$. To first order terms in κR^2, Equations (10) and (12) become

$$\frac{dS}{R} = \left\{ 1 - \kappa R^2 \left[\text{Cos } (\psi_o - \alpha) - (1 + \gamma) \text{ Cos } (\psi - \alpha) \right] \right\} d\psi \tag{13}$$

$$\frac{dS_o}{R} = \left\{ 1 - \kappa R^2 \left[\text{Cos } (\psi_o - \alpha) - \text{Cos } (\psi - \alpha) \right] \right\} d\psi \tag{14}$$

Making use of the boundary condition (8_1) and the fact that the integral over dS_o is equal to the original length $L_o = R\phi_o$, Equation (14) provides

$$\phi_o = \psi_o + \kappa R^2 \left[\text{Sin } (\psi_o - \alpha) + \text{Sin } \alpha - \psi_o \text{ Cos } (\psi_o - \alpha) \right] , \tag{15}$$

where ψ_o is the slope of the deformed curve at the end $S = L$ and is determined by this equation. Consistent with our first order theory, we now let

$$\psi_o = \phi_o - \theta \text{ , } \theta << 1 \text{ ,} \tag{16}$$

and Equation (15) gives

$$\theta = \kappa R^2 \left[\text{Sin } (\phi_o - \alpha) + \text{Sin } \alpha - \phi_o \text{ Cos } (\phi_o - \alpha) \right]. \tag{17}$$

Using the relations (Figure 1)

$$dx = \cos \psi dS \; , \; dz = \sin \psi dS$$

and Equation (13) together with the conditions $x = 0$, $z = 0$ at $S = 0$, integration provides

$$x_o = R \sin \psi_o + \frac{1}{4} \kappa R^3 \left[2(1 + \gamma) \psi_o \cos \alpha - (1 - \gamma) \sin \alpha - (1 - \gamma) \sin (2\psi_o - \alpha) \right]$$

$$z_o = R(1 - \cos \psi_o) + \frac{1}{4} \kappa R^3 \left[2(1 + \gamma) \psi_o \sin \alpha - 4 \cos (\psi_o - \alpha) + (1 - \gamma) \cos (2\psi_o - \alpha) + (3 + \gamma) \cos \alpha \right] \tag{18}$$

where (x_o, z_o) is the position of the endpoint $S = L$ of the deformed elastica. The displacements u_x, u_z are given by

$$u_x = x_o - R \sin \phi_o$$

$$u_z = z_o - R(1 - \cos \phi_o) \tag{19}$$

which, with (18) and (17) gives the displacements in the form

$$u_x = \frac{1}{4} \kappa R^3 (A \cos \alpha - B \sin \alpha)$$

$$u_z = \frac{1}{4} \kappa R^3 (-B \cos \alpha + C \sin \alpha) \tag{20}$$

where

$$A = 2(2 + \gamma)\,\phi_o + 2\phi_o \operatorname{Cos} 2\phi_o - (3 - \gamma) \operatorname{Sin} 2\phi_o$$

$$B = 4 \operatorname{Cos} \phi_o - (1 + \gamma) - 2\phi_o \operatorname{Sin} 2\phi_o - (3 - \gamma) \operatorname{Cos} 2\phi_o \tag{21}$$

$$C = 2(2 + \gamma)\,\phi_o - 2\phi_o \operatorname{Cos} 2\phi_o + (3 - \gamma) \operatorname{Sin} 2\phi_o - 8 \operatorname{Sin} \phi_o$$

The result (20) with (5) takes the form

$$u_x = \frac{R^3}{4EI}(AF - BV)$$

$$u_z = \frac{R^3}{4EI}(-BF + CV) \tag{22}$$

and we will use this result in the next section to obtain the linear elastic response of a woven fabric. We note that this result is consistent with the known existence of a Gibbs free energy $\mathcal{G}$ given by

$$\mathcal{G} = \frac{R^3}{8EI}(AF^2 - 2BFV + CV^2) \tag{23}$$

such that

$$u_x = \frac{\partial \mathcal{G}}{\partial F}, \quad u_z = \frac{\partial \mathcal{G}}{\partial V} \tag{24}$$

which follows directly from Castigliano's first theorem [17].

FABRIC ELASTIC CONSTANTS

The geometry of the woven fabric under consideration here is shown in Figure 2. With reference to Figure 2b, the usual geometrical weave parameters of pick spacing p, yarn length ℓ and crimp height h are represented in terms of the elastica parameters R and ϕ_o by

$$
\begin{aligned}
p &= 2R \sin \phi_o \\
\ell &= 2R\phi_o \\
h &= 2R\,(1 - \cos \phi_o).
\end{aligned}
\tag{25}
$$

To fix ideas we denote the x-direction as the warp direction and the warp yarns have mechanical properties and an initial geometry which we identify with a subscript x. Similarly the y-direction is the weft direction and the weft yarns have mechanical properties and an initial geometry which we will identify with a subscript y.

Under the action of applied forces f_x and f_y acting in the (x, y) plane, the woven fabric will deform with each yarn behaving like the extensible elastica analysed in the previous section. Equilibrium requires the transverse contact force V to be the same for both warp and weft yarns, and geometric compatibility of the displacements requires the displacements in the z or transverse directions to be the same. From Equations (22) for the warp or x-direction yarn

$$
\begin{aligned}
u_x &= \frac{R_x^{\,3}}{4E_x I_x}\,(A_x f_x - B_x V) \\
u_{zx} &= \frac{R_x^{\,3}}{4E_x I_x}\,(-B_x f_x + C_x V)
\end{aligned}
\tag{26}
$$

and for the weft or y-direction yarn

$$u_y = \frac{R_y^3}{4E_y I_y} (A_y f_y - B_y V)$$

$$u_{zy} = - \frac{R_y^3}{4E_y I_y} (-B_y V + C_y V) \tag{27}$$

with

$$u_{zx} = u_{zy} \; . \tag{28}$$

The compatibility condition (28) determines the contact force V as

$$V = \frac{L B_x f_x + B_y f_y}{(L C_x + C_y)} \tag{29}$$

where

$$L = \left(\frac{E_y I_y}{E_x I_x} \right) \left(\frac{R_x}{R_y} \right)^3 \tag{30}$$

and the in-plane displacements u_x, u_y are

$$u_x = \frac{R_x^3}{4E_x I_x (L C_x + C_y)} \left\{ \left[L(A_x C_x - B_x^2) + A_x C_y \right] f_x - B_x B_y f_y \right\}$$

$$u_y = \frac{R_y^3}{4E_y I_y (L C_x + C_y)} \left\{ \left[L A_y C_x + (A_y C_y - B_y^2) \right] f_y - L B_x B_y f_x \right\} \tag{31}$$

The results of Equation (31) represent the displacement force relations for this woven fabric.

The effective in-plane strain-stress relations for this woven fabric may be readily obtained from the following relations established from consideration of the weave geometry shown in Figure 2 and Equations (25):

$$
\begin{aligned}
u_x &= \varepsilon_{xx} R_x \operatorname{Sin} \phi_{ox} \\
u_y &= \varepsilon_{yy} R_y \operatorname{Sin} \phi_{oy} \\
f_x &= 2\sigma_{xx} R_y \operatorname{Sin} \phi_{oy} \\
f_y &= 2\sigma_{yy} R_x \operatorname{Sin} \phi_{ox} \quad .
\end{aligned} \tag{32}
$$

In Equation (32), ε_{xx} and ε_{yy} represent the material strains, and σ_{xx} and σ_{yy} the effective in-plane stresses with units of force per unit length. Using the relations of (32) in Equation (31) gives the strain-stress relations

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{R_x^{\,3}}{2E_x \mathcal{I}_x \, (\mathcal{L}C_x + C_y)} \left\{ \left[\mathcal{L}(A_x C_x - B_x^{\,2}) + A_x C_y \right] \left(\frac{R_y \operatorname{Sin} \phi_{oy}}{R_x \operatorname{Sin} \phi_{ox}} \right) \sigma_{xx} - B_x B_y \sigma_{yy} \right\} \\
\varepsilon_{yy} &= \frac{R_y^{\,3}}{2E_y \mathcal{I}_y \, (\mathcal{L}C_x + C_y)} \left\{ \left[\mathcal{L}A_y C_x + (A_y C_y - B_y^{\,2}) \right] \left(\frac{R_x \operatorname{Sin} \phi_{ox}}{R_y \operatorname{Sin} \phi_{oy}} \right) \sigma_{yy} - \mathcal{L}B_x B_y \sigma_{xx} \right\} .
\end{aligned} \tag{33}
$$

The effective Young's modulus $\hat{E}_x$, $\hat{E}_y$ in the x and y directions, respectively, are given by

$$
\begin{aligned}
\hat{E}_x &= \frac{2E_x \mathcal{I}_x \, (\mathcal{L}C_x + C_y) \operatorname{Sin} \phi_{ox}}{R_x^{\,2} R_y \left[\mathcal{L}(A_x C_x - B_x^{\,2}) + A_x C_y \right] \operatorname{Sin} \phi_{oy}} \\
\hat{E}_y &= \frac{2E_y \mathcal{I}_y \, (\mathcal{L}C_x + C_y) \operatorname{Sin} \phi_{oy}}{R_y^{\,2} R_x \left[\mathcal{L}A_y C_x + (A_y C_y - B_y^{\,2}) \right] \operatorname{Sin} \phi_{ox}} \quad .
\end{aligned} \tag{34}
$$

We note that since the terms A_x, A_y, B_x, ... all depend on the parameter γ_x or γ_y, the result (34) indicates that a significant coupling of bending and stretching effects occurs during fabric deformation even in the linear elastic regime.

RESULTS AND CONCLUSIONS

The effective fabric Young's moduli as given by Equation (34) have been used to estimate the moduli of a group of fabrics woven from a polymeric Vincel yarn as described by Leaf and Kandil [10]. The fabric group, group A of [10], was woven with identical warp and weft yarns of R60/2-tex Vincel with varying pick spacing in the warp and weft directions. A comparison of the theoretical moduli estimates with the experimentally determined moduli as reported in [10] for the three different fabrics of group A is shown in Table I. While the theoretical moduli are somewhat lower than the measured values, the lower theoretical values always correspond to the lower experimental values. This is particularly significant when we note that for fabrics A-1, A-2, the pick spacing in the x-direction is larger than in the y-direction while the reverse is true for A-3. Yet for all three weaves, the smaller modulus is $\hat{E}_x$ as predicted by the theory and verified experiementally.

In view of uncertainties as to the Young's modulus of the Vincel yarn itself and the degree of yarn flatening which directly effects the I, A, and γ of each yarn, comparison between the results of the theory developed here and experiment as shown in Table I appears to be quite good. Additional experimental comparisons with kevlar woven fabrics are underway. This linear theory based on the extensible elastica, which effectively couples yarn bending and stretching effects, readily extends to large non-linear deformations so long as the yarn force-displacement relation remains linear. This extension to non-linear deformations is thus particularly relevant to the large class of fabrics woven from kevlar yarns since these yarns exhibit an essentially linear force-displacement relation up to fracture.

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FIGURE CAPTIONS

Figure 1. The Extensible Elastica:

(a) Schematic of woven yarn, (b) The intrinsic coordinates and applied forces, (c) Forces acting on an element.

Figure 2. Geometry of the woven fabric:

(a) Schematic of woven yarn interlace, (b) Cross-section of weave in either X (warp) or y (weft) direction.

TABLE I

Comparison of Theory and Experiment for
Young's Moduli of Woven Vincel Fabric

Fabric Number	$\hat{E}_x$ (N/cm) Measured [10]	$\hat{E}_x$ (N/cm) Theoretical	$\hat{E}_y$ (N/cm) Measured [10]	$\hat{E}_y$ (N/cm) Theoretical
A-1	9.2	8.8	25.5	13.7
A-2	9.1	8.8	14.8	8.9
A-3	12.7	9.0	13.4	10.3

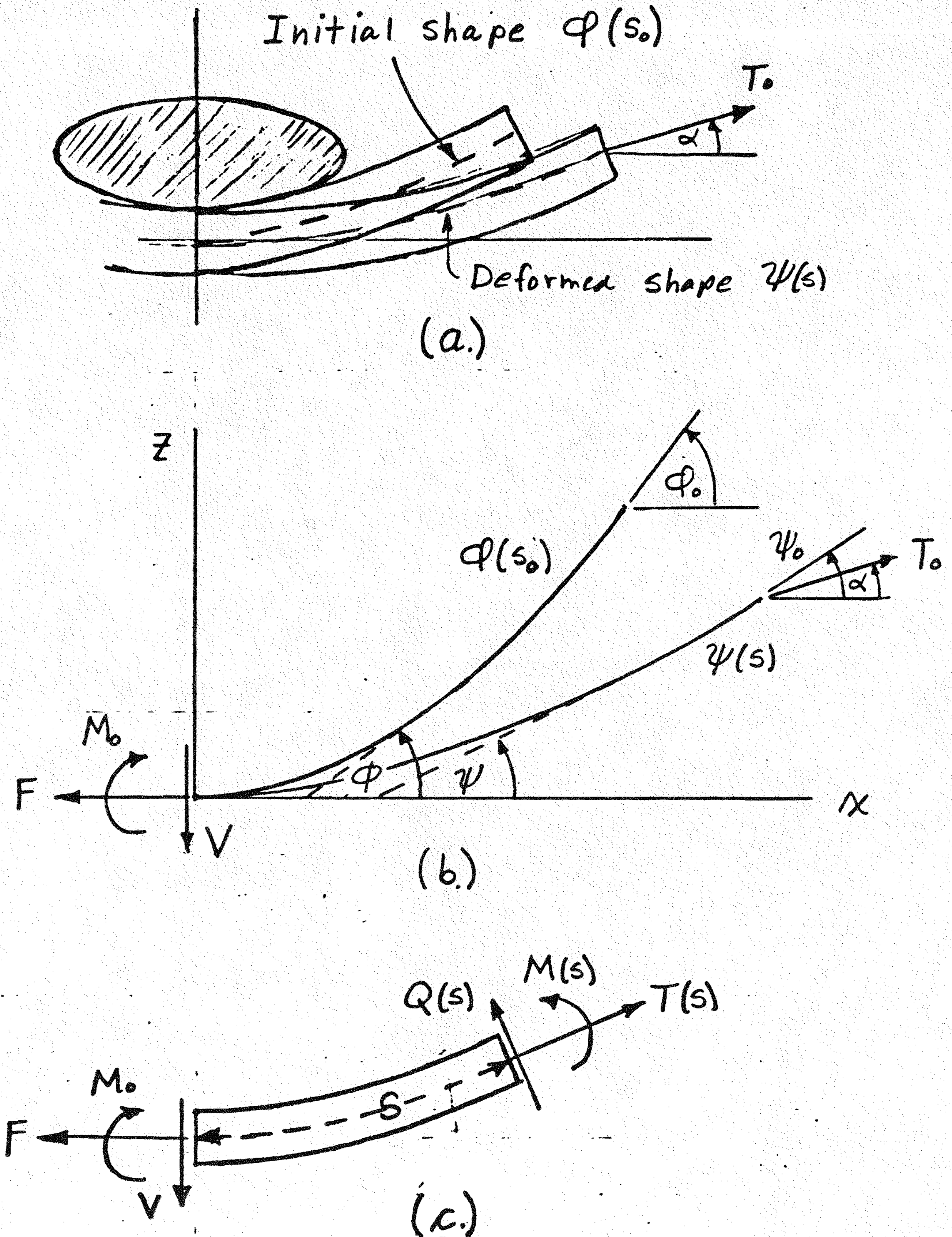
Initial shape φ(s₀)
T₀
α
Deformed shape ψ(s)
(a.)
z
φ₀
φ(s₀)
ψ₀
T₀
α
ψ(s)
M₀
F
φ
ψ
x
V
(b.)
Q(s)
M(s)
T(s)
M₀
S
F
V
(c.)

Fig. 1

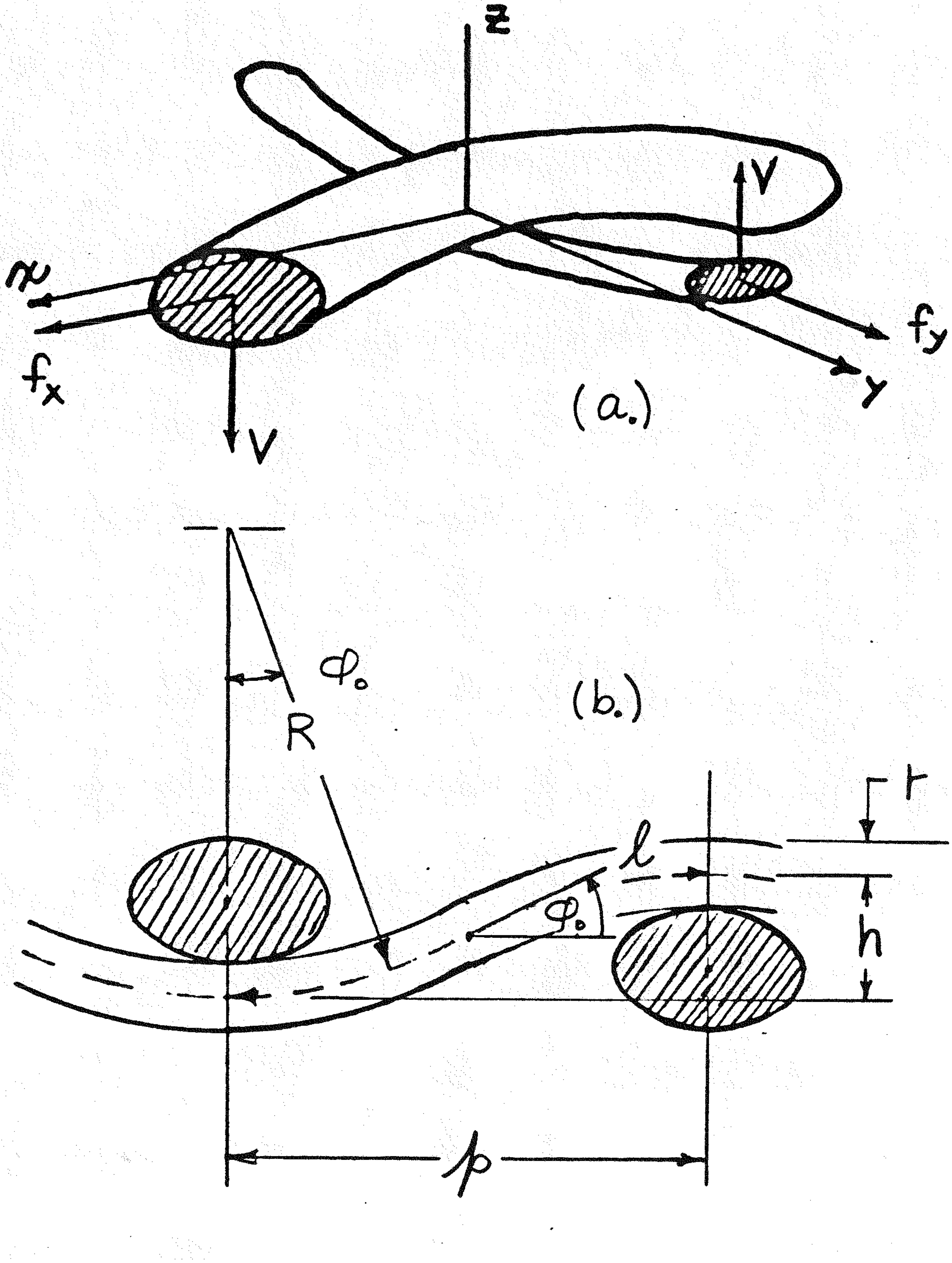

Fig. 2