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Abstract: Exponentially decaying "Gamow state" vectors are obtained from S-matrix poles in the lower half of the second sheet and are defined as functionals over a subspace of the nuclear space Φ. Exponentially growing "Gamow state" vectors are obtained from S-matrix poles in the upper half of the second sheet and are defined as functionals over another subspace of Φ. On functionals over these two subspaces the dynamical group of time development splits into two semigroups.

Von Neumann's Hilbert space (H.S.) formulation of quantum mechanics[1] assumes the one-to-one correspondence between (pure) physical states and elements (rays) of the Hilbert space, H, and between physical observables and linear (Hermitian) operators of H. This is a mathematical utopianization which cannot be justified by physical arguments, since up to an arbitrary small error of the measurement one can always approximate any element of H by an element of a dense linear subspace. All that is needed to describe physical states in quantum mechanics is a linear scalar-product space; how one completes it is a matter of convenience. Guided by the Dirac formalism[2] of quantum mechanics it has been suggested[3] to use instead of the Hilbert space (τ_H) completion a completion with respect to a stronger nuclear topology τ_Φ resulting in the rigged Hilbert space[4] (RHS) formulation of quantum mechanics. Though observationally indistinguishable from the HS formulation, this RHS formulation of quantum mechanics is mathematically enormously simpler and more practical; casually speaking, it makes the Dirac formalism mathematically rigorous. The RHS (Gelfand triplet) is the trinity of spaces

$$\Phi \subset H = H^{\times} \subset \Phi^{\times}$$

where $H^{\times}$ and $\Phi^{\times}$ denote the space of continuous antilinear functionals $F(\phi) = \langle\phi|F\rangle = \langle F|\phi\rangle^{*}$, $\phi \in \Phi$.

The bound states and other "physically preparable states" are in Φ and the scattering "states" or "Dirac states" (kets) are in $\Phi^{\times}$. If the topology in Φ is chosen sufficiently strong, then $\Phi^{\times}$ will not only contain the generalized eigenvectors of an essentially self-adjoint operator H, defined by

$$\langle H\phi|\omega\rangle = \langle\phi|H^{\times}|\omega\rangle = \omega\langle\phi|\omega\rangle, \quad \phi \in \Phi \quad |\omega\rangle \in \Phi^{\times} \tag{2}$$

which have a real eigenvalue $\omega = E$ (the "Dirac states"), but also the "Gamow states"[5], the generalized eigenvectors with complex eigenvalue $\omega = E - \frac{i\Gamma}{2}$ and $\omega^{*} = E + \frac{i\Gamma}{2}$, describing exponentially decaying and exponentially growing "states." However in distinction to the Dirac states $|E\rangle$ which are defined as functionals on the space of all physical states $\phi \in \Phi$, the decaying "Gamow states"

$|E - \frac{i\Gamma}{2}\rangle$ are defined as functionals on only half of the space of physical states $\phi \in \Phi \cap H_+$ and the growing Gamow states $|E + \frac{i\Gamma}{2}\rangle$ are defined as functionals on the other half $\Phi \cap H_-$. Here H_+ (H_-) is a space of vectors with an energy distribution $\langle E|\phi\rangle$ that is a Hardy-class function with respect to the upper (lower) complex energy plane.

Decaying Gamow states have been introduced from different points of view: (1) As eigenfunctionals of H with a complex generalized eigenvalue which is a continuation of a discrete eigenvalue in the continuous spectrum of the unperturbed Hamiltonian H - V.[6] (2) As generalized eigenvectors with a Breit-Wigner energy distribution,[7] that lead in a simple straightforward way to the decay rate formula. Since for many people the most satisfying definition of a resonance is a pole of the analytically continued S-matrix,[8] we will obtain the pair of decaying and growing Gamow states here from a pair of poles in the second sheet of the S-matrix immediately below and above the positive real energy axis, respectively.

We start with the S-matrix element which at any time t is written as:

$$(\psi^{out}(t), S\psi^{in}(t)) = (\Omega^- \psi^{out}(t), \Omega^+ \phi^{in}(t)) =$$
$$(\psi^-(t), \phi^+(t)) = \int_{\overline{Sp\,H}} dE' \langle\psi^-|E'^-\rangle S(E') \langle^+E'|\phi^+\rangle \qquad (3)$$

The notation here is standard:[7,9] $\phi^+(t)$ represents the state that develops from the prepared in-state ϕ^{in}; $\psi^-(t)$ represents the state that develops into the measured out-state ψ^{out}.

$$\left\{\begin{array}{l} \langle {}^{+}E|\phi^{+}\rangle = \langle E|\phi^{in}\rangle \\ \langle {}^{-}E|\psi^{-}\rangle = \langle E|\psi^{out}\rangle \end{array}\right\}$$ are the energy wave functions

representing the energy distribution of the

$\left\{\begin{array}{l} \text{prepared state } \phi^{in} \\ \text{measured state } \psi^{out} \end{array}\right\}$. The integration in (3) is along

the upper rim of the cut of the physical sheet for the S-matrix S(E) (indicated by $\overline{Sp\bar{H}}$). All extra quantum numbers besides the energy are ignored.

To insulate the property of a quasistationary state we will restrict ourselves to the simplest possible case and make the assumption:

S(ω) has no other singularity off the real axis but one pair of poles at $\omega = z_R$ and $\omega = z_R^*$ with $z_R = E_R - i\frac{\Gamma}{2}$, $\Gamma > 0$. The spectrum of $\bar{H}$ is the positive real line.

Then, depending upon the properties of ϕ^{+} and ψ^{-}, one can go with the path of integration in (3) into the second sheet of the complex energy pane. These properties depend upon the precise choice of the space of physical states Φ of which we have so far only required that it is nuclear (in order that the nuclear spectral theorem[10] holds). In the following we shall always assume that Φ is such that the analytically continued wave functions $\langle {}^{+}\omega|\phi^{+}\rangle = \langle\omega|\phi^{in}\rangle$ and $\langle {}^{-}\omega|\psi^{-}\rangle = \langle\omega|\psi^{out}\rangle$ on the Riemann energy surface have all the analyticity and asymptotic properties that we need. It needs to be checked that such a Φ is not empty. The path of integration in (3) is then deformed through the cut into the lower half second sheet resulting in

$$(\psi^-(t),\phi^+(t)) = \text{background} + \int_{-\infty_{II}}^{+\infty} dE' \; \langle\psi^-|E'^-\rangle \; \frac{s_{-1}}{E'-z_R} \; \langle^+E'|\phi^+\rangle \tag{4}$$

where we have denoted for reasons that will become clear later:

$$\text{background} = \int_0^{-\infty_{II}} dE' \; \langle\psi^-|E'^-\rangle \; S_{II}(E') \; \langle^+E'|\phi^+\rangle \tag{4a}$$

Here the integration is along the lower rim of the real axis in the second sheet, s_{-1} is the residium of the S-matrix in the second sheet, $S_{II}(\omega)$, at $\omega = z_R$ and the wave functions are the analytic continuations of the physical wave functions for $0 \leq E < \infty$. The crucial assumption that has been made in the derivation of (4) is that the integrants vanish along the infinite lower semicircle.

The complex conjugate of (3) is

$$(\phi^+(t),\psi^-(t)) = \int_{\underline{Sp}\Pi} dE \; \langle\phi^+|E^+\rangle \; S(E) \; \langle^-E|\psi^-\rangle \tag{3'}$$

where the integration runs now along the lower rim of the cut in the physical sheet (indicated by $\underline{Sp}\Pi$). The path of integration can under analogous assumptions be deformed into the upper half second sheet resulting in

$$(\phi^+(t),\psi^-(t)) = \text{background} + \int_{-\infty_{II}}^{+\infty} dE \; \langle\phi^+|E^+\rangle \; \frac{s_{-1}}{E - z_R^*} \; \langle^-E|\psi^-\rangle \tag{4'}$$

where

$$\text{background} = \int_0^{-\infty_{II}} dE \; \langle\phi^+|E^+\rangle \; S_{II}(E) \; \langle^-E|\psi^-\rangle \tag{4a'}$$

where s_{-1} is the residium of $S_{II}(\omega)$ at $z_R^* = E_R + i\frac{\Gamma}{2}$ and the integrals run along the upper rim of the real axis in the second sheet.

For $\langle\psi^-|E^-\rangle \in L^2_-(E_{II})$ = the space of Hardy class functions with respect to the lower plane, the integral in (4) can be evaluated using the Titchmarsh theorem.[11,7] And for $\langle\phi^+|E^+\rangle \in L^2_+(E_{II})$ = the space of Hardy class functions with respect to the upper plane, the integral in (4') can be evaluated using the same theorem. The results are

$$-2\pi i s_{-1}\langle\psi^-|z_R^-\rangle\langle^+z_R|\phi^+\rangle \text{ and } 2\pi i s_{-1}\langle\phi^+|z_R^{*+}\rangle\langle^-z_R^*|\psi^-\rangle \quad (5)(5')$$

But then $\langle^-E|\psi^-\rangle = \langle E|\psi^{out}\rangle \in L^2_+(E)$ or $\psi^- \in H_+ \cap \Phi$ (6)

and $\langle^+E|\phi^+\rangle = \langle E|\phi^{in}\rangle \in L^2_-(E)$ or $\phi^+ \in H_- \cap \Phi$. (6')

And according to the Paley-Wiener theorem[11]

$$\int_{-\infty_I}^{\infty} dE\, e^{-iEt}\langle\psi^-|E^-\rangle = \int_{-\infty_I}^{\infty} dE\langle\psi^-(t)|E^-\rangle = \int_{-\infty_I}^{\infty} dE\langle\psi^{out}(t)|E\rangle = 0$$

$$\text{for } t < 0 \qquad (7)$$

and

$$\int_{\infty_I}^{\infty} dE\, e^{-iEt}\langle\phi^+|E^+\rangle = \int_{\infty_I}^{\infty} dE\langle\phi^+(t)|E^+\rangle = \int_{-\infty_I}^{\infty} dE\langle\phi^{in}(t)|E\rangle = 0$$

$$\text{for } t > 0 \qquad (7')$$

(4) describes the transition amplitude from a state $\phi^+(t)$ (prepared as $\phi^{in}(t)$ for $t \to -\infty$) into a state $\psi^-(t)$ which is observed as $\psi^{out}(t)$ for $t \to \infty$. Under condition (6) the energy distribution of the detected state $\langle\psi^{out}(t)|E\rangle$ fulfills (7). Assuming sufficiently nice properties of the detected states ψ^{out} (i.e., of Φ), this must mean

that $\psi^{out}(t)$ is only detected for $t \geq 0$; i.e., (4) describes a transition process in which a state $\phi^+(t)$ starts decaying at (the arbitrarily chosen time) $t = 0$ and of which one does not ask how this state $\phi^+(t)$ was prepared at $t \to -\infty$. Similarly, (4') describes under condition (6') a transition process in which the state $\psi^-(t)$ ceases forming at $t = 0$ from the state $\phi^+(t)$ which is prepared as $\phi^{in}(t)$ only for $t \leq 0$. In such a process one does not ask how $\psi^-(t)$ is detected at $t \to +\infty$. Thus (4) ((4')) describes under the condition (6) ((6')) the later (earlier) half of the entire resonance scattering process, in which a state ϕ^{in} is prepared and develops through resonance formation into a state detected as ψ^{out}.

The comparison of (5) and the integral in (4) leads to the definition of the vector:

$$|z_R^-\rangle = -\frac{1}{2\pi i}\int_{-\infty}^{-\infty} dE|E^-\rangle \frac{1}{E-z_R} \quad \text{(integrated along the lower rim of the real axis in the second sheet)} \tag{8}$$

but only when it is considered as a functional over $\Phi \cap H_+$, not over the whole space of physical states Φ. This is the generalized complex energy eigenvector with Breit-Wigner energy distribution introduced in Ref. 7. Similarly, comparison of (5') and the integral in (4') leads to the definition of the vector:

$$|z_R^{*+}\rangle = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE|E^+\rangle \frac{1}{E-z_R^*} \quad \text{(integrated along the upper rim of the real axis in the second sheet)} \tag{8'}$$

only when it is considered as a functional over $\Phi \cap H_-$.

But then these functionals are also generalized eigenvectors of H:

$$H|z_R^-\rangle = z_R|z_R^-\rangle \quad \text{and} \quad H|z_R^{*+}\rangle = z_R^*|z_R^{*+}\rangle \qquad (9)(9')$$

and generalized eigenvectors of e^{-iHt}:

$$e^{-iHt}|z_R^-\rangle = e^{-iz_Rt}|z_R^-\rangle = e^{-iE_Rt}e^{-\frac{\Gamma}{2}t}|z_R^-\rangle \quad \text{for } t > 0 \text{ only} \qquad (10)$$

$$e^{-iHt}|z_R^{*+}\rangle = e^{-iz_R^*t}|z_R^{*+}\rangle = e^{-iE_Rt}e^{+\frac{\Gamma}{2}t}|z_R^{*+}\rangle \quad \text{for } t < 0 \text{ only} \qquad (10')$$

This is the case because not only is $\langle\psi^-|E^-\rangle \in L_-^2(E)$ and $\langle\phi^+|E^+\rangle \in L_+^2(E)$ but also $\langle\psi^-|E^-\rangle E \in L_-^2(E)$ and $\langle\phi^+|E^+\rangle E \in L_+^2(E)$ and further also

$$\langle\psi^-|E^-\rangle e^{-iEt} \in L_-^2(E) \quad \text{for } t > 0 \qquad (11)$$

and

$$\langle\phi^+|E^+\rangle e^{-iEt} \in L_+^2(E) \quad \text{for } t < 0 \qquad (11')$$

so that also for these functions the Titchmarsh theorem can be applied.

From (10) we see that (8) is the decaying Gamow state vector and from (10') we see that (8') is the growing Gamow state vector. Reinserting (8) and (4a) back into (4) we obtain

$$\phi^+ = \int_0^{-\infty_{II}} dE|E^-\rangle\, S_{II}(E)\, \langle^+E|\phi^+\rangle + |z_R^-\rangle(-2\pi i s_{-1})\langle^+z_R|\phi^+\rangle \qquad (12)$$

which is a new generalized eigenvector expansion of ϕ^+ with respect to a generalized basis system that contains in addition to the Dirac vectors $|E^-\rangle$ the decaying Gamow vector (in contrast to the usual basis system expansion in footnote 10). It is only valid for $\phi^+ \in H_+ \cap \Phi$ and $|z_R^-\rangle$ is a functional on $H_+ \cap \Phi$ only. For these ϕ^+ the time development is only given for t > 0. Analogously, reinserting (8') and (4a') into (4') we obtain

$$\psi^- = \int_0^{-\infty_{II}} dE|E^+\rangle\, S_{II}(E)\langle^- E|\psi^-\rangle \tag{14'}$$
$$+ |z_R^{*+}\rangle\, (2\pi i s_{-1})\langle^- z_R^*|\psi^-\rangle$$

which is a basis system expansion of $\psi^- \in H_- \cap \Phi$ with respect to a generalized basis system containing the growing Gamow vectors. Again, in this basis system expansion time development is only given for t < 0.

(12) and (12') show that a physical state (according to RHS formulation of quantum mechanics) contains in addition to the exponentially developing Gamow state a background term.

References and Footnotes

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