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In this talk I descibe some rather elegant mathematical properties of a simple cellular automaton model for self organized criticality. I will discuss how a subset of states in this model form an Abelian group. Then I will show how to construct the non-trivial state which represents the identity for this group [1]. The number of exact results known for this system suggests that it may ultimately be solvable.

While the model discussed below was first presented to illustrate self organized criticality [2], this talk is not directly on that subject. Nevertheless, I begin with a brief summary of the concept. Ref. [2] argued that strongly dissipative systems can drive themselves to a critical state. Unlike conventional critical phenomena, this should occur without any tuning of parameters to a critical value.

The prototypical example of this phenomenon is a sandpile. If sand is slowly added to a heap on a table, the pile will evolve towards a critical slope. If it is too steep, a catastrophic avalanche will flatten it, and if it is too flat, the sand will gradually pile up to steepen the pile. Ultimately, the size of an avalanche produced by the random addition of an additional grain of sand will be unpredictable. The expected distribution of avalanche sizes is a power law.

This concept of a dissipative system automatically becoming critical has been applied to many natural phenomena; indeed, it has been looked for in such diverse areas as earthquake structure [3] and economics [4]. The idea provides an alternative view of complex behavior in systems with many degrees of freedom. It complements the concept of "chaos," wherein simple systems with a small number of degrees of freedom can display quite complex behavior. Here systems with many degrees of freedom develop coherent behavior involving all length scales.

To illustrate self organized criticality, Ref. [2] introduced a simple cellular automaton model for sand flow. The model uses a finite two dimensional square lattice with open boundaries. On each cell i is a non-negative integer z_i representing the local slope of the sand. The dynamics of this system involves an instability threshold value for this slope, which I take to be $z_c = 4$. All cells in the system are updated simultaneously in discrete temporal steps. The updating rule is that if any cell has $z_i \geq z_c$, then that cell has its value decreased by four and each of its neighbors is increased by one. Such an event is referred to as a "tumbling," many of which can occur simultaneously in one updating step. The essential features discussed below are independent of the dimensionality and detailed layout of the lattice.

Note that in the interior of the lattice this dynamics locally conserves the total "sand" or sum of the z_i. Sand is lost only on the open boundaries. Because the dynamics involves a spreading, it is easily shown that any configuration will eventually relax to a stable state with all slopes less than critical. Recently, interesting geometric structures were seen to arise from the relaxation of uniform initial states [5].

After sand has been added to the system for a while, there are some stable states which cannot be reached. For example, there is no way to completely clean a sandy table to recover the state with all slopes vanishing. For another example, two adjacent cells which are initially not empty can never be made so; when one tumbles to zero it adds a grain of sand to the other and vice versa. The states which can be obtained from a full table have been called "recursive" [6]. A recursive state is defined to be one that can be obtained by some addition of sand to any other recursive state followed by relaxation to stability. One such state is the minimally stable state C^* defined as having all $z_i = z_c - 1$; thus, a general recursive state is any state which can be obtained by adding sand to C^* and then relaxing.

Ref. [6] showed that the number of recursive states is given by the determinant of the lattice Laplacian. Whereas an N site system has 4^N stable states, for large N the number of recursive states approaches $3.2102\ldots^N$. Ref. [6] also showed that in the self organized critical ensemble, each recursive state is equally likely; this ensemble serves as the analog of the Boltzmann distribution for a conventional statistical system.

Define a_i to represent the operation of adding a grain sand to cell i followed by a relaxation of the system back to stability. Ref. [6] pointed out the remarkable fact that these operators all commute with each other. The proof uses the linearity of toppling on the slopes z_i and the fact that a toppling decreases the slope only at the active site. For a detailed discussion see Ref. [1].

Several exact results follow from this observation. In particular, Ref. [6] showed that if we restrict ourselves to the recursive set of states, then these operators a_i are invertable. Thus, given a recursive configuration C, there is a unique recursive C' such that $a_i C' = C$. Because of this property, the operations of adding sand generate an Abelian group.

I now define an operation of addition between states. Given stable states C and C' with corresponding slopes z_i and z'_i, I define the state $C \oplus C'$ to be that configuration obtained by relaxing the configuration with slopes $z_i + z'_i$. By construction, this definition is commutative.

Now consider restricting oneself to recursive states. Since the process of adding a state to another can be decomposed into a set of individual sand additions, and because those additions are invertable on recursive states, this addition of states is itself invertable. Indeed, under $\oplus$ the recursive states themselves form an Abelian group, which is isomorphic to the group generated by the a_i.

One of the fundamental properties of any group is the existence of an identity element. Thus, among the recursive states there must exist a unique configuration I which when added to any other recursive state C relaxes back C. This is a property also possessed by the state with all $z_i = 0$, but that is not a recursive state.

Intrigued with the existence of this special state, I set out to find it. To proceed, I use the identity that adding four grains of sand to any site forces a tumbling and is equivalent to adding one grain to each neighbor. Thus the operation on a recursive state of adding four grains to one site and then removing one from each of its neighbors leaves the state unchanged. In terms of the operators a_i we have the statememt

$$a_i \prod_{j \in n(i)} a_j^{-1} = 1$$

where $n(i)$ denotes the nearest neighbors of site i.

Now consider applying this combined operation to all sites of the lattice. Any site in the interior will receive four grains but then have them taken away when the operation is applied to the neighbors. Only at the edges will things not balance. Thus we are led to consider adding one grain to all edge sites and two to the corner sites. On any recursive state this addition will relax back to the starting state.

This leads me to consider the non-recursive state I_0 defined to have $z_i = 1$ on the edges, two on the corners, and zero elsewhere. This state when added to any other state C will leaves that state unchanged if and only if C is recursive. In adding it to a recursive state, it can also be shown that each site of the entire lattice tumbles exactly once.

The state I_0 itself is not recursive (except on the smallest lattices) because it has a large empty region in the center. I now consider combining this state with itself iteratively until it becomes recursive. Thus I define $I_n = I_{n-1} \oplus I_{n-1}$. For large enough n I will have $I_n = I_{n-1} = I$, the identity I am searching for. The resulting state has a rather interesting structure, with patterns exhibiting many different scales. For a picture of this state on a particular lattice, see Ref. [1].

I now briefly mention a few other exact results. First, consider some arbitrary addition of sand to a recursive state. After this addition, follow the progress of the avalanches, and define an "avalanche region" to be the set of sites which have tumbled. A rather remarkable result is that this region must be simply connected. That is, if we construct any closed curve of sites in the avalanche region, this can be deformed to a point without moving any part of the curve through sites which have not tumbled. This implies a rather subtle correlation in recursive states, and is not in general true for an arbitrary state. This result is independent of the amount and distribution of sand used to start the avalanche, and remains true if at later times more sand is used to enlarge the avalanch region. It makes an amusing computer game to take a recursive state and try to create an untumbled island in the midst of an avalanche. Such islands always die away before the avalanche reaches stability.

Second, if C is itself recursive, on adding I the amount of sand lost at the edges must equal the sand contained in I, and is independent of C. In fact, a much stronger result is true: the number of topplings on any given site i during this relaxation is itself independent of C.

Third, if we start an avalanche with a single grain of sand at a site k steps away from the boundary, then the ensuing avalanche can cause no more than k tumblings at any given site. A related result is that an avalanche started with a single grain at any site can give at most k tumblings to another site k steps from the edge. In particular, such an avalanche can tumble the edge sites at most once.

To conclude, I have presented some amusing mathematical results for a simple model of avalanches in a sandpile. While many interesting cellular automata models are known, it is quite rare that so many precise results can be obtained. While exact expressions for the critical exponents have yet to be found, this multitude of results suggests that the model may indeed be solvable. Indeed, this may become an "Ising model" for self-organized criticality.

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