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MORE ON APPROXIMATIONS OF POISSON PROBABILITIES

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1. Introduction

Calculation for the Poisson probabilities frequently involves calculating high factorials which becomes tedious and time-consuming with regular calculators. The usual way to overcome this difficulty has been to find approximations by making use of the table of standard normal distribution. As indicated in Johnson and Klotz (1969), by relating the Poisson probability to the Chi-squares and applying Wilson-Hilferty's normal-distribution approximation to the Chi-square distribution, one has an approximation for the Poisson probability. An often used method is to transform Poissons to the standard normal, then obtain the probabilities from the standard-normal table. Along this line Makabe and Morimura (1955) suggested an approximation by using the trivial transformation $(X-\lambda)/\sqrt{\lambda}$ (λ is the Poisson parameter) to transform a Poisson to a standard normal. In fact, Makabe and Morimura obtained approximation for the standard-normal probabilities in non-integral form, which lead to an approximation for the Poisson probabilities that does not require the use of the standard-normal table. In general, one may consider with any of the traditional transformations from Poisson to normal as proposed noticeably by Tukey (1957), Freeman and Turkey (1950), Anscombe (1948), and Bartlett (1936). Recently a new transformation was proposed by Kao (1978) that appears to perform better for this purpose. In this paper a class of approximation methods are stated and compared numerically, including an approximation method that utilizes a modified version of Kao's transformation.

Now let X be a random variable that has the probability density function f(x) defined by

$$f(x) = e^{-\lambda}\lambda^{x}/x!, \quad x = 0,1,2,\ldots \quad (\lambda>0). \tag{1}$$

Then X is said to have the Poisson distribution with parameter λ. Note that $EX = \lambda$, $E(X-\lambda)^2 = \lambda$, $E(X-\lambda)^3 = \lambda$, and $E(X-\lambda)^4 = 3\lambda^2+\lambda$, etc...

2. The Traditional Approximations of Poisson Probabilities

Since $(X-\lambda)/\sqrt{\lambda}$ converges in law to standard normal (denoted by N(0,1)) as $\lambda\to\infty$, it may be shown using Taylor's expansion that for most functions g admitting the first derivative and $g'(\lambda)\neq 0$, $(g(X)-g(\lambda))/(\sqrt{\lambda}g'(\lambda))$ converges in law to N(0,1) as $\lambda\to\infty$. Therefore, in consideration that g(X) and $\sqrt{\lambda}\,g'(\lambda)$ be functionally independent of λ, square-root type dunctions are proposed for g. Then $\sqrt{\lambda}\,g'(\lambda)$ converges to 1/4 as $\lambda\to\infty$. It should be noted here that the approximation is good in the sense that λ is large. The square-root transformations available in literature include $t_B(X)=2\sqrt{X+1/2}$ (Bartlett (1936)), $t_A(X)=2\sqrt{X+3/8}$ (Anscombe (1948)), $t_{FT}(X)=\sqrt{X}+\sqrt{X+1}$ (Freeman and Tukey (1950)), $t_{T1}(X)=2\sqrt{X}$ (Tukey (1957)). Such consideration on g(X) and $\sqrt{\lambda}\,g'(\lambda)$ as prescribed is needed for dealing with Poisson regression problems, but it is not needed for approximating the Poisson probabilities. In the latter case, transformations of the forms like $\log(X+\lambda)$ (Tukey (1957)), and $1.5X^{2/3}$ (Kao (1978)) are equally applicable. In what follows the transformations from Poisson to standard normal to be investigated are

(i) $T_{MM}(X) = (X-\lambda)/\sqrt{\lambda}$,

(ii) $T_{T1}(X) = 2(\sqrt{X} - \sqrt{\lambda} + 0.125/\sqrt{X})$,

(iii) $T_{FT}(X) = \sqrt{X} + \sqrt{X+1} - \sqrt{\lambda} - \sqrt{\lambda+1} + 0.25/\sqrt{\lambda}$,

(iv) $T_A(X) = 2(\sqrt{X+3/8} - \sqrt{\lambda+3/8} + 0.125/\sqrt{\lambda})$,

(v) $T_{T2}(X) = 2\sqrt{\lambda}\,[\log(X+\lambda) - \log 2\lambda + 0.125\,\lambda]$,

(vi) $T_{K_0}(X) = 1.5(X^{2/3} - \lambda^{2/3} + \frac{1}{q}\lambda^{-1/3})/\lambda^{1/6}$, and

(vii) $T_{K_{0.1}}(X) = 1.5[(X+\frac{1}{10})^{2/3} - (\lambda+\frac{1}{10})^{2/3} + \frac{1}{9}(\lambda+\frac{1}{10})^{-1/3}]/(\lambda+\frac{1}{10})^{1/6}$.

Let $\lambda>0$ be given and T be a transformation among (i) through (vii). Then for $k=0,1,2,\ldots$, let $a_k = T(k-0.5)$, $b_k = T(k)$, and $c_k = T(k+0.5)$. Denote the cumulative distribution function of the standard normal by Φ, then we take

$$\xi_k = \begin{cases} \Phi(c_k) & \text{for } k=0 \\ \Phi(c_k) - \Phi(a_k) & \text{for } k>0 \end{cases} \tag{2}$$

to be an approximation for the Poisson probability densities $P_r\{X=k\}$. The special feature in Makabe and Morimura's approximation is that after doing the trivial transformation, $T_{MM}(X) = (X-\lambda)/\sqrt{\lambda}$, they suggested using η_k instead of ξ_k, where η_k is defined by

$$\eta_k = [c_k - a_k + \frac{1}{6}(3b_k - b_k^3)]e^{-b_k^2/2}/\sqrt{2\pi} . \tag{3}$$

The method that gives approximation to Poisson probability by relating to the χ^2(Chi-square) distribution basically makes use of the relation that, for k>0,

$$P_r\{X\leq k\} = P_r\{\chi^2_{2(k+1)} > 2\lambda\}.$$

Then apply the Wilson-Hilferty (1931) approximation to the χ^2 distribution. It follows that

$$P_r\{X\leq k\} \doteqdot \frac{1}{\sqrt{2\pi}} \int_{Z_k}^{\infty} e^{-u^2/2}du = 1-\Phi(A_k), \tag{4}$$

where

$$Z_k = 3[(\frac{\lambda}{k+1})^{3/2} - 1 + \frac{1}{9(k+1)}]\sqrt{k+1} .$$

Accordingly, for the probability densities we have

$$P_r\{X=k\} \doteqdot \Phi(Z_{k-1}) - \Phi(Z_k). \tag{5}$$

For the Poisson cumulative probabilities $P_r\{X\leq k\}$. The approximation that corresponds to (2) with any transformations (i) through (vii) is

$$P_r\{X\leq k\} \doteqdot \Phi(c_k). \tag{6}$$

In addition to (3) that treats the Poisson probability densities, Makabe and Morimura also suggested the following approximation to Poisson cumulatives:

$$P_r\{X\leq k\} \doteq \Phi(c_k) - \Phi(a_o) + \frac{1}{6\sqrt{2\pi\lambda}}[(1-c_k^2)\exp(-\frac{1}{2}c_k^2) - (1-a_o^2)\exp(-\frac{1}{2}a_o^2)] \quad (7)$$

where the transformation that defined the a_k's and the c_k's was taken to be T_{MM}. Actually along the same line of (7) one may suspect that for any transformation among the transformations of (i) through (vii) the following approximation to Poisson cumulatives is applicable:

$$P_r\{X\leq k\} \doteq \Phi(c_k) - \Phi(c_o) + \frac{1}{6\sqrt{2\pi\lambda}}[(1-c_k^2)\exp(-\frac{1}{2}c_k^2) - (1-c_o^2)\exp(-\frac{1}{2}c_o^2)]. \quad (8)$$

Regarding the choice of a transformation, T_{K_0} was also shown in Kao (1978) to be more desirable than the others among transformations (i) through (vi). In fact, as a modified version of T_{K_0}, $T_{K_{0.1}}$ (i.e. transformation (vii)) is considered a better choice than T_{K_0}.

3. The Proposed Approximation

It was argued in Kao (1978) that for large λ, the power transformation $T(x) = (x+d)^r$ appeared to be most desirable when $r = 2/3$ was taken, assuming that $d \ll \lambda$ and $\lim_{\lambda\to\infty} d/\lambda = 0$. By simply taking $d = 0$, one obtains the transformation T_{K_0}. In addition to setting $r = 2/3$, it is now found that $b = 0.1$ should be considered for better approximation result when λ is small, and $T_{k_{0.1}}$ is thus proposed. The reasoning is to be described in what follows.

We now let $T(x) = (x+d)^r$ with $d \ll \lambda$ and $\lim_{\lambda\to\infty} d/\lambda = 0$. Then let $\mu = \lambda+d$. By the Taylor's expansion, it can be shown that for large λ we have

$$\begin{aligned} m &= ET(X) \simeq \mu^r + \frac{1}{2} r(r-1)\mu^{r-1} = (\lambda+d)^r + \frac{1}{2} r(r-1)(\lambda+d)^{r-1}, \\ \mu_2 &= Var(T(X)) \simeq r^2(\lambda+d)^{2(r-1/2)} + (1.5\, r^2(r-1)^2 - r^2 d)(r+d)^{2(r-1)}, \\ \mu_3 &= E(T(X)-m)^3 \simeq r^3(3r-2)(\lambda+d)^{4r-2}, \text{ and} \\ \mu_4 &= E(T(X-m)^4) \simeq 3r^4(\lambda+d)^{4r-2}. \end{aligned} \tag{9}$$

Therefore, the skewness γ_3 and the kurtosis γ_4 (cf. Kendall and Stuart (1958)) have the following approximates:

$$\gamma_3 = \mu_3/\mu_2^{3/2} \simeq (3r-2)(\lambda+d)^{-3/2} \text{ and } \gamma_4 = \mu_4/\mu_2^2 - 3 \simeq 0. \tag{10}$$

Actually it may be further shown that $\gamma_4 = O((\lambda+d)^{-1})$ irrespective of the value of r by a detailed Taylor expansion. Since the normally-distributed random variables have zero skewness and kurtosis, the most desirable vaule for r in this regard in 2/3. Furthermore, after setting $r = 2/3$, we then have

$$Var(T(X)) \simeq \frac{4}{9}(\lambda+d)^{1/3} + \frac{2}{9}(\frac{1}{3} - d)(\lambda+d)^{-2/3}.$$

It is therefore expected that, in view of the variance value, by taking $d = 1/3$ to obtain the corresponding transformation $T_{k_{0.33}}$ instead of taking $d = 0$ to obtain T_{K_0} one would have the transformation $T_{K_{0.33}}$ closer to standard normal.

Nevertheless, since such consideration is in the sense of approximation, d = 1/3 is not precisely the value of d to produce the best result. It is found numerically that the best value for d is around the neighboring value d = 0.1. Now, for a given d, we define the transformation T_{K_d} by

$$T_{K_d}(X) = 1.5[(X+d)^{2/3}-(\lambda+d)^{-1/3}]/(\lambda+d)^{1/6}. \tag{11}$$

Then, in line with (2) and (6), there corresponds the approximation A_{K_d} defined by

$$A_{K_d}(X=k) = \begin{cases} \Phi(T_{K_d}(k+0.5)) & \text{for } k=0 \\ \Phi(T_{K_d}(k+0.5)) - \Phi(T_{K_d}(k-0.5)) & \text{for } k>0 \end{cases} \tag{12}$$

From the foregoing discussion, it is proposed to have $A_{K_{0.1}}$ as the best approximation for the Poisson probability. Naturally, we define for the cumulatives corresponding to (12) by

$$A_{K_d}(X\leq k) = \Phi(T_{K_d}(k+0.5)). \tag{13}$$

4. Numerical Comparison of Approximations

To make comparison on the accuracy of the various approximations, we use measures d_s, D_s, d_{rs}, and D_{rs} as the accuracy indicators. For an approximation $\tilde{A}(X=k)$ to approximate $P_r\{X=k\}$, and value $\tilde{A}(X\leq k)$ to approximate $P_r\{X\leq k\}$, define

$$ds = \max_{0\leq k<\infty} |P_r\{X=k\} - \tilde{A}(X=k)|,$$

$$d_{rs} = \max_{0\leq k<\infty} \left|\frac{P_r\{X=k\} - \tilde{A}(X=k)}{P_r\{X=k\}}\right|, \tag{14}$$

$$D_s = \max_{0\leq k<\infty} |P_r\{X\leq k\} - \tilde{A}(X\leq k)|,$$

and

$$D_{rs} = \max_{0\leq k<\infty} \left|\frac{P_r\{X\leq k\} - \tilde{A}(X\leq k)}{P_r\{X\leq k\}}\right|$$

d_s and D_s are called the Smirnov errors (respectively for the density and the cumulative). d_{rs} and D_{rs} are correspondingly called the relative Smirnov errors.

The traditional approximation obtained by applying the Wilson-Hilferty approximation to the χ^2 distribution as indicated by (4) and (5) will be denoted by B_{WH}. The Makabe-Morimura's approximation as indicated by (3) and

$$P_r\{X\leq k\} \doteqdot \Phi(c_k) - \Phi(c_{k-1}) + \frac{1}{6\sqrt{2\pi\lambda}} (1-c_k^2) \exp(-\tfrac{1}{2}c_k^2) \tag{15}$$

will be denoted by B_{MM}. For the approximations that are derived by making a transformation from Poisson to normal and followed by using (2) and (6), they will be denoted by A and proper subscripts. For instance, A_{FT} denotes the approximation that corresponds to the transformation T_{FT}.

It should be noted that the Poisson-χ^2 relation immediately before (4) may be replaced by

$$P_r\{X\leq k\} = P_r\{X\leq k+0.5\} \doteqdot P_r\{\chi^2_{2k+3}>2\lambda\} \qquad (16)$$

However, the numerical results show that using the right-hand side of (16) a worse result is obtained. Such fact is not shown in any of the tables provided. Regarding the values of d_{rs}, one finds that with each of the approximation methods d_{rs} is attained by some k at the right tail of the Poisson distribution, while D_{rs} is attained at the left tail. They are thus not as practical as d_s and D_s for indicating the accuracy of approximations. Nevertheless, the consistence of almost everywhere between the facts as shown by (d_{rs},D_{rs}) and the facts as shown by (d_o,D_o) in all the tables reinforces the conclusions to be stated in what follows. In addition, it is found that A_{T2} is very poor compared with the approximations in Table 2, and is therefore not considered in that table. Furthermore, it should be noted that if (7) or (8) replaces (6) for the cumulatives with the A approximations, the accuracy is considerable decreased.

It appears in Table 1 that except for λ as small as 0.4, $A_{K_{0.1}}$ is definitely better than A_{K_0}, and $A_{K_{1/3}}$ is placed the third among the three approximations throughout the λ values. Therefore, $A_{K_{0.1}}$ is further compared with the other A approximations in Table 2 including A_{MM}, A_{FT}, A_{T1}, A_A. According to Table 2, the A approximations are all desirable and $A_{K_{0.1}}$ stands out to be the best for all cases. In Table 3, the three approximations B_{WH}, B_{MM}, and $A_{K_{0.1}}$ are compared. It is obvious that the Wilson=Hilferty approximation and the Madabe-Marimura approximation are very poor compared with $A_{K_{0.1}}$. In fact, they are also very poor approximations compared with any A transformations considered in Table 2. It may be claimed that B_{MM} is substantially better than B_{WH}. From Table 3, it is seen that the use of Wilson-Hilferty approximation should be prevented. Although in a less strong sense, the use of B_{MM} should also be prevented. In a list of Wilson-Hilferty approximation to the Poisson probability densities for various λ values, the support (i.e. the smallest set of k values that together attain almost the total probability) relatively has a very short range, even when λ is large. In general, when the true values of the Poisson probability densities are used, it shows a support that approximately has a range with center at λ and a radius of $3\sqrt{\lambda}$, and the support appears symmetric with the center at λ as λ becomes large. However, irrespective of the magnitude of λ, the support that corresponds to the Wilson-Hilferty approximation is always substantially shifted to the left and it has a radius of less than $\sqrt{\lambda}$ and it is unsymmetric even when λ is large. On the other hand, the support that corresponds to the Makabe-Morimura approximation is about the same as that corresponding to the true Poisson probability densities. Nevertheless, it is largely unsymmetric, and sometimes the Makabe-Morimura approximation becomes negative. The support that corresponds to any of the A approximations almost completely coincides with the true support. Such preceding facts provide some insight about the reasons of poor performance of the Wilson-Hilferty approximation and the Makabe-Morimura approximation.

In Table 4 a list of the true Poisson probability densities and their corresponding $A_{K_{0.1}}$ values for various λ values is presented to show the extent of accuracy of $A_{K_{0.1}}$. It undoubtedly proves that $A_{K_{0.1}}$ approximates the Poisson probabilities very closely.

5. Conclusions

Approximation to Poisson probabilities appears to be rather accurate if it is obtained by first using a power transformation to transform the Poissons to the standard normal, then refering to the standard-normal table to find the end result. Especially the approximation $A_{K_{0.1}}$ which is based on the power transformation $(X+0.1)^{2/3}$ outperforms those that are based on the square-root type transformations as proposed in literature. The traditional Wilson-Hilferty approximation and Makabe-Morimura approximation are extremely poor compared with $A_{K_{0.1}}$. The approximation $A_{K_{0.1}}$ is a very close approximation for the Poisson probabilities even for values of λ as small as 0.4. All of such facts are proved numerically with d_s, D_s, d_{rs}, and D_{rs} as the measures of error.

Table 1

Values of the Smirnov errors and the relative Smirnov errors of approximations using $(x+c)^{2/3}$ - type transformations ($c = 0, 0.1, 1/3$) for different λ's.

	$d_s(d_{rs})$			$D_s(D_{rs})$		
λ	A_{K_0}	$A_{K_{0.1}}$	$A_{K_{1/3}}$	A_{K_0}	$A_{K_{0.1}}$	$A_{K_{1/3}}$
0.4	.0157	.0246	.0487	.0087	.0246	.0487
	(.086)	(.133)	(.525)	(.013)	(.037)	(.073)
0.8	.0222	.0136	.0270	.0188	.0136	.0320
	(.147)	(.090)	(.674)	(.093)	(.030)	(.040)
2	.0126	.0031	.0163	.0126	.0035	.0163
	(.101)	(.049)	(.330)	(.093)	(.023)	(.121)
4	.0030	.0008	.0057	.0039	.0013	.0084
	(.199)	(.116)	(.249)	(.166)	(.042)	(.249)
10	.0007	.0003	.0014	.0014	.0006	.0033
	(1.000)	(1.000)	(.546)	(1.000)	(1.000)	(.546)
50	.0002	.0002	.0002	.0005	.0004	.0009
	(1.244)	(1.276)	(1.352)	(1.000)	(1.000)	(1.000)
100	.0001	.0001	.0001	.0004	.0004	.0006
	(2.072)	(2.440)	(2.493)	(1.000)	(1.000)	(1.000)

Table 2

Values of the Smirnov errors d_s and D_s in the form of $d_s(D_s)$ of the approximations derived by using transformation from Poisson to the standard normal.

λ	A_{MM}	A_A	A_{FT}	A_{T1}	$A_{K_{0.1}}$
0.4	.1275	.0470	.0472	.0371	.0246
	(.1073)	(.0235)	(.0256)	(.0371)	(.0246)
0.8	.0810	.0497	.0471	.0200	.0136
	(.0810	(.0436)	(.0408)	(.0179	(.0136)
2	.0535	.0183	.0198	.0410	.0030
	(.0446)	(.0275)	(.0288	(.0273)	(.0035)
4	.0260	.0118	.0119	.0139	.0008
	(.0325)	(.0181)	(.0190)	(.0150)	(.0013)
10	.0110	.0045	.0047	.0055	.0003
	(.0209)	(.0108)	(.0112)	(.0102)	(.0006)
50	.0020	.0010	.0010	.0009	.0002
	(.0095)	(.0047)	(.0047)	(.0048	(.0004)
100	.0010	.0005	.0005	.0005	.0001
	(.0067)	(.0034)	(.0034)	(.0034)	(.0004)

Table 3

Values of the Smirnov errors and the relative Smirnov errors of the Wilson-Hilferty approximation (B_{WH}), the Makabe-Morimura approximation (B_{MM}), and $A_{K_{0.1}}$ (using transformation $T_{K_{0.1}}$).

	$d_s(d_{rs})$			$D_s(D_{rs})$		
λ	B_{WH}	B_{MM}	$A_{K_{0.1}}$	B_{WH}	B_{MM}	$A_{K_{0.1}}$
0.4	.3295	.5016	.0246	.3295	.1320	.0246
	(1.000)	(.955)	(.133)	(.492)	(.166)	(.037)
0.8	.5491	.3875	.0136	.5491	.1117	.0136
	(1.222)	(1.408)	(.090)	(1.222)	(.249)	(.030)
2	.3079	.1425	.0030	.5791	.0622	.0035
	(2.004)	(2.096)	(.049)	(2.004)	(.366)	(.023)
4	.3314	.1062	.0008	.5269	.0362	.0013
	(1.951)	(3.220)	(.116)	(1.216)	(1.361)	(.042)
10	.3067	.0988	.0003	.4317	.0224	.0006
	(2.626)	(226.8)	(1.000)	(1.000)	(466.3)	(1.000)
50	.1825	.0934	.0002	.3696	.0101	.0004
	(3.306)	(1059)	(1.276)	(1.000)	(19657)	(1.000)
100	.1345	.0924	.0001	.3518	.0072	.0004
	(3.408)	(384.4)	(2.440)	(1.000)	(1823)	(1.000)

Table 4

Values of the Poisson probability density p and its approximation $\hat{p}$ by using $A_{K_{0.1}}$ at representative X's for different λ values.

λ						
	X	0	1	2		
0.4	p	.6703	.2681	.0536		
	$\hat{p}$	.6457	.2843	.0607		
	X	0	1	2	3	4
0.8	p	.4493	.3595	.1438	.0383	.0077
	$\hat{p}$	.4357	.3644	.1489	.0412	.0084
	X	0	1	2	3	6
2	p	.1353	.2707	.1804	.0902	.0120
	$\hat{p}$	.1322	.2706	.1812	.0913	.0126
	X	0	2	4	6	8
4	p	.0183	.1465	.1954	.1042	.0298
	$\hat{p}$	.0175	.1461	.1952	.1043	.0303
	X	4	8	11	14	17
10	p	.0189	.1126	.1137	.0521	.0128
	$\hat{p}$	.0192	.1124	.1138	.0519	.0130
	X	37	43	50	57	63
50	p	.0102	.0363	.0563	.0330	.0105
	$\hat{p}$	.0103	.0361	.0565	.0329	.0106

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