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NEURAL NETWORK MODELS FOR LINEAR PROGRAMMING*

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Neural Networks Models for Linear Programming

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Abstract: The purpose of this paper is to present a neural network that solves the general Linear Programming (LP) problem. In the first part, we recall Hopfield and Tank's circuit for LP and show that although it converges to stable states, it does not, in general, yield admissible solutions. This is due to the penalization treatment of the constraints. In the second part, we propose an approach based on Lagragrange multipliers that converges to primal and dual admissible solutions. We also show that the duality gap (measuring the optimality) can be rendered, in principle, as small as needed.

1. Introduction. The contribution of neural networks to Optimization Theory has been mainly dedicated to NP-complete problems so far, and in particular to the Travelling Salesman Problem [1] (see also [2] for an excellent account). Here, we consider "simpler" problems, like Linear Programming (LP) and its variants, for which low-order polynomial algorithms are already available [3]. Although some of these simpler problems are combinatorial by formulation (like the Assignment Problem [4]), their structure is inherently continuous and seems well adapted to a neural network solution.
The only attempts we are aware of, concerning continuous optimization problems are Hopfield and Tank's circuit for LP [5], its application to Analog Decoding [6], the work of Jeffrey and Rosner [7] for the solution of variational problems, and our recent application of sigmoidic functions to general optimization problems[8,9]. In the next Section, we give some properties of linear programs. In Section 3, we recall some background on Hopfield and Tank's LP network and discuss its connection with standard optimization methods. We note that it converges to stable states which are not, in general, admissible solutions of the problem. In Section 4, we propose a new network that, in handling the constraints, relies on duality instead of direct penalization. This network always converges to primal and dual admissible solutions, and the associated duality gap can be rendered as small as desired.

2. Linear Programming. We intend to solve the LP problem

$$(1) \qquad \min < c, x > \quad subject \ \ to \ \ Ax \geq b,$$

where c and x are vectors in R^n, b is a vector in R^m, and A is an $m \times n$ matrix, with $m \leq n$. The brackets $< .,. >$ denote the scalar product in R^n. We assume that problem (1) has a bounded solution x^* and, for the purpose of the forthcoming derivations, that the rank of A is m. We define the operation $[\,.\,]^- : [y]^- = y \ \ if \ \ y < 0, \quad [y]^- = 0 \ \ if \ \ y \geq 0$, and apply it componentwise, if y is a vector. In the following, a vector x such that $Ax \geq b$ will be called admissible for (1). It is common, when dealing with LP problems to introduce their dual problems. The dual problem associated with (1) is

$$(2) \qquad \max < b, p > \quad subject \ \ to \ \ A^T p = c, \quad and \quad p \geq 0,$$

where A^T denotes the transpose of matrix A. Note that we have also denoted the scalar product in R^m by $< .,. >$. The fundamental result of duality is (for a proof, see [10]):
Proposition 1. *If $\hat{x}$ is admissible for (1) and $\hat{p}$ is admissible for (2) (that is $A^T\hat{p} = c, \ \hat{p} \geq 0$), then the duality gap $\delta := < c, \hat{x} > - < b, \hat{p} >$ is positive. If $\delta = 0$, then $\hat{x}$ is a solution of (1), and $\hat{p}$ is a solution of (2).*

3. Hopfield and Tank's Neural Network for Linear Programming. The neural network proposed in [5] for the solution of (1), contains n neurons with internal states u_i, output values $x_i = g_\lambda(u_i)$, and time response τ. We denote them as neurons (u_i, x_i), $(i = 1, ..., n)$. The function g is assumed linear and increasing ($g_\lambda(u) = \lambda u, \quad \lambda > 0$). We also have m neurons (with no time response) with internal state $y_j = \sum_i A_{ji} x_i - b_j$ and output values $\psi_j = (y_j^-)^2$. There are no connections within the set of $\{(u_i, x_i)\}_i$ neurons nor within the set of $\{(y_j, \psi_j)\}_j$ neurons. On the other hand, each (u_i, x_i) neuron is connected, with connection strength A_{ji}, to the neuron (y_j, ψ_j) . The dynamics of the $\{(u_i, x_i)\}_i$ neurons is given by the equation (that we directly write in vector form for the whole set)

$$\frac{du}{dt} = -c - \frac{u}{\tau} - A^T[Ax - b]^-. \tag{3}$$

An energy functional is associated with (3)

$$E(x) = < c, x > + \frac{1}{2} ||[Ax - b]^-||^2 + \frac{1}{2\lambda\tau}||x||^2. \tag{4}$$

We now prove the convergence of the network and address the optimality of the procedure.
Proposition 2. The functional $E(x)$ is a Lyapunov functional for (3).
Proof: $E(x)$ is bounded below since it contains a quadratic term in $||x||$. Also, we have

$$\frac{dE(x)}{dt} = < c, \frac{dx}{dt} > + < A^T[Ax - b]^-, \frac{dx}{dt} > + \frac{1}{\lambda\tau} < x, \frac{dx}{dt} >$$

$$= < c + A^T[Ax - b]^- + \frac{1}{\lambda\tau}x, \frac{dx}{dt} > = - < \frac{du}{dt}, \frac{dx}{dt} > = -\lambda||\frac{du}{dt}||^2 \leq 0.$$

Thus $E(x)$ is decreasing along the trajectories of (3) and $\frac{dE(x)}{dt} = 0$ implies that $\frac{du}{dt} = 0$. ∎

The network driven by (3) is designed to solve the problem $\min_x E(x)$, where one attempts to satisfy the constraint by penalization. We will show that, in general, this does not imply that the network solves the original problem (1). Let $(\bar{u}, \bar{x}, \bar{y}, \bar{\psi})$ be a stable state. Then $\bar{x}$ is solution of the fixed point equation

$$c + \frac{1}{\lambda\tau}\bar{x} + A^T[A\bar{x} - b]^- = 0 \tag{5}$$

We can state
Proposition 3. *Two necessary conditions (on $\lambda\tau$) for $\bar{x}$ to be admissible are*

$$\lambda\tau \leq -\frac{< c, x^* >}{||c||^2}, \quad \textit{and} \quad \lambda\tau Ac \leq -b. \tag{6}$$

Proof: Indeed, if $\bar{x}$ is admissible, then we get $\bar{x} = -\lambda\tau c$ from (5). Since the solution x^* is such that $< c, x^* > \; \leq \; < c, x >$ for every admissible x, we must have $< c, x^* > \; \leq -\lambda\tau||c||^2$. By applying the matrix A to (5) and by using the admissibility of $\hat{x}$, we get the other condition. ∎
Corollary. *In general, the solution of (5) is not admissible for (1).*
Indeed, since $\lambda\tau$ is strictly positive, whenever $< c, x^* >$ is positive or equal to zero, one cannot obtain an admissible $\bar{x}$. Even if $< c, x^* >$ is negative, the second condition of (6) implies $\lambda\tau < Ac, b > \; \leq \; -||b||^2$, which is not satisfied whenever $< Ac, b > \; > 0$. Counterexamples can be easily constructed.

4. A Primal-Dual Neural Network for Linear Programming. We now propose a neural network named Primal-Dual because it provides admissible primal ($\bar{x}$) and dual ($\bar{p}$) vectors . Moreover, we will show that the duality gap can be made as small as desired. We consider n neurons of the type (u, x) and m neurons of the type (v, p). We assume that

$$x = R\ G_\lambda(u), \quad p = g_\mu(v), \quad \textit{with} \quad \lambda > 0, \quad R > 0, \quad G'_\lambda(u) > 0, \quad g_\mu(v) \geq 0, \quad g'_\mu(v) > 0, \tag{7}$$

where R is a large bound on $||x||$ which has not to be known precisely. Although the following derivations do not use their explicit form, the functions G_λ and g_μ can be chosen as $G_\lambda(u) = \tanh(\lambda u)$ (with λ not too large in order to prevent instabilities) and $g_\mu(v) = \frac{e^{\mu v}}{\mu}$. The latter choice of g_μ was proposed in [6]. It may have, however, a natural tendancy to produce numerical difficulties.
The evolution equations of these neurons are assumed to be

$$\frac{du}{dt} = -c + A^T p, \quad \frac{dv}{dt} = -\frac{p}{\mu\tau} - b + Ax. \tag{8}$$

The main differences between (8) and (3) are the following: (i) the neurons (u, x) have no time constant, and the relation between u and x is not totally linear, (ii) the constraints are "softly" penalized by the output values of the neurons (v, p), (iii) unlike the neurons (ψ, y) of the Hopfield and Tank model, the neurons (v, p) have an explicit evolution as independent state variables, and have a time constant. These differences do not affect significantly the feasibility of an analog implementation [11].
We now address the convergence of the network and the optimality of its fixed points.
We choose the Lyapunov functional

$$E(x, p) =< c, x > - < p, Ax - b > + \frac{||p||^2}{2\mu\tau}. \tag{9}$$

Convergence: $E(x, p)$ is bounded below because $||x||$ is bounded by $\sqrt{n}R$ and E contains a quadratic term in $||p||$. It decreases along the trajectories of (8). Indeed,

$$\frac{dE}{dt} =< c - A^T p, \frac{dx}{dt} > - < \frac{dp}{dt}, Ax - b - \frac{p}{\mu\tau} >= -\lambda \sum_i G'_\lambda(u_i)(\frac{du_i}{dt})^2 - \sum_j g'_\mu(v_j)(\frac{dv_j}{dt})^2 \leq 0.$$

Thus, $\frac{dE(x)}{dt} = 0$ implies $\frac{du}{dt} = 0$ and $\frac{dv}{dt} = 0$. ∎
Consequently, the network defined by (7) and (8) converges to a stable state $(\bar{u}, \bar{x}, \bar{v}, \bar{p})$.
The admissibility and optimality of the fixed points is addressed in
Proposition 4. *The fixed point $\bar{x}$ is admissible for (1) and the fixed point $\bar{p}$ is admissible for (2). The associated duality gap $\delta =< c, \bar{x} > - < b, \bar{p} >$ is equal to $\frac{||\bar{p}||^2}{\mu\tau}$. Moreover, if the rank of A is equal to m, $\bar{p}$ is bounded and the duality gap has a magnitude of order $O(\frac{1}{\mu\tau})$.*
Proof: From the fixed point equations, one gets $A\bar{x} - b = \frac{\bar{p}}{\mu\tau} \geq 0$. Thus $\bar{x}$ is admissible for (1). Also, we have $A^T\bar{p} = c$ and since $\bar{p} = g_\mu(\bar{v}) \geq 0$, $\bar{p}$ is admissible for (2). If we multiply $A^T\bar{p} = c$ by $\bar{x}$, we get $< c, \bar{x} > - < b, \bar{p} >= \frac{1}{\mu\tau}||\bar{p}||^2$.
If the rank of A is equal to m, then (AA^T) is an $m \times m$ positive invertible matrix. Its eigenvalues (ordered by increasing size) $a_1, ..., a_m$ are stricly positive, and from $A^T\bar{p} = c$, one gets $||\bar{p}|| \leq \frac{1}{a_1}||c|| \leq M < +\infty$.
Thus, $< c, \bar{x} > - < b, \bar{p} >\leq \frac{1}{\mu\tau}M^2$. ∎

Application. In order to get an approximation of the optimal cost, $< c, x^* >$, with an absolute precision ϵ , one can choose $\mu_\epsilon = \frac{M^2}{\epsilon\tau}$. One might expect, however, that due to the form of g_μ, some bifurcation behavior can appear (See [9] for typical examples). It will thus be advisable to start the network with a moderate μ and increase it progressively to the value μ_ϵ. This procedure would bear some analogy with an "annealing" technique [1,9].

5. Conclusion. In this paper, we have studied two neural networks models for Linear Programming, the Hopfield and Tank network, and the Primal-Dual network. We have shown that the Primal-Dual network converges to admissible solutions and can be used to get a very good approximation of the optimal cost. Throughout the paper, we also addressed some implementation issues. Extensive numerical simulations and analog circuit implementation will be our next focus [11].

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