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Author(s): G. Zweig, T-DOT

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Wavelet Transforms as Solutions of Partial Differential Equations

George Zweig*

Abstract

This is the final report of a three-year, Laboratory Directed Research and Development (LDRD) project at Los Alamos National Laboratory (LANL). Wavelet transforms are useful in representing transients whose time and frequency structure reflect the dynamics of an underlying physical system. Speech sound, pressure in turbulent fluid flow, or engine sound in automobiles are excellent candidates for wavelet analysis. This project focused on 1) methods for choosing the parent wavelet for a continuous wavelet transform in pattern recognition applications and 2) the more efficient computation of continuous wavelet transforms by understanding the relationship between discrete wavelet transforms and discretized continuous wavelet transforms. The most interesting result of this research is the finding that the generalized wave equation, on which the continuous wavelet transform is based, can be used to understand phenomena that relate to the process of hearing.

1 Background

A significant recent development in applied mathematics is the study of wavelet transforms, discrete and continuous. Like Fourier transforms, wavelet transforms express functions in terms of simple building blocks; in particular, wavelet transforms decompose functions into components with respect to a set of expansion functions that are dilations and translations of a single function called the "parent wavelet." Of greatest interest are parent wavelets that are "almost compact" in *both* time and frequency. This contrasts with the Fourier transform whose expansion functions are delta functions in frequency but spread undiminished in time. Such wavelet transforms are useful in representing "transients" whose time and frequency structure reflect the dynamics of an underlying physical system. Speech sound, pressure in turbulent fluid flow, or engine sound in automobiles are excellent candidates for wavelet analysis.

There are two main classes of wavelets: discrete and continuous. A class of particular interest are the discrete wavelet transforms that use an orthonormal set of expansion functions generated by a discrete set of dilations (typically, powers of 2) and translations of the parent wavelet. A second interesting class are the continuous wavelet transforms based on a set of nonorthogonal and highly overcomplete expansion functions generated with continuous dilations and translations. While there is great freedom in

*Principal Investigator (E-mail: zweig@lanl.gov)

choosing the parent wavelet for a continuous wavelet transform, the choice of parent wavelet for an orthonormal discrete wavelet transform is severely restricted; the orthonormality conditions translate into restrictions on the functional form of the parent wavelet.

The distinctions between the discrete and the continuous wavelet transforms have several implications for their speed of computation and range of applicability. Computation in the discrete case is typically vastly faster than in the continuous case. In terms of applications, discrete wavelets have been shown to be capable of greatly decreasing the amount of information needed to represent functions, and thus are well suited for data compression. On the other hand, the lattice structure underlying discrete wavelets does not reflect the continuous nature of time or frequency and so is ill suited to pattern recognition applications. Continuous wavelets transforms, by contrast, capture in a simple way similarities in certain signals, such as those created at different times, or emitted at different rates, and thus provide powerful tools for problems in pattern recognition.

Our work has focused on the continuous wavelet transform and its connection to the process of hearing. The continuous wavelet transform represents a generalization of the transformation of sound that takes place within the cochlea (Zweig, 1976).

1.1 The continuous wavelet transform defined

If $f(t)$ is a signal and $\hat{f}(\omega_c, t)$ is its wavelet transform, then the forward transform is defined by

$$\hat{f}(\omega_c, t) \equiv \int_{-\infty}^{\infty} d\tau \left\{ \sqrt{\omega_c} w[\omega_c(t - \tau)] \right\} f(\tau),$$

and the inverse transform is given by,

$$f(t) = \int_0^{\infty} d\omega_c \int_{-\infty}^{\infty} d\tau \left\{ \sqrt{\omega_c} w[\omega_c(\tau - t)] \right\} \hat{f}(\omega_c, \tau).$$

where w is called the parent wavelet and must satisfy

$$\int_{-\infty}^{\infty} w(t) dt = 0.$$

As long as $w(t)$ has zero area it defines a continuous wavelet transform.

More compactly,

$$\hat{f}(\omega_c, t) \equiv \sqrt{\omega_c} w(\omega_c t) \otimes f(t),$$

and

$$f(t) = \int_0^\infty d\omega_c \sqrt{\omega_c} w(-\omega_c t) \otimes \hat{f}(\omega_c, t).$$

2 Results

2.1 Choosing the parent wavelet

How should the parent wavelet be chosen for a particular pattern recognition application? Suppose the signal being analyzed represents the vibration of a reactor vessel or the sound of a phoneme in speech. Each of those signals is created by exciting the resonant modes of a source, each mode having its own frequency and time constant. The problem is to extract those modal parameters so that the mechanical state of the source can be reconstructed. Such reconstruction can be helpful in diagnosing reactor vessel problems or in characterizing speech for speech recognition systems.

We defined an “entropy” that characterizes the “concentration” of the continuous wavelet transform of a signal. The idea is to pick a wavelet that provides the greatest concentration (the least entropy) of the wavelet transform coefficients. The envelope of the continuous wavelet transform is a positive quantity whose volume over the time-frequency plane can be normalized to one. That normalized envelope can then be interpreted as a probability distribution and a corresponding entropy defined. A set of causal parent wavelets can be defined parametrically and the parameters varied until the wavelet with minimum entropy is found.

More precisely, the entropy $\mathcal{E}\{w\}$ of the continuous wavelet transform of a signal $f(t)$ is defined by

$$\mathcal{E}\{w\} \equiv - \int_{-\infty}^{\infty} dt \int_0^{\infty} d\omega_c e\{\hat{f}(\omega_c, t)\} \log e\{\hat{f}(\omega_c, t)\},$$

where $e\{\hat{f}(\omega_c, t)\}$ is the normalized envelope of the continuous wavelet transform $\hat{f}(\omega_c, t)$ of $f(t)$ with respect to the wavelet $w(t)$.

$$\int_{-\infty}^{\infty} dt \int_0^{\infty} d\omega_c e\{\hat{f}(\omega_c, t)\} = 1,$$

and

$$\hat{f}(\omega_c, t) = \int_{-\infty}^{\infty} dt' w[\omega_c(t - t')] f(t').$$

For speech applications, we found that a simple parent wavelet that almost minimizes the entropy $\mathcal{E}$ while still preserving the causal characteristics of the transform has a Fourier transform given by

$$W(s) = \frac{s}{s^2 + \delta s + 1}$$

where

$$W(s) = \int_{-\infty}^{\infty} e^{-st} w(t) dt.$$

and the constant δ is chosen to match the formant bandwidths of speech ($\delta \approx 1/8$).

This work extends the signal dependent discrete orthonormal transforms introduced by Coifman and his collaborators. They work with the magnitude of the wavelet transform rather than its envelope. In the discrete case the concept of the envelope is not well defined. In the continuous case either magnitude of the wavelet transform, or its envelope, can be used to define an entropy. We find that the envelope extracts the essential information in speech better than magnitude of the transform coefficients.

2.2 Speeding up the computation

A broad class of continuous wavelet transforms can be realized as solutions to the two-dimensional generalized wave-equation (Zweig, 1991). The function to be transformed defines the boundary condition which specifies the temporal dependence of a point-like excitation of a wave medium. The continuous wavelet transform of the function defining the boundary condition is the solution to the partial differential equation. The dispersive properties of the wave medium define the parent wavelet. A certain symmetry of the generalized wave-equation provides sufficient conditions that its solutions be wavelet transforms (Zweig, 1991).

The generalized wave-equation is defined by

$$\lambda^2(\hat{s}) \frac{\partial^2 f}{\partial \tau^2} - \frac{\partial^2 f}{\partial t^2} = 0,$$

where the wavelength in the ordinary wave equation is replaced by the operator $\lambda(\hat{s})$, with $\hat{s} \equiv \tau \frac{\partial}{\partial t}$.

An example of an interesting operator λ , inspired by mechanical measurements of the squirrel monkey cochlea, is given by

$$\lambda^2(\hat{s}) \equiv \epsilon (\hat{s}^2 + \delta \hat{s} + 1 + \rho e^{-\psi \hat{s}}),$$

where ϵ , δ , ρ , and ψ are constants. Typically ϵ is small so that an asymptotic analysis is possible. The functional form of λ^2 determines the shape of the wavelet. More precisely, the wavelet $w(\omega_c t)$ is the solution to the generalized wave equation driven at time $t = 0$ by an impulse $\delta(t)$.

We demonstrated that continuous wavelet transforms, like that used by the squirrel monkey, can be computed most efficiently by interpreting them as solutions to generalized wave-equations and computing the solution to those equations numerically. The trick is to exploit the parameter of smallness ϵ in the equation which implies that the waves propagate unreflected. This simplifies the numerical analysis and makes it possible to solve the equation with parallel processing algorithms. We developed a numerical algorithm for this task that works, but is not as robust as we would like. For long time, e.g., times longer than the duration of a phoneme in speech, the algorithm becomes unstable. We know how to correct this deficiency, and will do so in the future.

2.3 An unexpected finding

The most interesting and unexpected result of this research relates to the finding that the generalized wave equation, on which the continuous wavelet transform is based, can be used to understand phenomena that relate to the process of hearing. The results are published in "Zweig, G. and C. SHERA, "The Origin of Periodicity in the Spectrum of Evoked Otoacoustic Emissions," J. Acoust. Soc. Am. 98, 2018-2047, (1995). An abstract of that work follows.

Abstract: Current models of evoked otoacoustic emissions explain the striking periodicity in their frequency spectra by suggesting that it originates through the reflection of forward-traveling waves by a corresponding spatial corrugation in the mechanics of the cochlea. Although measurements of primate cochlear anatomy find no such corrugation, they do indicate a considerable irregularity in the arrangement of outer hair cells. It is suggested

that evoked emissions originate through a novel reflection mechanism, representing an analogue of Bragg scattering in nonuniform, disordered media. Forward-traveling waves reflect off random irregularities in the micromechanics of the organ of Corti. The tall, broad peak of the traveling wave defines a localized region of coherent reflection that sweeps along the organ of Corti as the frequency is varied monotonically. Coherent scattering occurs off irregularities within the peak with spatial period equal to half the wavelength of the traveling wave. The phase of the net reflected wave rotates uniformly with frequency at a rate determined by the wavelength of the traveling wave in the region of its peak. Interference between the backward-traveling wave and the stimulus tone creates the observed spectral periodicity. Ear-canal measurements are related to cochlear mechanics by assuming that the transfer characteristics of the middle ear vary slowly with frequency compared to oscillations in the emission spectrum. The relationship between cochlear mechanics at low sound levels and the frequency dependence of evoked emissions is made precise for one-dimensional models of cochlear mechanics. Measurements of basilar-membrane motion in the squirrel monkey are used to predict the spectral characteristics of their emissions. And conversely, noninvasive measurements of evoked otoacoustic emissions are used to predict the width and wavelength of the peak of the traveling wave in humans.

Details of these results are found in the paper by Zweig and Shera. Here we mention a few of the highlights.

- Random inhomogeneities create coherent backscattering. This is an analogue of Bragg scattering for a nonuniform random medium. The backscattering leads to otoacoustic emissions.
- Otoacoustic emissions are detectable and relatable to fundamental mechanical properties of the cochlea. To describe this relation we define an intermediate quantity, the traveling wave ratio R at the stapes of the backward and forward traveling waves:

$$R \equiv \frac{P_-}{P_+} \bigg|_{\text{stapes}} .$$

R is related to a measurable quantity ρ , the ratio of two pressures measured in the ear canal:

$$\begin{aligned}
\rho(\omega, R) &\equiv \frac{P_{ec}(\omega; R)}{P_{ec}(\omega, 0)} \\
&= 1 + mR(1 + rR + r^2R^2 + \dots) \\
&= 1 + \frac{mR}{1 - rR},
\end{aligned}$$

where m is a slowly varying function of frequency that depends on the properties of the middle ear, and r is the reflection coefficient of the stapes. The quantity $P_{ec}(\omega, 0)$ is found from measurements at high sound pressure levels where the pressure in the ear canal is essentially independent of the backward traveling wave (the ear can pump only a limited amount of energy into traveling waves, so that its contribution to the energy in waves becomes negligible at high sound pressure levels). The connection of R to the internal mechanics of the cochlea is given by:

$$R(\nu_o) \propto \int_{-\infty}^{\infty} V(\nu_o - \nu) D^2(\nu) d\nu,$$

where V is a scattering potential that is proportional to the fluctuations in Z , the impedance of the organ of Corti.

$$V \propto Z - \langle Z \rangle.$$

and

$$\nu \equiv \nu_o - x/l,$$

with

$$\nu_o \equiv \ln(\omega/\omega_o),$$

where x is the position along the cochlea. l is a constant, and ω_o is the maximum frequency of hearing.

- Hearing threshold curves must oscillate:

Theory implies $R \approx |R|e^{-i(\omega\tau + \phi)}$, where τ is the round trip travel time of a wave of frequency ω traveling to and from the position of maximum reflection.

It is convenient to define an intermediate quantity η , related to the measurable ρ , from which $|rR|$ and τ can be determined:

$$\eta = \Re\{\eta\} + i \Im\{\eta\} \equiv \frac{d \ln(\rho - \langle \rho \rangle)}{d(\omega\tau)}.$$

This implies

$$\begin{aligned} \Re\{\eta\} &= -|rR| \sin(\omega t + \phi) \\ \Im\{\eta\} &= -1 - |rR| \cos(\omega t + \phi). \end{aligned}$$

Using measurements of ρ which we made, we find that

$$\tau = 11.8 \pm 0.1 \text{ msec}, \quad |rR| = 0.12 \pm 0.01.$$

From this we deduce that

$$\lambda_x \approx 1 \text{ mm}; \quad Q \equiv \omega_c / 2\Delta\omega \approx 8,$$

where λ_x is the wavelength of the traveling wave and Q is its dimensionless width.

- Tinnitus exists with the spacing between emissions being multiples of Δf , where

$$\frac{\Delta f}{f} = \frac{\hat{\lambda}_x}{2l} \approx \frac{1}{15}.$$

3 Publications

Zweig, G. and C. Shera," The Origin of Periodicity in the Spectrum of Evoked Otoacoustic Emissions," J. Acoust. Soc. Am. 98. 2018-2047, (1995)

4 References

1) Zweig, G. (1976), "Basilar membrane motion," in Cold Spring Harbor Symposia on Quantitative Biology, Vol. XL, 619-633.

2) Zweig, G. (1991), "Finding the impedance of the organ of Corti," J. Acoust. Soc. Am. 89(3), 1229-1254.