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DISPERSION FUNCTION AND CLOSED ORBIT DISTORTION IN ACCELERATOR RINGS

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A. Introduction

1.) Dispersion Function Distortion (DFD) affects accelerator operation and thus deserves attention, somehow as Closed Orbit Distortion (COD) does. Consequently, DFD correction schemes under computer control have been successfully developed and adopted in many rings such as PEP.[1)-4)]

It was realized during the author's study on the problem, however, that the existing DFD correction schemes ignore those terms that arise from bending magnets and their edges. Being of first order of correcting strength, the terms are significant in small rings, though really not important in big machines. This reminds of what has been noticed in the chromaticity calculation. A comparison between this note and the existing schemes shows a difference in DFD sensitivity matrix that is significant for sub-GeV machines and appears not negligible even for SPEAR.

Many storage rings at energy around 0.7 - 3 GeV are being proposed, constructed or operated everywhere as synchrotron radiation generators. Vertical DFD correction should be an important part of their operation, because vertical DFD enlarges beam height and hence reduces light source brightness. This is the purpose the author had in mind when beginning to study the problem. In addition, a correct DFD analysis along with COD analysis can hopefully help spot magnet misalignments and remove them.

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2.) The similarity of DFD correction to the chromaticity calculation also lies in that, for both of them, one has to start with the second order particle motion equations though what he looks for is only a first order dependence of a parameter on particle momentum. A previous note, Ref.5), presents the complete expressions of the first and second order particle motion equations, in either continuous field or "hard edges", and thereby a sound ground for this note to start from. The second topic of Ref.5) is the chromaticity calculation, resulting in some formulae and conclusions. A comparision of them with those obtaind in this note seems worthwhile, for the similarity mentioned above will be made clear.

3.) The first step in this note is to introduce the first order equations of COD and DFD by handling the first and second order particle motion equations. Then a general analytical solution to the equations is derived. After a discussion on COD correction to see if its schemes have to be modified in some cases, DFD correction is treated and new expressions of its sensitivity matrix presented. The last part will discuss the difference of the new expressions with the existing ones.

4.) Major assumptions taken throughout this note are:

a) Single particle (zero current) model. So any interaction between particles or between a particle and its environment is ignored.

b) No electric field in the part of orbit being studied, no particle energy change and no time dependence of magnetic field.

c) The ideal orbit lies in the median (symmetric) plane of magnetic field, so the natural orthogonal coordinate system x-y-z (curvilinear system), with y-axis fixed vertically, can be referred to and the ideal vertical dispersion is zero anywhere.

d) Magnetic field is piecewisely constant in all the magnets and is described by hard edge approximation at their edges.

e) Particle parameters x, x', y, y' and δ are small quantities.

f) Correcting magnetic field is much smaller than normal bending field, so the field deviation, g_u , is also regarded as a small quantity and its influence on beta function as well as the coupling between two correctors is neglected.

B. The First Order Equations of COD and DFD

1.) Symbol convention throughout this note is as follows:

For the coordinate system adopted, z is the azimuthal coordinate and ' is d/dz ; x and x' are the horizontal displacement and slope, y and y' the vertical displacement and slope, respectively. p is particle momentum and p_o the nominal momentum, so momentum deviation $\delta = (p-p_o)/p_o$ which is also the energy deviation for highly relativistic electrons.

u is used to denote x or y ; $u_c(\delta)$ the closed orbit of the particles with momentum deviation δ ; $u_c = u_c(0)$ the central closed orbit, namely, the COD ; η the ideal horizontal dispersion function ; $\tilde{\eta}_u$ the DFD on u plane ; W either u_c or $\tilde{\eta}_u$. β_u, α_u, and γ_u are the Twiss functions and ν_u the tune on u plane.

A few subscripts are used to denote special positions, among which "i" denotes the midpoint of the i-th corrector, "j" the j-th monitor, "m" the midpoint of the m-th magnet, "e" a magnet edge, "b" the point just before the edge and "a" the point just after the edge. For a magnet, define some numerical subscripts as : at the entrance, e = 1, b = 0, a = 1 ; while at the exit, e = 2, b = 2, a = 3. The definitions of subscripts 0, 1, m, 2 and 3 are illustrated in Fig.1 . Note that "e" always denotes the edge point inside the magnet.

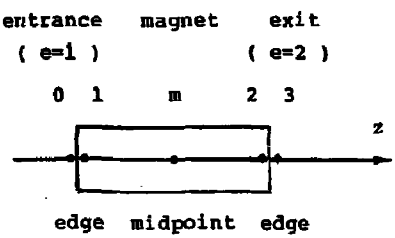

Fig.1 Definitions of subscripts 0, 1, m, 2 and 3

Let s_1 and s_2 be two such subscripts. The phase advance between s_1 and s_2 , $\psi_u(s_2/s_1)$, is defined by

$$\psi_u(s_2/s_1) = \int_{s_1}^{s_2} (1/\beta_u(z))\, dz$$

and $M_u(s_2/s_1)$ is the u plane transfer matrix from s_1 to s_2 .

The magnetic components are piecewise constants, such as.

$$B_y/(B\rho)_o = \frac{1}{\rho} - g_x \; ; \qquad B_x/(B\rho)_o = g_y \; ;$$

$$\frac{1}{(B\rho)_o}\left(\frac{\partial B_y}{\partial x}\right) = K \; ; \qquad \frac{1}{(B\rho)_o}\left(\frac{\partial^2 B_y}{\partial x^2}\right) = \lambda \qquad \text{(B-1)}$$

If not specially explained, the other components are all ignored. In Eq. (B-1), $(B\rho)_o = p_o/e$ is particle rigidity, ρ the curvature radius of ideal orbit ; $(1/\rho)$ is the dipole component, which is not zero in and only in bending magnets ; K the quadrupole component which is not zero in quadrupoles and in combined function bending magnets, where $K = -n/\rho^2$ with n the magnetic field index ; λ the sextupole component which is not zero in and only in sextupoles ; g_u the field deviation which is not zero in and only in misaligned magnets or the currently applied correctors. As stated before, $|g_u| \ll (1/\rho)$.

A magnet edge is characterized by the inclining angle θ_e and the edge face curvature radius r_e with the sign conventions shown in Fig.2 .

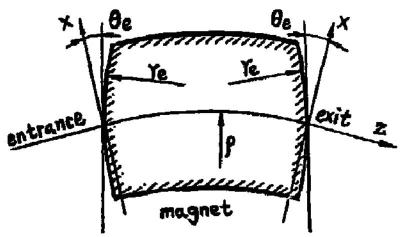

Fig.2 Sign conventions of θ_e, r_e and ρ are all positive as shown above.

2.) The up-to-second-order differential equations, for particle motion within a magnet other than a solenoid or a rotated quadrupole, can be found in many papers, Ref.5) as an example, as

$$x'' = -(K + \frac{1}{\rho^2})\,x + \frac{\delta}{\rho} + g_x + \frac{2}{\rho} g_x x - g_x \delta - (\frac{\lambda}{2} + \frac{2K}{\rho} + \frac{1}{\rho^3})\,x^2$$
$$+ (-K + \frac{2}{\rho^2})\,\delta x - \frac{\delta^2}{\rho} + \frac{1}{2}(\lambda + \frac{K}{\rho})\,y^2 + \frac{1}{2\rho} x'^2 - \frac{1}{2\rho} y'^2 + \dots \; ;$$
$$y'' = K y + g_y + \frac{2}{\rho} g_y x - g_y \delta + (\lambda + \frac{2K}{\rho})\,xy$$
$$- K\delta y + \frac{1}{\rho} x' y' + \dots \qquad \text{(B-2)}$$

Eq.(B-2) differs from the well-known up-to-second-order motion equations in two aspects: g_u is a nonzero first order small quantity since one is considering the distortion and correction caused by g_u ; $(1/\rho)'$ is zero due to the piecewisely constant field assumption. The field changes take place at hard edges only and are treated seperately.

A description of the up-to-second-order hard edge effects in terms of parameter increments is presented in Ref.5) as

$$x_a = x_b + \Delta_e \frac{1}{2\rho} x_b^2 t_e^2 - \Delta_e \frac{1}{2\rho} y_b^2 (1 + t_e^2) + \dots \ ;$$

$$y_a = y_b - \Delta_e \frac{1}{\rho} x_b y_b t_e^2 + \dots \ ;$$

$$x'_a = x'_b - K_e x_b - \frac{1}{2} (\lambda_e + \lambda^{(1)}_{xe}) x_b^2 + \frac{1}{2} (\lambda_e + \lambda^{(1)}_{xye}) y_b^2$$

$$+ K_e \delta x_b - \Delta_e \frac{1}{\rho} x_b x'_b (1 + t_e^2) + \Delta_e \frac{1}{\rho} y_b y'_b t_e^2 + \dots \ ;$$

$$y'_a = y'_b + K_e y_b + (\lambda_e + \lambda^{(1)}_{ye}) x_b y_b - K_e \delta y_b$$

$$- \Delta_e \frac{1}{\rho} x_b y'_b (1 - t_e^2) + \Delta_e \frac{1}{\rho} x'_b y_b (1 + t_e^2) + \dots \qquad \text{(B-3)}$$

where the symbols are defined by

Δ_e may be: $\Delta_1 = -1$ (at an entrance) or $\Delta_2 = 1$ (at an exit) ;

$t_e = \tan \theta_e$ (e = 1 or 2) ;

edge quadrupole component $K_e = - \frac{1}{\rho} t_e$;

edge sextupole component $\lambda_e = - 2 K t_e - 1/(\rho r_e \cos^3 \theta_e)$ (B-4)

and six special sextupole components

$$\lambda^{(1)}_{x1} = - \frac{2}{\rho^2} t_1 \ ; \qquad \lambda^{(1)}_{x2} = \frac{1}{\rho^2} t_2^3 \ ;$$

$$\lambda^{(1)}_{y1} = - \frac{1}{\rho^2} t_1 \ ; \qquad \lambda^{(1)}_{y2} = \frac{1}{\rho^2} t_2 (1 + t_2^2) \ ;$$

$$\lambda^{(1)}_{xy1} = \frac{1}{\rho^2} t_1 (1 + 2 t_1^2) \ ; \qquad \lambda^{(1)}_{xy2} = - \frac{1}{\rho^2} t_2^3 \qquad \text{(B-5)}$$

Here ρ and K are those inside the magnet in question. The different expressions for $\lambda^{(1)}_{ue}$ and $\lambda^{(1)}_{xye}$ suggest that edge sextupole effects vary both with transverse planes (x or y) and with edges (entrance or exit).

3.) $u_c(\delta)$ is the periodic solution of Eqs.(B-2) and (B-3). Since COD is nothing but $u_c = u_c(0)$, the first order equation of COD is readily found as

$$u_c'' + F_u u_c = g_u \tag{B-6}$$

$$u_{ca} - u_{cb} = 0 \quad ; \qquad u'_{ca} - u'_{cb} = - f_{ue} u_{cb} \tag{B-7}$$

where the focusing strengths are given by

$$F_x = K + \frac{1}{\rho^2} \quad ; \quad F_y = - K \quad ; \quad f_{xe} = - f_{ye} = K_e = -\frac{1}{\rho} \tan\theta_e \tag{B-8}$$

COD is, in the sense "to first order", the periodic solution of Eqs.(B-6) and (B-7), with Eq.(B-6) for the behavior of COD within any magnet and Eq.(B-7) for that at magnet edges. Note that, because of the first order terms in the edge effect increments of x' and y', u_c' is discontinuous at an inclined edge and so are β_u' and η'. This requires a clear distinction between the functions with subscript "a" and those with "b" for such "derivative functions".

4.) Before deriving the equations of DFD, one must make clear what DFD is. Let the definition of DFD be introduced as

$$\tilde{\eta}_x = \lim_{\delta \to 0} \left(\left(x_c(\delta) - x_c(0) \right) / \delta \right) - \eta \quad ;$$

$$\tilde{\eta}_y = \lim_{\delta \to 0} \left(\left(y_c(\delta) - y_c(0) \right) / \delta \right) \tag{B-9}$$

In other words,

$$x_c(\delta) = x_c + \delta (\eta + \tilde{\eta}_x) + \delta^2 \cdot (\ldots\ldots) \quad ;$$

$$y_c(\delta) = y_c + \delta \tilde{\eta}_y + \delta^2 \cdot (\ldots\ldots) \tag{B-10}$$

And η is the periodic function which satisfies

$$\eta'' + F_x \eta = \frac{1}{\rho} \quad ; \quad \eta_a - \eta_b = 0 \quad ; \quad \eta'_a - \eta'_b = - f_{xe} \eta_b \tag{B-11}$$

Combining Eqs.(B-2), (B-3), (B-8) through (B-11), and dropping the second order terms finally, one comes to the DFD equations without much difficulty:

$$\tilde{\eta}_u'' + F_u\ \tilde{\eta}_u = \left(\frac{2}{\rho}\eta - 1\right) g_u + Q_u\ u_c + R_u\ u_c' \qquad \text{(B-12)}$$

$$\tilde{\eta}_{ua} - \tilde{\eta}_{ub} = \overline{P}_{ue}\ u_{ce}\ ;$$

$$\tilde{\eta}'_{ua} - \tilde{\eta}'_{ub} = -\ f_{ue}\ \tilde{\eta}_{ub} + \overline{Q}_{ue}\ u_{ce} + \overline{R}_{ue}\ u'_{ce} \qquad \text{(B-13)}$$

where the functions Q_u, R_u, etc. are given by

$$Q_x = K + \frac{2}{\rho^2} - \left(\lambda + \frac{4K}{\rho} + \frac{2}{\rho^3}\right)\eta\ ; \qquad Q_y = -\ K + \left(\lambda + \frac{2K}{\rho}\right)\eta\ ;$$

$$R_x = R_y = \frac{1}{\rho}\ \eta' \qquad \text{(B-14)}$$

and $\overline{P}_{ue} = \overline{sg}_u \cdot \Delta_e \frac{1}{\rho}\ \eta_e\ t_e^2$;

$$\overline{Q}_{ue} = \overline{sg}_u \cdot \left(K_e - \left(\lambda_e + \lambda^{(2)}_{ue} \right) \eta_e - \Delta_e \frac{1}{\rho}\ \eta'_e \left(1 + t_e^2 \right)\right)\ ;$$

$$\overline{R}_{ue} = -\ \Delta_e \frac{1}{\rho}\ \eta_e \left(1 + \overline{sg}_u \cdot t_e^2 \right) \qquad \text{(B-15)}$$

The symbol $\overline{sg}_u$ is defined as : $\overline{sg}_x = 1$; $\overline{sg}_y = -\ 1$. The special sextupole components are revised as $\lambda^{(2)}_{ue}$'s, given by

$$\lambda^{(2)}_{x1} = \frac{2}{\rho^2}\, t_1^3 \ ; \qquad \lambda^{(2)}_{y1} = \frac{1}{\rho^2}\, t_1 \ ; \qquad \lambda^{(2)}_{u2} = \lambda^{(1)}_{u2} \qquad \text{(B-16)}$$

Note that η'_e and u'_{ce} are evaluated inside the magnet, so their subscripts are "e" (1 or 2) instead of "b". To express the functions by inside magnet values will help formula uniformity and simplification. Eqs.(B-7) and (B-11) are useful when the replacements of u'_{cb} (by u'_{ce}) and of η'_b (by η'_e) are needed, that is, at the entrance.

Similarly to COD, DFD is the periodic solution of Eqs.(B-12) and (B-13). Eqs.(B-12) and (B-14) describe DFD within any magnet while its behavior at magnet edges is represented by Eqs.(B-13) and (B-15).

5.) From the equations above, several conclusions can be drawn such as:

a) A change of the field in any corrector (Δg_u) will cause both COD and DFD to change all around the ring due to their periodicity.

b) A COD change in any magnet where K, λ and/or ($1/\rho$) are not zero or at any edge where K_e, λ_e and/or ($1/\rho$) are not zero will cause DFD change as effectively as Δg_u in the corrector does.

c) In the sense "to first order", the contribution from one corrector to COD change anywhere and thus to DFD change anywhere is proportional to the g_u change of that corrector.

d) On either COD or DFD, no first order coupling effect between the two transverse planes exists, i.e. any change in g_x and hence in x_c does not affect y_c or $\tilde{\eta}_y$ and neither in the other way, though nonzero ideal horizontal dispersion η has an important role in the equation of $\tilde{\eta}_y$. This is true for magnets other than solenoids or rotated quadrupoles.

6.) For completeness, the equations describing COD and DFD in solenoids or rotated quadrupoles, along with a few comments, are also presented here. But no further attention will be paid to them. The particle motion equations in these cases can also be found in Ref.5) .

a) For rotated quadrupoles. Let $h_R = \frac{1}{(B\rho)_o}(\frac{\partial Bx}{\partial x}) = -\frac{1}{(B\rho)_o}(\frac{\partial By}{\partial y})$. The particle motion equations are

$$x'' = h_R (1 - \delta) y \quad ; \qquad y'' = h_R (1 - \delta) x \qquad (B\text{-}17)$$

A treatment similar to that in the previous subsections finds the COD and DFD equations, respectively, as

$$x_c'' = h_R y_c \quad ; \qquad y_c'' = h_R x_c \quad ;$$

$$\tilde{\eta}_x'' = - h_R (y_c - \tilde{\eta}_y) \quad ; \qquad \tilde{\eta}_y'' = h_R (\eta - x_c + \tilde{\eta}_x) \qquad (B\text{-}18)$$

b) For solenoids. Let $g_S = B_z / (B\rho)_o$. Then, for the particle motion inside the solenoid, one has

$$x'' = g_S (1 - \delta) y' \quad ; \qquad y'' = - g_S (1 - \delta) x' \qquad (B\text{-}19)$$

and, for that at its edges,

$$x_a = x_b \quad ; \qquad x'_a = x'_b - \frac{1}{2}\Delta_e\, g_S\,(1-\delta)\, y_b \quad ;$$

$$y_a = y_b \quad ; \qquad y'_a = y'_b + \frac{1}{2}\Delta_e\, g_S\,(1-\delta)\, x_b \qquad \text{(B-20)}$$

Therefore, the equations of COD and DFD are obtained as

$$x_c'' = g_S\, y_c' \quad ; \qquad\qquad y_c'' = -\, g_S\, x_c' \quad ;$$

$$u_{ca} = u_{cb} \;; \quad x'_{ca} = x'_{cb} - \frac{1}{2}\Delta_e\, g_S\, y_{cb} \;; \quad y'_{ca} = y'_{cb} + \frac{1}{2}\Delta_e\, g_S\, x_{cb} \;;$$

$$\tilde{\eta}_x'' = -\, g_S\,(\, y_c' - \tilde{\eta}_y'\,) \quad ; \qquad \tilde{\eta}_y'' = -\, g_S\,(\, \eta' - x_c' + \tilde{\eta}_x'\,) \;;$$

$$\tilde{\eta}_{ua} = \tilde{\eta}_{ub} \;; \quad \tilde{\eta}'_{xa} = \tilde{\eta}'_{xb} + \frac{1}{2}\Delta_e\, g_S\,(\, y_{cb} - \tilde{\eta}_{yb}\,) \quad ;$$

$$\tilde{\eta}'_{ya} = \tilde{\eta}'_{yb} + \frac{1}{2}\Delta_e\, g_S\,(\, \eta_b - x_{cb} + \tilde{\eta}_{xb}\,) \qquad \text{(B-21)}$$

c) The following comments, resulting from a review of Eqs.(B-18) and (B-21), are valid for both solenoids and rotated quadrupoles :

Nonzero ideal horizontal dispersion η at their locations makes them significant sources of vertical DFD.

They give rise to coupling between x and y planes. An x_c and $\tilde{\eta}_x$ correction will influence y_c and $\tilde{\eta}_y$ if there are solenoids and/or rotated quadrupoles in the ring, and vice versa.

However, if the magnetic field in solenoids is much smaller than that in bending magnets ($|g_S| \ll (1/\rho)$) and the gradient in rotated quadrupoles much smaller than that in normal quadrupoles ($|h_R| \ll |K|$), they do not affect first order COD and DFD correction schemes, although they may be partly responsible for why the correction is needed.

These points are important not only because solenoids and/or rotated quadrupoles may be installed in a ring, but also because magnet errors produce solenoid and/or rotated quadrupole effects in normal magnets.

C. The General Analytical Solution to COD or DFD Equation

1.) Now the question is how to find the periodic solution to equations

$$W'' + F_u W = H(z) \tag{C-1}$$

$$W_a - W_b = \bar{I}_e \quad ; \qquad W'_a - W'_b = - f_{ue} W_b + \bar{J}_e \tag{C-2}$$

which are a general expression of Eqs.(B-6),(B-7),(B-12) and (B-13). H(z) is a z-dependent function that is zero outside magnets, while $\bar{I}_e$ and $\bar{J}_e$ functions evaluated at edge e.

2.) It is unnecessary to say that the first order transverse motion equations of a particle without energy deviation are

$$u'' + F_u u = 0 \tag{C-3}$$

$$u_a - u_b = 0 \quad ; \qquad u'_a - u'_b = - f_{ue} u_b \tag{C-4}$$

Define the piecewise cosine-like and sine-like solutions of Eq.(C-3) as $C_u(z)$ and $S_u(z)$ respectively. Since F_u is a piecewise constant, one can give, within a magnet,

$$C_u(z) = \sum_{n=0}^{\infty} (-F_u)^n z^{2n} / (2n)! = \begin{cases} \cos(\sqrt{F_u}\, z) , & \text{if } F_u > 0 \\ 1 , & \text{if } F_u = 0 \\ \cosh(\sqrt{-F_u}\, z) , & \text{if } F_u < 0 \end{cases} \tag{C-5}$$

$$S_u(z) = \sum_{n=0}^{\infty} (-F_u)^n z^{2n+1} / (2n+1)! = \begin{cases} \sin(\sqrt{F_u}\, z)/\sqrt{F_u} , & \text{if } F_u > 0 \\ z , & \text{if } F_u = 0 \\ \sinh(\sqrt{-F_u}\, z)/\sqrt{-F_u} , & \text{if } F_u < 0 \end{cases} \tag{C-6}$$

Functions C_u and S_u provide the convenience that the expressions involved will not depend on the sign of F_u, and have many other valuable properties. A detailed description for them can be found in the appendix of Ref.6). The following is a brief list of those properties to be made use of in the next subsection :

$C_u'(z) = - F_u S_u(z)$; $S_u'(z) = C_u(z)$;

$C_u(-z) = C_u(z)$; $S_u(-z) = - S_u(z)$;

$C_u^2(z) + F_u S_u^2(z) = 1$; $C_u(0) = 1$; $S_u(0) = 0$;

$C_u(z_1 + z_2) = C_u(z_1) C_u(z_2) - F_u S_u(z_1) S_u(z_2)$;

$S_u(z_1 + z_2) = S_u(z_1) C_u(z_2) + C_u(z_1) S_u(z_2)$

Eq.(C-3) and thereby Eqs.(C-5) and (C-6) are valid within a magnet, that is, in the region between points "1" and "2". Let L_m be the effective length of the m-th magnet. In terms of transfer matrix theory, the solution of Eq.(C-3) for the m-th magnet is represented by

$$M_u(2/1) = \begin{pmatrix} C_u(L_m) & S_u(L_m) \\ - F_u S_u(L_m) & C_u(L_m) \end{pmatrix} \qquad \text{(C-7)}$$

while Eq.(C-4) is represented by

$$M_u(a/b) = \begin{pmatrix} 1 & 0 \\ - f_{ue} & 1 \end{pmatrix} \qquad \text{(C-8)}$$

3.) Then, with the aid of δ-function theory, a solution of Eq.(C-1) can be expressed by functions C_u and S_u as

$$W_H(z) = \int_{z_1}^{z} S_u(z - \bar{z}) H(\bar{z}) d\bar{z} \qquad \text{(C-9)}$$

with $W_H'(z) = \int_{z_1}^{z} C_u(z - \bar{z}) H(\bar{z}) d\bar{z}$; $W_H(z_1) = W_H'(z_1) = 0$.

Therefore, with arbitrary initial conditions, the solution to Eq.(C-1) is

$$W(z) = W_1 C_u(z - z_1) + W'_1 S_u(z - z_1) + W_H(z)$$

It follows that function W which obeys Eq.(C-1) between points "1" and "2" can be described by a matrix equation

$$\begin{pmatrix} W_2 \\ W'_2 \end{pmatrix} = M_u(2/1) \begin{pmatrix} W_1 \\ W'_1 \end{pmatrix} + \begin{pmatrix} W_H(z_2) \\ W_H'(z_2) \end{pmatrix} \qquad \text{(C-10)}$$

And Eq.(C-2) is equivalent to

$$\begin{pmatrix} W_a \\ W'_a \end{pmatrix} = M_u(a/b) \begin{pmatrix} W_b \\ W'_b \end{pmatrix} + \begin{pmatrix} \overline{I}_e \\ \overline{J}_e \end{pmatrix} \qquad \text{(C-11)}$$

A complete description of the behavior of W function from "0" to "3", or through the m-th magnet, is then given by

$$\begin{pmatrix} W_3 \\ W'_3 \end{pmatrix} = M_u(3/0) \begin{pmatrix} W_0 \\ W'_0 \end{pmatrix} + \begin{pmatrix} \overline{I}_2 \\ \overline{J}_2 \end{pmatrix} + M_u(3/2) \begin{pmatrix} W_H(z_2) \\ W_H'(z_2) \end{pmatrix} + M_u(3/1) \begin{pmatrix} \overline{I}_1 \\ \overline{J}_1 \end{pmatrix} \qquad \text{(C-12)}$$

It is advisable to concentrate the effects on W function from H(z), $\overline{I}_e$ and $\overline{J}_e$ to the midpoint "m", as if W suddenly "jumped" at "m" with ΔW_m and $\Delta W'_m$ as the increments of W and W', respectively, but it satisfied Eqs.(C-3) and (C-4) anywhere else. Then the behavior of W looked like

$$\begin{pmatrix} W_3 \\ W'_3 \end{pmatrix} = M_u(3/0) \begin{pmatrix} W_0 \\ W'_0 \end{pmatrix} + M_u(3/m) \begin{pmatrix} \Delta W_m \\ \Delta W'_m \end{pmatrix} \qquad \text{(C-13)}$$

Since

$$\begin{pmatrix} W_H(z_2) \\ W_H'(z_2) \end{pmatrix} = \begin{pmatrix} -C_u(L_m/2) & S_u(L_m/2) \\ F_u\, S_u(L_m/2) & C_u(L_m/2) \end{pmatrix} \begin{pmatrix} \int_{z_1}^{z_2} S_u(\overline{z} - z_m)\, H(\overline{z})\, d\overline{z} \\ \int_{z_1}^{z_2} C_u(\overline{z} - z_m)\, H(\overline{z})\, d\overline{z} \end{pmatrix},$$

one finds by matrix multiplcation

$$\Delta W_m = -\int_{z_1}^{z_2} S_u(\overline{z} - z_m)\, H(\overline{z})\, d\overline{z} + C_u(L_m/2)\ (\overline{I}_2 + \overline{I}_1)$$
$$- S_u(L_m/2)\ (\overline{J}_2 + f_{u2}\, \overline{I}_2 - \overline{J}_1) \quad ;$$
$$\Delta W'_m = \int_{z_1}^{z_2} C_u(\overline{z} - z_m)\, H(\overline{z})\, d\overline{z} + F_u\, S_u(L_m/2)\ (\overline{I}_2 - \overline{I}_1)$$
$$+ C_u(L_m/2)\ (\overline{J}_2 + f_{u2}\, \overline{I}_2 + \overline{J}_1) \qquad \text{(C-14)}$$

4.) Then it is well known how to express the periodic function W that satisfies Eqs.(C-3) and (C-4) along the ring, but has a finite number of jumps. Suppose the "jump points" are designated by "p". The values of W and W' at point "s" are found by solving the equation

$$\begin{pmatrix} W_s \\ W'_s \end{pmatrix} = M_u(s/s) \begin{pmatrix} W_s \\ W'_s \end{pmatrix} + \sum_p M_u(s/p) \begin{pmatrix} \Delta W_p \\ \Delta W'_p \end{pmatrix} \qquad \text{(C-15)}$$

where $M_u(s/s)$ is the transfer matrix for one turn from s to s. Let I be the unit matrix. Since

$$(I - M_u(s/s))^{-1} = \frac{1}{2 \sin \pi \nu_u} \left(I \sin \pi \nu_u + \begin{pmatrix} \alpha_{us} & \beta_{us} \\ -\gamma_{us} & -\alpha_{us} \end{pmatrix} \cos \pi \nu_u \right)$$

one finds

$$\begin{pmatrix} W_s \\ W'_s \end{pmatrix} = \sum_p B_u(p,s) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \Delta W_p \\ \Delta W'_p \end{pmatrix} \qquad \text{(C-16)}$$

where $M_{11} = (\sin \phi_u(s/p) + \alpha_{up} \cos \phi_u(s/p)) / \beta_{up}$;

$M_{12} = \cos \phi_u(s/p)$;

$$M_{22} = (\sin \phi_u(s/p) - \alpha_{us} \cos \phi_u(s/p)) / \beta_{us} \qquad \text{(C-17)}$$

and

$$B_u(p,s) = \sqrt{\beta_{up} \beta_{us}} / (2 \sin \pi \nu_u) \qquad \text{(C-18)}$$

$$\phi_u(s/p) = \pi \nu_u - \psi_u(s/p) \qquad \text{(C-19)}$$

the expression of M_{21} will not be used in this note. Establishing the relation between function jumps at "p" and function values at "s", Eq. (C-16) is useful in both determing u_c and u_c' anywhere caused by given g_u and finding $\tilde{\eta}_u$ which is in turn generated by g_u, u_c and u_c' around the ring.

In principle, the problem of COD and DFD correction is almost solved by Eqs.(C-14) and (C-16), since all the functions required are given in section B.

D. About COD Correction

1.) For the COD equation, $\overline{I}_e = \overline{J}_e = 0$ and $H(z) = g_u$ which is constant in the applied corrector and is zero anywhere else. So the expression of the COD sensitivity matrix is easy to give. Since

$$\int_{z_1}^{z_2} S_u(\bar{z} - z_m)\, d\bar{z} = 0 \quad ; \qquad \int_{z_1}^{z_2} C_u(\bar{z} - z_m)\, d\bar{z} = 2\, S_u(L_m/2)$$

one can simply give the COD jump in the i-th corrector as

$$\Delta u_{ci} = 0 \quad ; \qquad \Delta u'_{ci} = \theta_{ui} \tag{D-1}$$

where θ_{ui}, usually referred to as "correcting strength" or "kick", is

$$\theta_{ui} = 2\, g_{ui}\, S_u(L_i/2) = \begin{cases} 2\, g_{ui} \sin(\sqrt{F_u}\, L_i/2)/\sqrt{F_u}\ ; & \text{if } F_u > 0 \\ g_{ui}\, L_i & \text{if } F_u = 0 \\ 2\, g_{ui} \sinh(\sqrt{-F_u}\, L_i/2)/\sqrt{-F_u}\ ; & \text{if } F_u < 0 \end{cases} \tag{D-2}$$

with L_i the effective length of the corrector, g_{ui} the field deviation given by Eq.(B-1), F_u the focusing strength given by Eq.(B-8) in which K and ρ are those in the corrector. Eq.(D-2) has taken any possible types of correctors into consideration.

Suppose N correctors are applied. Inserting Eq.(D-1) into Eq.(C-16), one obtains the COD change at the j-th monitor due to the correction as

$$u_{cj} = \sum_{i}^{N} C_u(i,j)\, \theta_{ui} \tag{D-3}$$

$$\text{with} \quad C_u(i,j) = B_u(i,j)\, \cos\phi_u(j/i) \tag{D-4}$$

$B_u(i,j)$ and $\phi_u(j/i)$ are given by Eqs.(C-18) and (C-19) respectively. Note that β_{ui} and $\psi_u(j/i)$ are evaluated at the midpoint of the i-th corrector.

$C_u(i,j)$ is just the (i,j)-th element of the u plane COD sensitivity matrix. To the accuracy of first order, the changes in u_{cj} caused by any given g_{ui} changes can be found by the sensitivity matrix very readily.

2.) At any point "s", the COD change caused by the correction is

$$u_{cs} = \sum_{i}^{N} B_u(i,s)\ \theta_{ui} \cos \phi_u(s/i) \quad ;$$

$$u'_{cs} = \sum_{i}^{N} B_u(i,s)\ \theta_{ui} \ (\sin \phi_u(s/i) - \alpha_{us} \cos \phi_u(s/i))/\beta_{us} \qquad \text{(D-5)}$$

As mentioned before, u_{cs} and u'_{cs} in magnets will give rise to DFD change.

3.) Eqs.(D-3) and (D-4) are widely adopted for a long time. The only thing new about COD correction is in Eq.(D-2) and is discussed here.

The existing schemes calculate θ_{ui} by[1)-4)]

$$\theta_{ui} = g_{ui} L_i \qquad \text{(D-6)}$$

And Eq.(D-2) can be written as a power series in L_i as

$$\theta_{ui} = g_{ui} L_i \ (1 - \frac{1}{24} F_u L_i^2 + \frac{1}{1920} F_u^2 L_i^4 + \dots) \qquad \text{(D-7)}$$

Let $k = |F_u| L_i^2$, where F_u is the u plane focusing strength in that corrector, evaluated by Eq.(B-8). How Eq.(D-6) differs from Eq.(D-2) is measured by k .

Up to first order, no mathematical approximation is made in the process where Eq.(D-2) results from. In this sense, Eq.(D-2) is supposed the precise formula for θ_{ui} in any case, Eq.(D-6) an approximate expression to first order of L_i, and k the criterion to judge how much the error is. By the way, k is also the criterion for thin-lens approximation in transfer matrix multiplication.

Obviously, Eqs.(D-2) and (D-6) mean the same for a "dedicated corrector", that is installed in a drift section with k = 0 . But, if the corrector has a relatively long length and is simultaneously functioning as a bending magnet or a quadrupole, i.e. the correcting field is produced by magnet trim coils, the relative difference between Eqs.(D-6) and (D-2) may be a few percent. Usually the difference is very slight. A numerical example is: the error is 1 percent when k = 0.24, that is, $|K|$ is around 1 m^{-2} for a 0.5 m long quadrupole-corrector, or the bending angle around 30 degrees for a non-gradient bend-corrector.

E. New Expressions of DFD Sensitivity Matrix Elements

1.) As for COD correction, sensitivity matrix method is widely adopted for DFD correction. When N correctors are on, the DFD change at the j-th monitor due to the correction is expected to be

$$\tilde{\eta}_{uj} = \sum_{i}^{N} D_u(i,j) \quad \theta_{ui} \tag{E-1}$$

where $D_u(i,j)$ is the (i,j)-th element of the u plane DFD sensitivity matrix. The question is how to evaluate $D_u(i,j)$ by known parameters.

2.) First, one can give the DFD jumps in each magnet by a rewritten form of Eq.(C-14) as

$$\Delta\tilde{\eta}_{um} = -\int_{z_1}^{z_2} S_u(\bar{z}- z_m)H_u(\bar{z})\,d\bar{z} + C_u(\frac{L_m}{2})(\bar{I}_{u2}+ \bar{I}_{u1}) - S_u(\frac{L_m}{2})(\bar{J}_{u2}- \bar{J}_{u1}) \quad ;$$

$$\Delta\tilde{\eta}'_{um} = \int_{z_1}^{z_2} C_u(\bar{z}- z_m)H_u(\bar{z})\,d\bar{z} + F_u S_u(\frac{L_m}{2})(\bar{I}_{u2}- \bar{I}_{u1}) + C_u(\frac{L_m}{2})(\bar{J}_{u2}+ \bar{J}_{u1}) \tag{E-2}$$

where $H_u = (\frac{2}{\rho}\eta - 1)\, g_u + Q_u\, u_c + \frac{1}{\rho}\eta'\, u_c'$ (E-3)

with Q_x and Q_y given by Eq.(B-14) ; and

$$\bar{I}_{ue} = \bar{P}_{ue}\, u_{ce} = \overline{sg}_u \cdot \Delta_e \frac{1}{\rho}\, t_e^2\, \eta_e\, u_{ce} \tag{E-4}$$

$$\bar{J}_{u1} = \bar{Q}_{u1}\, u_{c1} + \bar{R}_{u1}\, u'_{c1} \quad ;$$

$$\bar{J}_{u2} = \bar{Q}_{u2}\, u_{c2} + \bar{R}_{u2}\, u'_{c2} + f_{u2}\, \bar{P}_{u2}\, u_{c2} \tag{E-5}$$

Inserting Eqs.(B-8) and (B-15) into Eq.(E-5), one will find a uniform expression of $\bar{J}_{ue}$ as

$$\bar{J}_{ue} = \overline{sg}_u \cdot (K_e - (\lambda_e + \lambda_{ue})\, \eta_e - \Delta_e \frac{1}{\rho} (1 + t_e^2)\, \eta'_e)\, u_{ce}$$

$$- \Delta_e \frac{1}{\rho} (1 + \overline{sg}_u \cdot t_e^2)\, \eta_e\, u'_{ce} \tag{E-6}$$

with $\lambda_{xe} = \frac{2}{\rho^2} t_e^3 \quad ; \quad \lambda_{ye} = \frac{1}{\rho^2} t_e$ (E-7)

Remember that the edge sextupole effects appeared asymmetric in Eqs. (B-3) and (B-5), where the parameters u'_b were evaluated at asymmetric points. Now it turns out from Eq.(E-6) that the edge effects on DFD are perfectly symmetric. "Symmetric" here means that, if all the functions (η and u_c) and characteristics (θ_e and r_e) involved in a magnet are mirror symmetric about its midpoint, the contributions to $\Delta\tilde{\eta}'_{um}$ from the two edges, or more generally, from the two halves are exactly doubled and their contributions to $\Delta\tilde{\eta}_{um}$ entirely cancel out each other. This comes from the fact that the two edges are of exactly the same importance in their effects on DFD and is true for both x and y planes.

Eq.(E-2) tells that $\overline{J}_{ue}$ can be regarded as $\int H_u\,dz$ at the edge, and that a bending magnet edge behaves like a quadrupole-sextupole combined magnet, with K_e and λ_e the main quadrupole and sextupole components respectively. The effect varies with transverse planes (x or y) as seen from the explicit difference in λ_{ue} and $\overline{sg}_u$; but it does not vary with edges (entrance or exit). It is interesting that a study on the linear chromaticity calculation[5], though the mechanism is different, draws the same conclusion and gives a few similar formulae. For example, the "plane dependent sextupole components" λ_{ue} are given by exactly the same formulae as Eq.(E-7).

In Eq.(E-6), the terms $-\overline{sg}_u \cdot \Delta_e \frac{1}{\rho}\eta'_e u_{ce}$ and $-\Delta_e \frac{1}{\rho}\eta_e u'_{ce}$ do not depend on θ_e or r_e , so their roles are determined by the bending magnet only, regardless of what the edges are like. They are "normal edge terms", representing the effect of a normal edge (sector magnet edge), or rather, a part of the effect of the bending magnet in question.

Since "e" denotes the inner side of edges, the normal edge terms can be eliminated easily by integrating Eq.(E-2) by parts. Divide H_u into

$$H_u(z) = H_{Ju}(z) + H'_{Iu}(z) \qquad \text{(E-8)}$$

where $H_{Jx} = (\frac{1}{\rho}\eta - 1)\, g_x + (K - \lambda\eta + \frac{1}{\rho}(\frac{1}{\rho} - 2K\eta))\, x_c - \frac{1}{\rho}\eta'\, x_c'$;

$$H_{Jy} = (\frac{1}{\rho}\eta - 1)\, g_y + (-K + \lambda\eta + \frac{1}{\rho^2}(1 - \frac{1}{\rho}\eta))\, y_c + \frac{1}{\rho}\eta'\, y_c' \ ;$$

$$H_{Iu} = \frac{1}{\rho}(\overline{sg}_u \cdot \eta'\, u_c + \eta\, u_c') \qquad \text{(E-9)}$$

and let

$$\overline{J}_{Hue} = \overline{J}_{ue} + \Delta_e \, H_{Iu}(z_e)$$

$$= \overline{sg}_u \cdot ((K_e - (\lambda_e + \lambda_{ue}) \eta_e) \, u_{ce} - \Delta_e \frac{1}{\rho} t_e^2 (\eta'_e u_{ce} + \eta_e u'_{ce})) \qquad \text{(E-10)}$$

Then one gets a new equation which is equivalent to Eq.(E-2):

$$\Delta\tilde{\eta}_{um} = \int_{z_1}^{z_2} (C_u(\overline{z} - z_m) \, H_{Iu}(\overline{z}) - S_u(\overline{z} - z_m) \, H_{Ju}(\overline{z})) \, d\overline{z}$$

$$+ C_u(L_m/2) (\overline{I}_{u2} + \overline{I}_{u1}) - S_u(L_m/2) (\overline{J}_{Hu2} - \overline{J}_{Hu1}) \quad ;$$

$$\Delta\tilde{\eta}'_{um} = \int_{z_1}^{z_2} (C_u(\overline{z} - z_m) \, H_{Ju}(\overline{z}) + F_u \, S_u(\overline{z} - z_m) \, H_{Iu}(\overline{z})) \, d\overline{z}$$

$$+ C_u(L_m/2) (\overline{J}_{Hu2} + \overline{J}_{Hu1}) + F_u \, S_u(L_m/2) (\overline{I}_{u2} - \overline{I}_{u1}) \qquad \text{(E-11)}$$

3.) In principle, an accurate evaluation of $D_u(i,j)$ can be worked out in this way : Let θ_{ui} be unit and the kicks in other correctors be zero . Then Eq.(D-5) gives the closed orbit (u_c and u_c') at any points, the DFD jumps ($\Delta\tilde{\eta}_{um}$ and $\Delta\tilde{\eta}'_{um}$) in every magnet are found by the equations in last subsection, and Eq.(C-16) yields the DFD value at the j-th monitor ($\tilde{\eta}_{uj}$), which equals to $D_u(i,j)$ in this case.

Because of the involvement of integrals in Eq.(E-11), Eq.(D-5) has to be used many times to get a very accurate result. This is not difficult for a computer program, but may be unnecessarily time-consuming.

In practice, it is only a quite rough accuracy to which DFD measurement can be done and on which the DFD correction is based. Although no mathematical approximation is made in this note so far, what one needs for this problem is a good approximate method.

A choice to be considered is Simpson's integral calculation method. It means a quadratic approximation of the integrand functions and evaluates them at only three points, "1", "m" and "2" for each magnet in this case.

Since Simpson's method calculates an integral by

$$\int_{z_1}^{z_2} f(z)\, dz \doteq \frac{1}{6}(f_1 + 4 f_m + f_2) L_m$$

one can evaluate

$$\overline{J}^{*}_{Hue} = \overline{J}_{Hue} + \frac{1}{6} H_{Ju}(z_e)\cdot L_m \; ; \qquad \overline{I}^{*}_{ue} = \overline{I}_{ue} + \frac{1}{6} H_{Iu}(z_e)\cdot L_m \qquad \text{(E-12)}$$

at each edge, then replace Eq.(E-11) by

$$\Delta\tilde{\mathfrak{I}}_{um} \doteq \frac{2}{3} H_{Iu}(z_m)\cdot L_m + C_u(L_m/2)\,(\overline{I}^{*}_{u2} + \overline{I}^{*}_{u1}) - S_u(L_m/2)\,(\overline{J}^{*}_{Hu2} - \overline{J}^{*}_{Hu1}) \; ;$$

$$\Delta\tilde{\mathfrak{I}}'_{um} \doteq \frac{2}{3} H_{Ju}(z_m)\cdot L_m + C_u(L_m/2)\,(\overline{J}^{*}_{Hu2} + \overline{J}^{*}_{Hu1}) + F_u S_u(L_m/2)\,(\overline{I}^{*}_{u2} - \overline{I}^{*}_{u1}) \qquad \text{(E-13)}$$

and go through the $D_u(i,j)$ evaluation procedure. So Eq.(D-5) is used only thrice for each magnet.

4.) The method to be recommended in this note is: to expand $D_u(i,j)$ as well as all the functions involved into power series in L_m and express them approximately by function values at the midpoint "m" and L_m of every magnet. For example, one has

$$C_u(L_m/2) = 1 - \frac{1}{8} F_u L_m^2 + \ldots \; ; \qquad S_u(L_m/2) = \frac{1}{2} L_m\left(1 - \frac{1}{24} F_u L_m^2 + \ldots\right) \; ;$$

$$\int_{z_1}^{z_2} C_u(\bar{z} - z_m)\, H(\bar{z})\, d\bar{z} = H(z_m)\cdot L_m + \frac{1}{24}\left(H''(z_m) - F_u\, H(z_m)\right) L_m^3 + \ldots \; ;$$

$$\int_{z_1}^{z_2} S_u(\bar{z} - z_m)\, H(\bar{z})\, d\bar{z} = \frac{1}{12} H'(z_m)\cdot L_m^3 + \ldots \qquad \text{(E-14)}$$

Usually, it is sufficient for practical use to keep only the terms of first power in L_m so as to save computer time and preserve a satisfactory accuracy, that is not worse than the DFD measurement accuracy. Note that (L_m/ρ) and $(\sqrt{|F_u|}\, L_m)$ are usually much less than unit. But some extreme conditions are exceptional, such as a 90 degree bending magnet. Under those conditions, the quantitative results of the following formulae are not sufficiently convincing and a better approximation is needed.

Suppose the approximation "to first power in L_m" is good enough. It will be referred to as "first power approximation" later. Note that the power of edge angle θ_e is also counted as power of L_m, since θ_e is always no greater than L_m/ρ. Then the equations are simplified as

$$\overline{J}_{Hue} \doteq \overline{sg}_u \cdot (K_e - (\lambda_e + \lambda_{ue}) \eta_e) u_{ce} \quad ;$$

$$\overline{I}_{ue} \doteq 0 \quad ; \qquad \lambda_{xe} \doteq 0 \quad ; \qquad \lambda_{ye} = \frac{1}{\rho^2} t_e \tag{E-15}$$

$$\Delta\tilde{\eta}_{um} = H_{Iu}(z_m) \cdot L_m + \frac{1}{2} (\overline{J}_{Hu1} - \overline{J}_{Hu2}) L_m + \ldots \quad ;$$

$$\Delta\tilde{\eta}'_{um} = H_{Ju}(z_m) \cdot L_m + \overline{J}_{Hu1} + \overline{J}_{Hu2} + \ldots \tag{E-16}$$

with the functions at "e" calculated by

$$\eta_e = \eta_m + \frac{1}{2} \Delta_e \eta'_m L_m + \ldots \quad ; \qquad u_{ce} = u_{cm} + \frac{1}{2} \Delta_e u'_{cm} L_m + \ldots$$

All the functions have to be evaluated at only one point, "m", for each magnet. And define a new symbol

$$\tau_e = 1 / (r_e \cos^3 \theta_e) \tag{E-17}$$

A magnet edge is now characterized by t_e (tan θ_e) and τ_e instead of θ_e and r_e. Because τ_e/ρ is a zeroth power (in L_m) term of the edge sextupole component, the difference in τ_e at the edges of a magnet influences DFD as a first power term and should be taken into account.

5.) Introduce a few abbreviations

$$A_m = (K - \lambda\eta_m) L_m - (\frac{1}{\rho} - 2 K \eta_m)(t_1 + t_2)$$
$$+ \frac{1}{\rho} \eta_m (\tau_1 + \tau_2) - \frac{1}{2\rho} \eta'_m (\tau_1 - \tau_2) L_m \quad ;$$

$$A_x = \frac{1}{\rho} (\frac{1}{\rho} - 2 K \eta_m) L_m + A_m \quad ;$$

$$A_y = \frac{1}{\rho^2} ((1 - \frac{1}{\rho} \eta_m) L_m + \eta_m (t_1 + t_2)) - A_m \tag{E-18}$$

and $A_{u1} = \overline{sg}_u \cdot \frac{1}{\rho} (\eta'_m + \frac{1}{2} \eta_m (\tau_1 - \tau_2)) L_m$;

$$A_{u2} = \frac{1}{\rho} (\eta_m / \beta_{um}) L_m \qquad \text{(E-19)}$$

Note that , since $A_m \doteq (K - \lambda \eta_m) L_m + \sum_{e=1,2} (K_e - \lambda_e \eta_e)$, A_m is the main effect term for a magnet.

Eq.(E-16) then becomes

$$\Delta \tilde{\eta}_{um} \doteq A_{u1} u_{cm} + A_{u2} \beta_{um} u'_{cm} \; ;$$

$$\Delta \tilde{\eta}'_{um} \doteq (\frac{1}{\rho} \eta_m - 1) \theta_{um} + A_u u_{cm} - A_{u1} u'_{cm} \qquad \text{(E-20)}$$

where θ_{um} is not zero in and only in applied correctors, and is given by Eq.(D-2) or (D-6), between which there is no meaningful defference under the first power approximation.

Inserting Eqs.(D-5) and (E-20) into Eq.(C-16) and comparing the result with Eq.(E-1), one arrives at the expressions for $D_u(i,j)$, whjch reads

$$D_u(i,j) \doteq B_u(i,j) \; ((\frac{1}{\rho} \eta_i - 1) \cos \phi_u(j/i) + \frac{1}{2 \sin \pi \nu_u} \sum_m T_{uijm}) \qquad \text{(E-21)}$$

where η_i / ρ is evaluated at the midpoint of the i-th corrector, $B_u(i,j)$ and $\phi_u(j/i)$ given by Eqs.(C-18) and (C-19), and the sum added up for all the magnets with T_{uijm} as

$$T_{uijm} = (\beta_{um} A_u + 2 \alpha_{um} A_{u1} - \alpha^2_{um} A_{u2}) \cos \phi_u(m/i) \cos \phi_u(j/m) + (A_{u1} - \alpha_{um} A_{u2}) \sin(\phi_u(j/m) - \phi_u(m/i)) + A_{u2} \sin \phi_u(m/i) \sin \phi_u(j/m) \qquad \text{(E-22)}$$

All the parameters in Eq.(E-22) are only related to the m-th magnet mid-point except for the phase differences that are also related to the i-th corrector or the j-th monitor.

6.) A question one may concern is how to find the function values at a magnet midpoint, as most programs calculate them at the magnet edges. An easy way is, obviously, dividing each magnet into two halves so that the midpoint becomes an "edge". Some programs give function average values in every magnet. The averages can be used in place of the midpoint values, making a difference which is of higher than first power in L_m and is hence of no more importance than the approximation adopted.

Of course, the midpoint values of any function can be expressed by the edge values. A few useful formulae are presented in Ref.6). Since any first power approximation is allowable, the formulae can be simplified to

$$\beta_{um} \doteq \frac{1}{2} (\beta_{u1} + \beta_{u2} + \frac{1}{2} (\alpha_{u2} - \alpha_{u1}) L_m) \quad ;$$

$$\alpha_{um} \doteq \frac{1}{2} (\beta_{u1} - \beta_{u2}) / L_m \quad ;$$

$$\eta_m \doteq \frac{1}{2} (\eta_1 + \eta_2) \quad ; \qquad \eta'_m \doteq (\eta_2 - \eta_1) / L_m \qquad \text{(E-23)}$$

Let subscript "o" denote the origin point of z coordinate. Most programs calculate $\psi_u(e/o)$ for every magnet edge. Then one can give

$$\psi_u(m/o) = \tan^{-1} \left(\frac{\sqrt{\beta_{u1}} \sin \psi_u(1/o) + \sqrt{\beta_{u2}} \sin \psi_u(2/o)}{\sqrt{\beta_{u1}} \cos \psi_u(1/o) + \sqrt{\beta_{u2}} \cos \psi_u(2/o)} \right) + n\pi \qquad \text{(E-24)}$$

as the midpoint phase advance, where n is an integer which is so chosen that $\psi_u(1/o) < \psi_u(m/o) < \psi_u(2/o)$. And, as well known,

$$\phi_u(s/p) = \begin{cases} \psi_u(p/o) - \psi_u(s/o) + \pi\nu_u \quad , & \text{if } z_s > z_p \\ \psi_u(p/o) - \psi_u(s/o) - \pi\nu_u \quad , & \text{if } z_s < z_p \end{cases} \qquad \text{(E-25)}$$

where "s" and "p" are any points, but $s \neq p$.

A combined function corrector is also counted as a magnet in calculating $\sum_m T_{uijm}$, with $\cos \phi_u(m/i) = \cos \pi\nu_u$, but $\sin \phi_u(m/i) = 0$. This is equivalent to taking averages of u_c and u_c' near the midpoint, where u_c' is discontinuous.

An analytical expression for DFD sensitivity matrix elements is thus worked out and presented by Eqs.(E-18) through (E-22). Please remember that it is obtained under the first power approximation.

F. Differences with the Existing Expressions

1.) The results in last section differ obviously from the existing DFD sensitivity matrix expressions which are widely accepted.

As a contrast, the COD sensitivity matrix is the same, though a new formula to calculate the kicks (θ_{ui}), Eq.(D-2), is introduced and discussed in section D. The difference with the existing formula is usually very slight, unless the task of correction is fulfilled by high strength quadrupoles or by bending magnets with large bending angles, say greater than 30 degrees.

2.) The existing expressions of $D_u(i,j)$ is[1)-4)]

$$D_u(i,j) = B_u(i,j) \; (- \cos \phi_u(j/i) + \frac{1}{2 \sin \pi \nu_u} \sum_m \beta_{um} A_u \cos \phi_u(m/i) \cos \phi_u(j/m) \;) \qquad \text{(F-1)}$$

$$\text{where} \quad A_x = - A_y = (K - \lambda \eta_m) \, L_m \qquad \text{(F-2)}$$

If one lets all the ($1/\rho$) or t_e terms be zero, that is, ignores the effects of bending magnet and magnet edges, Eq.(E-21) becomes the same as Eq.(F-1). So the difference between these two expressions is essentially negligible for big machines, where ρ is, say, greater than 20 m.

3.) But, a modification of Eq.(F-1) seems necessary if one wants it to be appropriate to machines of any size with correctors of any type. As a result from the complete up-to-second-order particle motion equations and the first power in element length approximation, Eq.(E-21) is recommended to be used in DFD correction schemes. A discussion on the differences between Eqs.(F-1) and (E-21) will show how much the error of Eq.(F-1), if used for small rings, may be.

One of the major differences is in the coefficient of the first term, if a trim coil of a bending magnet is used as a corrector, either horizontally or vertically. The difference term η_i/ρ is quite significant for small rings, where ρ may be around 2 m and a typical value of η is 1.5 m.

What may be more important is mentioned before: Eq.(F-1) takes into account only the effects of quadrupoles and sextupoles, for which the terms in Eq.(E-21) are the same. It is then no surprising that magnet edges come into the expressions, since they are well known to affect particle motion with quadrupole and/or sextupole components. It is seen that K_e and λ_e of an edge act just like $K\,L_m$ and $\lambda\,L_m$ of quadrupoles or sextupoles, respectively. And, in their own way, bending magnets bring about their effects, which are reasonably proportional to L_m/ρ .

How much a magnet element affects DFD can be estimated by comparing the terms produced by elements of various types, with the cosine/sine function factors (phase difference factors) dropped for the moment:

The effect of a quadrupole is proportional to $K\,\beta_{um}\,L_m$; and by almost the same ratio, that of a sextupole proportional to $\lambda\beta_{um}\,\eta_m\,L_m$; that of a bending magnet proportional to $\beta_{um}\,L_m\,/\,\rho^2$ in A_u , and $\eta'_m\,L_m\,/\rho$ in A_{u1} , and $\eta_m\,L_m\,/(\,\rho\,\beta_{um}\,)$ in A_{u2} : and that of a magnet edge proportional to $(\,\beta_{um}\,/\rho\,)$ tan θ_e or $\beta_{um}\,\eta_m\,/(\rho\,r_e\,)$. Usually A_{u1} and A_{u2} are less influential than A_u (A_x or A_y), because β_{um} is greater than η_m or $\eta'_m\rho$ for most (but not all) magnets in a ring.

It is clear that the exclusion of bending magnets and magnet edges from consideration is not safe in small rings, where L_m/ρ or η_m/ρ or β_{um}/ρ is not very small.

4.) A special case in which all the bending magnets are rectangular, i.e. of parallel sides and of straight edge face, is met more frequently and hence given more attention.

Since $(\,1/\,r_e\,)\,=\,0$ and $\tan\theta_e \doteq L_m\,/\,(\,2\,\rho\,)$ here, one has

$$A_x \doteq (\,K\,-\,\lambda\,\eta_m\,)\,L_m\,;$$

$$A_y \doteq (\,-\,K\,+\,\lambda\,\eta_m\,)\,L_m\,+\,\frac{2}{\rho^2}\,L_m\,-\,\frac{2}{\rho}\,K\,\,\eta_m\,L_m \qquad \text{(F-3)}$$

Because A_u is usually the most influential term, the conclusion in this case is, the first order effect of bending magnets almost vanishes on x plane but remains active on y plane. This agrees with the fact that a magnet of this type functions roughly like a drift on x plane but has a focusing effect on y plane, as A.W.Chao once expected in a discussion.

5.) To conclude the discussion, two numerical examples are presented as follows.

a) Some typical values of the parameters in SPEAR ($E = 1.5 \sim 4$ GeV) are:

In the quadrupoles,

$$L_m = 0.5 \text{ m}, \quad |K| = 0.18 \sim 0.65 \text{ m}^{-2}, \quad \beta_{um} = 3.7 \sim 21 \text{ m},$$

so $|K| \beta_{um} L_m$ ranges from 0.78 to 6.8 .

In the bending magnets,

$$L_m = 2.37 \text{ m}, \quad \rho = 12.8 \text{ m}, \quad \beta_{ym} = 11.8 \text{ m}, \quad \beta_{xm} = 7.6 \text{ m},$$

$$\eta_m = 1.9 \text{ m}, \quad \eta'_m = \pm 0.28, \quad \alpha_{ym} = \pm 1.96, \quad \alpha_{xm} = \pm 1.17,$$

so $2 \beta_{ym} L_m / \rho^2 = 0.34$,

and the terms such as $2 \alpha_{um} A_{u1}$ and $\alpha_{um}^2 A_{u2}$ may be up to 0.2 .

This means that the effect of a bending magnet is roughly of the same order of magnitude as that of one of the less influential quadrupoles. It follows that the ignorance of a large number of bending magnets seems questionable.

b) As an example for sub-GeV rings, the design values of a possible configuration of the Hefei 800 MeV storage ring are:

In the quadrupoles,

$$L_m = 0.3 \text{ m}, \quad |K| = 1.51 \sim 3.07 \text{ m}^{-2}, \quad \beta_{ym} = 4.2 \sim 11.4 \text{ m},$$

$$\beta_{xm} = 1.5 \sim 17.8 \text{ m},$$

so $|K| \beta_{ym} L_m = 2.4 \sim 7.8$, and $|K| \beta_{xm} L_m = 1.0 \sim 9.5$.

In the bending magnets,

$L_m = 1.16$ m , $\rho = 2.22$ m , $\beta_{ym} = 0.93$ or 15.3 m ,

$\beta_{xm} = 2.6$ or 2.8 m , $\eta_m = 1.3$ or 0.08 m , $\eta'_m = 0$ or ± 0.26 ,

$\alpha_{ym} = 0$ or ± 0.39 , $\alpha_{xm} = 0$ or ± 1.12 ,

so $2 \beta_{ym} L_m / \rho^2$ may be as high as 7.2 , and A_{y2} may be 0.73 .

The effect on $\tilde{\eta}_y$ of one bending magnet is fairly close to that of the most influential quadrupole. Note that, in this example, β_y function reaches its maximum in bending magnets. This is not very unusual in small machines, because the y plane edge focusing effect is relatively strong.

One can conclude that the bending magnet effects must be considered if the L_m/ρ value is comparable with unit. In the Hefei 800 MeV ring, it is $0.524 = \pi / 6$. It is even larger in some other rings, for example, the Brookhaven NSLS VUV ring, where it is $\pi / 4$.

6.) It was reported that the DFD correction schemes based on Eqs.(E-1) and (F-1) work quite well in some multi-GeV rings, though sometimes not very satisfactorily, i.e. many iterations being required, if DFD rms value is relatively great. There seemed to be suggestions of absence of unknown first order factors[2], but it is not time yet to make a definite conclusion. For smaller rings things might be worse, if the theoretical similarity of the DFD problem to the chromaticity calculation, as seen above, is believable.

The results of this note are to be implemented into a program. The author hopes it would, as a complete first order approximation, give a better performance.

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