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Toward More Accurate Loss Tangent Measurements in Reentrant Cavities

Robert D. Moyer

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ABSTRACT

Karpova has described an absolute method for measurement of dielectric properties of a solid in a coaxial reentrant cavity. His cavity resonance equation yields very accurate results for dielectric constants. However, he presented only approximate expressions for the loss tangent. This report presents more exact expressions for that quantity and summarizes some experimental results.

CONTENTS

	Page
Introduction	7
Summary of Previous Results and Formulation of the Problem	7
Expressions for Stored Energies in the Resonator	13
Total Energy Stored in Region A	13
Total Energy Stored in Region B	15
Expressions for Power Losses in the Resonator	16
Power Losses in the Walls of the Resonator	16
Computation of P_C	24
Computation of P_J	25
Experimental Results	26
References	27
APPENDIX A -- Tabulation of Closed-Form Expressions	29
APPENDIX B -- Derivation of Equations	33

ILLUSTRATIONS

Figure		
1	Cross Section of Reentrant Resonator	8
2	Reentrant Cavity	12
3	Incremental Annular Area	18
4	z-Directed Current Flowing across an Incremental Band on a Cylindrical Surface	20

TABLE

Table		
1	Results of Dielectric Properties Measurements	27

TOWARD MORE ACCURATE LOSS TANGENT MEASUREMENTS IN REENTRANT CAVITIES

Introduction

Karpova has described an absolute method for measurement of dielectric properties of a solid in a coaxial reentrant cavity.[1] The usefulness of his technique has been recognized by another author[2] and verified at Sandia Laboratories. Some of the advantages of the technique are

- The solution is "exact" so that a wide range of dielectric constants may be accommodated.
- No standards are required to calibrate the cavity.
- The technique is convenient for very-high-frequency (VHF) and ultra-high-frequency (UHF) measurements because the largest dimension of the cavity is of the order of a quarter wavelength.
- The required sample shape is a simple right circular cylinder whose diameter-to-length ratio is not critical. For example, a 1:1 ratio is very convenient when preparing sample specimens from brittle materials.

The resonance equation given by Karpova[1] yields quite accurate results for the dielectric constant; however, he presented only approximate expressions for loss tangent. The purpose of this report is to give more exact expressions. A drawback of this measurement technique is the extensive computation required, but, for those with access to a computer, this drawback becomes rather insignificant.

Summary of Previous Results and Formulation of the Problem

For the sake of completeness, the resonance equation derived by Karpova is repeated in Eq. (1), and its symbolism is defined. With few exceptions, the notation of Karpova is retained in this paper, the main exception being that all work here is in the mks system of units.

The resonance equation is

$$\frac{\pi^2}{2} R_0^A + \sum_{n=1}^{\infty} \frac{R_n^A \sin^2(\pi a n)}{a^2 n^2} = \frac{\pi^2}{2a} R_0^B + \text{SUM} , \tag{1}$$

where

$$\text{SUM} = \sum_{m=1}^{\infty} \left\{ \frac{\left(\sum_{n=1}^{\infty} R_n^A \frac{\sin^2(\pi an)}{a^2n^2 - m^2} \right)^2}{\sum_{n=1}^{\infty} \frac{R_n^A \, a^2n^2 \sin^2(\pi an)}{(a^2n^2 - m^2)^2} - \frac{\pi^2}{4a} R_m^B} \right\} \tag{2}$$

In these equations, a is the ratio of sample length d to cavity length L, as shown in Figure 1, which is a cross section of the reentrant cavity.

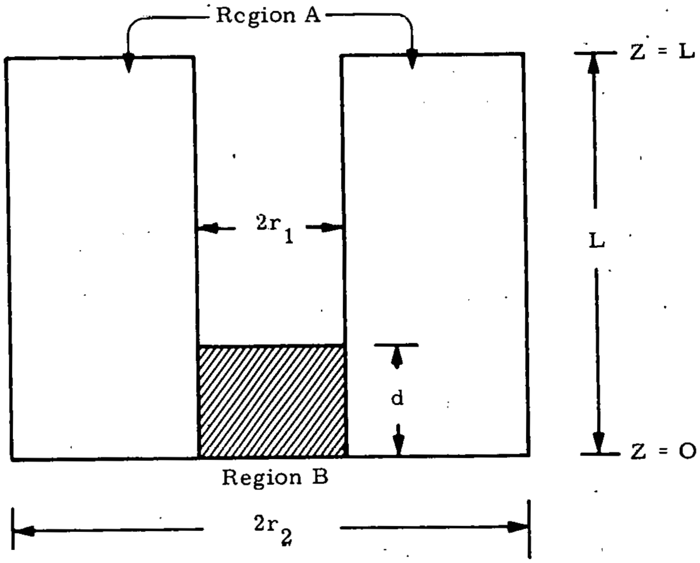

Figure 1. Cross Section of Reentrant Resonator

$$a = \frac{d}{L} \tag{3}$$

As suggested by Figure 1, the unknown material occupies region B while region A is filled with air.

$$R_0^A = j \frac{G_1(k_0r_1)}{G_0(k_0r_1)} \tag{4}$$

$$R_n^A = j \frac{k_0}{\kappa_n^A} \frac{D_{1n}\left(\kappa_n^A r_1\right)}{D_{0n}\left(\kappa_n^A r_1\right)} \tag{5}$$

$$R_0^B = j \sqrt{\varepsilon_{xr}} \frac{J_1(kr_1)}{J_0(kr_1)} \tag{6}$$

$$R_m^B = j \frac{k}{\kappa_m^B} \sqrt{\varepsilon_{xr}} \frac{I_1\left(\kappa_m^B r_1\right)}{I_0\left(\kappa_m^B r_1\right)} \tag{7}$$

In Eqs. (4) through (7)

$$j = \sqrt{-1} \ , \tag{8}$$

$$G_0(k_0r) = J_0(k_0r)N_0(k_0r_2) - J_0(k_0r_2)N_0(k_0r) \ , \tag{9}$$

$$G_1(k_0r) = J_1(k_0r)N_0(k_0r_2) - J_0(k_0r_2)N_1(k_0r) \ , \tag{10}$$

where

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} \tag{11}$$

ω = radian resonant frequency ,

$\mu_0 = 4\pi \times 10^{-7}$ H/m ,

$\varepsilon_0 = 8.8542 \times 10^{-12}$ F/m ,

J_i and N_i are Bessel functions of the first and second kind, respectively, of order i,

r_1 and r_2 are radii defined in Figure 1,

$$\left(\kappa_n{}^A\right)^2 = -\left(k_{cn}{}^A\right)^2 = \left(\frac{n\pi}{L}\right)^2 - k_0{}^2 \ , \tag{12}$$

$$D_{0n}\left(\kappa_n{}^A r\right) = I_0\left(\kappa_n{}^A r\right)K_0\left(\kappa_n{}^A r_2\right) - I_0\left(\kappa_n{}^A r_2\right)K_0\left(\kappa_n{}^A r\right), \tag{13}$$

$$D_{1n}\left(\kappa_n{}^A r\right) = I_1\left(\kappa_n{}^A r\right)K_0\left(\kappa_n{}^A r_2\right) + I_0\left(\kappa_n{}^A r_2\right)K_1\left(\kappa_n{}^A r\right), \tag{14}$$

I_i and K_i are modified Bessel functions of the first and second kind, respectively, of order i,

ε_{xr} is the relative dielectric constant of the material in Region B,

$$k = \sqrt{\varepsilon_{xr}}\,k_0 \ , \text{ and} \tag{15}$$

$$\left(\kappa_m{}^B\right)^2 = -\left(k_{cm}{}^B\right)^2 = \left(\frac{\pi m}{aL}\right)^2 - \varepsilon_{xr}k_0{}^2 \ . \tag{16}$$

Note that the factor $j = \sqrt{-1}$ cancels out of Eq. (1) so that it is a purely real equation. The wave numbers $k_{cn}{}^A$ and $k_{cm}{}^B$ appear only in Eqs. (12) and (16). However, they are quite frequently encountered in other literature,[3] so their relationships to the quantities $\kappa_n{}^A$ and $\kappa_m{}^B$, which are used extensively in the present report, are given. This completes the definition of all notation in resonance Eq. (1).

In the sections which follow, expressions for the fields in the two regions are needed. These are reproduced in Eqs. (17) through (22).

In region A,

$$E_z^A = aM_0 \frac{G_0(k_0 r)}{G_0(k_0 r_1)} + \sum_{n=1}^{\infty} C_n \frac{D_{0n}\left(\kappa_n^A r\right)}{D_{0n}\left(\kappa_n^A r_1\right)} \cos\left(\frac{\pi n z}{L}\right), \tag{17}$$

$$E_r^A = \frac{\pi}{L} \sum_{n=1}^{\infty} \frac{nC_n}{\kappa_n^A} \frac{D_{1n}\left(\kappa_n^A r\right)}{D_{0n}\left(\kappa_n^A r_1\right)} \sin\left(\frac{\pi n}{L} z\right), \tag{18}$$

$$H_\phi^A = j\frac{aM_0 G_1(k_0 r)}{\eta_0 G_0(k_0 r_1)} + j \sum_{n=1}^{\infty} \frac{C_n}{\eta_0} \frac{k_0}{\kappa_n^A} \frac{D_{1n}\left(\kappa_n^A r\right)}{D_{0n}\left(\kappa_n^A r_1\right)} \cos\left(\frac{\pi n}{L} z\right) . \tag{19}$$

In region B,

$$E_z^B = M_0 \frac{J_0(kr)}{J_0(kr_1)} + \sum_{m=1}^{\infty} M_m \frac{I_0\left(\kappa_m^B r\right)}{I_0\left(\kappa_m^B r_1\right)} \cos\left(\frac{\pi m}{aL} z\right) , \tag{20}$$

$$E_r^B = \sum_{m=1}^{\infty} \frac{\pi m}{aL\kappa_m^B} M_m \frac{I_1\left(\kappa_m^B r\right)}{I_0\left(\kappa_m^B r_1\right)} \sin\left(\frac{\pi m}{aL} z\right), \tag{21}$$

$$H_\phi^B = j\frac{\sqrt{\epsilon_{xr}}}{\eta_0} M_0 \frac{J_1(kr)}{J_0(kr_1)} + $$

$$+ j\frac{\sqrt{\epsilon_{xr}}}{\eta_0} \sum_{m=1}^{\infty} \frac{k}{\kappa_m^B} M_m \frac{I_1\left(\kappa_m^B r\right)}{I_0\left(\kappa_m^B r_1\right)} \cos\left(\frac{\pi m z}{aL}\right) , \tag{22}$$

where

$$C_n = 2M_0\left[\frac{\sin(\pi a n)}{\pi n} + \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^m \frac{M_m}{M_0} n\frac{\sin(\pi a n)}{\left(n^2 - \frac{m^2}{a^2}\right)}\right] \tag{23}$$

and

η_0 = 376.73 ohms = characteristic impedance of free space.

M_0 will turn out to be arbitrary and

$$\frac{M_m}{M_0} = \frac{(-1)^m \sum_{n=1}^{\infty} R_n^A \frac{\sin^2(\pi a n)}{a^2 n^2 - m^2}}{m^2 \left[\frac{\pi^2 R_m^B}{4a} - \sum_{n=1}^{\infty} \frac{R_n^A a^2 n^2 \sin^2(\pi a n)}{(a^2 n^2 - m^2)^2} \right]} . \tag{24}$$

Finally, the loss tangent (TAND) of the sample is given by

$$\text{TAND} = \left(\frac{W_A + W_B}{W_B Q} \right) - \left(\frac{P_W + P_C + P_J}{\omega W_B} \right) , \tag{25}$$

where

W_A = maximum energy stored in A region of resonator

W_B = maximum energy stored in B region of resonator

Q = measured, loaded Q of the resonator with sample in place

P_W = power dissipated in the conducting resonator walls

P_C = power dissipated in the external source and receiver circuits, and

P_J = power dissipated in the joint between the center conductor and base plate of the resonator.

The P_J term did not appear in Karpova's analysis. It is added here because of the manner in which our resonators were constructed.

To (a) relax the required tolerances on dimension d (see Figure 1) of the sample, and (b) permit samples of different lengths to be measured, the center conductor post of the cavity can slide in the z direction (see Figure 2). Once the sample is in place and the cover is attached (at the z = 0 plane), a screw is used to force intimate contact between the center conductor, the sample, and the cover of the cavity. This is done to minimize the air gap errors which are particularly troublesome when measuring samples with a large dielectric constant. Unfortunately, small losses occur in the sliding joint; these are accounted for by the P_J term. P_C is also computed differently from Karpova; this is described under "Expressions for Power Losses in the Resonator," later in this report.

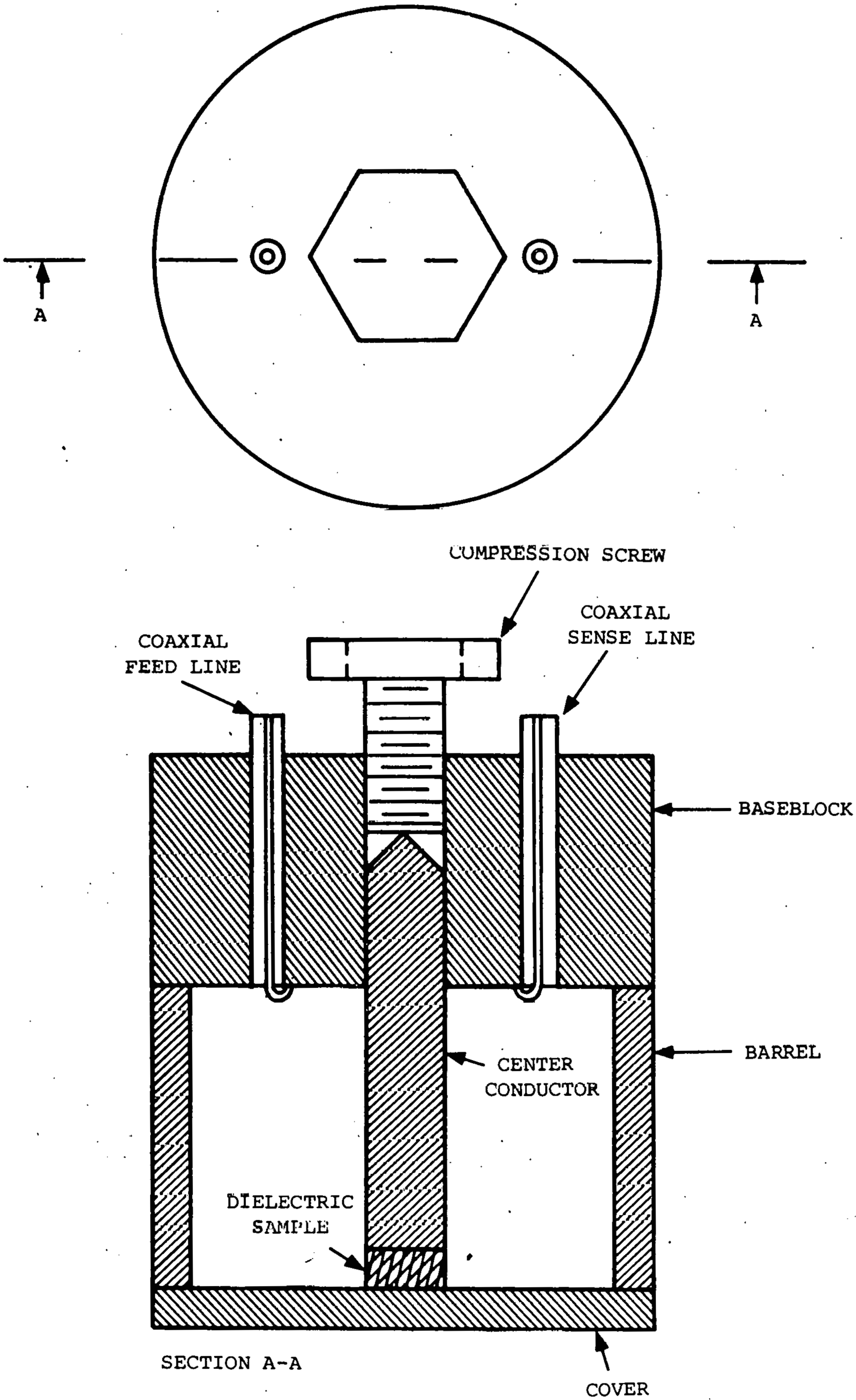

Figure 2. Reentrant Cavity

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All necessary information is now at hand to proceed with the "exact" computations of the terms W_A, W_B, P_W, P_C, and P_J on the right side of Eq. (25). "Exact" is enclosed in quotation marks because the fields given by Eqs. (17) through (22) were derived under the assumption that the cavity walls were lossless. Nevertheless, they will be used to calculate the effects of losses which do occur in the walls.

Expressions for Stored Energies in the Resonator

Total Energy Stored in Region A

The total energy stored in region A, W_A is given by

$$W_A = W_{Az} + W_{Ar} , \tag{26}$$

where W_{Az} and W_{Ar} are the energies stored in the z and r (radial) E fields, respectively. The term W_{Az} is computed first.

Computation of W_{Az} --

$$W_{Az} = \frac{\varepsilon_0}{2} \int_{r_1}^{r_2} \left[\int_0^L \left|E_z{}^A\right|^2 dz \right] 2\pi r dr \tag{27}$$

By using Eq. (17), Eq. (27) may be rewritten as

$$\begin{aligned} W_{Az} = \pi\varepsilon_0 \int_{r_1}^{r_2} r \int_0^L & \left[aM_0 \frac{G_0(k_0 r)}{G_0(k_0 r_1)} + C_1 \frac{D_{01}(\kappa_1^A r)}{D_{01}(\kappa_1^A r_1)} \cos\left(\frac{\pi z}{L}\right) + \right. \\ & \left. + \ldots \right] \left[aM_0 \frac{G_0(k_0 r)}{G_0(k_0 r_1)} + C_1 \frac{D_{01}(\kappa_1^A r)}{D_{01}(\kappa_1^A r_1)} \cos\left(\frac{\pi z}{L}\right) + \ldots \right] dz dr \end{aligned} \tag{28}$$

Considering first the integration over z, it is seen that each term will be of the form

$$\int_0^L \cos\left(\frac{\pi i z}{L}\right) \cos\left(\frac{\pi j z}{L}\right) dz ,$$

where i and j have, independently, any integer value between 0 and ∞. If $i \neq j$, it is easily shown that the integral vanishes. If $i = j \neq 0$,

$$\int_0^L \cos\left(\frac{\pi i z}{L}\right) \cos\left(\frac{\pi i z}{L}\right) dz = \frac{L}{2} . \tag{29}$$

If $i = j = 0$,

$$\int_0^L 1 \cdot 1 \, dz = L. \tag{30}$$

From these orthogonality results, Eq. (28) may be rewritten as

$$W_{Az} = \pi\varepsilon_0 \int_{r_1}^{r_2} r \left\{ a^2 L M_0^{\,2} \frac{G_0^{\,2}(k_0 r)}{G_0^{\,2}(k_0 r_1)} + \frac{L}{2} \sum_{n=1}^{\infty} \left[C_n \frac{D_{0n}\left(\kappa_n^A r\right)}{D_{0n}\left(\kappa_n^A r_1\right)} \right]^2 \right\} dr \tag{31}$$

Now, rewrite Eq. (31) as

$$W_{Az} = I_{A0z} + \sum_{n=1}^{\infty} I_{Anz} \,. \tag{32}$$

Comparing Eqs. (31) and (32) and incorporating Eq. (9) gives

$$I_{A0z} = \frac{\pi\varepsilon_0 a^2 L M_0^{\,2}}{G_0^{\,2}(k_0 r_1)} \left[N_0^{\,2}(k_0 r_2) \int_{r_1}^{r_2} J_0^{\,2}(k_0 r) r dr - \right.$$
$$- 2N_0(k_0 r_2) J_0(k_0 r_2) \int_{r_1}^{r_2} J_0(k_0 r) N_0(k_0 r) r dr +$$
$$\left. + J_0^{\,2}(k_0 r_2) \int_{r_1}^{r_2} N_0^{\,2}(k_0 r) r dr \right] . \tag{33}$$

Eq. (33) is now rewritten, using the notation which will be employed throughout the remainder of this report:

$$I_{A0z} = \frac{\pi\varepsilon_0 a^2 L M_0^{\,2}}{G_0^{\,2}(k_0 r_1)} \left[N_0^{\,2}(k_0 r_2) \mathrm{QJJR}(r_1, r_2, 0, k_0, k_0) - \right.$$
$$- 2N_0(k_0 r_2) J_0(k_0 r_2) \mathrm{QJNR}(r_1, r_2, 0, k_0, k_0) +$$
$$\left. + J_0^{\,2}(k_0 r_2) \mathrm{QNNR}(r_1, r_2, 0, k_0, k_0) \right] . \tag{34}$$

The rationale for this notation is as follows: Throughout this report, integrals such as those appearing in Eq. (33) appear repeatedly. Each integral can be expressed in closed form so function subroutines were written to evaluate them in the FORTRAN program KARPOVA, which implements this technique. The form of the calls to three of those function subroutines appears in Eq. (34). For example, the significance of the call QJNR(r_1, r_2, 0, k_0, k_0) is as follows: The last three letters of the function name indicate that the integrand is the product of a J Bessel function, an N Bessel function, and the independent variable r. The first two arguments of the function specify the range of the integration to be $r_1 \le r \le r_2$. The third argument specifies the order of both Bessel functions to be 0. The fourth function argument specifies that the argument of the first Bessel function is $k_0 r$. The fifth function argument indicates that the argument of the second Bessel function is also (in this case) $k_0 r$. Since one of the purposes of this report is to provide documentation for the KARPOVA program, the functional notation is used in this report. Explicit expressions for each definite integral appear in Appendix A.

If one compares Eqs. (31) and (32) and incorporates Eq. (13), Eq. (35) results.

$$I_{Anz} = \frac{\pi \varepsilon_0 L C_n^2}{2 D_{0n}^2 \left(\kappa_n^A r_1\right)} \left[K_0^2 \left(\kappa_n^A r_2\right) QIIR\left(r_1, r_2, 0, \kappa_n^A, \kappa_n^A\right) - \right.$$

$$- 2K_0 \left(\kappa_n^A r_2\right) I_0 \left(\kappa_n^A r_2\right) QKIR\left(r_1, r_2, 0, \kappa_n^A, \kappa_n^A\right) +$$

$$\left. + I_0^2 \left(\kappa_n^A r_2\right) QKKR\left(r_1, r_2, 0, \kappa_n^A, \kappa_n^A\right) \right] . \qquad (35)$$

Eqs. (32), (34) and (35) may now be combined to determine W_{Az}.

Computation of W_{Ar} --

$$W_{Ar} = \frac{\varepsilon_0}{2} \int_{r_1}^{r_2} \left[\int_0^L \left|E_r^A\right|^2 dz \right] 2\pi r dr \qquad (36)$$

By applying Eq. (18) and the orthogonality properties already discussed, Eq. (36) becomes

$$W_{Ar} = \frac{\varepsilon_0 \pi^3}{2L} \sum_{n=1}^{\infty} \left[\frac{nC_n}{\kappa_n^A D_{0n}\left(\kappa_n^A r_1\right)} \right]^2 \int_{r_1}^{r_2} D_{1n}^2 \left(\kappa_n^A r\right) r dr . \qquad (37)$$

Finally, by using Eq. (14) and the integral notation, the expression for W_{Ar} becomes

$$W_{Ar} = \frac{\varepsilon_0 \pi^3}{2L} \sum_{n=1}^{\infty} \left\{ \left[\frac{nC_n}{\kappa_n^A D_{0n} \kappa_n^A r_1} \right]^2 \cdot \right.$$

$$\left[K_0^2 \left(\kappa_n^A r_2\right) QIIR\left(r_1, r_2, 1, \kappa_n^A, \kappa_n^A\right) + \right.$$

$$+ 2K_0 \left(\kappa_n^A r_2\right) I_0 \left(\kappa_n^A r_2\right) QKIR\left(r_1, r_2, 1, \kappa_n^A, \kappa_n^A\right) +$$

$$\left.\left. + I_0^2 \left(\kappa_n^A r_2\right) QKKR\left(r_1, r_2, 1, \kappa_n^A, \kappa_n^A\right) \right] \right\} . \qquad (38)$$

W_A is now determined by Eqs. (26), (32), (34), (35), and (38).

Total Energy Stored in Region B

The total energy stored in region B, W_B, is given by

$$W_B = W_{Bz} + W_{Br} , \tag{39}$$

where W_{Bz} and W_{Br} are the energies stored in the z and r directed E fields, respectively.

Computation of W_{Bz} --

$$W_{Bz} = \frac{\varepsilon_0 \varepsilon_{xr}}{2} \int_0^{r_1} \left[\int_0^d \left| E_z^B \right|^2 dz \right] 2\pi r dr \tag{40}$$

By using Eq. (20) and the orthogonality relations and recalling that d = aL, one obtains

$$W_{Bz} = \frac{\varepsilon_0 \varepsilon_{xr} M_0^2 \pi d}{J_0^2(kr_1)} QJJR(0, r_1, 0, k, k) + \frac{\varepsilon_0 \varepsilon_{xr} M_0^2 \pi d}{2} \sum_{m=1}^{\infty} \left(\frac{M_m}{M_0}\right)^2 \frac{QIIR\left(0, r_1, 0, \kappa_m^B, \kappa_m^B\right)}{I_0^2\left(\kappa_m^B r_1\right)} . \tag{41}$$

Computation of W_{Br} --

$$W_{Br} = \frac{\varepsilon_0 \varepsilon_{xr}}{2} \int_0^{r_1} \left[\int_0^d \left| E_r^B \right|^2 dz \right] 2\pi r dr \tag{42}$$

Employing Eq. (21) and the orthogonality relations yields

$$W_{Br} = M_0^2 \varepsilon_0 \varepsilon_{xr} \pi \frac{d}{2} \sum_{m=1}^{\infty} \left(\frac{\pi m}{aL\kappa_m^B}\right)^2 \left(\frac{M_m}{M_0}\right)^2 \cdot \frac{QIIR\left(0, r_1, 1, \kappa_m^B, \kappa_m^B\right)}{I_0^2\left(\kappa_m^B r_1\right)} . \tag{43}$$

Equations (39), (41), and (43) are combined to yield W_B.

Expressions for Power Losses in the Resonator

Power Losses in the Walls of the Resonator

The total power dissipated in the resonator walls, P_W, is broken out into six components, i.e.,

$$P_W = P_{cB} + P_{cA} + P_{oc} + P_b + P_{ic} + P_e , \tag{44}$$

where

P_{cB} is the power dissipated in region B of the cover (at z = 0)

P_{cA} is the power dissipated in region A of the cover

P_{oc} is the power dissipated in the outer conductor of the resonator at $r = r_2$

P_b is the power dissipated in the base plate at $z = L$

P_{ic} is the power dissipated on the periphery of the inner conductor at $r = r_1$

P_e is the power dissipated on the end of the center conductor at $z = d$.

Computation of P_{cB}-- By referring to Figure 3, it is readily seen that the power dP_{cB} dissipated in an incremental annular area on the cover plate is given by

$$dP_{cB} = \frac{|I_{rms}|^2 R_{sc}\, dr}{2\pi r} . \tag{45}$$

In Eq. (45), I_{rms} is the total rms current which flows radially across the annular region bounded by r and $r + dr$. (Note: Since the only H fields present are ϕ-directed, the only currents which can flow in the cover are radially directed.)

R_{sc} is the surface resistivity of the cover and is given by[3]

$$R_{sc} = \sqrt{\pi f \mu_1 \rho_1} , \tag{46}$$

where f is the loaded cavity resonant frequency in Hz, μ_1 is the permeability of the cover material, and ρ_1 is its bulk resistivity.

I_{rms} is related to the H field as

$$|I_{rms}| = 2\pi r \frac{|H_\phi^B|}{\sqrt{2}} , \tag{47}$$

so that Eq. (45) becomes

$$dP_{cB} = \pi R_{sc} |H_\phi^B|^2 r dr . \tag{48}$$

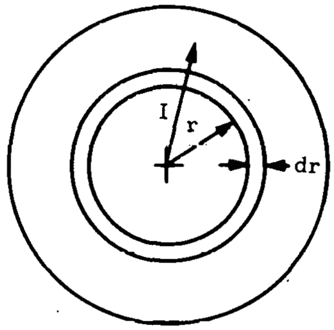

Figure 3. Incremental Annular Area

From Eq. (22),

$$\left|H_\phi^B\right|^2 = \frac{\varepsilon_{xr}}{\eta_0^2} M_0^2 \left[\frac{J_1^2(kr)}{J_0^2(kr_1)} + \right.$$

$$+ 2 \frac{k}{J_0(kr_1)} \sum_{m=1}^{\infty} \left(\frac{M_m}{M_0}\right) \frac{J_1(kr) I_1(\kappa_m^B r) S_1}{I_0(\kappa_m^B r_1)\kappa_m^B} +$$

$$\left. + k^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{M_i}{M_0}\right)\left(\frac{M_j}{M_0}\right) \frac{I_1(\kappa_i^B r) I_1(\kappa_j^B r) S_2}{\kappa_i^B \kappa_j^B I_0(\kappa_i^B r_1) I_0(\kappa_j^B r_1)} \right] , \qquad (49)$$

where $S_1 = 1 = \cos(0)$ for this case where $z = 0$. Also, in this case, $S_2 = 1 = \cos(0) \cdot \cos(0)$.

To find the total value (P_{cB}), Eq. (48) is integrated from $r = 0$ to $r = r_1$. To keep the algebra to manageable proportions, this integral is broken out into three parts, as shown in Eq. (50),

$$P_{cB} = \int_0^{r_1} \pi R_{sc} \left|H_\phi^D\right|^2 r dr = \pi R_{sc}(Q_4 + Q_5 + Q_6) . \qquad (50)$$

Here, Q_4, Q_5, and Q_6 represent the definite integrals of r times the first, second, and third terms, respectively, of Eq. (49); therefore,

$$Q_4 = \frac{M_0^2 \varepsilon_{xr}}{J_0^2(kr_1)\eta_0^2} \, QJJR(0, r_1, 1, k, k) , \qquad (51)$$

$$Q_5 = \frac{2M_0^2 \varepsilon_{xr} k}{J_0(kr_1)\eta_0^2} \sum_{m=1}^{\infty} \left(\frac{M_m}{M_0}\right) \frac{QIJR(0, r_1, 1, \kappa_m^B, k) S_1}{\kappa_m^B I_0(\kappa_m^B r_1)} , \qquad (52)$$

$$Q_6 = \frac{M_0^2 \varepsilon_{xr} k^2}{\eta_0^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{M_i}{M_0}\right)\left(\frac{M_j}{M_0}\right) \frac{QIIR(0, r_1, 1, \kappa_i^B, \kappa_j^B) S_2}{\kappa_i^B \kappa_j^B I_0(\kappa_i^B r_1) I_0(\kappa_j^B r_1)} . \qquad (53)$$

P_{cB} is therefore given by Eqs. (50), (51), (52), and (53).

Computation of P_{cA} -- The computation of P_{cA} is quite similar to that of P_{cB} so that Eq. (54) follows immediately.

$$dP_{cA} = \pi R_{sc} \left|H_\phi^A\right|^2 rdr \tag{54}$$

From Eq. (19),

$$\begin{aligned}\left|H_\phi^A\right|^2 = \frac{1}{n_0^2}\Bigg\{ & \left[\frac{aM_0 G_1(k_0 r)}{G_0(k_0 r_1)}\right]^2 + \\ & + 2\sum_{n=1}^{\infty} aM_0 \frac{G_1(k_0 r)}{G_0(k_0 r_1)} C_n \frac{k_0}{\kappa_n^A} \frac{D_{1n}(\kappa_n^A r)}{D_{0n}(\kappa_n^A r_1)} S_1 + \\ & + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_i \frac{k_0}{\kappa_i^A} \frac{D_{1i}(\kappa_i^A r)}{D_{0i}(\kappa_i^A r_1)} C_j \frac{k_0}{\kappa_j^A} \frac{D_{1j}(\kappa_j^A r)}{D_{0j}(\kappa_j^A r_1)} S_2 \Bigg\} .\end{aligned} \tag{55}$$

Once again, from Eq. (54), let

$$P_{cA} = \int_{r_1}^{r_2} \pi R_{sc} \left|H_\phi^A\right|^2 rdr = \pi R_{sc}(Q_1 + Q_2 + Q_3) , \tag{56}$$

where, as before, Q_1, Q_2, Q_3 represent the definite integrals of r times the first, second, and third terms, respectively, on the right side of Eq. (55). Now, by using Eqs. (55), (56), and (10), the value of Q_1 is found to be

$$\begin{aligned}Q_1 = \frac{a^2}{n_0^2} \frac{M_0^2}{G_0^2(k_0 r_1)} \Big[& N_0^2(k_0 r_2) QJJR(r_1, r_2, 1, k_0, k_0) - \\ & - 2N_0(k_0 r_2) J_0(k_0 r_2) QJNR(r_1, r_2, 1, k_0, k_0) + \\ & + J_0^2(k_0 r_2) QNNR(r_1, r_2, 1, k_0, k_0) \Big] .\end{aligned} \tag{57}$$

Taking the product of Eqs. (10) and (14) shows that

$$\begin{aligned}G_1(k_0 r) D_{1n}(\kappa_n^A r) = & \Big[J_1(k_0 r) N_0(k_0 r_2) - \\ & - J_0(k_0 r_2) N_1(k_0 r)\Big] I_1(\kappa_n^A r) K_0(\kappa_n^A r_2) + \\ & + \Big[J_1(k_0 r) N_0(k_0 r_2) - J_0(k_0 r_2) N_1(k_0 r)\Big] I_0(\kappa_n^A r_2) K_1(\kappa_n^A r) .\end{aligned} \tag{58}$$

Therefore, substituting Eq. (58) into Eq. (55) and using Eq. (56) gives

$$Q_2 = \frac{2}{\eta_0^{\,2}} \frac{aM_0 k_0}{G_0(k_0 r_1)} \sum_{n=1}^{\infty} \left\{ \frac{C_n S_1}{\kappa_n^A D_{0n}(\kappa_n^A r_1)} \cdot \right.$$

$$\cdot \Big[N_0(k_0 r_2) K_0(\kappa_n^A r_2) QIJR(r_1,\ r_2,\ 1,\ \kappa_n^A,\ k_0) -$$

$$- J_0(k_0 r_2) K_0(\kappa_n^A r_2) QINR(r_1,\ r_2,\ 1,\ \kappa_n^A,\ k_0) +$$

$$+ N_0(k_0 r_2) I_0(\kappa_n^A r_2) QKJR(r_1,\ r_2,\ 1,\ \kappa_n^A,\ k_0) -$$

$$\left. - J_0(k_0 r_2) I_0(\kappa_n^A r_2) QKNR(r_1,\ r_2,\ 1,\ \kappa_n^A,\ k_0) \Big] \right\} . \tag{59}$$

From Eq. (14), it is found that

$$D_{1i}(\kappa_i^A r) D_{1j}(\kappa_j^A r) =$$

$$\left[I_1(\kappa_i^A r) K_0(\kappa_i^A r_2) + I_0(\kappa_i^A r_2) K_1(\kappa_i^A r) \right] I_1(\kappa_j^A r) K_0(\kappa_j^A r_2) +$$

$$+ \left[I_1(\kappa_i^A r) K_0(\kappa_i^A r_2) + I_0(\kappa_i^A r_2) K_1(\kappa_i^A r) \right] I_0(\kappa_j^A r_2) K_1(\kappa_j^A r) \ . \tag{60}$$

Substituting Eq. (60) into Eq. (55) yields

$$Q_3 = \frac{k_0^{\,2}}{\eta_0^{\,2}} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left\{ \frac{C_i S_2 C_j}{\kappa_i^A \kappa_j^A D_{0i}(\kappa_i^A r_1) D_{0j}(\kappa_j^A r_1)} \cdot \right.$$

$$\cdot \Big[K_0(\kappa_i^A r_2) K_0(\kappa_j^A r_2) QIIR(r_1,\ r_2,\ 1,\ \kappa_i^A,\ \kappa_j^A) +$$

$$+ I_0(\kappa_i^A r_2) K_0(\kappa_j^A r_2) QKIR(r_1,\ r_2,\ 1,\ \kappa_i^A,\ \kappa_j^A) +$$

$$+ K_0(\kappa_i^A r_2) I_0(\kappa_j^A r_2) QKIR(r_1,\ r_2,\ 1,\ \kappa_j^A,\ \kappa_i^A) +$$

$$\left. + I_0(\kappa_i^A r_2) I_0(\kappa_j^A r_2) QKKR(r_1,\ r_2,\ 1,\ \kappa_i^A,\ \kappa_j^A) \Big] \right\} . \tag{61}$$

Then P_{cA} is calculated from Eqs. (56), (57), (59), and (61).

Computation of P_{oc} -- Since the only H fields in the resonator are those directed in the ϕ direction, the only currents which can flow in the resonator walls at $r = r_1$ and $r = r_2$ are those in the z direction (Figure 4).

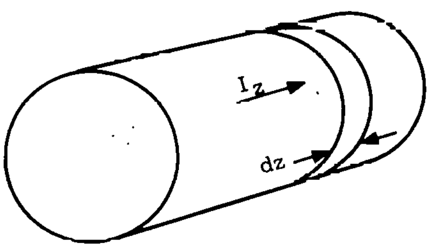

Figure 4. z-Directed Current Flowing across an Incremental Band on a Cylindrical Surface

With reference to Figure 4, it is evident that the incremental power (dP_{oc}) dissipated in the incremental band of length dz will be given by

$$dP_{oc} = \frac{\left|I_{rms}\right|^2 R_{soc}\, dz}{2\pi r_2} \, . \tag{62}$$

I_{rms} is the total rms current flowing across the incremental band, and R_{soc} is the surface resistivity of the outer conductor at $r = r_2$. R_{soc} is given by

$$R_{soc} = \sqrt{\pi f \mu_2 \rho_2} \, . \tag{63}$$

Values μ_2 and ρ_2 are the permeability and bulk resistivity, respectively, of the outer conductor metal.

Since

$$\left|I_{rms}\right| = \frac{2\pi r_2 \left|H_\phi^A\right|_{r_2}}{\sqrt{2}} \, , \tag{64}$$

Eq. (62) may be rewritten as

$$dP_{oc} = \pi \left|H_\phi^A\right|_{r_2}^2 r_2 R_{soc} dz \, , \tag{65}$$

so that the total power (P_{oc}) is given by

$$P_{oc} = \pi r_2 R_{soc} \int_0^L \left|H_\phi^A\right|_{r_2}^2 dz \, . \tag{66}$$

By using Eq. (19) and the orthogonality relations, one gets

$$P_{oc} = \frac{\pi r_2 R_{soc} L}{\eta_0^2} \left\{ \left[\frac{aM_0 G_1(k_0 r_2)}{G_0(k_0 r_1)}\right]^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{C_n k_0 D_{1n}(\kappa_n^A r_2)}{\kappa_n^A D_{0n}(\kappa_n^A r_1)}\right]^2 \right\} . \tag{67}$$

Computation of P_b -- The computation of P_b, the power dissipated in the base plate of the resonator at $z = L$, is very similar to that for P_{cA} as described earlier. In fact,

$$P_b = \pi R_{sb}(Q_1 + Q_2' + Q_3') \, , \tag{68}$$

where R_{sb} is the surface resistivity of the base plate and is given by

$$R_{sb} = \sqrt{\pi f \mu_3 \rho_3} \, . \tag{69}$$

Values μ_3 and ρ_3 are the permeability and bulk resistivity, respectively, of the base plate. Q_2' is the same as Q_2 in Eq. (59), provided that

$$S_1 = \cos(\pi n) = (-1)^n \quad . \tag{70}$$

Similarly, Q_3' is the same as Q_3 in Eq. (61) with the provision that

$$S_2 = \cos(\pi i)\ \cos(\pi j) = (-1)^{i+j} \quad . \tag{71}$$

Thus, P_b is computed using Eqs. (68), (69), (57), (59), (70), (61), and (71).

<u>Computation of P_{ic}</u> -- The computation of P_{ic}, the power dissipated on the periphery of the inner conductor, is somewhat similar to that for P_{oc} so that, from Eq. (66),

$$P_{ic} = \pi r_1 R_{sic} \int_d^L \left|H_\phi{}^A\right|_{r_1}^2 dz \tag{72}$$

can be derived. Here, R_{sic} is the surface resistivity of the inner conductor and is given by

$$R_{sic} = \sqrt{\pi f \mu_4 \rho_4} \quad . \tag{73}$$

Values μ_4 and ρ_4 are the permeability and bulk resistivity of the inner conductor.

Using Eq. (19) to express the value of $\left|H_\phi{}^A\right|_{r_1}^2$ and looking forward a little, it becomes apparent that the values of two definite trigonometric integrals will be required. These will now be evaluated in advance.

$$\int_d^L \cos\left(\frac{n\pi z}{L}\right) dz = \frac{L}{n\pi} \left[\sin\left(\frac{n\pi z}{L}\right)\right]_{z=d}^{z=L} \quad , \tag{74}$$

so

$$\int_d^L \cos\left(\frac{n\pi z}{L}\right) dz = -\ \frac{L}{n\pi}\ \sin\left(\frac{n\pi d}{L}\right) \quad . \tag{75}$$

The second integral is

$$\int_d^L \cos\left(\frac{\pi i z}{L}\right)\ \cos\left(\frac{\pi j z}{L}\right) dz$$

$$= \frac{L}{2\pi(i+j)} \int_d^L \left[\cos \frac{\pi}{L}(i+j)z\right]\left[\frac{\pi(i+j)dz}{L}\right] +$$

$$+ \frac{L}{2\pi(i-j)} \int_d^L \left[\cos \frac{\pi}{L}(i-j)z\right]\left[\frac{\pi}{L}(i-j)dz\right] \quad . \tag{76}$$

Therefore, if $i \neq j$

$$\int_d^L \cos\left(\frac{\pi i z}{L}\right) \cos\left(\frac{\pi j z}{L}\right) dz = \frac{L}{2\pi} \left[\frac{\sin\frac{\pi}{L}(i - j)d}{j - i} - \frac{\sin\frac{\pi}{L}(i + j)d}{j + i}\right] . \qquad (77)$$

If, however, $i = j$, then

$$\int_d^L \cos\left(\frac{\pi i z}{L}\right) \cos\left(\frac{\pi j z}{L}\right) dz = \frac{L}{\pi i} \int_d^L \left[\cos^2\left(\frac{\pi i z}{L}\right)\right]\left[\frac{\pi i dz}{L}\right]$$

$$= \frac{L}{\pi i} \int_{\frac{\pi i d}{L}}^{\pi i} \cos^2(u) du$$

$$= \frac{L}{\pi i} \left[\frac{u}{2} + \frac{\sin(2u)}{4}\right]_{\frac{\pi i d}{L}}^{\pi i} . \qquad (78)$$

So, if $i = j$, then

$$\int_d^L \cos\left(\frac{\pi i z}{L}\right) \cos\left(\frac{\pi j z}{L}\right) dz = \frac{L}{\pi i}\left[\frac{\pi i}{2} - \frac{\pi i d}{2L} - \frac{1}{4} \sin\left(\frac{2\pi i d}{L}\right)\right] . \qquad (79)$$

Now, define

$$H_{ij} \equiv \begin{cases} \frac{L}{2\pi}\left[\frac{\sin\frac{\pi}{L}(i - j)d}{j - i} - \frac{\sin\frac{\pi}{L}(i + j)d}{j + i}\right] \text{ if } i \neq j \\ \frac{L}{\pi i}\left[\frac{\pi i}{2}\left(1 - \frac{d}{L}\right) - \frac{1}{4} \sin\frac{2\pi i d}{L}\right] \text{ if } i = j \end{cases} \qquad (80)$$

From Eqs. (72), (19), (75), and (80),

$$P_{ic} = \frac{\pi r_1 R_{sic}}{\eta_0^2} \left\{ \left[\frac{aM_0 G_1(k_0 r_1)}{G_0(k_0 r_1)}\right]^2 (L - d) - \right.$$

$$- 2Lk_0 \frac{aM_0 G_1(k_0 r_1)}{\pi G_0(k_0 r_1)} \sum_{n=1}^{\infty} \frac{C_n D_{1n}\left(\kappa_n^A r_1\right)}{\kappa_n^A D_{0n}\left(\kappa_n^A r_1\right)} \sin\left(\frac{\pi n d}{L}\right) \left(\frac{1}{n}\right) +$$

$$\left. + k_0^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{C_i C_j}{\kappa_i^A \kappa_j^A} \frac{D_{1i}\left(\kappa_i^A r_1\right) D_{1j}\left(\kappa_j^A r_1\right)}{D_{0i}\left(\kappa_i^A r_1\right) D_{0j}\left(\kappa_j^A r_1\right)} H_{ij} \right\} . \qquad (81)$$

Computation of P_e -- The computation of P_e, the power dissipated on the end of the center conductor, is essentially the same as that described for P_{cB}. Therefore,

$$P_e = \pi R_{sic}(Q_4 + Q_5' + Q_6') \ . \tag{82}$$

R_{sic} and Q_4 are given by Eqs. (73) and (51), respectively. Q_5' is given by Eq. (52), provided that

$$S_1 = \cos(m\pi) = (-1)^m \ , \tag{83}$$

and Q_6' is the same as Q_6 in Eq. (53), provided that

$$S_2 = \cos(\pi i)\cos(\pi j) = (-1)^{i+j} \ . \tag{84}$$

All wall losses have now been computed.

Computation of P_C

The resonator is excited by driving a small loop (of area F) which is located in the base (at z = L) at a radius of $r_L = (r_1 + r_2)/2$. The plane of the loop is in the radial direction. The response of the resonator is sensed by a second identical loop which is diametrically opposite to the first one. Both the exciting source and the sensing receiver present a 50-ohm impedance to their respective loops. The rms voltage (V_{rms}) induced in each loop is given by

$$V_{rms} = \omega\mu_0 \frac{|H_L|}{\sqrt{2}} F \ . \tag{85}$$

Here, H_L is the magnetic field at the loop and normal to it. Therefore, the power P_{ex}, coupled from each loop to the external 50-ohm load is given by

$$P_{ex} = \frac{\omega^2 \mu_0^2 |H_L|^2 F^2}{(2)(50)} \ . \tag{86}$$

Since there are two identical loops, the total power (P_C) coupled out of the resonator is

$$P_C = 2P_{ex} = \frac{2\omega^2 \mu_0^2 |H_L|^2 F^2}{(2)(50)} \ . \tag{87}$$

Using Eq. (19), $|H_L|^2$ is found to be

$$|H_L|^2 = \left|H_\phi^A\right|^2_{r_L} = \frac{1}{\eta_0^2} \left\{ \left[aM_0 \frac{G_1(k_0 r_L)}{G_0(k_0 r_1)} \right]^2 + \right.$$

$$+ 2aM_0 k_0 \frac{G_1(k_0 r_L)}{G_0(k_0 r_1)} \sum_{n=1}^{\infty} \frac{C_n}{\kappa_n^A} \frac{D_{1n}\left(\kappa_n^A r_L\right)}{D_{0n}\left(\kappa_n^A r_1\right)} S_1 +$$

$$+ k_0^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{C_i D_{1i}(\kappa_i^A r_L)}{\kappa_i^A D_{0i}(\kappa_i^A r_1)} \frac{C_j D_{1j}(\kappa_j^A r_L)}{\kappa_j^A D_{0j}(\kappa_j^A r_1)} S_2 \Bigg\} , \tag{88}$$

where S_1 and S_2 are given by Eqs. (70) and (71), respectively. P_C is computed directly from Eq. (87) in place of the technique proposed by Karpova. Discussion of this last sentence and the means for determining the value of F^2 appear at the end of the section which follows.

Computation of P_J

As described earlier, the resonator was constructed so that the center conductor can slide in the base block to accommodate unknown samples of various lengths. The resistance in the joint (R_J) is of the order of 1 mΩ which becomes significant when measuring samples with very small loss tangents. Since all interior surfaces of the cavity are either copper- or silver-plated, the power dissipated in the joint (P_J) must be accounted for.

If I_{rms} is the rms current flowing across the joint, then

$$P_J = \left| I_{rms} \right|^2 R_J , \tag{89}$$

where

$$I_{rms} = 2\pi r_1 \left| \frac{H_\phi^A}{\sqrt{2}} \right|_{r_1, L} , \tag{90}$$

so, using Eqs. (19), (89), and (90),

$$P_J = \frac{2\pi^2 r_1^2 R_J}{\eta_0^2} \Bigg\{ \left[a M_0 \frac{G_1(k_0 r_1)}{G_0(k_0 r_1)} \right]^2 +$$

$$+ 2 a M_0 k_0 \frac{G_1(k_0 r_1)}{G_0(k_0 r_1)} \sum_{n=1}^{\infty} \frac{C_n}{\kappa_n^A} \frac{D_{1n}(\kappa_n^A r_1)}{D_{0n}(\kappa_n^A r_1)} S_1 +$$

$$+ k_0^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{C_i D_{1i}(\kappa_i^A r_1)}{\kappa_i^A D_{0i}(\kappa_i^A r_1)} \frac{C_j D_{1j}(\kappa_j^A r_1)}{\kappa_j^A D_{0j}(\kappa_j^A r_1)} S_2 \Bigg\} . \tag{91}$$

At this point, expressions for all quantities needed to compute TAND from Eq. (25) are available. It remains only to discuss an accurate means for determining the squared area (F^2) in Eq. (87). Since the loops are small and usually of irregular shape, it is difficult to perform an accurate physical measurement of F. It is, however, easy to determine the effective value of F^2 empirically. Once F^2 has been determined, it does not change so long as the coupling loops are not physically changed or moved; hence, after F^2 is once determined, there is no further need to make empty resonator measurements. (With Karpova's technique, this would be necessary every time the sample length (d) is changed.)

The determination of the effective value of F^2 proceeds as follows:

1. The empty cavity resonant frequency (F_0) and Q_0 are measured for an arbitrary but known (nonzero) value of d.
2. An estimate of the joint resistance (R_J) is obtained from dc measurements.
3. Various values of F^2 are substituted into Eq. (25) (along with the values of F_0, Q_0, and R_J which were obtained in steps 1 and 2) until a value for TAND which is as close to 0.0 as desired results.

Thus, the effective value of F^2 may be first determined as accurately as desired. That value will, of course, be valid for measurements on any subsequent dielectric sample of any length d. The effective value of F^2 thus derived will be quite consistent with an estimate based on physical measurement.

Finally, it may be noted that the valuc of M_0, which appears in numerous equations, is arbitrary, so it can conveniently be assigned a value of unity. This is true because each of the W and P terms in the expression for TAND, Eq. (25), is proportional to ${M_0}^2$; therefore, M_0 cancels out since TAND involves the P and W terms only in ratios of one to the other. (To verify the asserted proportionality of each of the P and W terms to M_0, it is necessary to note from Eq. (23) that C_n is proportional to M_0.)

Experimental Results

A reentrant cavity of the type suggested by the sketch in Figure 2 has been constructed for use in the VHF region. Three different barrels of different lengths were constructed to permit measurements at three different frequencies. As mentioned earlier there is a snug, sliding fit between the center conductor and the base block, thereby allowing the measurement of samples of different lengths. The compression screw forces intimate contact between the dielectric sample and the center conductor and cover of the cavity. The results of measurements made on Teflon, polystyrene, nylon, and fused silica are shown in Table 1. In addition, results from Reference 4 are given to provide a baseline by which the results may be judged. It is apparent that there is still work to be done--particularly in the loss tangent results. It is believed that most of the problems are caused by the variability of the resistance (R_J) between the center conductor and base block of the cavity. Variations of 0.1 mΩ produce dramatic shifts when measuring low-loss materials such as fused silica, etc. Work to eliminate this variability is continuing.

A word about the number of terms used to approximate the several infinite series is in order. A careful study of the trade-offs between accuracy and number of terms has not been conducted, so it is quite probable that more terms were used to obtain the results in Table 1 than were necessary. Two different approximations were used. Nine terms were used in all summations over m. Nine terms were also used in all summations over n with one exception: In solving the resonance equation, Eq. (1), more than nine terms were used in the summations over n. Each summation over n was terminated at a value $n = n_1$ when the following two conditions were met:

1. The magnitude of $\sin(\pi a n_1)$ had to exceed 0.9.
2. The n_1th term of the series had to have a magnitude of less than 0.01% of the $(n_1 - 1)$th partial sum.

The value of n_1 varies with the length of the cavity barrel and the properties of the dielectric sample being measured.

TABLE 1

Results of Dielectric Properties Measurements

Material	Frequency (MHz)	Dielectric Constant Measured	Dielectric Constant von Hippel	Loss Tangent Measured	Loss Tangent von Hippel
Teflon	226	2.08	2.04	0.00029	0.00015
Teflon	368	2.07	2.04	0.00040	0.00015
Teflon	490	2.08	2.04	0.0013	0.00015
Polystyrene	224	2.54	2.55	0.00027	0.0002
Polystyrene	361	2.54	2.55	0.00032	0.0002
Polystyrene	477	2.55	2.55	0.00041	0.0002
Nylon	220	3.09	3.10	0.016	0.015
Nylon	353	3.08	3.10	0.014	0.015
Nylon	463	3.09	3.10	0.014	0.015
Fused silica	216	3.75	3.78	0.00008	0.00004
Fused silica	342	3.80	3.78	0.00033	0.00004
Fused silica	445	3.80	3.78	0.000032	0.00004

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APPENDIX A

Tabulation of Closed-Form Expressions

This appendix tabulates closed-form expressions for the definite integrals involving Bessel functions which appear in this report. Most of the expressions were obtained from Reference 5; however, Eqs. (A-9) and (A-10) were derived as discussed in Appendix B.

Expressions are given for definite integrals of the form

$$\int_y^z Y_\nu(kr) Z_\nu(\ell r)\, r\, dr \ , \tag{A-1}$$

where $\nu = 0, 1, 2, 3, \cdots$ and Y and Z represent any of the Bessel functions J, N, I, or K; k and ℓ represent real, positive constants. Two expressions are given for each integral; the first is in terms of Bessel functions of order ν and $\nu + 1$; the second is in terms of Bessel functions of order ν and $\nu - 1$. Each integral is given in two notations: The first is the standard notation of expression (A-1), the second is the functional notation used in the body of this report, e.g.,

$$\int_y^z I_1(kr) J_1'(\ell r)\, r\, dr = \text{QIJR}(y,\ z,\ 1,\ k,\ \ell) \ .$$

Integral	Expressions	
$\int_0^z J_\nu^2(kr)\, r\, dr$	$\frac{z^2}{2}\left[J_\nu^2(kz) - \frac{2\nu}{kz} J_\nu(kz) J_{\nu+1}(kz) + J_{\nu+1}^2(kz)\right]$ or	
QJJR(0, z, ν, k, k)	$\frac{z^2}{2}\left[J_\nu^2(kz) - \frac{2\nu}{kz} J_\nu(kz) J_{\nu-1}(kz) + J_{\nu-1}^2(kz)\right]$	(A-2)
$\int_0^z J_\nu(kr) J_\nu(\ell r)\, r\, dr$	$\frac{z}{k^2 - \ell^2}\left[k J_\nu(\ell z) J_{\nu+1}(kz) - \ell J_\nu(kz) J_{\nu+1}(\ell z)\right]$ or	
QJJR(0, z, ν, k, ℓ) $k \neq \ell$	$\frac{z}{k^2 - \ell^2}\left[\ell J_{\nu-1}(\ell z) J_\nu(kz) - k J_{\nu-1}(kz) J_\nu(\ell z)\right]$	(A-3)

Integral	Expressions
$\int_0^z J_\nu(kr)N_\nu(kr)rdr$	$\frac{z^2}{2}\left[N_\nu(kz)J_\nu(kz) - \frac{\nu}{kz}\left\{J_{\nu+1}(kz)N_\nu(kz) + J_\nu(kz)N_{\nu+1}(kz)\right\} + N_{\nu+1}(kz)J_{\nu+1}(kz)\right]$ or
QJNR(0, z, ν, k, k)	$\frac{z^2}{2}\left[N_\nu(kz)J_\nu(kz) - \frac{\nu}{kz}\left\{N_{\nu-1}(kz)J_\nu(kz) + N_\nu(kz)J_{\nu-1}(kz)\right\} + N_{\nu-1}(kz)J_{\nu-1}(kz)\right]$ (A-4)
$\int_0^z J_\nu(kr)N_\nu(\ell r)rdr$	$\frac{z}{k^2 - \ell^2}\left[kN_\nu(\ell z)J_{\nu+1}(kz) - \ell J_\nu(kz)N_{\nu+1}(\ell z)\right]$ or
QJNR(0, z, ν, k, ℓ) $k \neq \ell$	$\frac{z}{k^2 - \ell^2}\left[\ell N_{\nu-1}(\ell z)J_\nu(kz) - kJ_{\nu-1}(kz)N_\nu(\ell z)\right]$ (A-5)
$\int_0^z I_\nu(kr)J_\nu(\ell r)rdr$	$\frac{z}{k^2 + \ell^2}\left[kJ_\nu(\ell z)I_{\nu+1}(kz) + \ell I_\nu(kz)J_{\nu+1}(\ell z)\right]$ or
QIJR(0, z, ν, k, ℓ)	$\frac{z}{k^2 + \ell^2}\left[kJ_\nu(\ell z)I_{\nu-1}(kz) - \ell I_\nu(kz)J_{\nu-1}(\ell z)\right]$ (A-6)
$\int_y^z K_\nu(kr)J_\nu(\ell r)rdr$	$\frac{1}{k^2 + \ell^2}\left\{\left[\ell K_\nu(kz)J_{\nu+1}(\ell z) - kJ_\nu(\ell z)K_{\nu+1}(kz)\right]z - y\left[\ell K_\nu(ky)J_{\nu+1}(\ell y) - kJ_\nu(\ell y)K_{\nu+1}(ky)\right]\right\}$ or
QKJR(y, z, ν, k, ℓ) $y > 0$	$\frac{1}{k^2 + \ell^2}\left\{\left[kJ_\nu(\ell y)K_{\nu-1}(ky) + \ell K_\nu(ky)J_{\nu-1}(\ell y)\right]y - z\left[kJ_\nu(\ell z)K_{\nu-1}(kz) + \ell K_\nu(kz)J_{\nu-1}(\ell z)\right]\right\}$ (A-7)
$\int_0^z N_\nu^2(kr)rdr$	$\frac{z^2}{2}\left[N_\nu(kz) - \frac{2\nu}{kz}N_\nu(kz)N_{\nu+1}(kz) + N_{\nu+1}^2(kz)\right]$ or
QNNR(0, z, ν, k, k)	$\frac{z^2}{2}\left[N_\nu^2(kz) - \frac{2\nu}{kz}N_\nu(kz)N_{\nu-1}(kz) + N_{\nu-1}^2(kz)\right]$ (A-8)
$\int_y^z I_\nu(kr)N_\nu(\ell r)rdr$	$\frac{1}{k^2 + \ell^2}\left\{\left[kN_\nu(\ell z)I_{\nu+1}(kz) + \ell I_\nu(kz)N_{\nu+1}(\ell z)\right]z - y\left[kN_\nu(\ell y)I_{\nu+1}(ky) - \ell I_\nu(ky)N_{\nu+1}(\ell y)\right]\right\}$ or

Integral	Expressions	
QINR(y, z, ν, k, ℓ) $y > 0$	$\frac{1}{k^2 + \ell^2} \{[kN_\nu(\ell z) I_{\nu-1}(kz) - \ell I_\nu(kz) N_{\nu-1}(\ell z)]z - y[kN_\nu(\ell y) I_{\nu-1}(ky) - \ell I_\nu(ky) N_{\nu-1}(\ell y)]\}$	(A-9)
$\int_y^z K_\nu(kr) N_\nu(\ell r) r dr$	$\frac{1}{k^2 + \ell^2} \{[\ell K_\nu(kz) N_{\nu+1}(\ell z) - kN_\nu(\ell z) K_{\nu+1}(kz)]z - y[\ell K_\nu(ky) N_{\nu+1}(\ell y) - kN_\nu(\ell y) K_{\nu+1}(ky)]\}$ or	
QKNR(y, z, ν, k, ℓ) $y > 0$	$\frac{1}{k^2 + \ell^2} \{[kN_\nu(\ell y) K_{\nu-1}(ky) + \ell K_\nu(ky) N_{\nu-1}(\ell y)]y - z[kN_\nu(\ell z) K_{\nu-1}(kz) + \ell K_\nu(kz) N_{\nu-1}(\ell z)]\}$	(A-10)
$\int_0^z I_\nu^2(kr) r dr$	$\frac{z^2}{2} \left[I_\nu^2(kz) - \frac{2\nu}{kz} I_\nu(kz) I_{\nu+1}(kz) - I_{\nu+1}^2(kz)\right]$ or	
QIIR(0, z, ν, k, k)	$\frac{z^2}{2} \left[I_\nu^2(kz) + \frac{2\nu}{kz} I_\nu(kz) I_{\nu-1}(kz) - I_{\nu-1}^2(kz)\right]$	(A-11)
$\int_y^z K_\nu(\ell r) I_\nu(\ell r) r dr$	$\frac{z^2}{2} \{I_\nu(\ell z) K_\nu(\ell z) - \frac{\nu}{\ell z} [K_\nu(\ell z) I_{\nu+1}(\ell z) - K_{\nu+1}(\ell z) I_\nu(\ell z)] + K_{\nu+1}(\ell z) I_{\nu+1}(\ell z)\} - \frac{y^2}{2} \{I_\nu(\ell y) K_\nu(\ell y) - \frac{\nu}{\ell y} [K_\nu(\ell y) I_{\nu+1}(\ell y) - K_{\nu+1}(\ell y) I_\nu(\ell y)] + K_{\nu+1}(\ell y) I_{\nu+1}(\ell y)\}$ or	
QKIR(y, z, ν, ℓ, ℓ) $y > 0$	$\frac{z^2}{2} \{I_\nu(\ell z) K_\nu(\ell z) + \frac{\nu}{\ell z} [K_\nu(\ell z) I_{\nu-1}(\ell z) - K_{\nu-1}(\ell z) I_\nu(\ell z)] + K_{\nu-1}(\ell z) I_{\nu-1}(\ell z)\} - \frac{y^2}{2} \{I_\nu(\ell y) K_\nu(\ell y) + \frac{\nu}{\ell y} [K_\nu(\ell y) I_{\nu-1}(\ell y) - K_{\nu-1}(\ell y) I_\nu(\ell y)] + K_{\nu-1}(\ell y) I_{\nu-1}(\ell y)\}$	(A-12)
$\int_y^z K_\nu(kr) I_\nu(\ell r) r dr$	$\frac{1}{k^2 - \ell^2} \{[kI_\nu(\ell y) K_{\nu+1}(ky) + \ell K_\nu(ky) I_{\nu+1}(\ell y)]y - z[kI_\nu(\ell z) K_{\nu+1}(kz) + \ell K_\nu(kz) I_{\nu+1}(\ell z)]\}$ or	
QKIR(y, z, ν, k, ℓ) $k \neq \ell$ $y > 0$	$\frac{1}{k^2 - \ell^2} \{[kK_{\nu-1}(ky) I_\nu(\ell y) + \ell I_{\nu-1}(\ell y) K_\nu(ky)]y - z[kK_{\nu-1}(kz) I_\nu(\ell z) + \ell I_{\nu-1}(\ell z) K_\nu(kz)]\}$	(A-13)

Integral	Expressions	
$\int_z^\infty K_\nu^2(kr)rdr$	$\frac{z^2}{2}\left[K_{\nu+1}^2(kz) - \frac{2\nu}{kz}K_\nu(kz)K_{\nu+1}(kz) - K_\nu^2(kz)\right]$ or	
QKKR(z, ∞, ν, k, k) $z > 0$	$\frac{z^2}{2}\left[K_{\nu-1}^2(kz) + \frac{2\nu}{kz}K_\nu(kz)K_{\nu-1}(kz) - K_\nu^2(kz)\right]$	(A-14)
$\int_z^\infty K_\nu(kr)K_\nu(\ell r)rdr$	$\frac{z}{k^2 - \ell^2}\left\{kK_\nu(\ell z)K_{\nu+1}(kz) - \ell K_\nu(kz)K_{\nu+1}(\ell z)\right\}$ or	
QKKR(z, ∞, ν, k, ℓ) $k \neq \ell$ $z > 0$	$\frac{z}{k^2 - \ell^2}\left\{kK_{\nu-1}(kz)K_\nu(\ell z) - \ell K_{\nu-1}(\ell z)K_\nu(kz)\right\}$	(A-15)

APPENDIX B

Derivation of Equations

This appendix gives the derivation of Eqs. (A-9) and (A-10), which appear in Appendix A. The author is grateful to D. E. Amos of the Numerical Mathematics Division 5642 for outlining this derivation.

From Eq. (2) on page 190 of Reference 5, the differential equation

$$y_1'' + \frac{y_1'}{z} + \left(\ell^2 - \frac{\nu^2}{z^2}\right)y_1 = 0 \qquad \text{(B-1)}$$

has a solution

$$y_1 = aJ_\nu(\ell z) + bN_\nu(\ell z) \ . \qquad \text{(B-2)}$$

Also, from Eq. (4) on page 190 of Reference 5, the differential equation

$$y_2'' + \frac{y_2'}{z} - \left(k^2 + \frac{\nu^2}{z^2}\right)y_2 = 0 \qquad \text{(B-3)}$$

has a solution

$$y_2 = cI_\nu(kz) + dK_\nu(kz) \ . \qquad \text{(B-4)}$$

In these equations, a, b, c, and d are arbitrary constants. Multiplication of Eq. (B-1) by z^2y_2 and Eq. (B-3) by z^2y_1 yields

$$y_2y_1''z^2 + zy_1'y_2 + (\ell^2z^2 - \nu^2)y_2y_1 = 0 \ , \qquad \text{(B-5)}$$

$$y_1y_2''z^2 + zy_2'y_1 - (k^2z^2 + \nu^2)y_2y_1 = 0 \ . \qquad \text{(B-6)}$$

Subtract Eq. (B-5) from Eq. (B-6) to get

$$z(y_1y_2'' - y_2y_1'') + (y_2'y_1 - y_1'y_2) = y_1y_2z(k^2 + \ell^2) \ . \qquad \text{(B-7)}$$

Observe that

$$\frac{d}{dz}\left[z(y_2'y_1 - y_1'y_2)\right] = z(y_1y_2'' - y_2y_1'') + y_2'y_1 - y_1'y_2 \ . \qquad \text{(B-8)}$$

Substituting Eq. (B-8) into Eq. (B-7) gives

$$\frac{d}{dz}\left[z(y_2'y_1 - y_1'y_2)\right] = y_1y_2z(k^2 + \ell^2) \ , \tag{B-9}$$

or

$$y_1y_2zdz = \frac{d\left[z(y_2'y_1 - y_1'y_2)\right]}{k^2 + \ell^2} \ . \tag{B-10}$$

Integrating from z_1 to z_2, where $0 < z_1 < z_2 < \infty$ yields

$$\int_{z_1}^{z_2} y_1y_2zdz = \left[\frac{z}{k^2 + \ell^2}(y_2'y_1 - y_1'y_2)\right]_{z_1}^{z_2} \ . \tag{B-11}$$

(Actually, if $b = d = 0$ in Eqs. (B-2) and (B-4), $z_1 = 0$ is permissible; also, if $c = 0$, then $z_2 = \infty$ is permissible.)

Choosing $a = d = 0$ in Eqs. (B-2) and (B-4) and substituting the resulting expressions into Eq. (B-11) results in

$$\int_{z_1}^{z_2} I_\nu(kz)N_\nu(\ell z)zdz = \left[\frac{z}{k^2 + \ell^2}\left\{N_\nu(\ell z)\frac{d}{dz}I_\nu(kz) - I_\nu(kz)\frac{d}{dz}N_\nu(\ell z)\right\}\right]_{z_1}^{z_2} \ . \tag{B-12}$$

Similarly, choosing $a = c = 0$ in Eqs. (B-2) and (B-4) and substituting the results into Eq. (B-11) yields

$$\int_{z_1}^{z_2} K_\nu(kz)N_\nu(\ell z)zdz = \left[\frac{z}{k^2 + \ell^2}\left\{N_\nu(\ell z)\frac{d}{dz}K_\nu(kz) - K_\nu(kz)\frac{d}{dz}N_\nu(\ell z)\right\}\right]_{z_1}^{z_2} \ . \tag{B-13}$$

From Eqs. 3.27(14a), 3.27(14b), 3.27(9), 3.27(10), 3.27(15a), and 3.27(15b) of Reference 3, one obtains

$$\frac{d}{dz}I_\nu(kz) = \frac{\nu}{z}I_\nu(kz) + kI_{\nu+1}(kz) \tag{B-14}$$

or

$$\frac{d}{dz}I_\nu(kz) = \frac{-\nu}{z}I_\nu(kz) + kI_{\nu-1}(kz) \ . \tag{B-15}$$

$$\frac{d}{dz}N_\nu(\ell z) = \frac{\nu}{z}N_\nu(\ell z) - \ell N_{\nu+1}(\ell z) \tag{B-16}$$

or

$$\frac{d}{dz}N_\nu(\ell z) = -\frac{\nu}{z}N_\nu(\ell z) + \ell N_{\nu-1}(\ell z) \quad . \tag{B-17}$$

$$\frac{d}{dz}K_\nu(kz) = \frac{\nu}{z}K_\nu(kz) - kK_{\nu+1}(kz) \tag{B-18}$$

or

$$\frac{d}{dz}K_\nu(kz) = -\frac{\nu}{z}K_\nu(kz) - kK_{\nu-1}(kz) \quad . \tag{B-19}$$

Substituting Eqs. (B-14), (B-15), (B-16), and (B-17) into Eq. (B-12) yields Eq. (A-9). Similarly, substituting Eqs. (B-18), (B-19), (B-16), and (B-17) into Eq. (B-13) yields Eq. (A-10).

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