ORO 5126-18

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U. of Md. TR #78-000
U. of Md. PP #78-002

August 1977

UNIVERSITY OF MARYLAND
DEPARTMENT OF PHYSICS AND ASTRONOMY
COLLEGE PARK, MARYLAND

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ABSTRACT

A once-subtracted form of the Low equation for the pion-nucleon scattering amplitude is derived, with PCAC used to define the amplitude when one pion is off the mass shell. The static approximation is not made and both the seagull terms and the antinucleon contribution (z-graphs) are retained. The theory is applied to calculate the S-wave amplitudes in the elastic scattering region. Good agreement is found with the phase shift fits to the data when we use $|g_\pi(4M^2)| = 11.69$ and 25.5 MeV for the πN σ-commutator. The implications of this work for the analysis of low-energy elastic scattering of pions from nuclei are discussed. In particular, we point out how this work establishes the presence of a Laplacian term in the pion-nucleus optical potential with a magnitude that is fixed from the value of the σ-commutator.

KEYWORD ABSTRACT

NUCLEAR REACTION Pion nucleon scattering, off-shell amplitude, σ-commutator, Laplacian in pion nucleus optical potential.

*Work supported in part by the U. S. Energy Research and Development Administration and the National Science Foundation.

I. INTRODUCTION

In a recent paper[1] we presented a brief description of a theory of the low-energy pion-nucleon interaction and its main results. The present article gives a detailed exposition of the rationale behind the theory, the method of calculation and a fuller description of our results for the S-wave πN amplitudes.

The work was motivated by our study[2] of the problem of low and intermediate energy pion-nuclear scattering which requires a knowledge of the pion-nucleon scattering amplitudes where all particles are off their mass shell. Currently several methods of constructing such amplitudes are popular. One of these, for example, is the Kisslinger[3] model where one represents the isoscalar, spin-independent part of the πN amplitude in the form $b(\omega) + c(\omega)\vec{k}\cdot\vec{k}'$. In the simplest model ω, $\vec{k}$ and $\vec{k}'$ are the pion energy, final and initial momenta, respectively, in the CM frame. The quantities $b(\omega)$ and $c(\omega)$ give the strengths of the S- and the P-wave amplitudes. There has been considerable discussion[4] as to whether one should use the Kisslinger form or the so-called Laplacian form, $b'(\omega) - c'(\omega)(\vec{k}-\vec{k}')^2$. Inevitably the question arises, why not a combination of both, $b''(\omega) - c''(\omega)(\vec{k}-\vec{k}')^2 + d(\omega)\vec{k}\cdot\vec{k}'$. Clearly, questions of this nature cannot be settled without a dynamical theory of the pion-nucleon interaction. For the P-wave there is the well-known work of Chew and Low[5] based on the static approximation and neglect of seagull terms. Similar approximate treatments have also been tried for the S-wave amplitudes.[6] The theory described in the present work is an extension and improvement of these earlier efforts in which both nucleon recoil and the seagull terms are included.

Another popular approach to describing the pion-nucleon interaction makes use of separable potentials. This method, originally proposed by Landau and Tabakin,[7] gives for the isoscalar, spin-independent amplitude the form $b(\omega)v_S(\vec{k}^2)v_S(\vec{k}'^2) + c(\omega)\vec{k}\cdot\vec{k}'v_P(\vec{k}^2)v_P(\vec{k}'^2)$. The method has the advantage that the quantities $b(\omega)$ and $c(\omega)$ and the form factors v_S and v_P can be determined from experimental data by an inversion procedure. Attempts have been made to improve this approach.[8] But the results are only as valid as the rationale for describing the dynamics with a potential in the first place, and then for a separable potential in every channel. The Chew-Low theory and the resonance dominance justifies a separable form for the P-wave amplitude. But it is hard to justify a separable S-wave amplitude and we do not know of any attempt at providing the needed justification.

There are three basic mechanisms in pion-nucleon elastic scattering which any model of the πN amplitude must include. First, there are exchanges of scalar-isoscalar bosons and vector-isovector bosons between the pion and nucleon. Second, a nucleon can absorb a pion and then emit it, or emit and then absorb a pion. Third, a pion can virtually dissociate into a nucleon-antinucleon pair, the nucleon going into the final state and the antinucleon being absorbed by the initial nucleon to form the final state pion (z-graph). In addition there are many other processes involving more bosons and baryons, but these are less important for the low-energy interaction. The most practical method of describing these important mechanisms is to use field theory, where the scalar quantity[9]

$$\bar{u}(p_f)(\not{p}_f - M)(k^2 - m_\pi^2)\int d^4x\, e^{ik\cdot x}\int d^4y\, e^{ip_f\cdot y}\int d^4z\, e^{-ip_i\cdot z}$$

$$\times \langle 0|T(\psi(y),\phi_\beta(x),\bar{\psi}(z),j_\alpha(0))|0\rangle(\not{p}_i - M)u(p_i)$$

is the general pion-nucleon scattering amplitude associated with the process depicted in Fig. 1. When any one of the four momenta p_i, p_f, k or $k' \equiv p_f + k - p_i$ is such that its square is equal to the square of the mass of the associated particle, the corresponding particle (or leg) is said to be on the mass shell.

Apart from the familiar symmetries imposed by Lorentz invariance and isospin independence the pion-nucleon scattering amplitude must satisfy the symmetry condition resulting from the self-charge-conjugate character of the pion. This is known as the crossing symmetry which says that the scattering amplitude is invariant under the change $k \longleftrightarrow -k'$ and $\alpha \longleftrightarrow \beta$. The importance of crossing symmetry in pion-nucleon scattering is well known. Its important role in pion-nuclear scattering has been discussed by us[10] earlier. Field theory provides a convenient framework in which to construct a crossing symmetric amplitude, though a potential formalism can also incorporate this symmetry.[10] There are, however, a number of difficulties in a potential theory approach to the πN amplitude.

Since potential theory is generally used in the analysis of pi-nuclear scattering, it is perhaps worthwhile to discuss the relationship between the potential and field-theoretic descriptions of the elementary pi-nucleon interaction. For the present discussion we ignore spin, so the πN amplitude becomes a function of six scalar variables. It is convenient to choose these to be the following six CM frame quantities: the total energy $W = p_{f0} + k_0$, the squares of initial and final momenta, $\vec{p}_f^{\,2} = \vec{k}^2$ and $\vec{p}_i^{\,2} = \vec{k}'^2$, the angle of scattering θ and the pion energies k_0 and k_0'. We note that usually a potential theory is used to describe an interaction which is instantaneous in the CM frame, so the resulting scattering amplitude cannot depend on k_0 or k_0'. It can be a function of the first four variables only. Nevertheless a subset of the relativistic scattering amplitudes can be described by an effective potential. The subset has $k_0 = k_0' = \frac{1}{2W}(W^2 - M^2 + m_\pi^2)$, which is the value when all particles are on mass shell. With this condition $k^2 - p_f^2 = k'^2 - p_i^2 = m_\pi^2 - M^2$, but not necessarily $k^2 = m_\pi^2$, $p_f^2 = M^2$, etc. These amplitudes are usually categorized as being on or off the energy shell. When $W = \sqrt{M^2 + \vec{p}_f^{\,2}} + \sqrt{m_\pi^2 + \vec{p}_f^{\,2}}$ and $\vec{p}_f^{\,2} = \vec{p}_i^{\,2}$ (CM frame) the amplitude is on the energy shell. When $W = \sqrt{M^2 + \vec{p}_f^{\,2}} + \sqrt{m_\pi^2 + \vec{p}_f^{\,2}}$ or $W = \sqrt{M^2 + \vec{p}_i^{\,2}} + \sqrt{m_\pi^2 + \vec{p}_i^{\,2}}$ and $\vec{p}_f^{\,2} \neq \vec{p}_i^{\,2}$ it is half off the energy shell, while if $W \neq \sqrt{M^2 + \vec{p}_f^{\,2}} + \sqrt{m_\pi^2 + \vec{p}_f^{\,2}}$ and $W \neq \sqrt{M^2 + \vec{p}_i^{\,2}} + \sqrt{m_\pi^2 + \vec{p}_i^{\,2}}$ and $\vec{p}_f^{\,2} \neq \vec{p}_i^{\,2}$ it is fully off the energy shell.

There are two steps in deriving the effective potential. First one writes down the Bethe-Salpeter[11] equation for the scattering amplitude. Next one uses the Blankenbecler-Sugar[12] prescription for converting the Bethe-Salpeter equation into the Lippmann-Schwinger form. In the process one replaces the product of the Feynman propagators for a nucleon and a pion appearing in the Bethe-Salpeter equation by the Lippmann-Schwinger propagator with a delta function which keeps the energy of the intermediate pion fixed at $\frac{1}{2W}(W^2 - M^2 + m_\pi^2)$. Now, if the external pion momenta are also fixed at this value all references to k_0 and k_0' are removed.

In practice it is very difficult to carry out the program described above.[13] It has proved to be quite difficult for nucleon-nucleon scattering. For pion-nucleon scattering the problem is further complicated by the existence of the absorption-emission mechanism and crossing symmetry.

Even if the equivalent potential could be found there would be still another difficulty in using the resulting amplitudes in the problem of pion-nuclear scattering. This can be seen from the following considerations. To construct the first order optical potential one needs the amplitude where, in the target nucleus rest frame, $k_0 = k_0'$ and $p_{f0} = p_{i0} = M - \varepsilon$ with ε the binding energy of an occupied single particle state. One also needs the amplitude for a wide range of values of $\vec{k}$, $\vec{p}_f$ and $\vec{p}_i$. So the required conditions for a potential description, namely $2k\cdot(k+p_f) = (k+p_f)^2 - M^2 + m_\pi^2$, etc., or equivalently $p_f^2 - k^2 = p_i^2 - k'^2 = M^2 - m_\pi^2$, cannot be fulfilled. (This problem is also present, in principle, in nucleon-nuclear scattering.) This problem is related to the non-instantaneous character of the interaction. The non-instaneity of the pion-nucleon interaction manifests itself through the dependence of the scattering amplitude on the variable $\nu = \frac{1}{2M}[W^2 - M^2 + |\vec{p}_i||\vec{p}_f|\text{Cos }\theta - k_0 k_0']$. The isoscalar (isovector) part of the scattering amplitude is an even (odd) function of ν which, we exphasize, depends explicitly on k_0 and k_0'.

Because of these difficulties with a potential theory description of the πN amplitude, we consider instead a field-theoretic description. We develop a theory for the amplitudes where only one pion is off the mass shell. Once these are known it is straightforward to construct amplitudes where both pions are off the mass shell. Construction of amplitudes where all particles are off the mass shell requires some approximations which will be discussed in a future paper. The present work is thus a necessary first step in the construction of the amplitudes required in the analysis of pion-nuclear scattering.

In the following we present a theory of πN scattering which is a logical extension of the work of Chew and Low.[5] We use the Low equation[14] obtained by LSZ reduction,[9] but in contrast to Chew-Low we do not use the static approximation, and we retain the seagull terms and the antinucleon intermediate state contribution. The definition of the scattering amplitude off the mass shell of one pion follows, in part, from the indentification of the interpolating pion field as the divergence of the axial vector current.

A soft-pion limit is used to eliminate the isoscalar part of the seagull term and obtain a once-subtracted form of the Low equation which suppresses contributions of high-mass intermediate states. This equation allows the evaluation of both the physical πN amplitude as well as the off-mass-shell[15] amplitude once the remaining dynamical inputs are specified. These include the isovector part of the seagull term, the pion nucleon form factor, and the sigma commutator term which appears in the soft-pion limit. We discuss the validity of each dynamical input to the theory.

The Low equation derived in Section II describes all πN partial waves and formally is valid for all energies. In Section III we present a covariant partial wave expansion of this equation and describe the method of numerical solution for the S-wave amplitudes using Padé approximants. Several new and useful techniques developed for solving the S-wave equation are discussed in sufficient detail so that they may be applied readily to other problems. In Section IV we comment on the various sets of "experimental" phase shifts, the method of searching for the best values of the parameters of the theory and our final results for the S-wave amplitudes. We find that our on-shell amplitudes agree well with experiment. Section V contains a brief discussion of the significance of the value of some of the parameters determined by our analysis, and Section VI includes a summary and concluding remarks.

II. DEVELOPMENT OF THEORY

A. Off-Mass-Shell Amplitude and the Low Equation

An expression for the off-mass-shell pion-nucleon amplitude is obtained by applying the LSZ reduction procedure[9] to the S-matrix element for πN scattering,

$$S_{fi} = \langle \pi_\beta(k),\ N(p_f);\text{out} | \pi_\alpha(k'),\ N(p_i);\text{in}\rangle ,$$

which gives

$$S_{fi} = \delta_{fi} + i(2\pi)^4\ \delta^4(k + p_f - k' - p_i)\ F_{\beta\alpha}(p_i, p_f, k)\Big|_{\substack{k_0=\sqrt{m_\pi^2+\vec{k}^2},\ k_0'=\sqrt{m_\pi^2+\vec{k}'^2} \\ p_{f0}=\sqrt{M^2+\vec{p}_f^2},\ p_{i0}=\sqrt{M^2+\vec{p}_i^2}}}$$

with

$$F_{\beta\alpha}(p_i, p_f, k) = \langle \pi_\beta(k),\ N(p_f);\ \text{out} | j_\alpha(0) | N(p_i)\rangle \Big|_{\substack{k_0=\sqrt{m_\pi^2+\vec{k}^2} \\ p_{f0}=\sqrt{M^2+\vec{p}_f^2} \\ p_{i0}=\sqrt{M^2+\vec{p}_i^2}}}$$

$$(\Box + m_\pi^2)\ \phi_\alpha(x) = j_\alpha(x) \qquad (1)$$

and where $\phi_\alpha(x)$ is the interpolating pion field. In the S-matrix element the four-momenta of all particles satisfy an appropriate mass shell constraint. The amplitude $F_{\beta\alpha}(p_i, p_f, k)$ has no explicit dependence on the four-momentum k' of the initial state pion. Defining the four momentum of this pion by the energy-momentum relation

$$k' = p_f + k - p_i, \qquad (2)$$

we do not necessarily have $k'^2 = m_\pi^2$. Thus with (2) giving the dependence of F on k', we take Eq. (1) as a definition of what can be called a "half-off-mass-shell" amplitude. In this work we are primarily concerned with the numerical evaluation of this amplitude. This is accomplished by solving the Low equation,[14] which is a nonlinear inhomogeneous integral equation for this off-mass-shell amplitude.

We develop the Low equation by reducing out the final state pion in (1) giving

$$F_{\beta\alpha}(k) = i\int d^4x\, e^{ik\cdot x}(\Box + m_\pi^2)\langle p_f|T(\phi_\beta(x)j_\alpha(0))|p_i\rangle\Big|_{p_{f0}=\sqrt{M^2+\vec{p}_f^{\,2}},\; p_{i0}=\sqrt{M^2+\vec{p}_i^{\,2}}} \qquad (3)$$

The symbol T denotes the usual time ordering of operators, and for brevity we henceforth indicate only k as the argument of F, and where no confusion can arise, we omit the particle symbols N and π. In (3) both initial and final nucleons are on their mass shell, but since the four-momentum of the final state pion occurs only in the exponential it can be taken as a freely variable parameter no longer restricted to satisfy the mass shell conditions $k^2 = m_\pi^2$, $k_0 > 0$. To guarantee a convergent integral, however, the four-vector k must be real. Thus with k' again given by (2), Eq. (3) defines what may be called a "fully-off-mass-shell" amplitude.

Allowing the Klein-Gordon operator to act on the matrix element in Eq. (3) gives

$$F_{\beta\alpha}(k) = \langle p_f|i\varepsilon_{\alpha\beta\lambda}(k+k')\cdot Y^\lambda(0)|p_i\rangle + \langle p_f|\delta_{\alpha\beta}\Sigma(0)|p_i\rangle$$

$$+\; i\int d^4x\, e^{ik\cdot x}\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle, \qquad (4)$$

where we have made the definitions

$$\langle p_f|i\varepsilon_{\alpha\beta\lambda}(k+k')\cdot Y^\lambda(0) + \delta_{\alpha\beta}\Sigma(0)|p_i\rangle = \int\langle p_f|\delta(x_0)k_0[\phi_\beta(x),j_\alpha(0)]$$

$$+\; i\delta(x_0)[\dot{\phi}_\beta(x),j_\alpha(0)]|p_i\rangle e^{ik\cdot x}d^4x \qquad (5)$$

with $\dot{\phi} = \frac{d}{dx_0}\phi$. This equation defines the so-called seagull terms. Σ is a scalar-isoscalar operator, while Y is a vector-isovector operator. Crossing symmetry requires that in an expansion of the seagull terms in powers of k and k' the isoscalar operator is associated with even powers while the isovector operator multiplies the odd powers. In phenomenological Lagrangian models[16] Σ and Y describe the t-channel exchanges of scalar-isoscalar and vector-isovector bosons, respectively. The $(k+k')$ factor in the vector term, for example, is also present in the expression corresponding to the Feynman diagram for ρ-meson exchange between a pion and a nucleon. From a simple study of phenomenological Lagrangians one can see that the quantum numbers of pions and nucleons exclude from Eq. (5) operators with transformation properties different from those of Σ and Y.

It is straightforward to obtain from (4) a nonlinear inhomogeneous equation for the half-off-mass-shell amplitude. One inserts a complete set of physical states between the current operators in both terms of the time ordered product. The $|\pi N\rangle$ state contribution gives terms quadratic in the half-off-mass-shell amplitude. The other intermediate state contributions plus the seagull terms constitute the inhomogeneous terms. Also setting $k_0 = \sqrt{m_\pi^2 + k^2}$ yields a half-off-mass-shell amplitude on the left-hand side of the equation.

B. Dynamical Input

Before presenting the details of the evaluation of Eq. (4), we list and discuss five dynamical features which determine both the off- and on-shell behavior of the πN amplitude.

1. The interpolating pion field is defined to be proportional to the divergence of the weak axial-vector current,

$$\phi_\beta(x) = \frac{\sqrt{2}}{f_\pi}\, \partial_\mu A^\mu_\beta(x), \tag{6}$$

where $f_\pi = 0.939 m_\pi^3$ is the charged pion decay constant. There are two consequences of this definition:

a. It fixes the <u>form</u> of the coupling of pions to nucleons in the soft-pion amplitude discussed below, and

b. it sets the strength of the coupling of pions to nucleons at zero momentum transfer by the Goldberger-Treiman relation[17] between the πN form factor $g_\pi(0)$ and the weak axial form factor $g_A(0)$; namely

$$g_\pi(0) = \sqrt{2}\, M\, m_\pi^2\, g_A(0)/f_\pi = 12.7, \qquad (7)$$

where M is the nucleon mass, and $g_A(0) = 1.25$.

The dynamics implicit in (6) has been tested before, for example, in the Adler-Weisberger sum rule,[18,19] which agrees with experiment to within 5%. Following conventional usage we refer to Eq. (6) as the hypothesis of the partial conservation of the axial-vector current (PCAC). It should be noted that in using (6) we will not need to assume "smoothness" in the behavior of matrix elements as pion variables are changed.

2. We assume that the isovector seagull term $Y^\lambda = 0$. This is basically equivalent to the assumption that there is no canonical ρ-meson field in the Lagrangian for πN scattering. We stress that this does not imply that our theory excludes the effects of vector meson exchanges Basdevant and Lee,[20] for example, using the σ-model of Gell-Mann and Lévy,[21] have shown that ρ and f_0 resonances can be dynamically generated from higher order iterations of a unitary theory even though the vector mesons are not included in the Lagrangian.

3. We must define the strengths and invariant momentum transfer dependences of two invariant form factors which enter our model independent analysis. One is the πN σ-commutator form factor $\Gamma_\sigma(t)$ (for $t \leq 0$) and the other is the πN form factor $g_\pi(t)$ (for $t \leq 0$ and $t \geq 4M^2$). We find that the most important features of these form factors are constrained by the on-shell data.

4. A necessary consequence of crossing symmetry and nucleon recoil in any theory of the πN amplitude is that the integral equation for each partial wave amplitude is coupled to all partial wave amplitudes. Thus to solve for the S-wave amplitudes from the Low equation it is necessary to specify the off-shell behavior of the $\ell \geq 1$ partial wave amplitudes. We have therefore introduced simple, separable forms for the P-, D- and F-wave off-shell amplitudes. We find that the effects of the D- and F-wave amplitudes are negligibly small over the entire elastic scattering region, and that the P-wave amplitudes have a small, but non-negligible, effect on the S-wave amplitudes.

5. Finally, we recall that to express Eq. (4) as an inhomogeneous equation for the off-shell amplitude a complete set of physical states is inserted between each term of the time ordered product. Naturally, this infinite sum must be truncated. The truncation is carried out in the CM frame. Because of truncation, the integral terms are no longer covariant. We include only those low mass states which are felt to be most important. These include the states $|N\rangle$, $|\pi N\rangle$ and the disconnected parts (z-graphs) arising from the $|\bar{N}NN\rangle$ terms, where $\bar{N}$ = antinucleon. The rationale for retaining only these states will be presented when we discuss the evaluation of the integral term of Eq. (4). But let us point out here that because our treatment of the isoscalar seagull term leads to an equation with a once-subtracted form, the effects of higher mass states will tend to be suppressed.

C. Evaluation of the Low Equation

We now consider the evaluation of the two remaining terms in Eq. (4), the isoscalar seagull and the time ordered product of currents terms, using only Lorentz covariance and the above-mentioned dynamics.

1. Isoscalar seagull term

The isoscalar seagull term can be eliminated from the Low equation by using a soft-pion limit of the fully-off-mass-shell amplitude. Rewriting Eq. (4) without the isovector seagull term

$$F_{\beta\alpha}(k) = \langle p_f|\delta_{\alpha\beta}\Sigma(0)|p_i\rangle + i\int d^4x\, e^{ik\cdot x}\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle, \qquad (8)$$

we recall that k is a freely variable, real parameter. Taking the limit in which all four components of k vanish yields a soft-pion amplitude

$$\lim_{k_0\to 0}[\lim_{\vec{k}\to 0} F_{\beta\alpha}(k)] = F_{\beta\alpha}(0) = \langle p_f|\delta_{\alpha\beta}\Sigma(0)|p_i\rangle + i\int d^4x\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle. \qquad (9)$$

Subtracting Eq. (9) from Eq. (8) formally eliminates the isoscalar seagull term

$$F_{\beta\alpha}(k) = F_{\beta\alpha}(0) + i\int d^4x\, e^{ik\cdot x}\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle - i\int d^4x\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle. \qquad (10)$$

For Eq. (10) to be an improvement over Eq. (8), which contains the unknown seagull term, we must of course know $F_{\beta\alpha}(0)$.

An exact expression for the soft-pion amplitude can be obtained by using PCAC, applying the generalized Ward-Takahashi identity[22] to the original equation for the fully-off-mass-shell amplitude, and taking the limit $k \to 0$. Using Eq. (6) and integrating by parts with the D'Alembertian in Eq. (3) gives

$$F_{\beta\alpha}(k) = i\int d^4x\, e^{ik\cdot x}(m_\pi^2 - k^2)\langle p_f|T\left(\frac{f_\pi}{\sqrt{2}}\,\partial_\mu A^\mu_\beta(x), j_\alpha(0)\right)|p_i\rangle.$$

With P denoting the energy-momentum four-vector operator, translational invariance implies the equations

$$A^\mu_\beta(x) = e^{iP\cdot x}A^\mu_\beta(0)e^{-iP\cdot x},\quad j_\alpha(x) = e^{iP\cdot x}j_\alpha(0)e^{-iP\cdot x}.$$

Thus we have

$$F_{\beta\alpha}(k) = i \int d^4 x\, e^{-ik'\cdot x}(m_\pi^2 - k^2)\langle p_f|T\left(\frac{f_\pi}{\sqrt{2}}\, \partial_\mu A^\mu_\beta(0), j_\alpha(x)\right)|p_i\rangle,$$

where $k' = (p_f + k - p_i)$ has been used. From the definition of the pion source current

$$j_\alpha(x) = (\Box + m_\pi^2)\phi_\alpha(x) = (\Box + m_\pi^2)\frac{f_\pi}{\sqrt{2}}\, \partial_\nu A^\nu_\alpha(x)$$

we get

$$\begin{aligned} F_{\beta\alpha}(k) &= i(m_\pi^2 - k^2)(m_\pi^2 - k'^2)\left(\frac{f_\pi}{\sqrt{2}}\right)^2 \int d^4 x\, e^{-ik'\cdot x}\langle p_f|T\left(\partial_\mu A^\mu_\beta(0), \partial_\nu A^\nu_\alpha(x)\right)|p_i\rangle \\ &= i(m_\pi^2 - k^2)(m_\pi^2 - k'^2)\left(\frac{f_\pi}{\sqrt{2}}\right)^2 \int d^4 x\, e^{ik\cdot x}\langle p_f|T\left(\partial_\mu A^\mu_\beta(x), \partial_\nu A^\nu_\alpha(0)\right)|p_i\rangle \end{aligned}$$

where a second translation was used to get the last line.

Upon integrating by parts the $\partial_\mu A^\mu_\beta(x)$ term we find

$$\begin{aligned} F_{\beta\alpha}(k) = i(m_\pi^2 - k^2)(m_\pi^2 - k'^2)\left(\frac{\sqrt{2}}{f_\pi}\right)^2 \Big[& i\langle p_f|\sigma(0)\delta_{\alpha\beta}|p_i\rangle \\ & - i \int d^4 x\, e^{ik\cdot x}\langle p_f|T\left(k\cdot A_\beta(x), \partial_\nu A^\nu_\alpha(0)\right)|p_i\rangle\Big]. \end{aligned} \quad (11)$$

The πN σ-commutator[23,24] is defined by

$$\begin{aligned} \langle p_f|\sigma(0)\delta_{\alpha\beta}|p_i\rangle &= i \int d^4 x\, \delta(x_0)\langle p_f|\,[A^0_\beta(x), \partial_\nu A^\nu_\alpha(0)]\,|p_i\rangle \\ &= \Gamma_\sigma(t)\bar{u}(p_f)u(p_i)\delta_{\alpha\beta} \end{aligned} \quad (12)$$

The last line follows from Lorentz covariance as the most general form for this matrix element, with $\Gamma_\sigma(t)$ an invariant function of the invariant momentum transfer $t = (p_f - p_i)^2$. That the σ-commutator is symmetric in isospin indices follows from the assumption that

$$\frac{\partial}{\partial y_\mu} \int \delta(x_0 - y_0)\,[A^\alpha_0(x), A^\beta_\mu(y)]\,d^4x = 0.$$

Though this relation is implied by the $SU(2) \times SU(2)$ algebra of currents,[25] the converse is not true. Thus the isoscalar nature of the σ-commutator follows from a weaker condition than the current algebra.

We parametrize the form factor $\Gamma_\sigma(t)$ as

$$\Gamma_\sigma(t) = \frac{\sigma(\pi N)}{\left(1 - \frac{t}{\mu_1^2}\right)^2\left(1 - \frac{t}{\mu_2^2}\right)} \quad (13)$$

The quantity $\sigma(\pi N)$ is an important parameter of low energy πN scattering. Its numerical value is especially significant since it provides a direct measure of the strength of the chiral-symmetry breaking part of the strong interaction

Hamiltonian.[24] As we discuss below, $\sigma(\pi N)$ is accurately determined from Eq. (10) by requiring that the on-shell amplitudes agree with experiment.

In the final term in Eq. (11) we insert a complete set of physical (in or out) states in both parts of the time ordered product, translate $A_\beta(x)$ to $A_\beta(0)$ and carry out the integration giving

$$-i \int d^4 x\, e^{ik\cdot x} \langle p_f | T\left(k\cdot A_\beta(x), \partial_\nu A^\nu_\alpha(0)\right) | p_i \rangle$$

$$= (2\pi)^3 \sum_n [\delta^3(\vec{k}+\vec{p}_f-\vec{n}) \frac{\langle p_f | k\cdot A_\beta(0) | n\rangle\langle n | \partial_\nu A^\nu_\alpha(0) | p_i\rangle}{k_0 + p_{f0} - n_0 + i\varepsilon}$$

$$- \delta^3(\vec{k}+\vec{n}-\vec{p}_i) \frac{\langle p_f | \partial_\nu A^\nu_\alpha(0) | n\rangle\langle n | k\cdot A_\beta(0) | p_i\rangle}{k_0 + n_0 - p^0_i - i\varepsilon}] \tag{14}$$

where, in an obvious notation, n denotes the total four-momentum of the state $|n\rangle$.

To form the soft-pion amplitude, we must let all four components of k approach zero. If we first let $\vec{k} \to 0$ then $k_0 \to 0$ on the right-hand side of (14) we have a well-defined limit

$$\sum_s [\frac{M}{p_{f0}} \langle p_f | A^0_\beta(0) | p_f s\rangle\langle p_f s | \partial_\nu A^\nu_\alpha(0) | p_i\rangle - \frac{M}{p_{i0}} \langle p_f | \partial_\nu A^\nu_\alpha(0) | p_i s\rangle\langle p_i s | A^0_\beta(o) | p_i\rangle], \tag{15}$$

i.e., only the nucleon intermediate state contributes. The sum over s denotes a sum over the nucleon spin states and the factors $\frac{M}{p_0}$ arise from our invariant normalization of states. The limit whereby $k_0 \to 0$ first, then $\vec{k} \to 0$ differs from (15) and does not appear to be useful in the present work.

The form of the matrix element of the axial current between nucleon states follows from Lorentz covariance as

$$\langle p' | A^\mu_\beta(0) | p\rangle = \bar{u}(p')[g_A(t)\gamma^\mu + g_P(t)(p'-p)^\mu]\gamma_5 \frac{\tau_\beta}{2} u(p) \tag{16}$$

with $g_A(t)$ and $g_P(t)$ the weak axial and induced pseudoscalar form factors, respectively. An expression for the matrix element of the divergence of the axial current is obtained using PCAC, the definition of the pion source current $j_\alpha(0)$ and Lorentz covariance:

$$\langle p_f | \partial_\nu A^\nu_\alpha(0) | p_i\rangle = \langle p_f | \frac{f_\pi}{\sqrt{2}} \phi_\alpha(0) | p_i\rangle = \frac{f_\pi}{\sqrt{2}} \frac{\langle p_f | j_\alpha(0) | p_i\rangle}{m^2_\pi - t}$$

$$= \frac{f_\pi}{\sqrt{2}} \frac{i g_\pi(t) \bar{u}(p_f)\gamma_5 \tau_\alpha u(p_i)}{m^2_\pi - t}. \tag{17}$$

Equation (15) then becomes

$$i\,\frac{g_A(0)g_\pi(t)}{4(m_\pi^2 - t)}\,\left(\frac{f_\pi}{\sqrt{2}}\right)\bar{u}(p_f)\left[\gamma^0\frac{(M-\not{p}_f)}{p_{f0}}\tau_\beta\tau_\alpha + \frac{(M-\not{p}_i)}{p_{i0}}\gamma^0\tau_\alpha\tau_\beta\right]u(p_i).$$

This represents a nucleon pole term in the soft-pion limit.

Taking the limit $k \to 0$ in the order noted above, and using $k'^2 = (p_f - p_i)^2 = t$ in that limit, we obtain from Eq. (11) the soft pion amplitude in the form

$$F_{\beta\alpha}(0) = \left(\frac{\sqrt{2}}{f_\pi}\right)^2 m_\pi^2 (t - m_\pi^2)\Gamma_\sigma(t)\bar{u}(p_f)u(p_i)\delta_{\alpha\beta}$$
$$- g_\pi(t)g_\pi(0)\bar{u}(p_f)\left[\gamma_0\frac{(M-\not{p}_f)}{4Mp_{f0}}\tau_\beta\tau_\alpha + \frac{(M-\not{p}_i)}{4Mp_{i0}}\gamma_0\tau_\alpha\tau_\beta\right]u(p_i), \tag{18}$$

where in getting the last line, we have used the Goldberger-Treiman relation. The factor $(t - m_\pi^2)$ in the σ-term is precisely what is required by the Adler consistency condition.[26] In Section VI we will comment on the role this factor plays in the pion-nucleus optical potential. We note that this expression for the soft-pion amplitude is exact, as there are no other terms in the sum over states which survive in the $k \to 0$ limit.

The remaining part of the isoscalar seagull term, the integral expression in Eq. (9), is evaluated in the same manner as the first integral term in (10), which we consider in the next section.

2. Time-ordered product of currents terms

To solve for the half-off-mass-shell amplitude from Eq. (10) we must evaluate two integral terms containing time-ordered products of pion source currents. In one term, the four momentum of the final state pion must satisfy $k_0 = \sqrt{m_\pi^2 + \vec{k}^2}$ (hard pion integral), while in the other $k = 0$ (soft pion integral). Since the latter can be obtained from the former by a trivial change of variables we concentrate on the evaluation of only the hard pion integral.

Inserting a complete set of physical states in both terms of the time-ordered product and carrying out the coordinate integration after translating the Hesienberg picture operators to the space-time coordinate origin gives

$$i\int d^4x\, e^{ik\cdot x}\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle = -(2\pi)^3\sum_n[\delta^3(\vec{k}+\vec{p}_f+\vec{n})\frac{\langle p_f|j_\beta(0)|n\rangle\langle n|j_\alpha(0)|p_i\rangle}{k_0+p_{f0}-n_0+i\varepsilon} - \delta^3(\vec{k}-\vec{p}_i+\vec{n})\frac{\langle p_f|j_\alpha(0)|n\rangle\langle n|j_\beta(0)|p_i\rangle}{k_0-p_{i0}+n_0-i\varepsilon}] \tag{19}$$

The first term or the right will be referred to as the direct (s-channel) part, while the second will be called the crossed (u-channel) part. The admissible states $|n\rangle$ consist of one or more particles with total baryon number +1. When $|n\rangle$ consists of one pion and one nucleon we get a contribution to the right-hand side of (19) which involves an integral quadratic in the off-shell amplitudes of interest. Other contributions, along with the isoscalar seagull term, form the inhomogeneous part of the integral equation for the off-shell amplitude.

We include in the inhomogeneous term the contributions from the nucleon and antinucleon intermediate states. The nucleon intermediate state contribution is straightforwardly evaluated. As in Eq. (17), Lorentz covariance implies that the matrix element of the pion source current between nucleon states has the form

$$\langle p'|j_\alpha(0)|p\rangle = ig_\pi((p'-p)^2)\ \bar{u}(p')\gamma_5\tau_\alpha u(p), \tag{20}$$

which leads to the following pole terms

$$\frac{g_\pi((p_f-p)^2)g_\pi((p_i-p)^2)}{2(k_0+p_{f0}-M)}\bar{u}(p_f)(1-\gamma_0)\tau_\beta\tau_\alpha u(p_i) + \frac{g_\pi((p_f-\ell)^2)g_\pi((p_i-\ell)^2)}{2\ell_0(k_0+\ell_0-p_{i0})}\bar{u}(p_f)[(\ell_0-p_{i0}-p_{f0})\gamma_0+M]\tau_\alpha\tau_\beta u(p_i) \tag{21}$$

as the direct and crossed nucleon state contributions to the right hand side of Eq. (19). Equation (21) is expressed in the πN center-of-mass frame, $\vec{k}+\vec{p}_f = 0$, and ℓ and p are four momenta of physical nucleons with $\vec{\ell} = -(\vec{k}+\vec{k}')$ and $\vec{p} = 0$. Figure 2a shows a diagrammatic representation of the nucleon pole terms.

Obtaining the antinucleon term is a bit more complicated. It is given by the fully disconnected piece of the product of matrix elements of currents appearing on the right of Eq. (19). The term "disconnected" can be defined rigorously using the reduction technique.[27] Roughly what it amounts to is as follows. In a matrix element like $\langle N_f|G|N_i x\rangle$, where x is any particle(s) and N denotes nucleons, one has a term in which N_i propagates freely, and the rest which does not. Thus we may write

$$\langle N_f|G|N_i x\rangle = \langle N_f|N_i\rangle\langle 0|G|x\rangle + \langle N_f|G|N_i x\rangle_c .$$

The first term is referred to as disconnected, and the second as connected (subscript c). Note that if G cannot connect the vacuum and the state $|x\rangle$, there is no disconnected term. When a product of two such matrix elements appears as in Eq. (19), we will then have a fully connected piece, two semi-connected (or semi-disconnected) pieces, and a fully disconnected piece. In the Appendix this decomposition is worked out explicitly for the terms of Eq. (19). We find for the fully disconnected part of Eq. (19) the expression

$$(2\pi)^3 \sum_n [\delta^3(\vec{k}-\vec{p}_i-\vec{n})\frac{\langle 0|j_\beta(0)|p_i,n\rangle\langle p_f,n|j_\alpha(0)|0\rangle}{k_0-p_{i0}-n_0+i\varepsilon}$$

$$+\ \delta^3(\vec{k}+\vec{n}+\vec{p}_f)\frac{\langle 0|j_\alpha(0)|p_i,n\rangle\langle p_f,n|j_\beta(0)|0\rangle}{k_0+p_{f0}+n_0-i\varepsilon}], \qquad (22)$$

where now the states n must have baryon number -1. Thus the anti-nucleon is the lowest mass particle in the sum.

From covariance we have

$$\langle 0|j_\alpha(0)|N(p_i),\bar{N}(\bar{p})\rangle = ig_\pi((p_i+\bar{p})^2)\bar{v}(\bar{p})\gamma_5\tau_\alpha u(p_i). \qquad (23)$$

That the <u>same</u> form factor appears in both the coupling of the pion current to a nucleon-antinucleon pair and the coupling of the current to two nucleons [Eq. (20)] follows from the reduction formalism. It should be noted that in Eq. (20), the argument of g_π is ≤ 0, while in Eq. (23) it is $\geq 4M^2$.

From Eq. (22) the antinucleon contribution from the hard pion integral to the inhomogeneous term is

$$\text{Re}[g_\pi^*((p_f+p)^2)g_\pi((p_i+p)^2)]\ \frac{\bar{u}(p_f)(\not{p}+\not{p}_f)\tau_\beta\tau_\alpha u(p_i)}{2p_0(k_0+p_{f0}+p_0)}$$

$$-\text{Re}[g_\pi^*((p_f+\tilde{\ell})^2)g_\pi((p_i+\tilde{\ell})^2)]\ \frac{\bar{u}(p_f)(\tilde{\not{\ell}}+\not{p}_i)\tau_\alpha\tau_\beta u(p_i)}{2\tilde{\ell}_0(p_{i0}+\tilde{\ell}_0-k_0)}\ , \tag{24}$$

where the four-vector p is the same as defined previously and $\tilde{\ell}$ is the four momentum of a physical nucleon with $\vec{\tilde{\ell}} = \vec{k}+\vec{k}'$. Again, Eq. (24) is expressed in the CM frame. Figure 2b shows a diagrammatic representation of the antinucleon pole terms, and from the shape of the nucleon lines it is clear why these are referred to as z-graphs. The factors $\text{Re}[g_\pi^*((p_f+\bar{N})^2)g_\pi((p_i+\bar{N}))]$ arise because in summing over intermediate states in (22) we have used $1/2(\sum_{\text{in states}} + \sum_{\text{out states}})$. Since there is considerable uncertainty[28] in the knowledge of $g_\pi(t)$ for $t \geq 4M^2$, in Eq. (24) we will make the simplifying assumption of replacing the real part of the product of two complex form factors by the product of two real functions, called $\bar{g}_\pi(t)$, and defined, for $t \geq 4M^2$, by

$$\bar{g}_\pi(t) = \bar{g}_\pi[1 + \frac{t-4M^2}{4m_0^2}]^{-1}, \tag{25}$$

where $\bar{g}_\pi = |g_\pi(4M^2)|$. This quantity will be determined from on-shell data as will the parameter m_0. For $g_\pi(t)$, $t \leq 0$, we use

$$g_\pi(t) = g_\pi(0)[1 + \frac{t(t-4M^2)}{4M^2m_0^2}]^{-1}, \tag{26}$$

with the same mass parameter m_0. The expressions (25) and (26) were obtained from a crude dispersion relation analysis of the πN form factor.[28]

Estimates of the magnitudes of terms arising from other intermediate states suggest they are small in comparison to those discussed so far for either of two reasons: the coupling constant for the process is relatively small or the phase space factors tend to suppress the contribution. Hence, combining these results, our approximation to the hard pion integral becomes

$$i\int d^4x\, e^{ik\cdot x}\langle p_f|T(j_\beta(x)j_\alpha(0))|p_i\rangle = \frac{g_\pi((p_f-p)^2)g_\pi((p_i-p)^2)}{2(k_0+p_{f0}-M)}\,\bar{u}(p_f)(1-\gamma_0)\tau_\beta\tau_\alpha u(p_i)$$

$$+\frac{g_\pi((p_f-\ell)^2)g_\pi((p_i-\ell)^2)}{2\ell_0(k_0+\ell_0-p_{i0})}\,\bar{u}(p_f)[(\ell_0-p_{i0}-p_{f0})\gamma_0+M]\tau_\alpha\tau_\beta u(p_i)$$

$$-\frac{\bar{g}_\pi((p_f+p)^2)\bar{g}_\pi((p_i+p)^2)}{2p_0(k_0+p_{f0}+p_0)}\,\bar{u}(p_f)(\not{p}+\not{p}_f)\tau_\beta\tau_\alpha u(p_i)$$

$$-\frac{\bar{g}_\pi((p_f+\tilde{\ell})^2)\bar{g}_\pi((p_i+\tilde{\ell})^2)}{2\tilde{\ell}_0(p_{i0}+\tilde{\ell}_0-k_0)}\,\bar{u}(p_f)(\tilde{\not{\ell}}+\not{p}_i)\tau_\alpha\tau_\beta u(p_i)$$

$$+\sum_\gamma\sum_s\int\frac{d^3q_\pi d^3q_N}{(2\pi)^3 2q_{\pi 0}}\,\frac{M}{q_{N0}}\,\frac{\langle p_f|j_\beta(0)|q_N s,q_\pi\gamma\rangle\langle q_N s,q_\pi\gamma|j_\alpha(0)|p_i\rangle}{q_{N0}+q_{\pi 0}-p_{f0}-k_0-i\epsilon}\,\delta^3(\vec{q}_\pi+\vec{q}_N+\vec{p}_f+\vec{k})$$

$$+\sum_\gamma\sum_s\int\frac{d^3q_\pi d^3q_N}{(2\pi)^3 2q_{\pi 0}}\,\frac{M}{q_{N0}}\,\frac{\langle p_f|j_\alpha(0)|q_N s,q_\pi\gamma\rangle\langle q_N s,q_\pi\gamma|j_\beta(0)|p_i\rangle}{q_{N0}+q_{\pi 0}-p_{i0}+k_0}\,\delta^3(\vec{q}+\vec{q}_N+\vec{k}-\vec{p}_i)\,. \tag{27}$$

The corresponding expression for the soft-pion integral is obtained from Eq. (27) by dropping the nucleon pole terms, setting $(k_0,\vec{k}) = 0$, and replacing the four-vectors p by $(p_{f0},-\vec{p}_f)$ and $\tilde{\ell}$ by $(p_{i0},-\vec{p}_i)$.

In summary, we have developed from Eq. (10) an inhomogeneous, nonlinear integral equation for the half-off-mass shell pion-nucleon amplitude which can be written in the schematic form

$$F \sim V + \int [F^\dagger F)_D + (F^\dagger F)_C]_H - \int [(F^\dagger F)_D + (F^\dagger F)_C]_S, \tag{28}$$

where V consists of the σ-commutator and soft-pion N-pole terms of Eq. (18), the hard-pion N-pole terms of Eq. (21) and the z-graphs of Eq. (24). The subscripts D and C denote direct and crossed terms, while H and S stand for the hard- and soft-pion integral, respectively, with appropriate energy denominators understood. By omitting the S-wave inelastic states from the complete sum in these integrals we have limited the range of applicability of the theory to the elastic region. Because the scattering amplitude is given by an equation with a once-subtracted form, the neglect of these high mass states will have a weaker effect on the low energy elastic amplitude than what occurs in an unsubtracted equation (e.g., the Chew-Low equation).

III. METHOD OF SOLUTION

In this section we present a method for the numerical solution of Eq. (28) for the partial wave components of the scattering amplitude. The method is illustrated by a detailed discussion of the solution for the S-wave amplitude.

A. Covariant Partial Wave Expansion

Our equation for the scattering amplitude is crossing symmetric and fully includes nucleon recoil. As discussed below a consequence of this is that the integral equation for each partial wave amplitude is coupled to all partial wave amplitudes. To handle the terms which couple the amplitudes, it is particularly convenient to make the partial wave expansion in terms of projection operators which can readily be expressed in any Lorentz frame, as will become clear subsequently.

Expressions for angular momentum projection operators for πN scattering in the CM frame are well known. Denoting the projector for a state with orbital momentum quantum number ℓ and total angular momentum j by $\mathcal{P}^j_\ell$, the S- and P-wave projectors in the center-of-mass frame are given by

$$
\begin{aligned}
\mathcal{P}^{1/2}_0(\vec{p}_f s_f,\vec{p}_i s_i) &= \frac{1}{4\pi}\,\chi^\dagger_f\,\chi_i \\
\mathcal{P}^{1/2}_1(\vec{p}_f s_f,\vec{p}_i s_i) &= \frac{1}{4\pi}\,\chi^\dagger_f\,\sigma\cdot\hat{p}_f\;\sigma\cdot\hat{p}_i\,\chi_i \\
\mathcal{P}^{3/2}_1(\vec{p}_f s_f,\vec{p}_i s_i) &= \frac{3}{4\pi}\,\chi^\dagger_f\,\hat{p}_i\cdot\hat{p}_f\,\chi_i-\mathcal{P}^{1/2}_1(\vec{p}_f s_f,\vec{p}_i s_i),
\end{aligned}
\tag{29}
$$

where $\hat{p}_i(\hat{p}_f)$ denotes a unit vector in the direction of the initial (final) CM momentum and $\chi_i(\chi_f)$ represents a Pauli spinor for the initial (final) nucleon's spin. These projectors satisfy the idempotency relation

$$
\sum_{s_f}\int d\Omega_{\hat{p}_f}\,[\mathcal{P}^{j'}_{\ell'}(\vec{p}_f s_f,\vec{p}^{\,\prime}_i s^{\prime}_i)]^\dagger\,\mathcal{P}^j_\ell(\vec{p}_f s_f,\vec{p}_i s_i) = \delta_{\ell\ell'}\delta_{jj'}\mathcal{P}^j_\ell(\vec{p}^{\,\prime}_i s^{\prime}_i,\vec{p}_i s_i). \tag{30}
$$

To express these projectors in a manifestly covariant form, we first write them in terms of Dirac spinors

$$\mathcal{P}_0^{1/2}(\vec{p}_f s_f,\vec{p}_i s_i) = \frac{M}{4\pi}\,\bar{u}(p_f s_f)\frac{1+\gamma_0}{[(M+p_{i0})(M+p_{f0})]^{1/2}}\,u(p_i s_i)$$

$$\mathcal{P}_1^{1/2}(\vec{p}_f s_f,\vec{p}_i s_i) = \frac{M}{4\pi}\,\bar{u}(p_f s_f)\frac{\gamma_0 - 1}{[(p_{io}-M)(p_{f0}-M)]^{1/2}}\,u(p_i s_i) \qquad (31)$$

$$\mathcal{P}_1^{3/2}(\vec{p}_f s_f,\vec{p}_i s_i) = \frac{M}{4\pi}\,\bar{u}(p_f s_f)\frac{3(1+\gamma_0)\vec{p}_i\cdot\vec{p}_f}{(M+p_{i0})(M+p_{f0})[(p_{i0}-M)(p_{f0}-M)]^{1/2}}\,u(p_i s_i) - \mathcal{P}_1^{1/2}(\vec{p}_f s_f,\vec{p}_i s_i).$$

It is now easy to express these projectors in an arbitrary frame. If, in a given frame, P denotes the total four momentum of the pion and nucleon and $s = P^2$, the angular momentum projectors in that frame are obtained from Eq. (31) by the replacements

$$\gamma_0 \to \not{P}/\sqrt{s}$$

$$p_{i0}, p_{f0} \to \frac{p_{i,f}\cdot P}{\sqrt{s}} \qquad (32)$$

$$\vec{p}_i\cdot\vec{p}_f \to -(p_i - \frac{p_i\cdot P}{s}P)\cdot(p_f - \frac{p_f\cdot P}{s}P) = \frac{(p_i\cdot P)(p_f\cdot P)}{s} - p_i\cdot p_f.$$

These results can be generalized to higher partial waves by first defining

$$R_\ell(\cos\theta) = \textstyle\sum_{n=0}^{[\ell/2]}(2\ell+1-4n)P_{\ell-2n}(\cos\theta), \qquad (33)$$

where $[x]$ = largest integer in x and $P_\ell(\cos\theta)$ is a Legendre polynomial in $\cos\theta$, with θ the CM scattering angle. We find, for any ℓ,

$$\mathcal{P}_\ell^{\ell-1/2} = R_{\ell-1}(\cos\theta)\mathcal{P}_1^{1/2} - R_{\ell-2}(\cos\theta)\mathcal{P}_0^{1/2}$$

$$\mathcal{P}_{\ell+1}^{\ell+1/2} + \mathcal{P}_{\ell-1}^{\ell-1/2} = (2\ell+1)\,\mathcal{P}_\ell(\cos\theta)\,\mathcal{P}_1^{1/2} \qquad (34)$$

$$\mathcal{P}_\ell^{\ell+1/2} + \mathcal{P}_\ell^{\ell-1/2} = (2\ell+1)\,\mathcal{P}_\ell(\cos\theta)\,\mathcal{P}_0^{1/2}$$

The projectors defined by Eq. (31)-(34), of course, also satisfy the idempotency relation (30).

Since each of the terms in Eq. (28) can be written in the form

$$\frac{M}{4\pi}\,\bar{u}(p_f)[A(\cos\theta) + \gamma_0 B(\cos\theta)]u(p_i),$$

with A and B appropriate functions of $\cos\theta$, we will need the partial wave expansion of this expression. Writing

$$A(\cos\theta) = \Sigma_\ell(2\ell+1)A_\ell P_\ell(\cos\theta),$$

with a similar expansion for B, and using Eqs. (31) and (34) we find

$$\frac{M}{4\pi}\,\bar{u}(p_f)[A(\cos\,\theta)+\gamma_0\,B(\cos\,\theta)]\,u(p_i) = \sum_\ell\{\tfrac{1}{2}(A_\ell+B_\ell)[(p_{i0}+M)(p_{f0}+M)]^{1/2}$$
$$-\tfrac{1}{2}(A_{\ell+1}-B_{\ell+1})[(p_{i0}-M)(p_{f0}-M)]^{1/2}\}\mathcal{P}_\ell^{\ell+1/2}+\sum_\ell\{\tfrac{1}{2}(A_\ell+B_\ell)[(p_{i0}+M)(p_{f0}+M)]^{1/2}$$
$$-\tfrac{1}{2}(A_{\ell-1}-B_{\ell-1})[(p_{i0}-M)(p_{f0}-M)]^{1/2}\}\mathcal{P}_\ell^{\ell-1/2}. \qquad (35)$$

Using this result, the expansion of the terms denoted by V in Eq. (28) in partial waves is trivially carried out. For the other terms, we expand the scattering amplitude as

$$F_{\beta\alpha}(k) = 4\pi\sum_{\ell,j,I}f_{2I,2j}(q_i,q_f)\Pi^I(\beta,\alpha)\mathcal{P}_\ell^j(\vec{p}_i s_i,\vec{p}_f s_f) \qquad (36)$$

where $q_i = |\vec{p}_i|$ ($q_f = |\vec{p}_f|$) is the magnitude of the initial (final) momentum in the CM frame and Π^I is a projector for a state of total isotopic spin quantum number I, and is given by

$$\Pi^{1/2}(\beta,\alpha) = \frac{1}{3}\,\xi_f^\dagger\tau_\beta\,\tau_\alpha\,\xi_i$$
$$\Pi^{3/2}(\beta,\alpha) = \delta_{\beta\alpha} - \Pi^{1/2}(\beta,\alpha), \qquad (37)$$

where ξ is a Pauli spinor for isospin and τ_α (α = 1,2,3) is the usual Pauli 2×2 matrix. These projectors obey the idempotency relation

$$\sum_\gamma \Pi^I(\beta,\gamma)\Pi^{I'}(\gamma,\alpha) = \delta_{II'}\Pi^I(\beta,\alpha). \qquad (38)$$

B. Partial Wave Expansion of Integral Terms

Next we discuss the evaluation of the integrals in (28) with the expansion (36) substituted for the half-off-mass shell amplitudes. A typical integral is of the form

$$\sum_\gamma\sum_s\int\frac{d^3\vec{q}_\pi d^3\vec{q}_N M}{(2\pi)^3 2q_{\pi 0}q_{N0}}\langle p_f|j_\mu(0)|q_N s,q_\pi\gamma\rangle\langle q_N s,q_\pi\gamma|j_\nu(0)|p_i\rangle\,\frac{\delta(\vec{q}_N+\vec{q}_\pi\;\;\vec{L})}{q_{N0}+q_{\pi 0}-\varepsilon}. \qquad (39)$$

The values of μ, ν, $\vec{L}$ and ε in the CM frame for the four integrals DH, CH, DS and CS are given in Table I. Whenever $\vec{L}\neq 0$ the integrations over the angles of $\vec{q}_\pi$ and $\vec{q}_N$ are difficult. The difficulty is removed by rewriting the integral in the following manner:

$$\int \frac{d\sqrt{W^2+\vec{L}^2}}{\sqrt{W^2+\vec{L}^2}-\varepsilon}\left[\sum_\gamma \sum_s \int \frac{d^3\vec{q}_\pi d^3\vec{q}_N M}{(2\pi)^3 2q_{\pi 0}q_{N0}} <p_f|j_\mu(0)|q_N\ s, q_\pi\gamma>\right.$$

$$\left.\times <q_N\ s, q_\pi\gamma|j_\nu(0)|p_i>\delta(\vec{q}_N+\vec{q}_\pi-\vec{L})\delta\left(q_{N0}+q_{\pi 0}-\sqrt{W^2+\vec{L}^2}\right)\right]. \qquad (40)$$

W is the total energy of the intermediate state in its own CM frame. Its minimum value is $M+m_\pi$. The expression within the square brackets is a Lorentz invariant. Therefore one can evaluate it in a frame where the intermediate state is at rest. Upon doing so one gets

$$\int \frac{d\sqrt{W^2+\vec{L}^2}}{\sqrt{W^2+\vec{L}^2}-\varepsilon}\ [\sum_\gamma \sum_{s'} \frac{Mq}{2(2\pi)^3 W}\int d\Omega_{\hat{q}} <p_f'|j_\mu(0)|-\vec{q}\ s', \vec{q}\gamma><-\vec{q}\ s', \vec{q}\gamma|j_\nu(0)|p_i'>]. \qquad (41)$$

Here q is the CM frame momentum of the pion corresponding to the total energy W, i.e.,

$$W = \sqrt{M^2+\vec{q}^2} + \sqrt{m_\pi^2+\vec{q}^2}\ . \qquad (42)$$

As a result of the boost to the intermediate state CM frame (ICM) $\vec{p}_f$ changes to $\vec{p}_f'$, etc. Now we can substitute expansion (36) for the half-off-mass shell amplitudes.

$$\int \frac{d\sqrt{W^2+\vec{L}^2}}{\sqrt{W^2+\vec{L}^2}-\varepsilon}\left[\sum_\gamma \sum_{s'} \sum_{II'} \sum_{\substack{jj'\\ \ell\ell'}} \frac{Mq}{\pi W}\int d\Omega_{\hat{q}} f^*_{2I,2j}(q,|\vec{p}_f'|)f_{2I',2j'}(q,|\vec{p}_i'|)\right.$$

$$\left.\times \{\Pi^I(\gamma,\mu)\}^\dagger \Pi^{I'}(\gamma,\nu)\{\mathcal{P}^j_\ell(\vec{q}s',\vec{p}_f's_f')\}^\dagger \mathcal{P}^{j'}_{\ell'}(\vec{q}s',\vec{p}_i's_i')\right].$$

Summing over γ and s', integrating over $\Omega_{\hat{q}}$ and using the idempotency conditions (30) and (38) gives

$$\frac{1}{\pi}\int_{M+m_\pi}^\infty \frac{dW}{\sqrt{W^2+\vec{L}^2}-\varepsilon}\ \frac{Mq}{\sqrt{W^2+\vec{L}^2}} \sum_{Ij\ell} f^*_{2I,2j}(q,|\vec{p}_f'|)f_{2I,2j}(q,|\vec{p}_i'|)\Pi^I(\mu,\nu)\mathcal{P}^j_\ell(\vec{p}_f',s_f',\vec{p}_i's_i'). \qquad (43)$$

At this stage it is useful to illustrate the form of the operators $\mathscr{P}_\ell^j(\vec{p}_f' s_f', \vec{p}_i' s_i')$ by considering the case $j = 3/2$ and $\ell = 1$:

$$\mathscr{P}_1^{3/2}(\vec{p}_f' s_f', \vec{p}_i' s_i') = \frac{M}{4\pi}\, \bar{u}(p_f, s_f)\left[\frac{3\, \vec{p}_f' \cdot \vec{p}_i'}{(M + p_{f0}')(M + p_{i0}')[(p_{f0}' - M)(p_{i0}' - M)]^{1/2}} \left(\frac{\not{L}}{W} + 1\right) - \frac{1}{[(p_{f0}' - M)(p_{i0}' - M)]^{1/2}} \left(\frac{\not{L}}{W} - 1\right)\right] u(p_i, s_i).$$

Here we have used the covariant nature of the projection operators and the results (32). The four vector

$$L \equiv (\sqrt{W^2 + \vec{L}^2}, \vec{L}) \qquad (44)$$

replaces P in the equations (32). Thus

$$p_{i0}, p_{f0} \to p_{i0}', p_{f0}' = \frac{p_{i,f} \cdot L}{W},$$
$$\vec{p}_i' \cdot \vec{p}_f' = \frac{(p_f \cdot L)(p_i \cdot L)}{W^2} - (p_i \cdot p_f). \qquad (45)$$

From Table I we see that for the Direct Hard (DH) integral $\vec{L} = 0$. The projection operators in (43) are those for the final (or initial) state CM frame (FCM) and (43) has the proper form for the partial wave expansion of (28).

For the other three integrals $\vec{L} \neq 0$ and the projection operators appearing in (43) are not projection operators for the FCM frame. For each of these integrals the partial wave expansion in the FCM will have to be carried out. An immediate consequence is that the integral equation for any partial wave is coupled to all partial waves. The coupling arises not only through the crossed integral but also through the soft integrals. The isospin crossing, present for both crossed terms, is exactly the same as in the Chew-Low theory.

The FCM partial wave projection can be carried out with the help of Eq. (35). Expression (43) depends on the scattering angle θ between $\vec{p}_f$ and $\vec{p}_i$ through the kinematical factors as well as through $|\vec{p}_i'|$ and $|\vec{p}_f'|$ appearing as arguments in the partial wave amplitudes. The expressions for $|\vec{p}_i'|^2 = p_{i0}^2 - M^2$, etc., can be readily obtained from (45), since the nucleons are always on their mass shell. The Dirac matrix $\vec{\gamma}$ appearing in the projection operators can be eliminated by adding to $\not{L}$ an appropriate linear combination of $\not{p}_i - M$ and $\not{p}_f - M$, which then puts Eq. (43) in the form of Eq. (35).

In our study of the S-wave scattering amplitudes we include the recoil effects from P-wave only. Preliminary examination shows that D- and F-waves have individually small contributions to the S-wave equations and, furthermore, they tend to cancel one another.

To include the effects of the P-wave amplitudes we must make a model for the half-off-mass shell amplitudes in the four P-wave channels. We use the factorable form

$$f_\nu(q,p) = \frac{p}{q}\frac{\phi(p)}{\phi(q)} f_\nu(q,q), \qquad \nu \equiv (P11, P13, P31, P33), \tag{46}$$

where the on-shell elastic scattering amplitude

$$f_\nu(q,q) = -\frac{4\pi W}{M}\frac{1-\eta_\nu e^{2i\delta_\nu}}{2iq} \tag{47}$$

with η_ν the modulus of the S-matrix element and δ_ν the real phase shift. The form factor $\phi(p)$ is parametrized as

$$\phi(p) = \frac{1}{(1+\frac{\vec{p}^2}{\mu^2})^{5/2}} . \tag{48}$$

The power of 5/2 was chosen so that $p\phi(p) \xrightarrow[p\to\infty]{} p^{-4}$, which we felt was the desirable rate for damping at high momentum. So far as the low energy phase shifts are concerned the quantity of interest is $\frac{d}{dp^2}\phi(p)\Big|_{p=0} = -\frac{5}{2\mu^2}$, not the power or the value of μ^2 individually.

The form (46) is likely to be quite good for the P33 channel because of the resonance dominance, but for the other channels it may not be as good.

These amplitudes always appear in the form $f_\nu^*(q,p)f_\nu(q,p')$. In other words, the channel elastic cross sections determine their sizes. Inspection of Eq. (19) shows all P-wave inelastic channels can be included by simply extending the factorability ansatz to the inelastic amplitudes. Thus the complete P-wave contribution to expression (43) is

$$16\pi \int_{M+m_\pi}^{\infty} \frac{dW}{\sqrt{W^2+\vec{L}^2}-\varepsilon} \; \frac{W^2}{M\sqrt{W^2+\vec{L}^2}} \; \frac{p_f' p_i' \; \phi(p_f')\phi(p_i')}{\phi^2(q)} \sum_{\substack{I,j \\ (\text{P-wave})}} \frac{1-\eta_{2I,2j}\cos 2\delta_{2I,2j}}{2q^3}$$

$$\times \; \Pi^I(\mu,\nu)\,\mathcal{P}_1^j(\vec{p}_f{}',s_f',\vec{p}_i{}',s_i') . \tag{49}$$

To evaluate this expression, we will use the P-wave phases and inelasticities of the CERN theoretical fit.[29]

C. The S-Wave Equation

The nonlinear integral equation for S-wave amplitudes can be schematically represented in the style of (28),

$$f_\nu = D_\nu + \int [(f^*f)_D + (f^*f)_C]_H - \int [(f^*f)_D + (f^*f)_C]_S . \tag{50}$$

The driving term D_ν is the sum of all terms in the integral equation which do not contain the S-wave amplitudes. It includes the S-wave projections of the sigma commutator, nucleon pole, z-graphs and the P-wave contribution to the S-wave. The σ-commutator is a purely isoscalar, repulsive (negative) term. The z-graphs in the limit $|\vec{p}_i| = |\vec{p}_f| = 0$ have the form

$$\bar{g}_\pi^2 \frac{m_\pi}{4M^2}[\tau_\beta,\tau_\alpha] - \bar{g}_\pi^2 \frac{m_\pi^2}{4M^3}\,\delta_{\alpha\beta} .$$

One finds that the leading term of the hard-pion z-graphs, the isoscalar expression $-\frac{\bar{g}_\pi^2}{M}\,\delta_{\alpha\beta}$, is cancelled by the corresponding term of the soft-pion graphs. The soft-pion subtraction thus effects the familiar "pair suppression." After subtraction the isovector part of the z-graphs becomes the leading term. This is also the largest isovector term in D_ν.

The remaining parts of the driving term, the nucleon pole terms and the P-wave contribution, vanish when $|\vec{p}_i| = |\vec{p}_f| = 0$. Nevertheless, these terms have a significant effect on the S-wave phase shifts. The omission of either of these terms from D_ν causes the low energy S31 phase shifts to increase about 20%. Their effects on the S11 phase shifts are even larger because of the partial cancellation between the σ-term and the z-graphs.

Having excluded the S-wave inelastic channels our amplitudes satisfy elastic unitarity conditions

$$\text{Im } f_\nu(q,q) = \frac{Mq}{4\pi W}|f_\nu(q,q)|^2 \tag{51a}$$

$$\text{Im } f_\nu(q,p) = \frac{Mq}{4\pi W} f_\nu^*(q,q) f_\nu(q,p). \tag{51b}$$

From these equations it is easy to see that

$$\text{Im } \frac{f_\nu(q,p)}{f_\nu(q,q)} = 0. \tag{52}$$

These conditions are maintained exactly by the method of numerical evaluation of Eq. (50) which we now describe.

D. Method of Padé Approximants

In the schematic equation (50) we attach an order parameter λ with D_ν,

$$f_\nu = \lambda D_\nu + \int [(f^*f)_D + (f^*f)_H] - \int [(f^*f)_D + (f^*f)_C]_S \tag{53}$$

and iterate to obtain the power series

$$f_\nu = \sum_{n=1}^{\infty} \lambda^n f_\nu^{(n)}. \tag{54}$$

$$f_\nu^{(1)} \equiv D_\nu \tag{55}$$

$$f_\nu^{(n)} = \sum_{m=1}^{n-1} \{\int [(f^{(n-m)*} f^m)_D + (f^{(n-m)*} f^m)_C]_H - \int [(f^{(n-m)*} f^m)_D + (f^{(n-m)*} f^m)_C]_S. \tag{56}$$

The power series (54) satisfies the unitarity conditions (51) and (52) in every order. This may be verified with the help of the following observations. Since the driving terms are real an imaginary part can arise only from the integrals. If the numerator in the integrals are all real than the $i\pi\delta(W - k_0 - p_{f0})$ factor is the only source of an imaginary part, exactly as demanded by elastic unitarity. If all $f_\nu^{(m)}$ for $m < n$ satisfy the unitarity conditions, i.e.,

$$\text{Im } f_\nu^{(m)}(q,p) = \frac{Mq}{4\pi W} \sum_{\ell=1}^{m-1} f_\nu^{(\ell)*}(q,q) f_\nu^{(m-\ell)}(q,p); \; m < n \tag{57}$$

then in the process of evaluating $f_\nu^{(n)}$ one will find all the numerators to be real and that Im $f_\nu^{(n)}$ satisfies the unitarity condition of the form (57).

In many areas of physics a power series like that of (54) has often been approximated by a rational function of λ.[30] This is known as the method of Padé approximants. It has the advantage that the rational function may be a good representation of the initial function even when $\lambda = 1$ is outside the radius of convergence of the series. We should point out that f_ν is not an analytic function of λ. Rather, it can be made an analytic function of two variables, viz., the real and imaginary parts of λ. Of course, this in no way precludes the possibility that a rational function in the two variables can serve as a good approximation for f_ν.

We found no tendency of convergence in our power series expansion of the S-wave equation, not even of logarithmic type. We have therefore tried the method of Padé approximants to obtain a solution. Though the method of Padé approximants has been studied extensively in the context of linear integral equations, there have been few applications to nonlinear equations.[30] Nevertheless, we find that it is possible to construct a solution of the nonlinear S-wave equation using Padé approximants. The validity of the solution is established by putting it into the equation and verifying that it accurately reproduces itself.

In the context of scattering theory it has been customary to approximate the half-off-mass shell amplitude with a rational function[30]

$$f_{ij} \equiv f_\nu(k_i,k_j) = \frac{P_{ij}^{[N]}(\lambda)}{1+Q_{ij}^{[M]}(\lambda)} , \tag{58}$$

where i and j label the on-mass-shell and the off-mass-shell momenta, respectively. The polynomials are

$$P_{ij}^{[N]}(\lambda) = \sum_{n=1}^{N} \lambda^n p_{ij}^{(n)}$$

$$Q_{ij}^{[M]}(\lambda) = \sum_{n=1}^{M} \lambda^n q_{ij}^{(n)} . \tag{59}$$

It is well known that the unitarity condition imposes the restriction $N \leq M$. We found that [N,M] Padé approximants with $N \leq M$ are quite unsatisfactory for the S11 amplitude. We therefore circumvented the unitarity restriction by making Padé approximants not for f_{ij} but for the amplitude

$$K_{ij} = \frac{f_{ij}}{1 + i\,\frac{Mk_i}{4\pi W_i}\, f_{ii}} , \tag{60}$$

where $W_i = \sqrt{M^2 + \vec{k}_i^2} + \sqrt{m_\pi^2 + \vec{k}_i^2}$. The on-shell element

$$K_{ii} = \frac{4\pi W_i}{Mk_i} \tan \delta_i . \tag{61}$$

The inverse of (60) is

$$f_{ij} = \frac{K_{ij}}{1 - i\,\frac{Mk_i}{4\pi W_i}\, K_{ii}} . \tag{62}$$

The unitarity conditions (51) are identically satisfied when the K_{ij}'s are real and, conversely, when the f_{ij}'s satisfy (51) the K_{ij}'s are real. As we have noted before, the iteration procedure satisfies the constraints of (51) as a series of identities. Thus maintaining the reality of the K_{ij}'s is not a problem. We are therefore free to try Padé approximants for K_{ij} with real polynomials. Our procedure is as follows. First we find the Padé approximants for the on-mass-shell elements

$$K_{ii} = \frac{f_{ii}}{1 + i\frac{Mk_i}{4\pi W_i} f_{ii}} = \frac{P_{ii}^{[N]}}{1 + Q_{ii}^{[M]}} . \tag{63}$$

Then for the half-off-mass shell elements we use the form

$$K_{ij} = \frac{f_{ij}}{1 + i\frac{M k_i}{4\pi W_i} f_{ii}} = \frac{P_{ij}^{[N]}}{1 + Q_{ii}^{[M]}} \tag{64}$$

where the denominator is the one determined for the on-mass-shell element. This procedure is dictated by practical considerations. $1 + i\frac{Mk_i}{4\pi W_i} f_{ii} = e^{i\delta_i}\cos\delta_i$. Hence both K_{ii} and K_{ij} have poles when $\delta = \pi/2$. In other words, $1 + Q_{ii}^{[M]}(\lambda)$ and $1 + Q_{ij}^{[M]}(\lambda)$ should both be zero at the same time for a value of k_i for which $\delta_i = \pi/2$. Because of numerical inaccuracies it is not possible to achieve this exactly. So we enforce the condition with the form (64). As a result, for K_{ij}, $i \neq j$, we have to determine only the numerator polynomial.

E. Details of Numerical Calculation

1. Evaluation of integrals

The entire range of momentum, from 0 to ∞, was divided into five sectors with $1.25m_\pi$, $2.5m_\pi$, $9m_\pi$ and $24m_\pi$ as the interval dividing points. Integration over each of the first four sectors was carried out with 5-point Gaussian quadrature. The sector $24m_\pi \leq p \leq \infty$ was mapped into $-1 \leq y = \frac{p-50m_\pi}{p} \leq 1$ and the integration over y was also carried out with 5-point Gaussian quadrature.

Evaluation of the CH, CS and DS integrals require angle integrations. A certain amount of care is necessary in carrying out these integrals because of the considerable cancellation between a hard and the corresponding soft term. The angle dependence always arises in the form $p_i p_f \cos\theta$. So the number of Gaussian mesh points used to span the range of $\cos\theta$ was increased as the value of $p_i p_f$ increased. The entire range of $p_i p_f$ from 0 to ∞ was divided into seven sectors with $4m_\pi^2$, $12m_\pi^2$, $36m_\pi^2$, $108m_\pi^2$, $324m_\pi^2$ and $972m_\pi^2$ as the internal dividing points. The number of Cos θ mesh points for each sector was increased from 6 to 24 in steps of 3.

From (43) one also sees that the integrals CH, DS and CS refer to the quantities $|\vec{p}_i^{\,\prime}| = \sqrt{p_{i0}^{\prime 2} - M^2}$, etc., where $p_{i0}^{\prime}$ is defined by (45). The latter depends on p_i, p_f, q and $\cos\theta$. In general, the value of $p_i^{\prime}$ and $p_f^{\prime}$ is not equal to any of the momentum mesh points. The values of $f_\nu^{(n)}(q,|\vec{p}_i^{\,\prime}|)$, etc., were obtained from the calculated values at the mesh points by using three-point interpolation or extrapolation. For $|\vec{p}_i^{\,\prime}| > 25m_\pi$ the reciprocal of the momentum was used as the basic variable for the inter-(extra-)polation.

The principle value integral in the DH integral was evaluated by rearranging it in the form

$$\mathcal{P}\int_{M+M_\pi}^{\infty} \frac{H(W)}{W(W-W_0)}\,dW = W_0 H(W_0)\,\mathcal{P}\int_{M+m_\pi}^{\infty} \frac{dW}{W^2(W-W_0)} + \int_{M+m_\pi}^{\infty} \frac{dW}{W(W-W_0)}\left\{H(W) - \frac{W_0}{W}H(W_0)\right\}$$

$$= -\left[\frac{1}{M+m_\pi} + \frac{1}{W_0}\,\ell n\left(\frac{W_0}{M+m_\pi} - 1\right)\right] H(W_0) + \int_{M+m_\pi}^{\infty} \frac{dW}{W(W-W_0)}\left\{H(W) - \frac{W_0}{W}H(W_0)\right\}.$$

The value of the quantity in curly brackets at the point $W = W_0$ was obtained by interpolation from its values at the two nearest points. Whenever the on-mass-shell momentum was greater than $25m_\pi$ the reciprocal of the momentum was chosen the variable for the purpose of interpolation.

Because of the large intervals between the momentum mesh points at the upper range, the calculation of $f_\nu^{(n)}(q,p)$, $n \geq 2$, becomes increasingly unreliable as q and p increase. So instead of actually calculating for the last two mesh points at $104m_\pi$ and $512m_\pi$, we evaluated the amplitudes by an extrapolation procedure assuming a quadratic form $a_1 + a_2/q(p) + a_3/q^2(p^2)$.

The accuracy of our calculation is severely restricted by the number of momentum mesh points. The time for calculating a complete set of elements $f_\nu^{(n)}(q,p)$ goes as N^η where N is the number of momentum mesh points and $4 > \eta > 3$. It also increases linearly with n. For example, a Univac 1140 computer takes 7.5 minutes to calculate all second order terms and 11 minutes for all sixth order terms. Thus it was not practical for us to increase the number of momentum mesh points. It is obvious that a more efficient numerical approach is required.

2. Construction of solution

As stated earlier the accuracy of the calculation of $f_\nu^{(n)}(q,p)$ worsens as q and p increase. The problem first appears in the calculation of the second order term. As we calculate higher order terms the errors in the lower order terms flow down to the lower momentum region, which makes it very difficult to check the convergence of the Padé approximants. In Table II the S11 and S31 phase shifts from [N,N] Padé approximants with $1 \leq N \leq 4$ are listed. The results from [N+1,N] Padé approximants are listed in Table III. The [4,4] Padé results are quite different from those of all lower order approximants for both isospins. The

[5,4] Padé results for I = 1/2 exhibit the same feature.

Since a convergent Padé approximant was not feasible we decided to use the amplitudes obtained from the various Padé approximants as trial solutions. These solutions were put into the right-hand side of the integral equation (50) and the output was compared with the input. We required that the output and input agree well for low values of p_i and p_f. After many trials we found that the quality of the agreement could be improved vastly by taking linear combinations of amplitudes from two Padé approximants. The combination which worked best was of the form

$$K^I_{ij} = \frac{x_I\ P^I_{ij}(N_1,M_1) + (1-x_I)P^I_{ij}(N_2,M_2)}{1 + x_I\ Q^I_{ii}(N_1,M_1) + (1-x_I)Q^I_{ii}(N_2,M_2)} \qquad (65)$$

where $P^I_{ij}(N,M)$ and $Q^I_{ii}(N,M)$ are the numerator and denominator polynomials for [N,M] Padé approximants, defined by (63) and (64). Our best result is obtained by combining the amplitudes from the [3,2] and [3,3] Padé approximants. For $|\vec{p}_i|$, $|\vec{p}_f| \leq 2m_\pi$ the average percentage difference between the output and the input amplitudes $<200|f^{output}_\nu - f^{input}_\nu|/(f^{output}_\nu + f^{input}_\nu)>$ is 36 for I = 1/2 and 21 for I = 3/2 when we use pure [3,2] and [3,3] amplitudes for I = 1/2 and 3/2, respectively. Both these numbers reduce to 3 when we combine the two sets of amplitudes with $x_I = -0.05$ for I = 1/2 and $x_I = 1.195$ for I = 3/2. These improvements occur mainly in the off-mass-shell amplitudes. An inspection of Tables II and III shows that the phase shifts for the two sets of values of $x_{1/2}$ and $x_{3/2}$ differ by less than 1%.

The most plausible explanation of these results is that had it not been for the limited accuracy of the calculation the [N+1,N] Padé approximants would have converged for I = 1/2 and the [N,N] for I = 3/2. It appears possible to suppress the effects of errors by a linear combination of the amplitudes from two different approximants.

IV. NUMERICAL RESULTS

A. Experimental Phase Shifts

Before discussing the numerical results a few comments on the status of the low-energy S-wave phase shifts are necessary. There are two general types of analysis by which phase shifts are obtained from cross section data. One is the so-called energy-dependent fit, where a certain reasonable, smooth dependence of the phase shifts on energy is assumed, and the other is the energy-independent fit, where data is analyzed separately at each energy. Several recent energy-dependent fits are available.[29] For the present discussion we consider a preliminary set of energy-dependent phase shifts due to Zidell, Roper, and Arndt[31] (ZRA). We also consider the energy-independent fit of Carter, Bugg, and Carter[32] (CBC). The two sets are shown in Fig. 3.

For I = 1/2 there is a remarkable difference between the two sets. The CBC phase shifts as a function of T_π, the pion lab energy, have a gentle curvature. The ZRA phase shifts, on the other hand, start with a larger slope and then around $T_\pi \sim 70$ MeV the slope rapidly decreases to a very small value. The entire change occurs in an interval of 20 MeV. Other energy-dependent analyses, for example, the CERN theoretical fit,[29] give S11 phase shifts with a similar energy dependence. For I = 3/2 both types of analysis appear to give phase shifts with the same general qualitative behavior. The qualitative nature of the energy dependence of the experimental phase shifts is a particularly important consideration in the present work, since we find that when our theoretically calculated phase shifts have the experimentally observed energy dependence, getting high quality agreement with experiment is then a matter of carefully searching the parameter values.

Our theory contains no scale parameter comparable to 20 MeV, so it is not surprising that we failed to find a set of parameters which can reproduce the energy dependence of the S11 phase shifts of ZRA. We therefore simply assume that the CBC phase shifts for I = 1/2 are the correct experimental data. For I = 3/2 we take the ZRA set as the experimental data. These authors have a high level of confidence in these phase shifts. The ZRA and the CBC sets for I = 3/2 differ slightly as T_π increases. But in that energy region our present calculations are also not reliable as we have not included S-wave inelasticity.

From the standpoint of our theory we prefer the CBC set for I = 1/2. An additional reason for preferring these CBC phase shifts is that when our theoretical S11 phase shifts agree with the CBC fit we get for the charge exchange scattering length, $a^{(-)}$, the value $0.0793 m_\pi^{-1}$ in excellent agreement with the current algebra prediction[23] of $0.0786 m_\pi^{-1}$. An inspection of Fig. 3 shows that the energy-dependent fit gives a larger value of $a^{(-)}$.

B. Parameter Search for Best Results

In its present form the theory has six parameters: $\sigma(\pi N)$, μ_1 and μ_2 (σ-commutator), $\bar{g}_\pi$ (z-graphs), μ (P-wave), and m_0 (nucleon pole terms and z-graphs). It should be noted that $g_\pi(0)$ is fixed by $g_A(0)$ and f_π through the Goldberger-Treiman relation. The numerical results are most sensitive to the values of $\sigma(\pi N)$ and $\bar{g}_\pi$. As stated previously the phase shifts are less sensitive to the four form factor mass parameters.

We searched for the best values of the mass parameters by examining the qualitative behavior of the [1,1] Padé phase shifts. The values we settled on are

$$\mu_1 = 8.24 m_\pi, \ \mu_2 = 7.5 m_\pi, \ m_0 = 8.6 m_\pi \text{ and } \mu = 8 m_\pi. \tag{66}$$

Obviously some variations in these parameters are possible. But we think that μ_1, μ_2 and m_0 are within $1 m_\pi$ of their best values. Since the entire role of the P-waves is smaller than that of the σ-commutator term or the z-graphs, the parameter μ has a larger uncertainty when we try to fix it from the low-energy S-wave data only. The P-wave work is in progress. When it is complete we will have a better knowledge of the off-mass-shell P-wave amplitude.[33] However, as subsequent discussion will show a better value of μ is about $10m_\pi$.

The preliminary search of the parameters $\sigma(\pi N)$ and $\bar{g}_\pi$ was carried out also with the [1,1] Padé results. The final search was made in terms of the solutions constructed in the manner described at the end of Section III. In Fig. 3 we present the phase shifts due to three sets of values of $\sigma(\pi N)$ and $\bar{g}_\pi$ given in Table IV, with the other parameters given by Eq. (66). The parameters were chosen to have the S31 phase shifts in very good agreement with the ZRA results. It was possible to accomplish this for $T_\pi \leq 100$ MeV. As we have not included S-wave inelasticity we have no justification for demanding good agreement for larger values of T_π. But since we have a once-subtracted Low equation, and the effects of S-wave inelasticity are quite small near the elastic threshold, the theory is required to do well at low energy.

The values of $\sigma(\pi N)$ and $\bar{\bar{g}}_\pi$ for the sets (a) and (c) differ from those of set (b) by 2% and 1.3%, respectively, while the corresponding S11 phase shifts differ by 10% in each case. For I = 1/2 there is considerable cancellation between the attraction from the isovector terms produced by the z-graphs and the repulsive isoscalar σ-commutator term. The two terms are both repulsive in the I = 3/2 channel. This explains the sensitivity of the S11 phase shifts on the two parameters. At present we consider the set (b) our best result for $\mu = 8m_\pi$. The various order Padé results listed in Tables II and III, the percentage differences between the input and output solutions and the values of $x_{1/2}$ and $x_{3/2}$ quoted in Section III are all for set (b). The corresponding parameters of the effective range expansion

$$k \operatorname{Cot} \delta(k) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + P k^4$$

are listed in Table V.

V. DISCUSSION OF RESULTS

If the true S11 phase shifts have the gentle energy dependence of the CBC fit[32] then the theory discussed in this paper can explain the low-energy S-wave phase shifts. If the true phase shifts, while having the desired energy dependence, are numerically different from the CBC fit the major parameters $\sigma(\pi N)$ and $\bar{g}_\pi = |g_\pi(4M^2)|$ will be commensurately different from the values 25.5 MeV and 11.69, respectively, obtained by us. It is very likely that such changes would be small.

It is desirable to have independent checks of $\sigma(\pi N)$ and $\bar{g}_\pi$. Since the introduction of the notion of the σ-commutator as a measure of chiral symmetry breaking, there has been a large amount of work on the evaluation of $\sigma(\pi N)$. The situation in regards to $\bar{g}_\pi$ is quite the opposite; there are no reliable theoretical estimates of $g_\pi(t)$ for $t \simeq 4M^2$, so there is nothing to compare with our result.

Reya[24] has reviewed the work prior to 1974 on the determination of $\sigma(\pi N)$. Though there is a wide spread in the values obtained by various authors in the earlier work, more recent evaluations[34] appear to have converged to $\sigma(\pi N) = 65 \pm 5$ MeV, in violent disagreement with our result. In a recent paper[35] we have analyzed this problem carefully. We concluded that the large value results from errors of extrapolation. With $\sigma(\pi N) = 25.5$ MeV our theory gives reasonably good quantitative agreement with the basic amplitudes which are used for extraction of $\sigma(\pi N)$ by extrapolation to the unphysical value $t = 2m_\pi^2$. The agreement is improved if we increase μ, the P-wave form factor mass, from its present value of $8m_\pi$ to $10m_\pi$. With the larger value of μ we can essentially reproduce the set (b) S-wave phase shifts discussed earlier if we change $\sigma(\pi N)$ to 24.9 MeV and $\bar{g}_\pi$ to 11.90, all other parameters remaining the same.[35] We find that only the smaller value of $\sigma(\pi N) \sim 25$ MeV appears to be consistent with the experimental data and theoretical constraints.

Before ending this section we make a few comments on the consequences of nucleon recoil in the S-wave equations. The nucleon pole terms and the P-wave contributions are entirely due to recoil and so they vanish in the static limit. To illustrate the role of nucleon recoil we have calculated the S-wave phase shifts by dropping these terms one at a time but keeping all other parameters exactly the same as those of curves (b) of Fig. 3. These results along with curves (b) are shown in Fig. 4. The accuracy of these new solutions are not as good as that of solution (b). But this does not affect the conclusion one can draw from inspection of Fig. 4, namely, that the recoil terms are of considerable importance.

VI. SUMMARY AND CONCLUDING REMARKS

We have discussed a theory of pion-nucleon scattering based on the dynamics of boson exchange, the absorption-emission process and the z-graphs. The dynamics enter through PCAC, the σ-commutator term, the coupling constants g_π and $\bar{g}_\pi$ and the various form factor masses. We have constructed solutions of the nonlinear integral equations for the S-wave amplitudes using Padé approximants. We succeed in reproducing the energy dependence of the low-energy S31 and S11 (CBC) phase shifts. In the process we evaluated the major parameters $\sigma(\pi N)$ and $\bar{g}_\pi$, g_π being fixed by the Goldberger-Treimann relation. Our low value of 25.5 MeV for $\sigma(\pi N)$ has been justified in a previous paper.[35]

The complete numerical evaluation of the amplitude required for pion-nucleon scattering must await completion of work on the P-wave amplitudes, which is in progress. However our present work already establishes a very important feature of the theoretical description of pion-nuclear scattering. The presence of the factor $(t - m_\pi^2)$ in Eq. (18) tells us that the pion-nucleus optical 'potential' (for use in the Klein-Gordon equation) will contain a Laplacian term of well-defined magnitude. From Eq. (18) we determine this term to be $-(\frac{\sqrt{2}}{f_\pi})^2 m_\pi^2 \sigma(\pi N)\nabla^2\rho = -0.414 m_\pi^{-3} \nabla^2\rho$, where ρ is the nuclear density. A preliminary study[36] of the role of this term in pion-nuclear scattering has shown it to be of great importance at low energies.

ACKNOWLEDGEMENTS

The authors wish to thank V. S. Zidell, L. D. Roper, and R. A. Arndt for advance communication of their phase shifts. We thank the Computer Science Center of the University of Maryland for providing part of the computer time used for this work. Part of this work was completed while one of us (JBC) participated in the Visitors Program of the University of Maryland's Nuclear Theory Group.

APPENDIX

We illustrate the separation into connected and disconnected parts of a product of matrix elements of the pion current by considering the first term on the right of Eq. (19). Denoting by $P = (H,\vec{P})$ the energy-momentum four-vector operator, we have

$$\sum_n \frac{\langle p_f|j_\beta(0)|n\rangle\langle n|j_\alpha(0)|p_i\rangle}{k_0 + p_{f0} - n_0 + i\varepsilon}\,\delta^3(\vec{k}+\vec{p}_f-\vec{n}) = \langle p_f|j_\beta(0)\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}\,j_\alpha(0)|p_i\rangle$$

$$= \langle 0|a_f(\text{out})j_\beta(0)\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}j_\alpha(0)a_i^\dagger(\text{in})|0\rangle, \qquad \text{(A.1)}$$

where we have arbitrarily chosen the second quantized creation (annihilation) operator for the initial (final) state nucleon to be an in (out) operator. Equation (A.1) can be expressed as

$$\langle 0|\{[a_f(\text{out}),j_\beta(0)] + j_\beta(0)a_f(\text{out})\}\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}\{[j_\alpha(0),a_i^\dagger(\text{in})] + a_i^\dagger(\text{in})j_\alpha(0)\}|0\rangle$$

$$= \langle 0|[a_f(\text{out}),j_\beta(0)]\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}[j_\alpha(0),a_i^\dagger(\text{in})]|0\rangle$$

$$+ \langle 0|[a_f(\text{out}),j_\beta(0)]\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}\,a_i^\dagger(\text{in})j_\alpha(0)|0\rangle$$

$$+ \langle 0|j_\beta(0)a_f(\text{out})\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}[j_\alpha(0),a_i^\dagger(\text{in})]|0\rangle$$

$$+ \langle 0|j_\beta(0)a_f(\text{out})\frac{\delta^3(\vec{k}+\vec{p}_f-\vec{P})}{k_0+p_{f0}-H+i\varepsilon}\,a_i^\dagger(\text{in})j_\alpha(0)|0\rangle. \qquad \text{(A.2)}$$

Using the operator identities, valid for either in or out operators,

$$P\,a^\dagger_{p_{i,f}} = a^\dagger_{p_{i,f}}(P + p_{i,f})$$

$$a_{p_{i,f}}P = (P + p_{i,f})a_{pi,f}, \qquad \text{(A.3)}$$

and inserting complete sets of physical states between all terms in (A.2) we obtain

$$\sum_n \frac{\langle p_f|j_\beta(0)|n\rangle_c \langle n|j_\alpha(0)|p_i\rangle_c}{k_0 + p_{f0} - n_0 + i\varepsilon}\,\delta^3(\vec{k}+\vec{p}_f-\vec{n}) + \sum_n \frac{\langle p_f|j_\beta(0)|p_i,n\rangle_c \langle n|j_\alpha(0)|0\rangle}{k_0 + p_{f0} - p_{i0} - n_0 + i\varepsilon}$$

$$\times\ \delta^3(\vec{k}+\vec{p}_f-\vec{p}_i-\vec{n}) + \sum_n \frac{\langle 0|j_\beta(0)|n\rangle\langle n,p_f|j_\alpha(0)|p_i\rangle_c}{k_0 - n_0}\,\delta^3(\vec{k}-\vec{n})$$

$$- \sum_n \frac{\langle 0|j_\beta(0)|p_i,n\rangle\langle p_f,n|j_\alpha(0)|0\rangle}{k_0 - p_{i0} - n_0}\,\delta^3(\vec{k}-\vec{p}_i-\vec{n}) . \qquad \text{(A.4)}$$

The subscript c denotes a connected matrix element. The first term is thus the fully connected part, the second and third are the semi-connected (semi-disconnected) parts, and the final term is the fully disconnected part. The minus sign in front of this term comes from commuting $a_f(\text{out})$ and $a_i^\dagger(\text{in})$ in (A.2).

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TABLE CAPTIONS

Table I: CM frame expressions for the variables μ, ν, $\vec{L}$ and ε in Eq. (39) for each of the four integrals of Eq. (28).

Table II: S11 and S31 phase shifts in degrees corresponding to the set (b) parameters listed in Section IV for various order [N,N] Padé approximants.

Table III: Same as Table II, except for [N+1,N] Padé approximants.

Table IV: Values of the main parameters of solutions (a), (b) and (c). All other parameters for each of these solutions are fixed at the values listed in Eq. (66).

Table V: Parameters of the effective range expansion corresponding to solution (b).

TABLE I

	μ	ν	$\vec{L}$	ε
DIRECT-HARD DH	β	α	$\vec{p}_f + \vec{k} = 0$	$p_{f0} + k_0 + i\eta$
CROSSED-HARD CH	α	β	$\vec{p}_i - \vec{k} = \vec{p}_i + \vec{p}_f$	$p_{i0} - k_0$
DIRECT-SOFT DS	β	α	$\vec{p}_f$	p_{f0}
CROSSED-SOFT CS	α	β	$\vec{p}_i$	p_{i0}

TABLE II

ENERGY	S11 [1,1]	S11 [2,2]	S11 [3,3]	S11 [4,4]	S31 [1,1]	S31 [2,2]	S31 [3,3]	S31 [4,4]
1078.4	.38	.38	.44	.86	-.30	-.29	-.32	-.45
1084.6	1.85	1.85	2.14	4.60	-1.48	-1.43	-1.61	-2.36
1107.2	3.83	3.82	4.50	17.34	-3.47	-3.34	-3.78	-6.34
1141.7	5.45	5.43	6.54	-37.63	-5.92	-5.66	-6.42	-17.57
1170.2	6.29	6.24	7.71	-2.19	-7.87	-7.53	-8.49	-14.88
1186.0	6.64	6.55	8.29	.39	-8.94	-8.53	-9.64	.18
1218.9	7.16	6.96	9.26	4.66	-11.18	-10.87	-12.02	-3.49
1271.0	7.64	7.11	10.50	6.60	-14.73	-14.62	-15.71	-10.33
1326.8	7.95	7.18	12.37	7.17	-18.50	-18.50	-19.60	-1.25
1366.7	8.28	7.84	15.12	6.60	-21.18	-21.08	-21.97	-17.69
1432.6	11.46	10.04	15.89	7.53	-25.45	-24.99	-26.28	-24.15
1667.6	-1.37	6.60	14.61	7.49	-36.78	-35.81	-38.29	-23.47
2049.7	-14.11	-6.56	-6.56	-1.69	-43.16	-42.53	-53.71	-48.09
2462.3	-30.20	-17.47	-17.44	-18.76	-46.73	-46.19	-70.32	-56.38
2756.2	-41.52	-32.83	-7.77	-31.69	-48.92	-48.92	-67.87	-57.02

TABLE III

ENERGY	S11 [2,1]	S11 [3,2]	S11 [4,3]	S11 [5,4]	S31 [2,1]	S31 [3,2]	S31 [4,3]	S31 [5,4]
1078.4	.38	.47	.51	.49	-.31	-.34	-.38	-.38
1084.6	1.88	2.34	2.51	2.41	-1.55	-1.72	-1.90	-1.88
1107.2	3.90	4.94	5.31	5.01	-3.63	-4.03	-4.40	-4.31
1141.7	5.60	7.28	7.96	7.31	-6.19	-6.80	-7.23	-6.98
1170.2	6.55	8.60	9.24	7.95	-8.21	-9.00	-9.64	-9.43
1186.0	6.97	9.23	9.78	8.12	-9.32	-10.13	-10.51	-9.96
1218.9	7.68	10.32	10.78	7.45	-11.58	-12.48	-12.72	-12.02
1271.0	8.43	11.29	11.41	5.81	-15.05	-16.11	-16.24	-15.23
1326.8	8.90	11.11	11.23	2.79	-18.41	-19.64	-19.64	-18.87
1366.7	9.08	10.33	11.28	2.37	-20.44	-21.92	-21.92	-16.67
1432.6	9.02	8.20	10.45	-3.65	-26.48	-25.45	-25.60	-18.85
1667.6	5.83	5.72	7.61	7.62	-34.66	-28.31	-35.42	-40.69
2049.7	-4.88	-6.96	-5.23	-4.39	-46.69	-16.16	-41.45	-44.96
2462.3	-18.08	-17.83	-18.03	-18.29	-50.44	-27.80	-42.17	-57.11
2756.2	-17.76	-23.99	-22.90	-32.79	-48.83	-32.81	-43.86	-56.51

TABLE IV

Set	$\bar{g}_\pi$	$\sigma(\pi N)$ in MeV
a	11.85	25
b	11.69	25.5
c	11.54	26

TABLE V

Quantity	S11	S31
a (in m_π^{-1})	-0.143	0.095
r_0 (in m_π^{-1})	0.981	5.349
P (in m_π^{-3})	0.036	-0.768

FIGURE CAPTIONS

Fig. 1: Diagrammatic representation of the πN scattering amplitude for $\pi_\alpha(k') + N(p_i) \to \pi_\beta(k) + N(p_f)$, where α,β denote isospin components (1,2,3).

Fig. 2: (a) Nucleon pole diagrams corresponding to Eq. (21). The intermediate state nucleon is in a positive energy state only, and thus these are not Feynman diagrams. (b) z-graphs corresponding to Eq. (24). The intermediate state particle is an antinucleon, described by a negative energy propagator.

Fig. 3: The dashed lines are our results with (a), (b) and (c) representing different choices for $\sigma(\pi N)$ and $\bar{g}_\pi$, as discussed in the text. The solid lines give Ref. 31 phase shifts, while flagged circles are from Ref. 32.

Fig. 4: Illustration of the sensitivity of phase shifts to nucleon recoil terms in the S-wave equation. Curves labelled (1) result when the P-wave contributions are dropped, while those labelled (2) result when the N-pole terms are omitted. In each calculation all other terms in the S-wave equation are exactly the same as in solution (b) of Fig. 3, which is redrawn here for comparison.

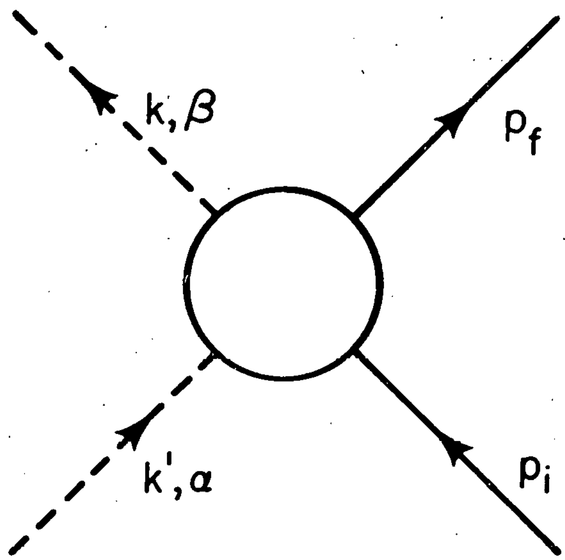


Fig. 1

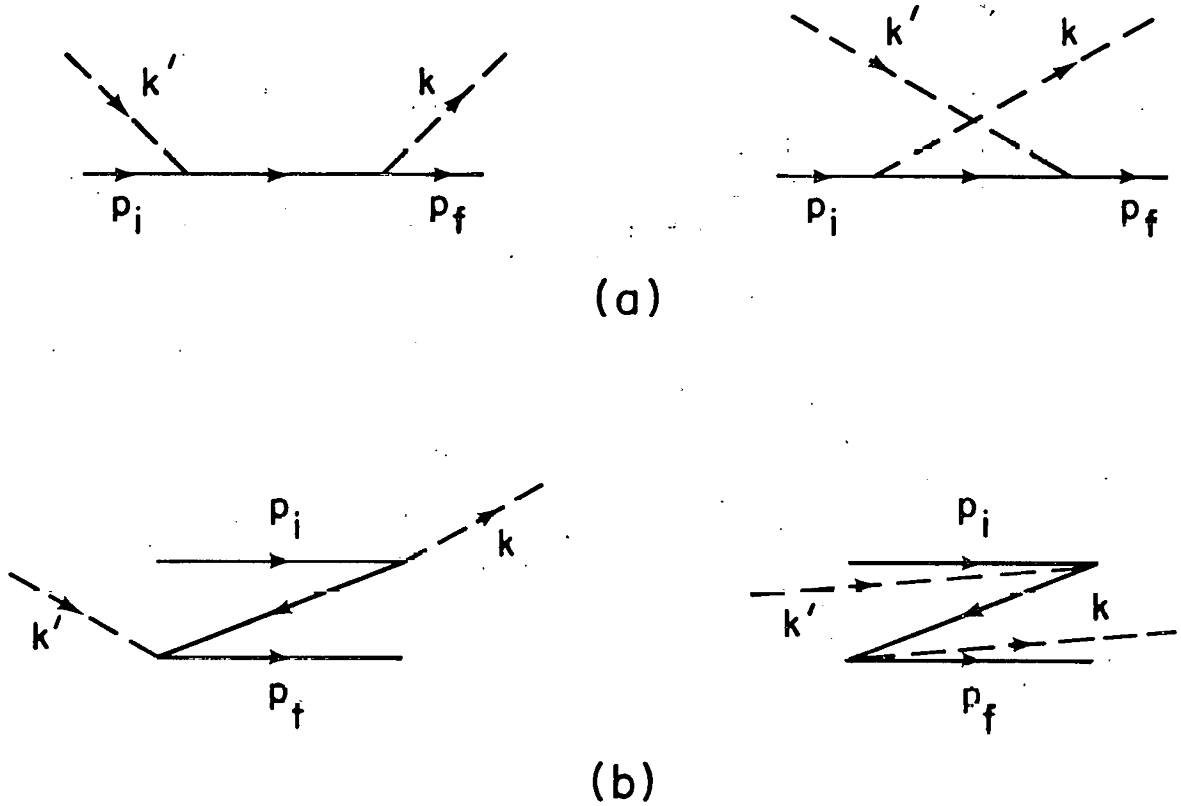
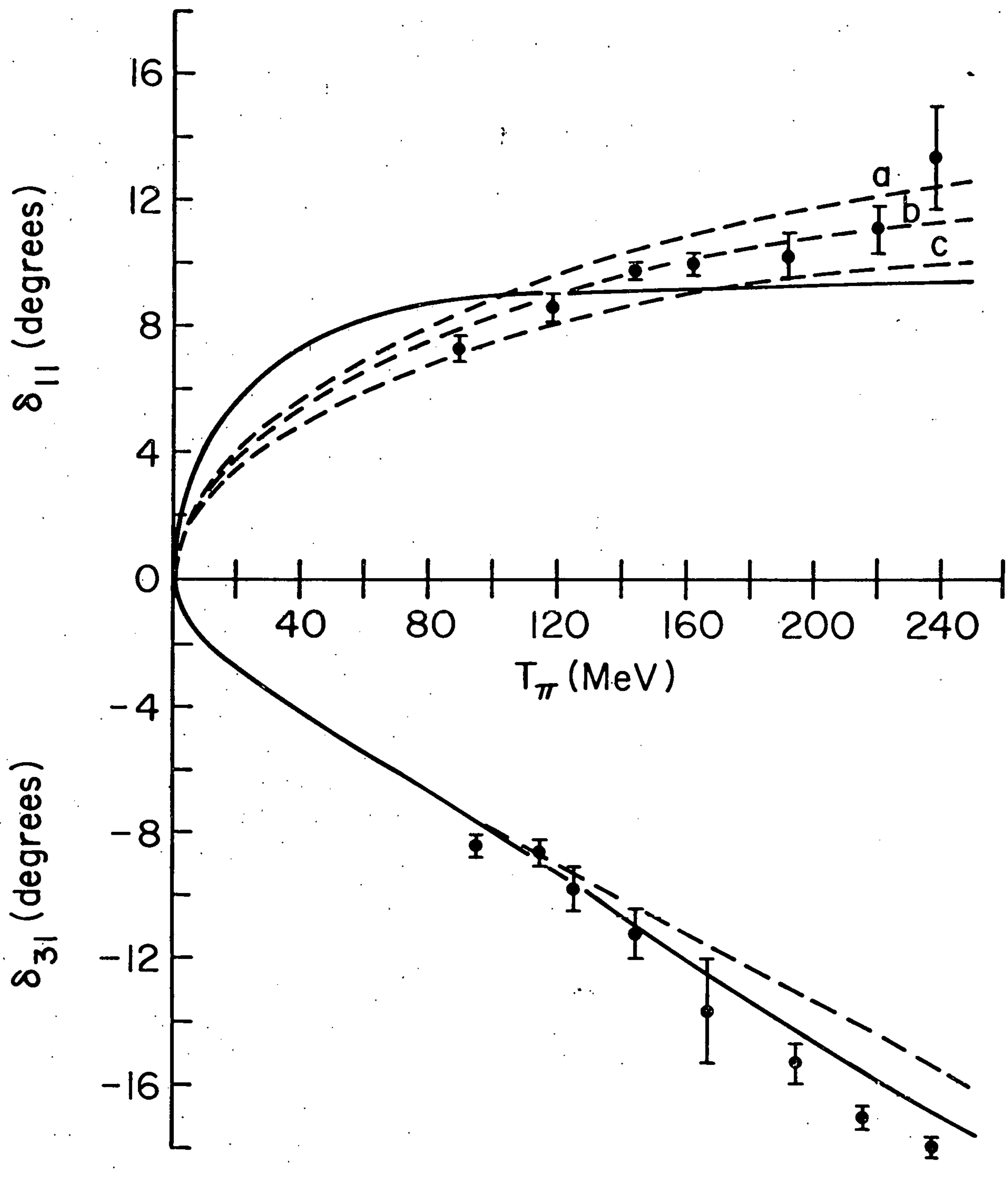
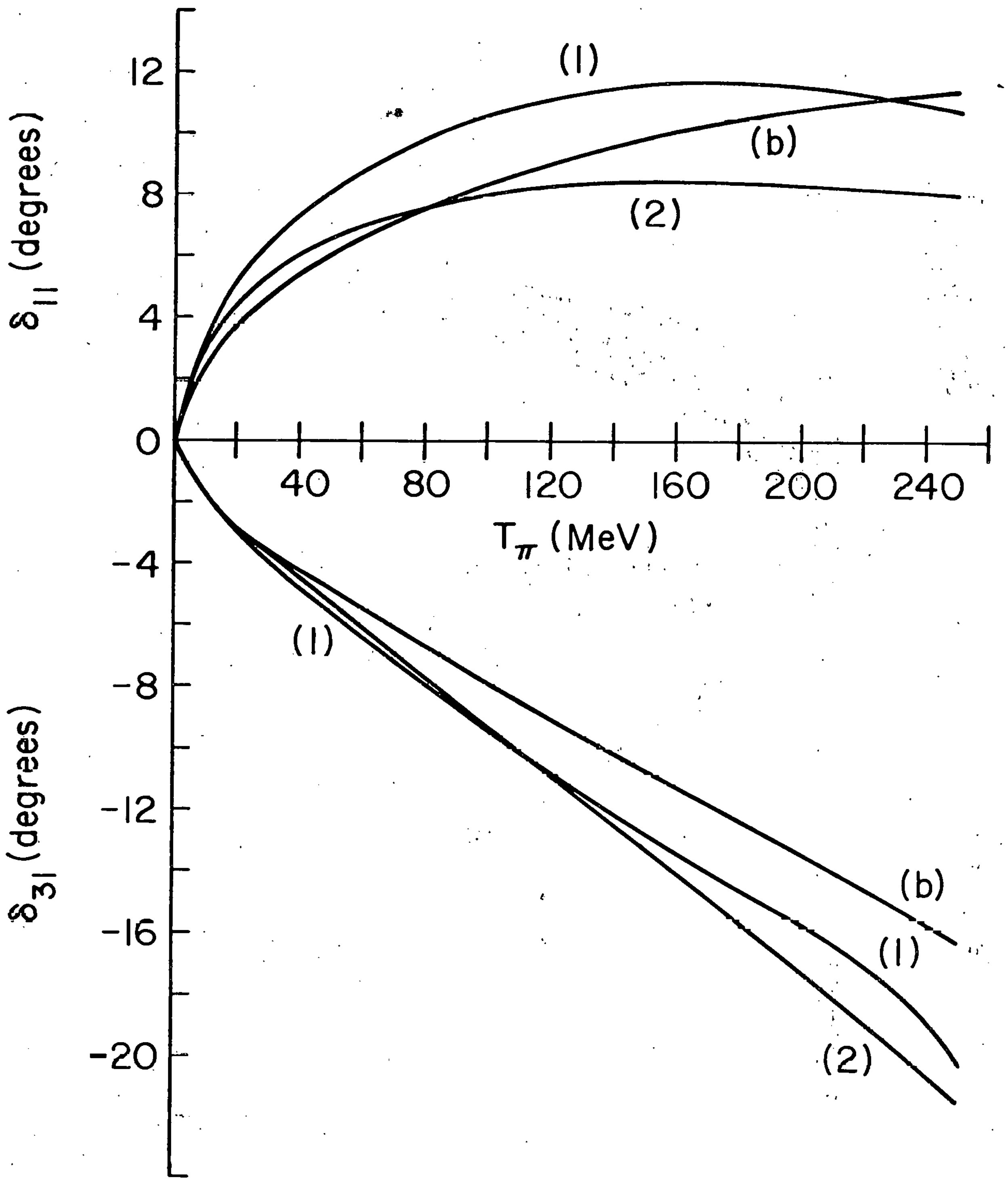


Fig. 2



16
12
8
4
0
-4
-8
-12
-16
δ_{11} (degrees)
δ_{31} (degrees)
a
b
c
40
80
120
160
200
240
T_π (MeV)

Fig. 3


(1)
(b)
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12
8
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0
-4
-8
-12
-16
-20
δ_{11} (degrees)
δ_{31} (degrees)
40
80
120
160
200
240
T_π (MeV)
(1)
(b)
(1)
(2)

Fig. 4