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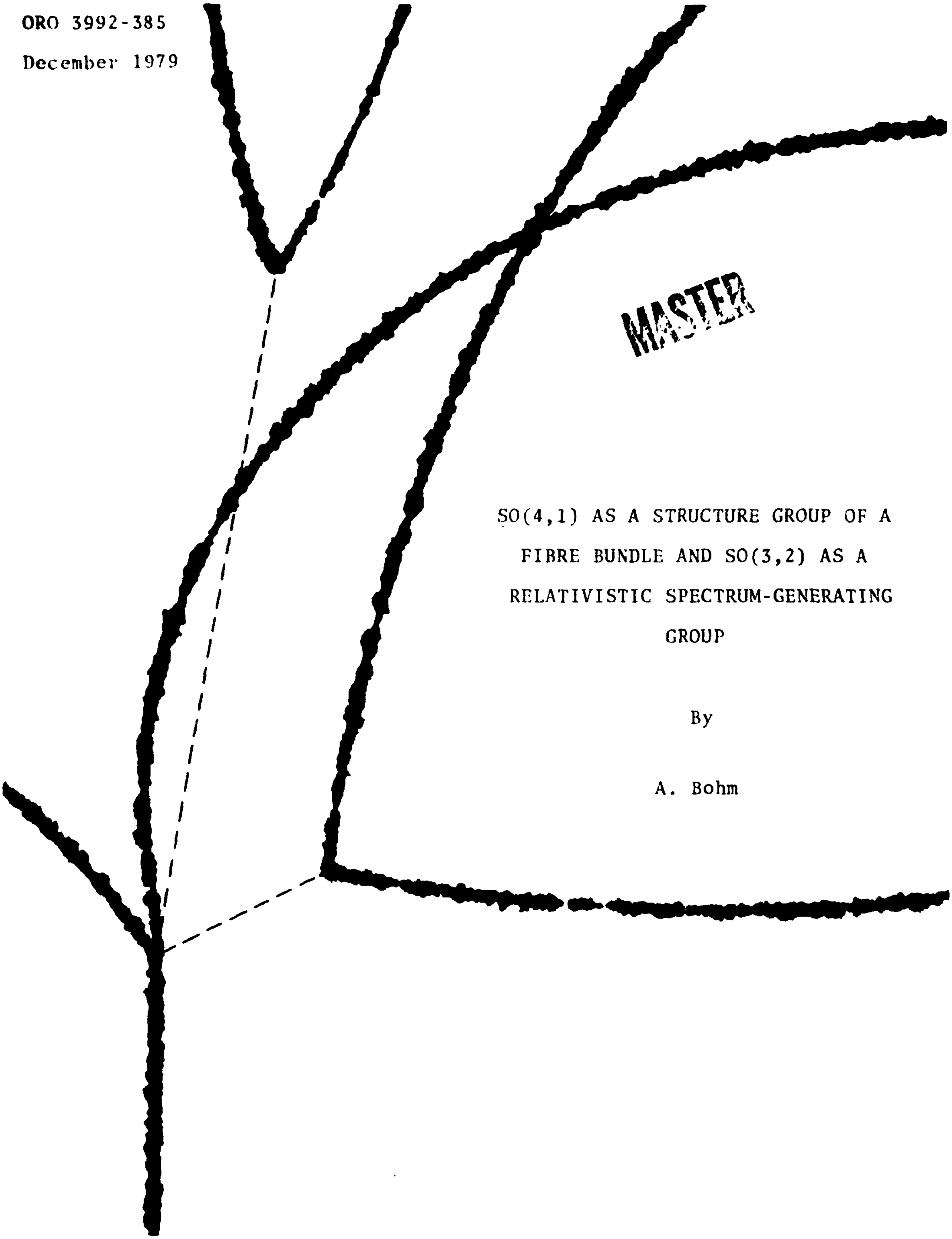

MASTER

SO(4,1) AS A STRUCTURE GROUP OF A FIBRE BUNDLE AND SO(3,2) AS A RELATIVISTIC SPECTRUM-GENERATING GROUP

By

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SO(4,1) as a Structure Group of a Fibre Bundle and SO(3,2) as a Relativistic Spectrum-Generating Group*

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The model that I want to present and which may be called a "collective model" for hadrons has two aspects:

1. The description of non-local objects
2. The construction of spectrum-generating groups in a relativistic theory.

The reason that I chose this subject for the conference--rather than the subject that was erroneously announced in the Preliminary Program--is that it combines the two areas in which the Chairman, M. A. Markov, and the Vice Chairman, V. I. Man'ko, have done pioneering work, and many contributors to these fields are in the audience.

The first aspect has a long history[1] in which the (4+1) de Sitter group played a prominent role[2] and which led to the use of the de Sitter space for the momentum coordinates.[3] Better known is the use of the de Sitter space for the position coordinates,[4] in which case, however, the de Sitter radius does not have the interpretation of a fundamental length in particle physics. I shall use the de Sitter space neither for momentum nor for position coordinates but in another way, which expands the concept of space-time beyond its conventional form: The fundamental length is the radius of micro-de Sitter spaces that are attached to the space-time points,[5] the underlying geometric stratum is Drechsler's (4+1) de Sitter bundle of Cartan type.[6]

The second aspect, the use of groups that are not related to symmetry transformations, and which were therefore called Dynamical Groups and later Spectrum-Generating Groups, has also a considerable history.[7] Combining this idea with the idea of the Relativistic Symmetries[8] led to SO(3,2) as a relativistic spectrum-generating group[9] and by different arguments to the concept of the dynamical stability group[10] of the velocity operator $\hat{P}_\mu = P_\mu/M$, which obeys the Werle relation $[P_\mu, G] = 0$.

The model that I want to present remains unaccessible if approached from the atomistic point of view which dominates the thinking in hadron physics. Therefore I have to premise a few general remarks:

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By understanding one means reduction to the simpler. This reduction to the simpler is usually understood atomistically as a reduction from a more complicated object to simpler objects, the constituents. One asks the question: What does it consist of? The best example for this is the understanding of the molecule: One can understand it by reducing it into its constituents, electrons and nuclei. But another way of understanding, which is closer to the experimental analysis, is the understanding of the molecule in terms of rotators, oscillators, Kepler systems. This holistic way of understanding is a reduction to simpler structures. One asks the question: What does it do? There are two reasons in favor of the holistic understanding: 1) a classical reason: the term "consists of" has no clear meaning if the binding energy is of the same order as the rest energy; 2) a quantum physical reason: for quantum physical systems there exist observables that are incompatible with all observables of its subsystems (holistic observables).

It is the holistic understanding that I wish to try out on the hadrons. In particular, I want to check whether the hadron contains a rotator structure.

Theoretical ideas, however beautiful, are worthless for physics if they do not relate to experimental data. The experimental data for this model is the mass and spin spectrum of hadron towers with corresponding internal quantum numbers. Each tower is characterized by a "system constant" α (in the same way as each hadron is characterized by the system constant mass m). Examples of such towers are given in columns 4 and 5 of the Table. The mass formula that will be derived for our model is rotator spectrum:

$$m^2 = \lambda^2(\alpha^2 - \frac{9}{4}) + \lambda^2 s(s+1) \qquad (1)$$

where $R = 1/\lambda$ is the radius of the micro-de Sitter spaces, which is universal (independent of the particular tower). We will consider two cases determined by the choice of the irreducible representation of the "relativistic" spectrum-generating group SO(3,2). In the first case, λ^2 can only be a constant $= \lambda_1^2 + \frac{1}{4}\lambda_2^2$ (rigid rotator) and the spin spectrum is the one that is underlined in column 1 of the Table. In the second case a new quantum number n will be introduced and R^2 is an operator with nontrivial spectrum: $\lambda^2_{(n,s)} = \lambda_1^2 - \lambda_2^2(s^2 + s - n^2)$. Column 2 and 3 of the table give the predictions of (1) for two selected sets of the universal parameters λ_1, λ_2 or the radius. These predictions should be compared with the experimental data in column 4 or 5.

A rotator spectrum is, of course, realized by numerous nonrelativistic quantum physical systems, so that a relativistic rotator spectrum for hadrons should not be surprising. And though the

Table: Comparison of (1) with experimental data.

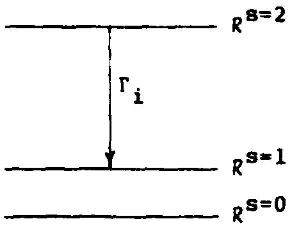
n,s^{PC}	$m_{(n,s)}$ [McV] for $\lambda_1^2 = 0.298$ for $\lambda_2^2 = 0.005$	 for $\lambda_1^2 = 0.300$ for $\lambda_2^2 = 0.0061$	Experimental data from Particle Data Table or later references	
$1,1^{--}$	765	762	$\rho(770)$	$\omega(783)$
$0,2^{++}$	1269	1260		f(1270)
$2,2^{++}$	1313	1318	$A_2(1310)$	
$1,3^{--}$	1705	1670	g(1680)	$\omega(1670)$
$3,3^{--}$	1840	1850	a $\bar{p}n(1890)$ has been seen in the D-wave,[18a] with $c = -1$ follows $s^P = 3^-$.	
$0,4^{++}$	1990	1925	S(1940)	
$2,4^{++}$	2090	2050		h(2040)
$4,4^{++}$	2360	2350	$(p\bar{p}\to\pi\eta)(2320)$[18c,g]	U(2350)[18c,d]
$1,5^{--}$	2140	1940		
$3,5^{--}$	2440	2280	$N\bar{N}(2480)$ with $s^{PC} = 5^{--}$[18d]	
$5,5^{--}$	2860	2850	NN(2850)[17e] 2950[17f] s^{PC} not known	

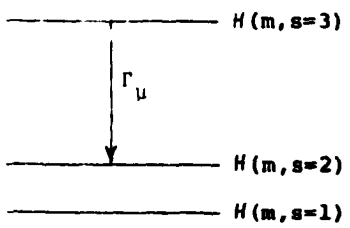
present experimental evidence for (1) is good, the superiority of (1) over any other spectrum (e.g., linearly rising Regge trajectories, $m^2 = \alpha_0 + \alpha_1 s$) can only be shown by further experimental data.

We will justify the rotator spectrum (1) in two steps: First we will explain the origin of the spin spectrum using a relativistic spectrum-generating SO(3,2). Then we turn to the first aspect above from which we postulate a constraint relation[10] from which (1) can be calculated.

II. Relativistic Spectrum-Generating SO(3,2)

The rotating molecule is a tower of elementary rotators whose space of physical states is an irreducible representation space R^s of the symmetry group $SO(3)_{S_{ij}}$[11] $(i,j = 1,2,3)$.





In analogy, the relativistic rotator should be a tower of elementary particles whose space of physical states is an irreducible representation space $H(m,s)$ of the symmetry group $P_{P_\mu L_{\mu\nu}}$, the Poincaré group, generated by the observables momentum P_μ and angular momentum and Lorentz generators $L_{\mu\nu}$ ($\mu,\nu = 0,1,2,3$). Transitions between various energy levels of the rotating molecule (e.g. $A^* \to A\gamma$) are described by SO(3)-vector operators Γ_i. In analogy, transitions between various mass levels of the relativistic rotator (e.g. $A_2 \to \rho\gamma$) should be described by Lorentz-vector operators Γ_μ, the "vector currents." The total spectrum of the nonrelativistic rotator is described by an irreducible representation space of, e.g., the spectrum-generating group[7] $SO(3,1)_{S_{ij},\Gamma_i}$

$$H \cong \sum_{s = 0,1,2\cdots} \oplus R^s \tag{2}$$

In analogy, the total spectrum of the relativistic rotator should be described by an irreducible representation space of the spectrum-generating group $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ that acts on the indices of the spinor basis; $S_{\mu\nu}$ is called the spin part of the Lorentz generators $L_{\mu\nu}$. This leads to a representation space (of the "relativistic symmetry")

$$H \cong \sum_{n,s} \oplus H^n(m,s) \tag{3}$$

which is the direct sum of irreducible representation spaces $H^n(m,s)$ of the Poincaré group when the representation of $SO(3,1)_{S_{\mu\nu}}$ is "induced to a representation of P. The detailed construction for some particular irreducible representations of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$ is given in references 9 and 10. The reduction with respect to (s, n = eigenvalue of $\Gamma_\mu \hat{P}^\mu$) is determined by the $SO(3)_{S_{ij}} \times SO(2)_{\Gamma_0}$-reduction of the particular irreducible representation of $SO(3,2)_{S_{\mu\nu},\Gamma_\mu}$.

A particular 4-dimensional version is obtained if in addition to the SO(3,2)-commutation relations one requires the relation $\{\Gamma_\mu,\Gamma_\nu\} = \frac{1}{2}\eta_{\mu\nu}$; ($\eta_{\mu\mu} = -1,1,1,1$). Then the matrix elements $\langle|\Gamma_\mu|\rangle = 1/2\gamma_\mu$ (Dirac matrices), $\langle|S_{\mu\nu}|\rangle = \frac{1}{2}\sigma_{\mu\nu}$ and (3) is precisely $H \cong H^{n=-1/2}(m,s=1/2) \oplus H^{n=1/2}(m,s=1/2)$, the space of solutions of the Dirac equation. The relativistic spectrum-generating SO(3,2) is thus an infinite dimensional generalization of the well known Dirac theory; one obtains this generalization if one does not require the Dirac anti-commutation relation but replaces it by the condition of Hermiticity of Γ_μ and $S_{\mu\nu}$.

If one requires instead of this Dirac anti-commutation relation the relation

$$\{\Gamma_\mu,\Gamma_\nu\} + \{S_{\mu\rho},S_\nu{}^\rho\} = -\eta_{\mu\nu} \tag{4}$$

then one obtains another special case, the infinite dimensional Majorana representation called "remarkable representation" in reference 10. For this representation the reduction (3) is given precisely by

$$H \doteq \sum_{\substack{s = 0,1,2\cdots \\ |n| = s+1/2}} H^n(m,s) \tag{5}$$

This representation will then give the spin spectrum underlined in column 1 of the Table.

Due to the vector operator character the Γ_i will change the spin and transform between different irreducible representation spaces $H^n(m,s)$ of P, describing transitions between different hadrons.

Whether Γ_i will also change the mass and in what way it will change the mass depends upon the postulated constraint-relation. Such constraints can--according to reference 10--be arbitrarily imposed: $M^2 = f(s)$, and Biedenharn and van Dam favor the one that leads to hadron Regge bands $m^2 = \alpha_0 + \alpha_1 s$, though they do not give a theoretical justification for it. If one assumes the constraint relation $[\partial_\mu P^\mu, \Gamma_\nu] = 0$, one obtains a degenerate mass spectrum and a particular case of the relativistic symmetry.[8] The constraint that we shall impose follows from the idea of hadrons as de Sitter fibers, as we shall discuss now.

III. Nonlocal Objects and SO(4,1)

The (4,1) de Sitter bundle $T^R(V_4)$ is in a certain sense a union of de Sitter fibers $V_4'(x)$

$$T^R(V_4) = \bigcup_{x \in V_4} V_4'(x) \tag{6}$$

where $V_4'(x)$ for every $x \quad V_4$ (the base space for which we will in particular choose the Minkowski space) is related to the "standard fiber" $V_4' = SO(4,1)/SO(3,1)$ = de Sitter space by a map

$$V_4'(x) \rightarrow V_4' = SO(4,1)/SO(3,1)$$

Two maps

$$V_4'(x) \rightarrow V_4' \quad \text{and} \quad V_4'(x') \rightarrow V_4'$$

differ from each other by an element of the group SO(4,1) which is the structure group of the de Sitter bundle, or in physical terms, the gauge group. Thus the de Sitter bundle is an expansion of the usual Minkowski space V_4 by attaching one copy of the de Sitter space V_4' to each point x of V_4. It is not quite the direct product of base space V_4 and de Sitter space V_4' but sort of the direct product modulo a transformation of SO(4,1) acting on V_4': going from one point in the base space to another point one also makes a transformation in the fiber.

For Drechsler's de Sitter bundles[6] the attachment of de Sitter fiber to the base space is not arbitrary but the fiber is "soldered" to the base space in such a way that the fiber over $x \in V_4$ is tangent to V_4 for every $x \in V_4$. This can be achieved by the identification:

$$T_{\overset{\circ}{\xi}}(V_4'(x)) = T_x(V_4) \qquad (7)$$

where $T_x(V_4)$ is the tangent space to V_4 at x and $T_{\overset{\circ}{\xi}}(V_4'(x)$ is the tangent space to $V_4'(x)$ at the point of contact $\overset{\circ}{\xi} \in V_4'(x)$, $\overset{\circ}{\xi} \equiv x$. See Fig. 1

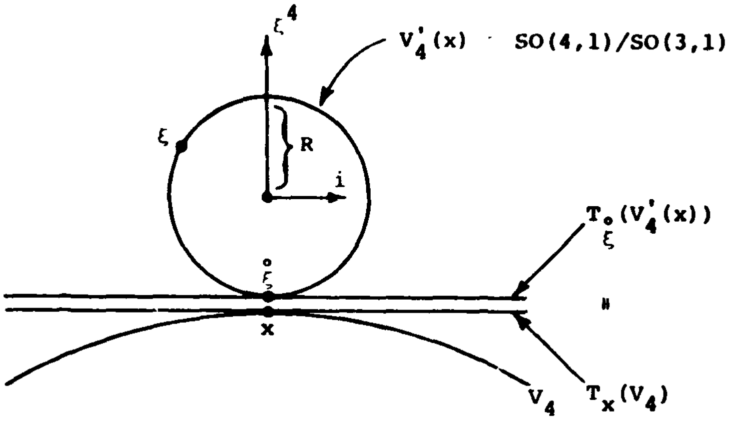


Fig. 1. de Sitter fiber soldered to the base space, V_4 (drawn here as curved). Of the base space, only one space dimension, x, is drawn. For the time dimension the circle goes over into a hyperbola.

(Soldering can be done in general when certain conditions are fulfilled for fiber F and base space B: a cross section must exist, F must be the homogeneous space G/H where G is the structure group and H the stability group (residual gauge group) of the point $\overset{\circ}{\xi} \in F$, dim B = dim F. Fiber bundles with soldering are called Cartan type bundles. The usual non-Abelian gauge theories are described by fiber bundles that are not of Cartan type.)

The purpose of the soldering is to give the distance in the fiber a physical meaning. Without soldering the radius R of the de Sitter space has no physical meaning, distance has only a meaning in Minkowski space where it is measured in cm. Soldering transfers the meaning of distance to the de Sitter fiber so that R is again measured in cm. From comparison of the resulting mass spectrum (1) with experimental data (Table) we obtain empirically that $R = 1/\lambda$ is of the order of 10^{-13} cm. This, however, is not the "size" of the physical object represented by the de Sitter space. Its "size" is only its trace that we see in Minkowski space V_4 and this is the point $\overset{\circ}{\xi}$ or the small area around it which is affected by the interaction of observation.

The consequence of the soldering is that the SO(3,1) subgroup of the structure group SO(4,1) (the residual gauge group) is identified with the Lorentz subgroup $SO(3,1)_{L_{\mu\nu}}$ of the Poincaré group $P_{P_\mu L_{\mu\nu}}$.

So far we have described the de Sitter group SO(4,1) and the Poincare group P as they arise in a classical geometrical picture of hadrons as the traces of de Sitter fibers in Minkowski space. A quantum physical system is not described by a geometrical picture (though such a picture may be helpful) but by an algebra of operators in the space of physical states. To the group of motion in Minkowski space correspond physical systems described by the irreducible unitary representation spaces $H(m,s)$ of the Poincaré group. The generators of the Poincaré group are represented by Hermitian operators P_μ, $L_{\mu\nu}$. And, as I am a follower of Wigner, I shall use these representation spaces and the algebra of observables generated by P_μ, $L_{\mu\nu}$ as the basis for my description of hadrons: The quantum physical system "elementary particle" has as its mathematical image the irreducible representation space $H(m,s)$ of the Poincaré group P.

To introduce some interaction one may try to use the substitution in analogy to the introduction of the electromagnetic interaction ("minimal coupling"). In classical mechanics the electromagnetic interaction is introduced by the substitution

$$\vec{p} \rightarrow \vec{\pi} = \vec{p} + e\vec{A} \ . \tag{8}$$

In quantum mechanics the numbers should be replaced by operators so that

$$\vec{p} \rightarrow \vec{\Pi}^{op} = \vec{P} + e\vec{A}^{op} \ . \tag{9}$$

In the position representation of quantum mechanics where

$$<x|P_i|\phi> = \frac{1}{i}\frac{\partial}{\partial x_i}<x|\phi> = \frac{1}{i}\frac{\partial}{\partial x_i}\phi(x) \qquad i = 1,2,3 \ . \tag{10}$$

(9) can be written as

$$<\vec{x}|\Pi_i^{op}|\phi> = \frac{1}{i}\frac{\partial}{\partial x_i}<\vec{x}|\phi> + e<\vec{x}|\vec{A}^{op}(Q_j)|\phi> \tag{11}$$

or

$$\frac{1}{i}D_i<\vec{x}|\phi> = \left(\frac{1}{i}\frac{\partial}{\partial x_i} + eA_i(x_j)\right)<\vec{x}|\phi> \tag{12}$$

Here it was assumed that the operator $A_i^{op}(Q_j)$ is only a function of the position operators Q: $Q_i|\vec{x}> = x_i|\vec{x}>$. In this form the description of the interaction is generalized and leads to the general definition of the covariant derivative for non-Abelian gauge groups[12]

$$D_\mu = \partial_\mu + ieA_\mu^a(x_\nu)U(I_a) \qquad \mu = 0,1,2,3 \tag{13}$$

where I_a are the generators of the non-Abelian gauge group G and $U(I_a)$ are their Hermitian representatives in a unitary representation: G → g → U(g). Here the $A_\mu^a(x_\nu)$ are numbers, which are functions of the numbers x_ν. (13) is the relativistic non-Abelian analogue of (12) which is only identical with (9) in the position representation, i.e. when applied to the "wave function in the position representation." Independent of the question what the meaning of a position representation should be in a relativistic quantum theory, (13) is certainly not the full analogue of (9) if one wants to use it in a more general way than just applying it to functions $\phi(x_\nu)$ of the coordinates x_ν. In general, the analogue of (9) for the introduction of interaction in a relativistic theory with a non-Abelian gauge group would be the substitution:

$$P_\mu \to B_\mu = P_\mu + \frac{1}{2}\{b^a, U(I_a)\} \tag{14}$$

where the b^a_μ are not numbers but operators and are the quantum mechanical analogues of the numbers eA^a_μ. This general form of the "minimal coupling" would then hold not only when applied to fields (functions of x_ν) but also as a general operator expression that can be applied to general vectors of the space of physical states.

In our case we choose for the $U(I_a)$ the representatives $L_{\rho\sigma}$ of the generators for the residual gauge group, which after soldering are identical with the observables generating the Lorentz group. The problem is then to find for a particular interaction the operators b^a_μ. We make the ansatz:

$$b^{\rho\sigma}_\mu = \lambda \eta^\rho_\mu \hat{P}^\sigma \qquad \text{where } \hat{P}_\sigma = M^{-1}P_\sigma , \quad M = (P_\mu P^\mu)^{1/2}$$

and λ is a constant of the dimension MeV. Then the substitution (14) has the form

$$P_\mu \to B_\mu = P_\mu + \frac{\lambda}{2} M^{-1}\{P^\sigma, L_{\mu\sigma}\} \quad . \tag{15}$$

From this it follows that

$$[B_\mu, B_\nu] = i\lambda^2 L_{\mu\nu} \tag{16}$$

so that the $B_\mu, L_{\mu\nu}$ generate a representation of SO(4,1), which is the group of motion in a de Sitter space with radius $R = 1/\lambda$. By the Inönü-Wigner contraction[13a]:

$$B_\mu \xrightarrow[\lambda\to 0]{} P_\mu$$

the de Sitter group goes into the Poincare group [13b] and the de Sitter bundle goes into the affine tangent bundle.[13c]

Mathematical formulas similar to (15) have reappeared several times in the literature over the past three decades,[14,2,4] however the operator corresponding to the B_μ had the entirely different meaning of a center-position[14] and when it was realized that they are related to SO(4,1), the de Sitter space was chosen to have the radius of the universe.[4]

IV. The SO(4,1) Constraint Relation for the Relativistic Spectrum-Generating SO(3,2)

We are now in the position to formulate the constraint relation. We define the Casimir operator of the SO(4,1) structure group

$$Q = \frac{1}{\lambda^2} B_\mu B^\mu - \frac{1}{2} L_{\mu\nu} L^{\mu\nu} \tag{17}$$

where B_μ is given by (15). The constraint relation is then formulated as

$$[Q, \Gamma_\nu] = 0 \tag{18}$$

As Q commutes already with all the other generators except Γ_ν, the constraint (18) makes the invariant operator of the de Sitter space Q into the invariant operator of the whole algebraic structure which describes the de Sitter bundle. Its eigenvalue α^2 will then characterize the physical system (in the same way as the eigenvalue of $M^2 = P_\mu P^\mu$ characterizes the elementary particle). From this constraint relation (18) the mass formula (1) is derived by a straightforward but lengthy calculation

$$m^2_{(n,s)} = \lambda^2(\alpha^2 - 9/4) + \lambda^2 s(s + 1) \qquad \lambda = 1/R . \qquad (19) = (1)$$

The spin spectrum is given by (5), λ is a constant and if one chooses as an example the tower of I = 0 mesons, one obtains from the experimental values of $m_\omega(s = 1^-)$ and $m_U(s = 4^+)$ the empirical value for λ and R:

$$\lambda^2 \approx 0.29 \text{ Gev}^2 \Rightarrow R \approx \frac{1}{3}\cdot 10^{-13} \text{ cm} \quad (R = \frac{\hbar c}{\lambda}) \quad . \tag{20}$$

(19) is a simple relation, and I have often been asked whether this simple result (which is not even the full SU(6) mass formula) warrants the enormous mathematical apparatus that was necessary to derive it. The answer is of course that this is an entirely different type of result from the SU(3) or SU(6) "mass" formulas. These are obtained from the assertion that an operator with a certain transformation property is the mass operator. That this "mass operator" has something to do with the observable $P_\mu P^\mu$ which is measured as the mass-square and that its expectation values (and what is called spin in the SU(6) mass formula) are the quantities that characterize the elementary particles, is never even attempted to be shown. Here, the quantities m and s in (19) are exactly the quantities that characterize (according to Wigner) the elementary particles and (19) is, therefore, of much greater depth than those mass formulas that provide a relation between two quantities which are just called "mass" and "spin." Therefore, the derivation of (19) requires more mathematics but in return also provides a much deeper insight leading to the picture of hadrons as micro-de Sitter spaces attached to the space-time points.

V. Generalization of the "Remarkable" Representation and Generalization of the de Sitter Fiber Bundle-- The General Relativistic Rotator

The mass spectrum (19) is that of a rigid rotator. Nonrelativistic rotators are realized in nature by numerous molecules and nuclei. From these examples one knows that the model of a rigid rotator is really of limited applicability and that nonrigidity has to be taken into account if one wants a more accurate description, especially for problems that involve higher angular momenta. In the classical picture of the molecule, the origin of this nonrigidity can be explained as the effect of centrifugal forces which change the moment of inertia (or radius). In the quantum mechanical description of the rotating molecule this can be described as replacing the number for the inverse moment of inertia by an operator.

For the relativistic rotator this would mean clasically, that the radius of the de Sitter fiber is not constant[15] and quantum mechanically, that the number λ should be replaced by an operator Λ.

Therefore, in order to obtain a quantum mechanical description of de Sitter fibers with varying radii, we have to construct a nontrivial Lorentz-invariant operator Λ, which commutes with the observables that specify the states of a single hadron (otherwise the radius would depend upon, e.g. the polarization of the hadron). In the "remarkable" representation used by van Dam and Biedenharn[10] this does not seem to be possible.

However, there exist even more remarkable representations[16] than the Majorana representation which have the additional advantage that they are not only representations of the quantum mechanical Poincare group but also representations of the quantum mechanical Poincare group extended by P, T and C.[17] I call these representations Generalized Dirac Representations (GDR), because for the half-integer spin case the 4 × 4 upper corners of the infinite matrices of Γ_μ are equivalent to the Dirac γ-matrices. Therefore, these representations appear to be ideally suited for the description of an infinite baryon tower whose lowest states are proton and antiproton.

There is also an integer spin GDR with the Poincare group reduction[16]

$$H(\mathrm{GDR}) = \sum_{\substack{s=0,1,2,\cdots \\ |n|=s,s-2,\cdots 0 \text{ or } 1}} \oplus H^{n}(m_{(n,s)},s) \tag{21}$$

which therefore describes a meson tower containing

$$\begin{aligned}(n,s^{PC}) = {} & (0,0),(1,1^{--}),(0,2^{++}),(2,2^{++}),(1,3^{--}),(3,3^{--}),\\ & (0,4^{++}),(2,4^{++}),(4,4^{++}),(1,5^{--})\ ,\end{aligned} \tag{22}$$

In these GDR one can form a nontrivial Lorentz-invariant operator Λ defined by

$$\Lambda^2 = \lambda_1^2 - \lambda_2^2(\hat{W} - (\Gamma_\mu \hat{P}^\mu)^2)$$

where (23)

$$\hat{W} = -\hat{w}_\mu \hat{w}^\mu \qquad \hat{w}_\mu = \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma} L^{\nu\rho} \hat{P}^\sigma \qquad \hat{P}_\mu = P_\mu M^{-1}$$

λ_1^2 and λ_2^2 are constants with dimension MeV^2. In the Majorana representation the operator Λ^2 is the constant $\Lambda^2 = \lambda_1^2 + \frac{1}{4}\lambda_2^2$. In the GDR the operator Λ^2 has the following spectrum:

$$\lambda^2_{(s,n)} = \lambda_1^2 - \lambda_2^2(s^2 + s - n^2) \ . \tag{24}$$

Thus the radius of the de Sitter fiber $\frac{1}{\lambda_{(s,n)}}$ increases with increasing spin, as one would intuitively expect for a rotator. With this operator Λ one can define the operator B_μ (the generator of translation along the fiber) now by

$$B_\mu = P_\mu + \Lambda M^{-1} \frac{1}{2} \{P_\nu, L^\nu_{\cdot\mu}\} \ . \tag{25}$$

And because of the Lorentz invariance of Λ the $L_{4\mu} = \Lambda^{-1}B_\mu$ fulfill together with the $L_{\mu\nu}$ the commutation relation of SO(4,1). If one now again considers the constraint relation

$$[\Gamma_\mu, Q] = 0 \ ; \qquad Q = \Lambda^{-1}B_\mu\Lambda^{-1}B^\mu - \frac{1}{2} L_{\mu\nu}L^{\mu\nu} \tag{26}$$

then one obtains the mass spectrum

$$m^2_{(n,s)} = \lambda^2_{(s,n)}(\alpha^2 - 9/4) + \lambda^2_{(s,n)}s(s + 1) \tag{27}$$

where α^2 is again the eigenvalues of the invariant operator Q and, therefore, characterizes the physical system described by the algebraic structure.

The comparison of (27) with the experimental data[18]--using two suitably picked sets of values for (λ_1, λ_2)--is given in the Table. In the comparison in the table both the I = 1 and I = 0 resonances are given, because of their almost equal masses. It is also possible that the transition operators Γ_μ have nontrivial isospin properties in which case the isospins would intermingle in one hadron tower.

The radii of the de Sitter fibers calculated from the fitted values of λ_1 and λ_2 are approximately $R_{(n,s)} \approx 0.36 \times 10^{-13}$ cm and differ very little for small values of s because $\lambda_1^2 >> \lambda_2^2$. For large values of s however $\lambda_2^2(s^2 + s - n^2)$ can become large, even larger than λ_1^2 and as one would expect in analogy to the centrifugal forces for the nonrelativistic rotator (rotating molecule) the relativistic rotator ceases to exist.

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