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A NUMERICAL METHOD FOR SOLVING ELLIPTIC BOUNDARY VALUE PROBLEMS IN UNBOUNDED DOMAINS*

MASTER

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DISCLAIMER

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I. INTRODUCTION

In this paper we shall describe a numerical method for solving elliptic boundary value problems in unbounded domains. For a survey of various methods for solving such problems, see [1]. Here we shall describe a particular method obtained by first introducing an artificial boundary near infinity, say a sphere Γ_R of large radius R, and an approximate local boundary condition on Γ_R. A finite element method is then employed to discretize this approximate problem. Since R must be large in order to suitably estimate the error due to the artificial boundary, the resulting system of linear equations is usually quite large. We shall describe how to reduce the number of linear equations to the asymptotically optimal amount, while preserving optimal error estimates and the sparseness of the matrix, by grading the mesh systematically in such a way that the element mesh sizes are increased as the distance from the origin increases.

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II. THE MESH GRADING PROCEDURE

Let Ω denote a bounded subset of R^3 with smooth boundary, $\partial\Omega$. Denote the complement of $\bar{\Omega}$ by Ω^C and let $x = (x_1, x_2, x_3)$ denote an arbitary point in R^3. For simplicity, we consider the following model problem:

$$(-\Delta - K^2)\, u = f \text{ in } \Omega^C,\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

and (2.1)

$$\frac{\partial u(x)}{\partial r} + \left(\frac{1}{r} - iK\right) u(x) = o\left(\frac{1}{r}\right) \text{ as } r = |x| \to \infty,$$

where $\frac{\partial}{\partial n}$ denotes the outward directed normal to $\partial\Omega$, f and $\partial\Omega$ are smooth, K is real and f has bounded support. This problem is well-posed. When $K \neq 0$ $(K = 0)$, (2.1) is referred to as the exterior Helmholtz (Laplace) problem. Our method and results may be readily extended to other boundary value problems, including those with variable coefficients and more general boundary conditions.

We next formulate an approximate problem on a bounded domain. Define the sets S_R and Ω_R by $S_R = \{x \in R^3 : |x| \le R\}$ and $\Omega_R = S_R \cap \Omega^C$, where R is large. Now consider the following problem:

$$(-\Delta - K^2) u_R = f \text{ in } \Omega_R,\ \frac{\partial u_R}{\partial n} = 0 \text{ on } \partial\Omega$$

and (2.2)

$$\frac{\partial u_R}{\partial r} + \left(\frac{1}{r} - iK\right) u_R = 0 \text{ on } \Gamma_R = \partial S_R.$$

This problem is also well-posed. Note that there are various other approximate boundary conditions that may be imposed on Γ_R (see, e.g., [1]) for which our method applies.

Set $U(x) = e^{-iKr} u(x)$, where u satisfies (2.1). It may be proved that

$$|D^\alpha U(x)| \le \frac{C}{r^{|\alpha|+1}} \text{ for r sufficiently large,} \tag{2.3}$$

where $D^\alpha U$ denotes an arbitrary derivative of U of order $|\alpha|$. In view of (2.3), we shall approximately solve for U and $U_R = e^{-iKr}u_R(x)$, where u_R satisfies (2.2). Let $H_R^E = H^1(\Omega_R)$ denote the space of complex-valued functions whose derivatives of order less than two are square-integrable over Ω_R. It follows from integration by parts that U_R satisfies

$$A_R(U_R,V) = (F,V) = \int_{\Omega_R} F(x)\overline{V(x)}dx \text{ for each } V \text{ in } H_R^E, \qquad (2.4)$$

where

$$A_R(U_R,V) = a_R(u_R,v) = \int_{\Omega_R} (\nabla u_R \cdot \nabla\bar{v} - K^2 u_R\bar{v})\,dx$$

$$+ \oint_{\Gamma_R} (\frac{1}{R} - iK)u_R\bar{v}ds,$$

$$F(x) = e^{iKr}f(x) \text{ and } v(x) = e^{-iKr}V(x).$$

We shall approximately solve (2.4) using the finite element method. For a detailed discussion of the finite element method, see [2]. For simplicity we assume here that our one-parameter family of finite dimensional spaces, $S^h \subset H_R^E$, are defined for $h\in(0,\infty)$ as follows. Suppose that we have a partition of Ω_R into simple subsets (elements), denoted by t^h, with diameter of order $0(h)$. S^h then consists of all continuous complex-valued functions v^h such that $v^h|_{t^h} \in P_\ell$, where P_ℓ denotes the set of polynomials of degree less than ℓ for some integer ℓ greater than one. It may be readily seen that $\dim(S^h) = 0(R^3h^{-3})$. Since R is large, the number of linear equations will be large.

We next construct a new family of spaces $\tilde{S}^h$, defined for $h\in(0,1]$, satisfying

$$\dim(\tilde{S}^h) \le Ch^{-3}, \qquad (2.5)$$

where C is independent of R (as well as h). Set $S_j = \{x: 2^{j-1} < |x| \le 2^j\}$ and $\Omega_j^R = S_j \cap \Omega_R$, $j = 1,\ldots,J_R$, with

$J_R \approx \log_2 R$. Set $\Omega_0^R = \Omega_R - \bigcup_{j=1}^{J_R} \Omega_j^R$ and assume for simplicity that Ω and supp(f) are contained in the unit sphere and (2.3) holds outside of this sphere. We now construct a new partition of Ω_R into elements, t^{h_j}, with diameter of order $O(h_j)$ in each annular region Ω_j^R. We define the increasing sequence of positive numbers, h_j, by

$$h_0 = h \text{ and } h_j = 2^{\ell' j} h \text{ with } 1 < \ell' < \frac{\ell - \frac{1}{2}}{\ell - 1},\ j = 1, \ldots, J_R. \qquad (2.6)$$

We now define $\tilde{S}^h$ to consist of those continuous functions v^h such that $v^h|_{t^{h_j}} \in P_\ell$. The motivation for this construction and the proof of (2.5) are based on approximation theory and (2.3) and are given in [3].

III. ERROR ESTIMATES

Our main theoretical results establish the existence and uniqueness of the finite element approximation, U_R^h, satisfying

$$A_R(U_R^h, V^h) = (F, V^h) \text{ for each } V^h \text{ in } \tilde{S}^h, \qquad (3.1)$$

as well as error estimates for $U - U_R^h$. These results are easier to prove when $K = 0$ since the bilinear form, $A_R(\ ,\ )$, is non-negative definite. A detailed treatment of this case is given in [3]. When $K \neq 0$, the proofs are considerably more difficult and are given in [4] (where the case of variable coefficients and other extensions of Theorem 3.1 below are treated). Here we merely state some of our main results.

<u>Theorem 3.1</u>: Suppose that B is a fixed subset of Ω^C, $h \in (0,1]$ and R is sufficiently large. Then the following results hold.

(a) There exist positive constants, γ_1 and γ_2, such that if $KR \leq \gamma_1$ and $h_{J_R} \leq \gamma_2 R$, there exists a unique solution, U_R^h, of (3.1). Furthermore, the following estimates hold:

$$(\int_{\Omega_R} |\nabla(U-U_R^h)|^2 dx)^{\frac{1}{2}} \leq C(R^{-2} + h^{\ell-1}) \tag{3.2}$$

and

$$(\int_{\Omega_R} |U-U_R^h|^2 dx)^{\frac{1}{2}} \leq C(R^{-2} + h^{\ell} + (R^{-\frac{3}{2}} + h^{\ell-1})^2), \tag{3.3}$$

where C is independent of h and R.

(b) There are constants $C_{K,R}$ and Λ such that there exists a unique solution, U_R^h, of (3.1) provided

$$C_{K,R} K h_{J_R} \leq \Lambda. \tag{3.4}$$

Furthermore, estimates (3.2) and (3.3) hold in this case.

Observe that Theorem 3.1(b) implies that the largest mesh size h_{J_R} must satisfy the additional stability constraint, (3.4), when KR is not sufficiently small. The constant $C_{K,R}$ becomes large as $KR \to \infty$. Hence the number of equations must increase as the frequency K increases.

IV. NUMERICAL RESULTS

In collaboration with Alvin Bayliss, the author has begun a numerical investigation of the mesh grading procedure. We shall demonstrate typical results with respect to an axially symmetric problem analogous to (2.1) with $\Omega = \{x: |x| \leq .5\}$, $f = 0$ and the boundary condition on $\partial\Omega$ given by $\frac{\partial u}{\partial n} = g$. The data g corresponds to the solution, $u(x) = \frac{e^{iK|x-x_s|}}{4\pi|x-x_s|}$, where x_s is a fixed axial point such that $|x_s| = .4$. Introducing the artificial boundary Γ_R as before and employing spherical polar coordinates and axial symmetry, we consider the computational domain, $\Omega_R = \{(r,\theta): .5 \leq r \leq R,\ 0 \leq \theta \leq \pi\}$, for problem (2.2). We denote the last boundary condition in (2.2) by $B_1 u_R = 0$ on Γ_R and also consider the boundary condition

$$B_2 u_R = \frac{\partial^2 u_R}{\partial r^2} + (\frac{4}{r} - 2iK)\frac{\partial u_R}{\partial r} + (\frac{3}{r} - 4ik)\frac{u_R}{r} - K^2 u_R = 0 \text{ on } \Gamma_R.$$

The boundary operators, B_1 and B_2, correspond to the first two in a hierachy of approximate boundary operators developed in [5].

We employ continuous piecewise linear finite elements on a triangular partition in (r,θ) coordinates to obtain an approximate solution using both a uniform and a graded mesh. In the following computations, we keep the grid evenly spaced in the θ-direction with 41 grid points and use 6 grid points in the r-direction with $R = 1.125$. When employing a uniform mesh we use the grid points $r_j = r_0 + .125j$, $j = 0,\ldots,5$, in the r-direction with $r_0 = .5$. When mesh grading is employed, we factor out e^{iKr} as described in Section 2 and grade the mesh in the r-direction in accordance with $r_1 = r_0 + .047$ and $r_j = r_{j-1} + 5r_{j-1}^{\frac{3}{2}}$, $j = 2,\ldots,5$. Only a few changes were necessary to modify the computer program so as to incorporate the mesh grading. We measure the error E_j, defined by $E_j = \frac{\|u-u_R^h\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\partial\Omega)}}$, corresponding to the boundary operator B_j, $j=1,2$, where $u_R^h = e^{iKr}U_R^h$ is computed as described in Sections 2 and 3. In the following table, we compare the errors, E_j, obtained using the graded mesh with those obtained using a uniform mesh for various values of K.

Table I.

K	E_1(%)		E_2(%)	
	Graded	Uniform	Graded	Uniform
0	2.39	3.74	1.17	3.5
3	4.26	6.08	1.33	4.03
6	4.62	7.37	2.4	5.68
9	5.9	8.1	8.17	8.8
12	8.4	21.4	9.4	17.2

Observe the substantial improvements due to mesh grading using both B_1 and B_2. These results are typical of our numerical experiments. Note that as the frequency K increases, the error increases. As we indicated in Section 3, this is to be expected since we are keeping the number of equations fixed. We also observe that the main limitation on the number of equations in our computations is the storage requirements of the banded Gauss solver. We are attempting to circumvent this difficulty by using iterative methods. A more complete discussion of our numerical results will appear elsewhere, [6].

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