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REMARKS ON THE GENERALIZED TUKEY'S LAMBDA FAMILY OF DISTRIBUTIONS

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1. INTRODUCTION

We consider the family of curves generated by the mapping of a uniform density, namely

$$F^{-1}(\lambda) = \alpha + \beta[\lambda^{\gamma\delta} - (1-\lambda)^{\gamma}]/\gamma \quad , \quad 0 < \lambda < 1, \quad (1)$$

these being a reparametrization of the set introduced by Ramberg *et al* (1979), and a generalization of a mapping due to Tukey (1960).

Our main concern is the application of (1) to approximating theoretical distribution functions of test statistics such as S.D., skewness, and kurtosis under non-normality.

It is as well to remark at the outset that if (1) is fitted by using the first four moments, there may be more than one solution. This problem does not arise if we merely define the values of α, β, γ, δ and use the mapping to generate random samples.

Again, moments are straight forward to evaluate in terms of gamma functions, or polygamma functions in special cases. Thus when $\gamma = 0$, the m.g.f. is

$$M_X(t) = \frac{\Gamma(1+t\delta)\Gamma(1-t)}{\Gamma(t\delta-t+2)} \quad , \quad -\frac{1}{\delta} < t < 1 \quad , \quad (2)$$

and when $\gamma \neq 0$, the k^{th} non-central moments is

$$\mu'_k = \gamma^{-k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \frac{\Gamma(\delta\gamma k+1)\Gamma[\gamma(k-j)+1]}{\Gamma[\gamma\delta k+\gamma(k-j)+2]} \quad , (3)$$

$$\gamma\delta k+1 > 0 \quad , \quad \gamma(k-j)+1 > 0$$

2. SKEWNESS AND KURTOSIS

Let $\sqrt{\beta_1} = \mu_3/\sigma^3$, and $\beta_2 = \mu_4/\sigma^4$, where $\mu_k = E[(x-\mu'_1)^k]$ is the k^{th} central moment and $\sigma = \sqrt{\mu_2}$. $\sqrt{\beta_1}$ and β_2 are often discribed as measuring skewness and kurtosis. For the family, $\sqrt{\beta_1}$ and β_2 can be obtained from (2) or (3). It is clear that $\sqrt{\beta_1}$ and β_2 are functions of the shape parameters γ and δ, i.e.

$$\begin{aligned} \sqrt{\beta_1} &= \sqrt{\beta_1}(\gamma,\delta) \; , \\ \beta_2 &= \beta_2(\gamma,\delta) \; . \end{aligned} \quad (4)$$

For given values of γ and δ, $\sqrt{\beta_1}$ and β_2 can be easily obtained by using digital computer. Fig. 1 illustrates the contours of $\sqrt{\beta_1}$ and β_2 in the (γ,δ) plane.

In general it is impossible to obtain the inverse expression of (4) explicitly, i.e.

$$\gamma = \gamma(\sqrt{\beta_1}, \beta_2) \quad ,$$
$$\delta = \delta(\sqrt{\beta_1}, \beta_2) \quad .$$

A satisfactory numerical method is the finite difference Levenberg-Marquardt algorithm. Suppose $\sqrt{\beta_1}^*$ and β_2^* are two specified values of $\sqrt{\beta_1}$ and β_2 of the distribution to be fitted. We define

$$g_1(\gamma,\delta) = \sqrt{\beta_1}(\gamma,\delta) - \sqrt{\beta_1}^* \quad ,$$
$$g_2(\gamma,\delta) = \beta_2(\gamma,\delta) - \beta_2^* \quad ,$$

then the Levenberg-Marquardt algorithm can be applied to obtain the values of γ and δ which will minimize the sum of squares of g_1 and g_2. If this sum of squares is satisfactory close to zero, the γ, δ value just obtained can be considered as the approximate solution. Like most numerical algorithms, initial guesses of γ and δ are required and important; reasonable initial guesses will save considerable computer time. Figure 1 may serve as a guide in this respect.

3. APPROXIMATIONS TO DISTRIBUTIONS

A standard technique to fit a curve is by making the first four moments of the fitted curve agree with those of the distribution to be fitted (see Shenton and Bowman, (1979)). Pearson, Johnson and Burr (1978) investigated the degree of agreement of the percentage points of distributions with the same first four moments. Supplementing the percentage points of the generalized lambda family to their table I gives the following comparison of standardized 5 and 1 percentage points of six distinct families having $\sqrt{\beta_1}$ and β_2 values close to 0.8 and 4.2 respectively. There is satisfactory agreement between the six approximations.

In the rest of this section, we will fit the generalized family to the distribution of test statistics such as standard deviation, $\sqrt{b_1}$, b_2. Exact distribution, in some cases the exact

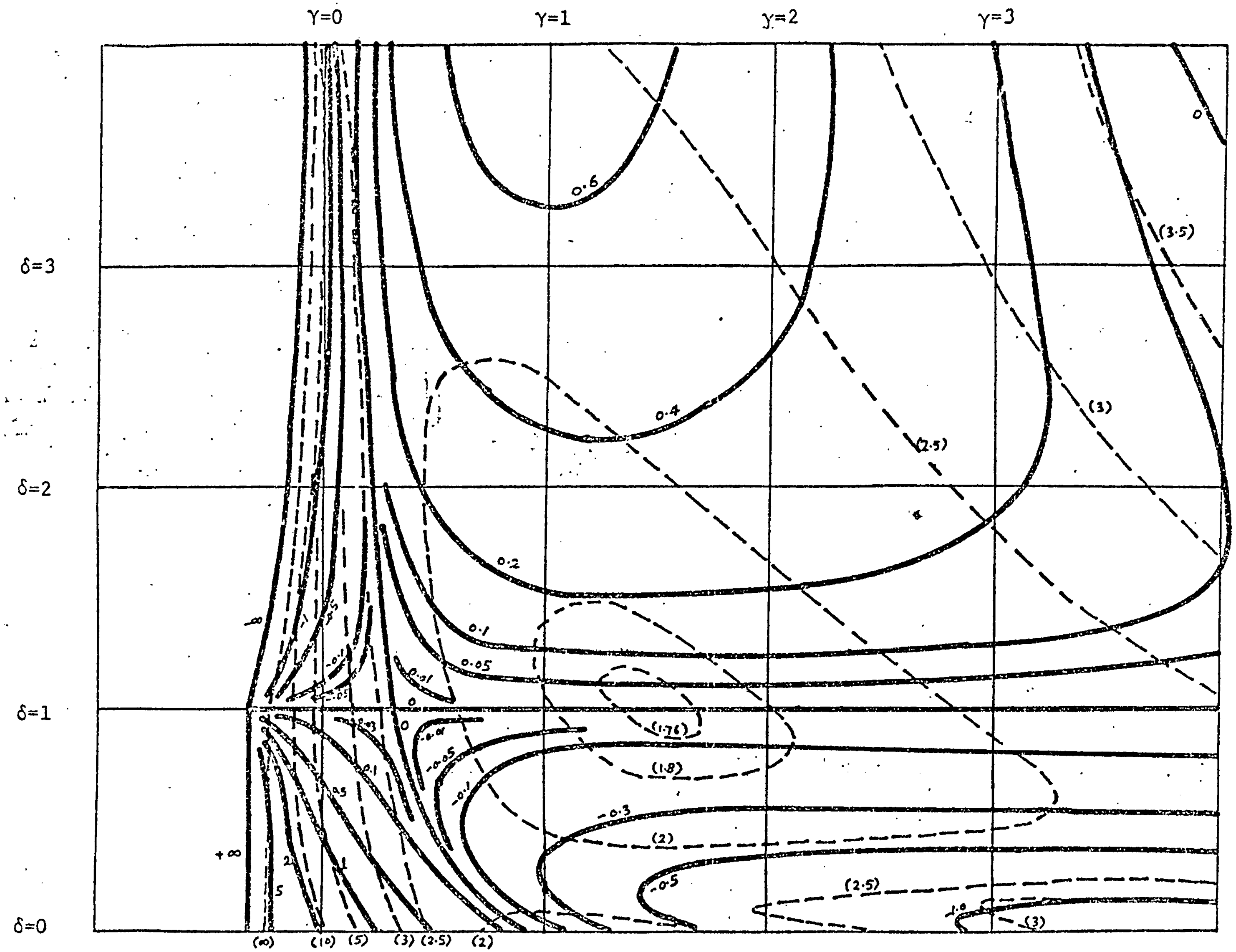

Fig. 1

Contours of $\sqrt{\beta_1}$ and β_2

$\sqrt{\beta_1}$

(β_2)

expression of the moments also, of these statistics in most cases are still unknown. However, we can approximate its moments, and thus using four of these, approximate its percentage points. The method of using moments of statistics which themselves are moment functions is described in detail in Shenton, Bowman and Reinfelds (1967), and Shenton, Bowman and Sheehan (1971).

Table 1. Comparisons of Percentile of Families having close Values of $\sqrt{\beta_1}$ and β_2.

Family	$\sqrt{\beta_1}$	β_2	1	5	95	99
$L(\gamma,\delta,\alpha,\beta)$	.800	4.200	-1.87	-1.38	1.85	2.95
Pearson Type	.800	4.200	-1.80	-1.40	1.82	2.90
Johnson S_U	.800	4.200	-1.80	-1.40	1.82	2.91
Non-central t	.780	4.229	-1.83	-1.42	1.81	2.89
Log-normal	.814	4.200	-1.78	-1.40	1.83	2.91
Log-χ^2	.780	4.188	-1.83	-1.41	1.81	1.90

Note: For the generalized lambda distribution, γ=.071578, δ=.35540, α=-.51833, β=.88360 No log-normal or log-χ^2 distributions have $\sqrt{\beta_1}$=.8, β_2=4.2 exactly and this beta-point is outside the non-central t area.

Denote the member of the generalized family by $L(\gamma,\delta,\alpha,\beta)$. For specified values of μ (mean), σ^2 (variance), $\sqrt{\beta_1}$ and β_2 , the values of γ and δ , say γ_0 and δ_0 are obtained by employing the finite different Levenberg-Marquardt algorithm. Denote the mean and variance of $L(\gamma_0,\delta_0,0,1)$ by u and u_2 , then $L(\gamma_0,\delta_0,\alpha_0,\beta_0)$, where $\beta_0 = \sqrt{(\sigma^2/u_2)}$ and $\alpha_0 = \mu - u\beta_0$, will have the specified moments μ , σ^2 , $\sqrt{\beta_1}$ and β_2 .

3.1 <u>Sample standard deviation ($\sqrt{m_2}$) in exponential sampling</u>

Lam(1978) derived the exact distribution of $\sqrt{m_2}$ in exponential sampling for $n \leq 4$. The exact distributions for $n \geq 5$ are still unknown. To fit the distribution, we need the first 4 moments of the statistic. Lam (1978) also gave the approach of obtaining the exact expression of the general moment of m_2 for all sample size. Hence exact value of the even moments of $\sqrt{m_2}$ are known. For the first two odd moments, we apply the Levin's algorithm to the first 28 and 13 terms of the series expansion in powers of n^{-1} of μ and μ_3 (see Bowman, Lam and Shenton (1979)). Table 2 shows the numerical values of the first four moment paraments. Distribution fitting and comparisons are given in Table 3.

Table 2. Moments of $\sqrt{m_2}$ in Exponential Sampling

n	μ	σ^2	$\sqrt{\beta_1}$	β_2
5	.77451	.20013	1.42955	6.27480
6	.80717	.18181	1.33699	5.95404
7	.83120	.16625	1.26517	5.70555
8	.84969	.15302	1.20691	5.50615
9	.86440	.14170	1.15815	5.34177
10	.87640	.13192	1.11641	5.20335
15	.91388	.09815	.96954	4.73839
20	.93367	.07826	.87632	4.46412
25	.94597	.06514	.80921	4.27772

Table 3. Percentage Point Comparison for fitting $\sqrt{m_2}$ in Exponential sampling

n		1%	5%	10%	90%	95%	99%	
5	L	.119	.249	.320	1.364	1.640	2.258	γ=.02216 δ=.17014
	M	.138	.235	.305	1.356	1.619	2.250	α=.43815 β=.41587
6		.150	.291	.366	1.369	1.626	2.200	.02393 .20367
		.179	.280	.351	1.359	1.608	2.178	.50641 .38860
7		.181	.327	.404	1.368	1.610	2.147	.02660 .22776
		.208	.316	.389	1.371	1.604	2.124	.55666 .36718
8		.209	.358	.436	1.365	1.593	2.100	.02951 .24634
		.237	.352	.423	1.366	1.583	2.086	.59569 .34950
9		.236	.386	.463	1.360	1.577	2.057	.03242 .26139
		.268	.376	.450	1.366	1.582	2.050	.62714 .33445
10		.260	.410	.486	1.355	1.562	2.018	.03523 .27399
		.294	.402	.471	1.360	1.557	2.006	.65317 .32138
15		.353	.497	.570	1.327	1.499	1.872	.04726 .31721
		.379	.490	.558	1.330	1.495	1.864	.73775 .27421
20		.418	.554	.622	1.303	1.453	1.773	.05653 .34456
		.442	.545	.611	1.301	1.443	1.750	.78537 .24370

L: $L(\gamma,\delta,\alpha,\beta)$
M: Monte-Carlo Simulation, 100000 cycles for n=15 and n=20; 50000 cycles for $n \leq 10$.

3.2 $\sqrt{b_1}$ in Normal Sampling

$\sqrt{b_1} = m_3/m_2^{3/2}$, where m_r is the r^{th} sample central moment, is used in testing for normality. The only exact distribution for $\sqrt{b_1}$ in normal sampling being known is the case n=4 due to Mckay (1933). Exact values of the first 3 even moments— the odd moments are, of course, zero— of normal $\sqrt{b_1}$ have been determined by R. A. Fisher (1930). Let n be the sample size, then

$$\mu_2 = \frac{6(n-2)}{(n+1)(n+3)}$$

$$\mu_4 = \frac{108(n-2)(n^2+27n-70)}{(n+1)(n+3)(n+5)(n+7)(n+9)}$$

Table 4 and Table 5 give the $L(\gamma,\delta,\alpha,\beta)$ fit to the distribution of $\sqrt{b_1}$ along with alternative model comparisons (see Shenton, Bowman and Lam (1979)).

Table 4. Percentage Points of Normal $\sqrt{b_1}$, n=4

	.900	.950	.975	.990	.995	.999
L	.792	.957	1.073	1.178	1.234	1.314
M	.831	.987	1.070	1.120	1.137	1.151
P	.793	.958	1.074	1.178	1.231	1.306
P_c	.829	.985	1.069	1.124	1.143	1.160
S_B	.793	.956	1.072	1.180	1.238	1.328
S_{Bc}	.832	.985	1.066	1.122	1.144	1.168

Note: L : $L(\gamma=.40738,\ \delta=1,\ \alpha=0,\ \beta=.56967)$
M : Mulholland (1977)
P : 4-moment Pearson
P_c: Conditional 4-moment Pearson
S_B, S_{Bc} : Johnson 4-moment & 4-moment conditional

Table 5. Percentage Points of Normal $\sqrt{b_1}$, $5 \leq n \leq 25$

n	$\mu_2(\sqrt{b_1})$	$\beta_2(\sqrt{b_1})$		.900	.950	.975	.990	.995	.999	γ	β
5	.3750	2.5714	L	.806	1.010	1.174	1.346	1.451	1.635	.23937	.48400
			M	.821	1.049	1.207	1.337	1.396	1.464		
6	.3810	2.8196		.799	1.019	1.204	1.411	1.544	1.798	.17202	.44488
				.795	1.042	1.239	1.429	1.531	1.671		
7	.3750	3.0000		.784	1.010	1.205	1.431	1.582	1.880	.13491	.41842
				.782	1.018	1.230	1.457	1.589	1.797		
8	.3636	3.1357		.766	.993	1.193	1.429	1.590	1.916	.11154	.39795
				.765	.998	1.208	1.452	1.605	1.866		
9	.3500	3.2398		.748	.973	1.175	1.416	1.582	1.926	.09570	.38113
				.746	.977	1.184	1.433	1.598	1.898		
10	.3357	3.3204		.729	.952	1.153	1.396	1.565	1.919	.08447	.36683
				.728	.954	1.159	1.407	1.578	1.906		
11	.3214	3.3833		.711	.931	1.131	1.373	1.543	1.903	.07628	.35440
				.710	.931	1.134	1.381	1.553	1.899		
12	.3077	3.4326		.694	.910	1.107	1.348	1.518	1.880	.07017	.34343
				.693	.910	1.109	1.353	1.526	1.882		
13	.2946	3.4711		.678	.890	1.085	1.323	1.492	1.853	.06558	.33363
				.677	.890	1.085	1.325	1.497	1.859		
14	.2824	3.5011		.663	.871	1.063	1.298	1.465	1.824	.06209	.32479
				.662	.870	1.061	1.298	1.468	1.832		
15	.2708	3.5245		.648	.853	1.041	1.273	1.438	1.794	.05944	.31675
				.648	.851	1.039	1.272	1.440	1.803		
16	.2601	3.5424		.635	.836	1.021	1.249	1.412	1.764	.05745	.30939
				.635	.834	1.018	1.247	1.412	1.773		
17	.2500	3.5559		.622	.819	1.001	1.226	1.386	1.733	.05596	.30262
				.622	.817	.997	1.222	1.385	1.744		
18	.2406	3.5659		.610	.804	.982	1.203	1.361	1.704	.05487	.29636
				.610	.801	.978	1.199	1.359	1.714		
19	.2318	3.5730		.598	.789	.964	1.181	1.337	1.674	.05410	.29054
				.599	.786	.960	1.176	1.334	1.685		
20	.2236	3.5778		.588	.774	.947	1.161	1.314	1.646	.05359	.28511
				.588	.772	.942	1.155	1.310	1.657		
21	.2159	3.5806		.577	.761	.931	1.141	1.291	1.618	.05329	.28003
				.578	.758	.925	1.134	1.287	1.628		
22	.2087	3.5818		.568	.748	.915	1.122	1.270	1.591	.05316	.27525
				.568	.746	.909	1.114	1.265	1.602		
23	.2019	3.5816		.558	.736	.900	1.103	1.249	1.565	.05317	.27076
				.559	.733	.894	1.096	1.243	1.575		
24	.1956	3.5804		.549	.724	.886	1.086	1.229	1.540	.05330	.26651
				.550	.722	.880	1.078	1.223	1.550		
25	.1896	3.5783		.541	.713	.872	1.069	1.210	1.515	.05353	.26249
				.542	.710	.866	1.060	1.203	1.526		

Note: L : L(γ, δ=1, α=0, β)
M : Mulholland (1977)

3.3 $\sqrt{b_1}$ and b_2 from Normal Mixture

Table 6. Percentage Points of $\sqrt{b_1}$ from N.M.

n	Population moment	Moment of $\sqrt{b_1}$		.010	.050	.100	.900	.950	.990	$L(\gamma,\delta,\alpha,\beta)$
50	$\sqrt{\beta_1}=0.1$	μ = .1027	L	-.536	-.340	-.239	.449	.555	.761	γ=.10540
		μ_2= .0739	C	-.532	-.340	-.241	.449	.552	.757	δ=.91729
	$\beta_2=1.2$	$\sqrt{\beta_1}$= .051	Z	-.532	-.340	-.241	.449	.552	.757	α=.09013
		β_2=3.205	M	-.534	-.339	-.240	.449	.551	.755	β=.18421
10	0.1	.0858		-1.109	-.728	-.538	.714	.910	1.304	.07945
		.2487		-1.132	-.744	-.533	.737	.952	1.299	.95271
	2.2	.034		-1.000	-.726	-.540	.714	.906	1.294	.07277
		3.374		-1.107	-.728	-.541	.711	.903	1.307	.31995
25	0.4	.4022		-.363	-.129	-.011	.836	.979	1.265	.09185
		.1139		-.354	-.135	-.019	.830	.972	1.259	.73205
	2.0	.214		-.355	-.131	-.016	.834	.973	1.258	.34581
		3.390		-.354	-.134	-.019	.834	.970	1.255	.24523
100	0.6	.5576		-.164	.055	.168	.950	1.066	1.289	.11714
		.0943		-.159	.053	.165	.948	1.060	1.286	.95425
	4.0	.025		-.160	.055	.167	.949	1.064	1.285	.54993
		3.118		-.158	.055	.166	.946	1.062	1.280	.20832
75	1.0	.9979		.518	.658	.729	1.286	1.381	1.565	.12298
		.0485		.524	.654	.725	1.283	1.375	1.561	.61403
	3.2	.287		.524	.654	.725	1.284	1.376	1.561	.94111
		3.263		.525	.653	.723	1.283	1.376	1.559	.17772
100	1.2	1.1909		.753	.880	.945	1.455	1.542	1.710	.12473
		.0406		.758	.876	.941	1.452	1.536	1.707	.60743
	4.0	.291		.759	.877	.941	1.452	1.537	1.707	1.1378
		3.257		.758	.876	.941	1.451	1.534	1.697	.16357
150	1.2	1.1945		.834	.938	.992	1.410	1.479	1.610	.13541
		.0270		.838	.936	.990	1.408	1.475	1.608	.64047
	4.0	.241		.838	.936	.989	1.408	1.475	1.608	1.15580
		3.167		.837	.934	.988	1.408	1.472	1.606	.13282

Note: L : $L(\gamma,\delta,\alpha,\beta)$; P : Pearson 4-moment approximant
C : Cornish-Fisher expansion with S_U as kernel (Shenton and Bowman (1975))
Z : Zeroth-order S_U approximant
M : Monte-Carlo approximant, 100000 cycles

Table 7. Percentage Points of b_2 from N.M.

n	Population moment	Moment of b_2		.010	.050	.100	.900	.950	.990	$L(\gamma,\delta,\alpha,\beta)$
50	$\sqrt{\beta_1}=0.6$	μ =2.6162	L	1.690	1.928	2.045	3.315	3.641	4.408	γ=-.02443
		μ_2= .3004	C	1.737	1.922	2.030	3.315	3.642	4.399	δ= .30808
		$\sqrt{\beta_1}$=1.341	Z	1.737	1.922	2.030	3.318	3.635	4.387	α=2.30819
	$\beta_2=2.6$	β_2=6.707	M	1.754	1.916	2.018	3.316	3.636	4.393	β= .43101
			P	1.755	1.921	2.026	3.320	3.635	4.379	
50		3.9062		2.070	2.565	2.802	5.248	5.895	7.478	-.05817
	0.8	1.1573		2.140	2.557	2.780	5.273	5.889	7.447	.34031
		1.473		2.140	2.557	2.780	5.254	5.889	7.447	3.34681
	4.2	7.933		2.276	2.584	2.775	5.274	5.913	7.488	.78282
				2.209	2.556	2.765	5.264	5.890	7.417	
100		3.9661		2.640	2.998	3.174	4.907	5.309	6.206	.01790
	1.0	.5255		2.692	2.884	3.156	4.902	5.296	6.184	.35899
		1.005		2.692	2.984	3.156	4.906	5.295	6.175	3.59648
	4.0	5.135		2.704	2.977	3.141	4.903	5.287	6.148	.59072
				2.705	2.982	3.147	4.907	5.292	6.167	
150		2.6892		1.627	1.906	2.045	3.457	3.778	4.480	.03813
	1.2	.3428		1.673	1.873	2.032	3.454	3.765	4.459	.33753
		.964		1.673	1.894	2.026	3.455	3.765	4.455	2.37495
		4.818		1.673	1.884	2.013	3.429	3.740	4.439	.49878
	2.6			1.679	1.893	2.024	3.456	3.764	4.452	
150		4.0306		2.762	3.107	3.280	4.899	5.236	5.937	.07424
	1.2	.4327		2.755	3.074	3.258	4.892	5.217	5.919	.39122
		.724		2.802	3.091	3.259	4.894	5.218	5.913	3.71684
	4.0	4.056		2.809	3.084	3.252	4.886	5.209	5.900	.56973
				2.805	3.090	3.257	4.894	5.217	5.911	
150		3.5403		2.009	2.414	2.615	4.648	5.128	6.202	.01519
	1.5	.7283		2.075	2.340	2.600	4.646	5.113	6.168	.34008
		1.059		2.075	2.399	2.588	4.648	5.112	6.166	3.08696
	3.4	5.296		2.045	2.400	2.589	4.614	5.133	6.190	.70099
				2.090	2.397	2.584	4.650	5.110	6.157	

3.4 $\sqrt{b_1}$ from Pearson Type I

Table 8. Percentage Points of $\sqrt{b_1}$ in Pearson Type I Sampling

n	Population moment	Moments of $\sqrt{b_1}$ *		.005	.010	.050	.950	.990	.995	$L(\gamma,\delta,\alpha,\beta)$
100	$\sqrt{\beta_1}$=.860663	μ = .84	L	.370	.416	.539	1.168	1.316	1.370	γ=.13768
		μ_2= .0365	$M^{(1)}$	.375	.418	.535	1.161	1.308	1.369	δ=.68805
	β_2=3.095	$\sqrt{\beta_1}$= .1953								α=.80217
		β_2=3.124	C-F	.378	.419	.536	1.163	1.313	1.371	β=.15104
100	.467707	.4558	L	.017	.060	.176	.750	.879	.925	.13671
		.0303	$M^{(2)}$	.019	.059	.172	.744	.875	.925	.80055
	2.625	.1108								.43524
		3.071	C-F	.022	.063	.175	.747	.876	.925	.13002

Note: L : $L(\gamma,\delta,\alpha,\beta)$

$M^{(1)}$: Monte-Carlo approximant, mean of 3 runs, 50000 cycles each.

$M^{(2)}$: Monte-Carlo approximant, mean of 3 runs, 30000 cycles each.

C-F : Cornish-Fisher approximant (Bowman and Shenton (1973)).

* : Moments by Cornish-Fisher approximant.

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