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NUCLEAR DIVISION

UNION CARBIDE

MASTER

K/OA-4748

ON GAS FLOW IN A CENTRIFUGE

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June 1980

Sponsor: J. T. Bradbury

OPERATED BY
UNION CARBIDE CORPORATION
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

ABSTRACT

The axisymmetric, linearized gas dynamic equations for flow in a centrifuge are rederived from Perturbation Theory viewpoint using limit processes. A commonly accepted boundary condition is carefully reconsidered.

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SUMMARY

The mathematical relationship between Onsager's pancake equation and the Ekman boundary layer equations is derived by methods of perturbation theory. The so-called "internal flow region" is shown to be simply the first term in an asymptotic series for the outer expansion. The complementary inner expansion is shown to be the Ekman layer. Some of the more subtle approximations and assumptions on boundary conditions are given attention herein.

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NOMENCLATURE

a	= Bowl radius
A^2	$= \Omega^2 a^2 / 2RT_o$
B	$= R_e S^{1/2} / 4A^6$
$c_1,\ c_2,\ c_3$	= Integration constants
$F(x,y)$	= Non homogeneity
h	$= \theta + 2(S-1)\omega$
k	= Thermal conductivity
ℓ_c	= Length scale of corner region
M	= Coordinate perturbation parameter
P	= Axial mass flow
p	= Perturbation pressure
P_r	= Prandtl number
Q	= Dimensionless parameter
r	= Local radius
R_e	= Reynolds number
S	$= 1 + \frac{\gamma - 1}{2\gamma} P_r A^2$
T_o	= Reference temperature
u, ω, w	= Perturbation velocity components
U^0	$= \varepsilon^{1/2} u^0$
x, y	= Cartesian coordinates
$\tilde{x}$	= Transformed coordinate
Y	= Inner variable
Y_2	$= (R_e/\varepsilon)^{1/2}$
γ	= Specific heat ratio

NOMENCLATURE
(Continued)

$\Delta(\ )$	=	$\frac{1}{\eta}\eta(\ )_{\eta}{}_{\eta} + (\ )_{yy}$
$\Delta'(\ )$	=	$4A^4(\ )_{xx} + (\ )_{yy}$
$\delta\chi$	=	First order variation in χ
δ_1	=	Gauge function
ε	=	$(4A^4)^{-1}$
η	=	r/a
θ	=	Perturbation temperature
Ξ	=	Streamfunction
ξ	=	εY
γ	=	$\frac{-R_e \frac{1}{2T_o}}{8A^6}\left(\frac{\partial T}{\partial y}\right)\Big\vert_{x=0}$
ρ_o	=	Wheel flow density, e^{-x}
ϕ	=	$\theta - 2\omega$
χ	=	Master potential
Ω	=	Rotor spin speed

SUPERSCRIPTS

0	=	Order $(\varepsilon/R_e)^0$ expansion
1	=	Order $(\varepsilon/R_e)^1$
o	=	Outer expansion
i	=	Inner expansion

SUBSCRIPTS

x,y	=	Partial derivatives

INTRODUCTION

Basic theoretical elements of modern linear centrifuge fluid modeling are primarily due to Onsager, Carrier and Maslen and have been implemented and extended by others, e.g., Wood and Morton [1]. These models were developed separately and provide a complete description of the flow field which determines the isotope separation when matched together.

Onsager's pancake equation for centrifuge gas dynamics

$$(e^{x}(e^{x}\chi_{xx})_{xx})_{xx} + B^{2}\chi_{yy} + F(x,y) = 0 \tag{1}$$

was originally derived from a functional for the energy dissipation associated with the flow. In that derivation it was assumed that diffusive transport of heat and momentum in the axial direction is negligible resulting in an expression like (1) in η. The further assumption of negligible curvature led to equation (1). It is a sixth order, anisotropic linear elliptic partial differential equation in two dimensions and for cases of interest here $B^2 >> 1$.

In reference [1] Onsager's equation was derived directly from the linearized Navier-Stokes equations in cylindrical coordinates based on a classical boundary layer analysis (regular order of magnitude considerations). Methodologically this treatment is similar to the way Prandtl's boundary layer equations were originally derived [2]. The procedure is summarily described as away from the ends retaining among the viscous terms only those most highly differentiated in the radial direction and setting $\eta = 1$ where it appears algebraically [1]. The region away from the ends is termed the internal flow region.

Likewise, the Ekman layer equations were derived by order of magnitude procedures keeping only the largest terms. Because curvature effects were fully retained there is some interest in rederiving these equations without curvature and thus provide a clear and consistent picture. This is important because one would probably not want to keep curvature in one region and leave it out of the other.

Approximate equations for the entire flow field are described briefly in [1]. The present work is intended to describe lucidly a unified theory through a somewhat "rigorous" analysis of the basic equations. For example, we will address

(1) the mathematical relationship between the internal flow region and the Ekman layers (matching)

(2) the auxiliary equation for h

(3) corner boundary regions.

The approach taken here (adopted from Singular Perturbation Theory) is to develop asymptotic expansion of the field variables in a manner like the formal derivation of Prandtl's equations [3]. This is a formal mathematical derivation of the approximate equations and a statement of the approximation. Furthermore, it provides a uniform view of the internal flow region and the Ekman boundary layer and their relation. An approach such as this might also prove useful in the derivation of approximate three-dimensional flow equations when axisymmetry is inappropriate.

The continuum assumption is implicit in this study and Figure 1 schematically illustrates the problem geometry.

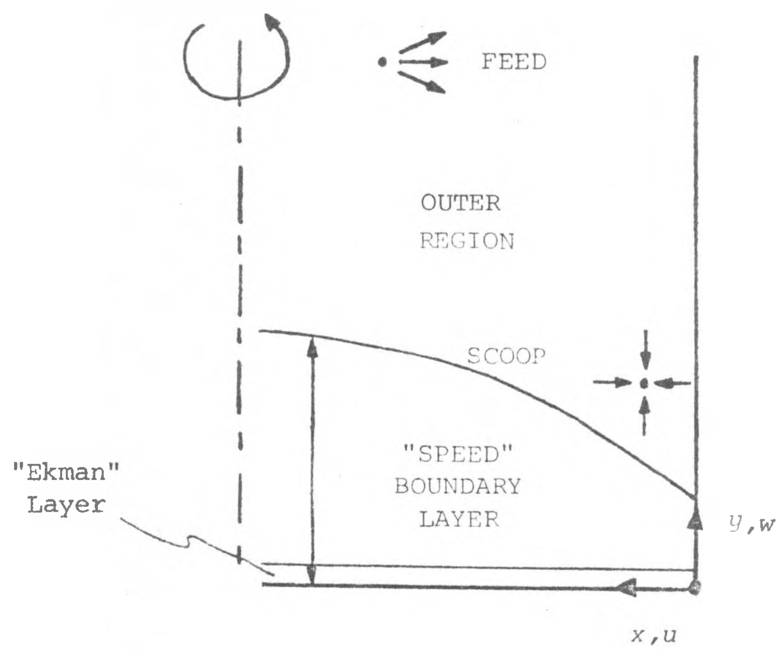

Figure 1

BASIC EQUATIONS

For small perturbations about rigid body rotation the Navier-Stokes equations for the rotating gas can be linearized thus eliminating convection. The simplified equation system in cylindrical coordinates for continuity, momentum and energy is developed in [1] and is included below for reference. In the absence of internal sources and sinks one simply imposes axisymmetry and the linearized ideal gas law to obtain (2) - (6) as the continuity, radial, tangential and axial momentum and energy equations.

$$(\eta\rho_o u)_\eta + \eta\rho_o w_y = 0 \tag{2}$$

$$-\eta\rho_o(\theta - 2\omega) + \eta p - \frac{1}{2A^2} p_\eta + \frac{1}{R_e}\left[\Delta u - \frac{u}{\eta^2} - \frac{2A^2}{3}(\eta u)_\eta\right] = 0 \tag{3}$$

$$-2\rho_o u + \frac{1}{R_e}\left[\Delta(\eta\omega) - \frac{\omega}{\eta}\right] = 0 \tag{4}$$

$$-p_y + \frac{2A^2}{R_e}\left[\Delta w - \frac{2A^2}{3}\eta u_y\right] = 0 \tag{5}$$

$$4R_e(S-1)\eta\rho_o u + \Delta\theta = 0 \tag{6}$$

This normalized system is characterized by the three-dimensionless parameters A, R_e and S. Because S is order 1 it does not play a major role in what follows. Both $R_e >> 1$ and $A^4 >> 1$ in the parameter range of interest, hence this analysis will entail multiple limits. Assume $R_e >> A^4$.

We will immediately neglect curvature and subsequently deal with first order boundary layer equations. The Laplacian in cylindrical two-space is

$$\Delta(\ ) = \frac{1}{\eta}\,(\eta(\ )_\eta)_\eta + (\ )_{yy}\ . \tag{7}$$

Let us define x as

$$A^2(1 - \eta^2) \tag{8}$$

Then the Laplacian becomes

$$\Delta(\ ) = 4A^4 \left[\left(1 - \frac{x}{A^2}\right)(\ )_x\right]_x + (\ )_{yy}\ . \tag{9}$$

Neglecting curvature or unfolding

$$\Delta(\ ) \rightarrow \Delta'(\ ) \equiv 4A^4(\ )_{xx} + (\ )_{yy} \tag{10a}$$

and

$$\Delta(\eta\omega) \rightarrow \Delta'\omega + \omega \tag{10b}$$

where $\eta \rightarrow 1$ or $1 - \frac{x}{A^2} \rightarrow 1$. Then the unfolded equations are:

$$-\ 2A^2(\rho_o u)_x + \rho_o w_y = 0 \tag{11}$$

$$\rho_o\phi = p + p_x + \frac{1}{R_e}\left(\Delta' u - u - \frac{4}{3}A^4 u_x\right) \tag{12}$$

$$\frac{1}{R_e}\,\Delta'\phi = -\ 4S\rho_o u \tag{13}$$

$$p_y = \frac{2A^2}{R_e}\left(\Delta' w - \frac{2}{3}A^2 u_y\right) \tag{14}$$

$$\Delta' h = 0 \tag{15}$$

In what follows we will tacitly retain the continuity equation and the pressure terms as they appear because a streamfunction will be introduced later that does away with (11) and pressure will be eliminated by cross differentiation.

Defining the small parameter ε as

$$\frac{1}{4A^4} \tag{16}$$

then (11) - (15) become

$$-(\rho_o u)_x + \varepsilon^{1/2}\rho_o w_y = 0 \tag{17}$$

$$\varepsilon\rho_o\phi = \varepsilon\left(p + p_x\right) + \frac{1}{R_e}\left(u_{xx} + \varepsilon u_{yy} - \varepsilon u - \frac{1}{3}u_x\right) \tag{18}$$

$$\frac{1}{R_e}\left(\phi_{xx} + \varepsilon\phi_{yy}\right) = -4S\rho_o u\varepsilon \tag{19}$$

$$\varepsilon p_y = \frac{2A^2}{R_e}\left(w_{xx} + \varepsilon w_{yy} - \frac{1}{3}\varepsilon^{1/2}u_y\right) \tag{20}$$

$$h_{xx} + \varepsilon h_{yy} = 0 \tag{21}$$

There are numerous boundary layers present in both x and y that interplay here, e.g., $1/R_e^{1/2}$, $\varepsilon^{1/2}$ and $\left(\varepsilon/R_e\right)^{1/2}$. One may view this as a system of linear perturbation equations in the very small parameter $\left(\varepsilon/R_e\right)$ and achieve great simplifications. Specifically, they are of the boundary layer type. Free shear layers that might develop in the neighborhood of an internal source or sink do not occur here.

AN ε/R_e OUTER EXPANSION

Make a regular asymptotic expansion [3] of the dependent variables w,ϕ,h,p in terms of ε/R_e, e.g.,

$$w = w^0(x,y) + \delta_1(\varepsilon/R_e)w^1(x,y) + h.o.t. \tag{22a}$$

where $\lim_{\varepsilon/R_e \to 0} \delta_1(\varepsilon/R_e) = 0$ and the gauge functions for w,ϕ,h,p being different.

From the continuity equation (17) it is reasonable to suppose

$$u = \varepsilon^{1/2}\left[u^0(x,y) + \delta_1(\varepsilon/R_e)u^1(x,y) + h.o.t.\right] \tag{22b}$$

as an expansion for the radial velocity component. Throw out the higher order terms ($h.o.t.$) for convenience and replace u,w,ϕ,h,p by their respective expansions and collect terms of like order.

$O(1)$:

$$-(\rho_o u^0)_x + \rho_o w^0_y = 0 \tag{23}$$

$$\rho_o \phi^0 = p^0 + p^0_x \tag{24}$$

$$\frac{1}{R_e}\phi^0_{xx} = -4S\rho_o u^0 \varepsilon^{3/2} \tag{25}$$

$$\varepsilon^{3/2} p^0_y = \frac{1}{R_e} w^0_{xx} \tag{26}$$

$$h^0_{xx} + \varepsilon h^0_{yy} = 0 \tag{27}$$

Notice that equation (24) is a result of premultiplying (12) with $\varepsilon^{1/2}$ and then taking the limit.

Higher order equations like $O(\varepsilon/R_e)$ are neglected here. Letting

$$U^O \equiv \varepsilon^{1/2} u^O \tag{28}$$

the far field is governed by

$$-\varepsilon^{-1/2}(\rho_o U^O)_x + \rho_o w_y^O = 0 \tag{29}$$

$$\rho_o \phi^O = p^O + p_x^O \tag{30}$$

$$\frac{1}{R_e}\phi_{xx}^O = -4S\rho_o \varepsilon U^O \tag{31}$$

$$\varepsilon^{3/2} p_y^O = \frac{1}{R_e} w_{xx}^O \tag{32}$$

$$h_{xx}^O + \varepsilon h_{yy}^O = 0 \tag{33}$$

Here we have dropped designation of the order of the expansion (O) for identification as an outer expansion (o). The order is assumed understood. This produces the equations of the internal flow field [1] without sources as an asymptotic expansion.

In this approach it is clear that the auxiliary equation (33) appropriately retains axial diffusion h_{yy} in the ε/R_e limit. That term only vanishes in the ε limit [4]. However, in Onsager's original derivation, axial diffusion of heat was neglected but this does not affect the equation for the potential χ since χ is independent of h. Neglection of axial heat diffusion is exactly the ε limit and is associated with a speed boundary layer. That is how the solution of (31) also can be simplified [4]. The propriety of this treatment can be seen from reference [2].

The remaining steps leading to (1) involve elimination of pressure, definition of a streamfunction and a potential χ. These details were glossed over here because the point considered crucial was the limit process.

Summarizing, in the internal flow region $O(1)$ terms are retained in the partial differential equation system and terms of $O(\varepsilon/R_e)$ and higher order are thrown out resulting in the usual singular perturbation problem. These higher order derivatives play an all important role of boundary layer corrections. The end boundary layers are captured by an inner expansion and a complete solution of the problem is achieved by matching the basic solution to the end boundary layers.

Retention of curvature [5] or sources and sinks [1] is handled in more or less the same manner as presented here.

EKMAN LAYER

The end boundary layer equations, named after Ekman, can be formally derived using ideas from Perturbation Theory as predilected by Carrier and Maslen [6]. However, it involves just a little more than a straightforward inner limit. Difficulty arises in the inner expansion process due to the previously mentioned multiple scales in this equation system. It is expected that the relation between Onsager's equation and the Ekman layers can be more clearly understood from a unified view as offered by formal asymptotic approximations.

Neglect curvature effects to ensure consistency with equation (1) and recall that the unfolded, two-dimensional linearized fluid flow equations are (16) - (21). To eliminate pressure differentiate (20) with respect to x,

$$\varepsilon p_{yx} = \frac{2A^2}{R_e}\left(w_{xxx} + \varepsilon w_{yyx} - \frac{1}{3}\varepsilon^{1/2} u_{yx}\right) \tag{34}$$

and differentiate (18) with respect to y,

$$\varepsilon\rho_o\phi_y = \varepsilon\left(p_y + p_{xy}\right) + \frac{1}{R_e}\left(u_{xxy} + \varepsilon u_{yyy} - \varepsilon u_y - \frac{1}{3}u_{xy}\right). \tag{35}$$

Combine equations (34) and (35) above to get

$$\varepsilon\rho_o\phi_y = \frac{1}{R_e}\left\{2A^2\left[\left(1 + \frac{\partial}{\partial x}\right)\left(w_{xx} + \varepsilon w_{yy} - \frac{1}{3}\varepsilon^{1/2}u_y\right)\right]\right.$$

$$\left. + \left(u_{xxy} + \varepsilon u_{yyy} - \varepsilon u_y - \frac{1}{3}u_{xy}\right)\right\}. \tag{36}$$

Define a streamfunction Ξ that identically satisfies the linearized continuity equation (17),

$$u = \Xi_y \tag{37a}$$

$$w = 2A^2(\Xi_x - \Xi) \tag{37b}$$

and substitute into (36),

$$\varepsilon\rho_o\phi_y = \frac{1}{R_e}\left\{2A^2\left(1+\frac{\partial}{\partial x}\right)\left[2A^2(\Xi_{xxx}-\Xi_{xx}) + 2A^2\varepsilon(\Xi_{xyy}-\Xi_{yy}) - \frac{1}{3}\varepsilon^{1/2}\Xi_{yy}\right]\right.$$

$$\left. + \left(\Xi_{xxyy} + \varepsilon\Xi_{yyyy} - \varepsilon\Xi_{yy} - \frac{1}{3}\Xi_{xyy}\right)\right\} . \tag{38}$$

Substitution for u in (19) gives

$$\frac{1}{R_e}(\phi_{xx} + \varepsilon\phi_{yy}) = - 4S\rho_o\varepsilon\Xi_y . \tag{39}$$

Equations (21), (38) and (39) constitute a complete closed set of equations equivalent to (17) - (21). It is interesting to observe that

$$Y = yR_e^{1/2}/\varepsilon \tag{40a}$$

and

$$Y_2 = y(R_e/\varepsilon)^{1/2} \tag{40b}$$

are "natural" local coordinates for Ξ and ϕ respectively in the ε/R_e limit and $Y >> Y_2$.

Assume the following inner expansion for the Ekman layer near the bottom end $(y = 0)$:

$$\Xi(x,y) = R_e^{-1/2}\Xi^0(x,Y) + h.o.t. \tag{41a}$$

$$\phi(x,y) = \phi^0(x,Y) + h.o.t. \tag{41b}$$

$$h(x,y) = h^0(x,Y) + h.o.t. \tag{41c}$$

Making these changes and dropping the higher order terms gives

$$\rho_o\phi^0_Y R_e^{1/2} =$$

$$(1+\frac{\partial}{\partial x})\left[\frac{1}{\varepsilon R_e^{3/2}}\left(\Xi^0_{xxx} - \Xi^0_{xx}\right) + \frac{1}{R_e^{1/2}\varepsilon^2}\left(\Xi^0_{xYY} - \Xi^0_{YY}\right) - \frac{1}{3R_e^{1/2}\varepsilon^2}\Xi^0_{YY}\right]$$

$$+ \frac{1}{R_e^{1/2}\varepsilon^2}\Xi^0_{xxYY} + \frac{R_e^{1/2}}{\varepsilon^3}\Xi^0_{YYYY} - \frac{1}{\varepsilon R_e^{1/2}}\Xi^0_{YY} - \frac{1}{3R_e^{1/2}\varepsilon^2}\Xi^0_{xYY} \tag{42}$$

$$\frac{1}{R_e}\phi^0_{xx} + \frac{1}{\varepsilon}\phi^0_{YY} = -4S\rho_o\Xi^0_Y \tag{43}$$

$$h^0_{xx} + \frac{R_e}{\varepsilon}h^0_{YY} = 0\ . \tag{44}$$

Rewriting one obtains

$$\rho_o\phi^0_Y\varepsilon^3 =$$

$$(1+\frac{\partial}{\partial x})\left[\left(\frac{\varepsilon}{R_e}\right)^2\left(\Xi^0_{xxx} - \Xi^0_{xx}\right) + \frac{\varepsilon}{R_e}\left(\Xi^0_{xYY} - \Xi^0_{YY}\right) - \frac{1}{3}\frac{\varepsilon}{R_e}\Xi^0_{YY}\right]$$

$$+\left(\frac{\varepsilon}{R_e}\Xi^0_{xxYY} + \Xi^0_{YYYY} - \frac{\varepsilon^2}{R_e}\Xi^0_{YY} - \frac{1}{3}\frac{\varepsilon}{R_e}\Xi^0_{xYY}\right) \tag{45}$$

$$\frac{\varepsilon}{R_e} \phi^0_{XX} + \phi^0_{YY} = - 4S\rho_o \varepsilon \Xi^0_Y \tag{46}$$

$$\frac{\varepsilon}{R_e} h^0_{XX} + h^0_{YY} = 0. \tag{47}$$

Taking the inner limit $\frac{\varepsilon}{R_e} \to 0$ with Y fixed, the consistent equations are

$$\rho_o \phi^i_Y \varepsilon^3 = \Xi^i_{YYYY} \tag{48}$$

$$\phi^i_{YY} = - 4S\rho_o \varepsilon \Xi^i_Y \tag{49}$$

$$h^i_{YY} = 0 \tag{50}$$

where order 0 is not designated.

Clearly enormous simplification occurs in this limit. Most notable is the loss of all x-wise derivatives which is tantamount to omission of the corner region where x is small and x and y variations are of the same order of magnitude. Therefore, it is valid for $x >> \ell_c$ where ℓ_c is the length scale associated with the corner boundary region. This formulation for the compressible rotating flow [1] agrees with Carrier's simplified analysis for the incompressible flow [7]. The corner region was given attention in the original analysis [6] but later discarded as a negligibly small boundary layer type correction.

Equations (48) - (50) may be thought of as representing yet another set of perturbation equations in the small parameter ε and are not yet uniformly ordered. The outer limit $\varepsilon \to 0$ gives entirely homogeneous equations,

$$\Xi^{i}_{YYYY} = 0 \tag{51a}$$

$$\phi^{i}_{YY} = 0 \tag{51b}$$

$$h^{i}_{YY} = 0 \tag{51c}$$

For the inner expansion let

$$\xi \equiv \varepsilon Y \ . \tag{52}$$

Since $\varepsilon << 1$ this is a shrinking transformation rather than the usual stretching kind. Making this coordinate change,

$$\rho_o \phi^{i}_{\xi} = \Xi^{i}_{\xi\xi\xi\xi} \tag{53a}$$

$$\phi^{i}_{\xi\xi} = - 4S\rho_o \Xi^{i}_{\xi} \tag{53b}$$

$$h^{i}_{\xi\xi} = 0 \ . \tag{53c}$$

These inner equations are exactly the equations for the Ekman layer given by Carrier and Maslen [1] with curvature neglected, i.e., $\eta = 1$.

One can get similar results by defining an asymptotic expansion in $1/R_e$ and ξ and taking the limit $\frac{1}{R_e} \to 0$, however, then the inner and outer expansions involve disparate limits. At the present time perhaps the most satisfying way to look at this is to consider equations (53a) - (53c) as the formal compressible Ekman layer equations under the ε/R_e limit and equation (52) as a coordinate transformation that reduces them to a normalized form with terms of $O(1)$. One may want to go deeper into the theory of perturbations for a clearer view of the inner expansion.

It follows directly that equation set (53a) - (53c) is equally valid at the other end near $y = y_o$.

BOUNDARY CONDITIONS ON OUTER EXPANSION

Strictly proper continuum boundary conditions on χ are at the machine centerline and rotor wall,

$$x = A^2 \quad \text{at} \quad \eta = 0 \tag{54}$$

$$x = 0 \quad \text{at} \quad \eta = 1 \tag{55}$$

and at the ends

$$y = 0 \tag{56}$$

$$y = y_o \ . \tag{57}$$

In reference [1] the distant x boundary was approximated by letting $x \to \infty$.

Define $\tilde{x}$ to include x. In addition, consider that $\tilde{x}$ lives everywhere else in the upper half plane, i.e., $\tilde{x}$ defines the semi-infinite domain

$$0 \leq \tilde{x} \leq \infty \ . \tag{58}$$

This is a commonplace mathematical trick used when the modified problem is simpler to solve than the original.

Alternately this process can be thought of as a coordinate perturbation. Allow the boundary $x = A^2$ to go to

$$\tilde{x} = A^2 + M \ . \tag{59}$$

Here the perturbed coordinate is designated by an overhead tilde. This gives us a regular perturbation problem. Letting $M \to \infty$ is quite obviously not a small coordinate perturbation but nonetheless its effect may be small depending on the sensitivity of the "true" solution. By this we mean

$$\chi(x,y) = \chi(\tilde{x},y) + \delta\chi \qquad (60)$$

where the correction $\delta\chi$ may be small if χ is insensitive to $x \to \tilde{x}$.

This is quite reasonable since χ has boundary layer character in x as well as y. Because this approximation cannot be rigorously ordered as depending on a small parameter to some power one may justifiably feel uneasy. Fortunately, it is possible to precisely evaluate the error in the hydrodynamic solution for the axial velocity in the special case of one-dimensional rod flow.

The governing equation for rod flow is [8]:

$$w_{xxx} + w_{xx} = -2 \frac{R_e}{8A^6} e^{-x} \phi_y \qquad (61)$$

subject to the boundary conditions

$$w = 0 \quad \text{at} \quad x = 0 \qquad (62)$$

$$w_x = 0 \quad \text{at} \quad x = A^2 \qquad (63)$$

and the integral equation

$$\frac{PA^2}{\pi a^2 \ \Omega a \rho_w} = Q = \int_0^{A^2} e^{-x} w \, dx. \qquad (64)$$

Equation (61) has the general solution

$$w = -2\lambda x e^{-x} + c_1(x-1) + c_2 + c_3 e^{-x}. \qquad (65)$$

Application of (62) to (65) yields

$$-c_1 + c_2 + c_3 = 0 \qquad (66)$$

and application of (63) to (65) yields

$$c_1 = 2\lambda(1 - A^2)e^{-A^2} + c_3 e^{-A^2} . \qquad (67)$$

Use of the integral equation for net flow (64) gives

$$Q = \frac{\lambda}{2}\, e^{-2A^2}(1 + 2A^2) - c_1 A^2 e^{-A^2} - c_2 e^{-A^2} - \frac{c_3}{2}\, e^{-2A^2} - \left(\frac{\lambda}{2} - c_2 - \frac{c_3}{2}\right). \quad (68)$$

Now equations (66), (67) and (68) constitute a set of three equations in the three unknown constants of integration which can be readily solved.

$$c_1 = 2\lambda(1 - A^2)\, e^{-A^2} + e^{-A^2} c_3 \quad (69)$$

$$c_2 = 2\lambda(1 - A^2)\, e^{-A^2} + (e^{-A^2} - 1)c_3 \quad (70)$$

$$c_3 = Q - \left\{\frac{\lambda}{2}\left[e^{-2A^2}(1 + 2A^2) - 1\right] - 2A^2\lambda\,(1 - A^2)e^{-2A^2} + 2\lambda(1 - A^2)e^{-A^2}(1 - e^{-A^2})\right\} \cdot \left[- A\, e^{-2A^2} + (e^{-A^2} - 1)(1 - e^{-A^2}) + \frac{1}{2}\,(1 - e^{-2A^2})\right]^{-1}. \quad (71)$$

Some notable differences appear when the mathematically correct boundary condition is used. Specifically $c_1 \neq 0$ but is ordered $O(e^{-\varepsilon^{-1/2}})$; likewise the same order terms are retained in c_2 and c_3. These are transcendentally small terms in ε not ε/R_e. Hence for uniformity in ε/R_e in the outer expansion, equation (54) describes to the correct order where the boundary condition should be applied. Use of (59) introduces errors transcendentally small in ε into the solution for the hydrodynamic field variables.

In the limit of the coordinate perturbation (59) one obtains

$$c_1 = 0 \quad (72)$$

$$c_2 = 2Q + \lambda \quad (73)$$

$$c_3 = - 2Q - \lambda \quad (74)$$

which reproduces the approximate solution given in [8] associated with the approximate boundary condition. Evidently, doing this particular boundary condition exactly would simply change the integration constants of (1) and require calculation of the three eigenfunctions which are unbounded as $x \to \infty$ [1].

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