CONF-771029--145

BENDING FREE TOROIDAL SHELLS FOR TOKAMAK FUSION REACTORS*

W. H. Gray
Fusion Energy Division

W. C. T. Stoddart
Engineering Division

Oak Ridge National Laboratory
Oak Ridge, Tennessee 37830

J. E. Akin
University of Tennessee
Knoxville, Tennessee 37916

Abstract

Several authors have suggested a novel shape for the toroidal field (TF) coils of a tokamak fusion reactor. Collectively, these magnet shapes have become referred to as the "Princeton D-coil." This coil shape can be derived by assuming that for a thin conductor to be in a state of "pure tension," its radius of curvature must be proportional to the toroidal radius. A principal disadvantage of this derivation is that out-of-plane support, a necessary feature in the design of a tokamak fusion reactor, is neglected.

In this paper, a derivation of a bending free toroidal shell for a tokamak fusion reactor is presented. The out-of-plane structure is considered to be an integral part of the fusion reactor and therefore its shape is optimized to produce a bending free stress distribution. This shape, which is nearly circular for aspect ratios greater than 2.5, is derived by solving the equilibrium, constitutive, and kinematic relationships for a uniform toroidal membrane. This membrane is subjected to a magnetic pressure which is inversely proportional to the square of the toroidal radius. A comparison between this bending free shape and the D-shape is presented.

Introduction

There have been numerous proposals for the shape of the TF coils for a tokamak fusion reactor.[1,2] Several of these shapes are described as being in a state of pure tension or bending free.[3-6] Toroidal field coils of this type have been collectively referred to as the Princeton D-shape, a name based upon the similarity of the coil shape to the capital letter D. These coil shapes are derived by considering the equilibrium between the Lorentz forces and the mechanical forces within the coil cross section. The simplest approach to this shape derivation assumes the magnetic field which generates the Lorentz forces to be that of a continuously wound toroidal shell. The magnetic field is then inversely proportional to toroidal radius.

Subsequently, detailed stress analyses which were performed upon this shape reported significant bending stresses.[7,8] These bending stresses were attributed to the difference between the simplified magnetic field variation and that calculated for a discrete coil system.

Modification of the D-shape to account for discrete coils, e.g., realistic magnetic field distributions, has been presented by several authors.[9-11] In their reports they cite stress analyses of their modified D-shape coils showing reduced (but not eliminated) bending stresses.

In general, little consideration has been given to the effect on coil shapes of loadings other than in the plane of the winding. Further, only limited attention has been directed toward the effect of coil support on coil shape out of the plane of the windings.[12] One report[13] has indicated from a cost (as well as an out-of-plane support) standpoint that a shell-type structure would be an acceptable alternative to some form of torque frame.

The following sections of this report present a derivation of the meridian shape for a toroidal bending free shell. The resulting shell structure is proposed for consideration as an in-plane as well as an out-of-plane support structure for TF coils. A comparison is presented between the meridian shapes calculated for various aspect ratios, as well as meridians calculated for D-shaped coils.

Solution for the Meridian of a Toroidal Bending Free Shell of Revolution Subjected to a Pressure Which Varies Inversely with Toroidal Radius

In this section, we present a derivation for the slope of the meridian of a toroidal shell of revolution which is in a bending free (membrane) state of stress. This shell is in equilibrium with a magnetic pressure distribution which varies inversely with the toroidal radius. Three equations are derived which govern the equilibrium and compatibility of the shell. These equations contain three undetermined coefficients we solve for by applying three physical boundary conditions.

The resulting meridian is nearly circular for toroids with aspect ratios greater than 2.5 and gradually transforms into an oval shape as the aspect ratio decreases.

Equations of Equilibrium for a Shell of Revolution Under Axisymmetric Loading

The equations of equilibrium for an axisymmetric shell of revolution in a state of membrane stress are[14]

$$\frac{d}{d\phi}(N_\phi r) - N_\theta \rho \cos\phi + P_Y \rho r = 0 \ , \qquad (1)$$

$$N_\phi r + N_\theta \rho \sin\phi + P_Z \rho r = 0 \ , \qquad (2)$$

where P_Y and P_Z are the surface components, N_θ and N_ϕ are the stress resultants, ρ is the meridianal radius of curvature, r is the radius of the hoop circle, and ϕ is the tangent angle of the meridian.

From Fig. 1 we may obtain the relationship

$$(\rho d\phi) \cos\phi = dr \ . \qquad (3)$$

*Research sponsored by the Department of Energy under contract with Union Carbide Corporation.

Substituting Eq. (3) into Eqs. (1) and (2) produces the working form of the equilibrium equations,

$$N_\theta = \frac{d}{dr}(N_\phi r) + P_Y r(\sec\phi) \tag{4}$$

and

$$\frac{d}{dr}(N_\phi r \sin\phi) = -P_Y r(\tan\phi) - P_Z r \ . \tag{5}$$

Compatibility Equations for an Axisymmetric Shell of Revolution

The equilibrium equations must be augmented by the compatibility equations, which represent the requirement for zero bending stresses, i.e., vanishing changes in curvature. For small strains of the middle surface, the changes in curvature of the meridian, χ_ϕ, and the curvature in a plane perpendicular to the meridian, χ_θ, are

$$\chi_\phi = \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{v}{\rho} + \frac{dw}{\rho d\phi}\right) \ ,$$

$$\chi_\theta = \frac{\cos\phi}{r}\left(\frac{v}{\rho} + \frac{dw}{\rho d\phi}\right) \ ,$$

where v and w are the displacement components of the middle surface (see Fig. 2).[15] The displacement, v, is in a direction tangent to the surface of the shell and the displacement, w, is in a direction perpendicular to the surface of the shell. It is evident that χ_ϕ and χ_θ vanish if, and only if,

$$v + \frac{dw}{d\phi} = 0 \ . \tag{6}$$

We must relate the displacements to the stress resultants in order to use Eq. (6). First, the strains of the middle surface in the direction of the meridian, ε_ϕ, and a plane perpendicular to the meridian, ε_θ, may be expressed in terms of displacements as

$$\varepsilon_\phi = \frac{1}{\rho}\left(\frac{dv}{d\phi} - w\right) \tag{7}$$

and

$$\varepsilon_\theta = \frac{1}{r}(v\cos\phi - w\sin\phi) \ . \tag{8}$$

The strain components ε_ϕ and ε_θ can be expressed in terms of the stress resultants, N_ϕ and N_θ, by application of a constitutive relationship (Hooke's law). This gives

$$\varepsilon_\phi = \frac{1}{Eh}(N_\phi - \nu N_\theta) \tag{9}$$

and

$$\varepsilon_\theta = \frac{1}{Eh}(N_\theta - \nu N_\phi) \ , \tag{10}$$

where E = Young's modulus, ν = Poisson's ratio, and h = the shell thickness.

By substituting Eqs. (7) and (8) into Eqs. (9) and (10), we obtain the relationships between the stress resultants and the displacements

$$N_\phi = \frac{Eh}{1-\nu^2}\left[\frac{1}{\rho}\left(\frac{dv}{d\phi} - w\right) + \frac{\nu}{r}(v\cos\phi - w\sin\phi)\right] \ , \tag{11}$$

$$N_\theta = \frac{Eh}{1-\nu^2}\left[\frac{1}{r}(v\cos\phi - w\sin\phi) + \frac{\nu}{\rho}\left(\frac{dv}{d\phi} - w\right)\right] \ . \tag{12}$$

Solving Eq. (11) and (12) for v and w, substituting into Eq. (6), and simplifying yields the compatibility condition[15]

$$\frac{d}{dr}(N_\phi + N_\theta) = (N_\theta - \nu N_\phi)\frac{1}{h}\frac{dh}{dr} + P_Y(\sec\phi) \ . \tag{13}$$

Electromechanical Force on a Toroidal Shell of Revolution

For the purpose of this analysis, we assume that the toroidal shell is uniformly wound and contains N turns, each of which carries a current, I_ϕ, in the meridianal direction. From elementary electromagnetic considerations, the magnetic field may be expressed as

$$B_\theta = \frac{\mu_o N I_\phi}{2\pi r} \ . \tag{14}$$

The magnetic pressure exerted by the cross produce of the current and the magnetic field may be calculated from

$$p_m = \frac{B_\theta^2}{2\mu_o} = \frac{\mu_o N^2 I_\phi^2}{8\pi^2 r^2} = \frac{F}{r^2} \ , \tag{15}$$

where

$$F = \text{constant} = \frac{\mu_o N^2 I_\phi^2}{8\pi^2} \tag{16}$$

and in the mks system, F has the units of newtons.

Further Simplification of the Equations for the Linear Membrane Theory of Shells of Revolution

Equations (4), (5), and (13) are the fundamental equations of the linear membrane theory of shells of revolution under axisymmetric loading. In order to obtain a solution for the meridian, the relationship between h and r must be known in advance. We will only consider solutions for constant thickness. Further, for the present solution, only the P_Z load component will be considered.

Stated mathematically, these assumptions are

$$\frac{dh}{dr} = 0 \ ,$$

$$P_Y = 0 \ ,$$

and

$$P_Z = -p_m \ .$$

Substituting into Eqs. (4), (5), and (13) yields the following reduced set of compatibility and equilibrium equations:

$$N_\theta = \frac{d}{dr}(N_\phi r) ,$$

$$\frac{d}{dr}(N_\phi r \sin \phi) = -P_z r = p_m r = \frac{F}{r} ,$$

$$\frac{d}{dr}(N_\phi + N_\theta) = 0 .$$

In order to normalize the radial coordinate, the following variable transformations are made:

$$R = r/r_{in}$$

and

$$Z = z/r_{in} .$$

The normalized compatibility and equilibrium equations become

$$N_\theta = \frac{d}{dr}(N_\phi R) ,$$

$$\frac{d}{dR}(N_\phi + N_\theta) = 0$$

$$\frac{d}{dR}(N_\phi R \sin \phi) = \frac{F}{r_{in} R} .$$

Integration of these three equations yields

$$N_\phi R \sin \phi = \frac{F}{r_{in}} \ln R + C_1 ,$$

$$N_\phi = \frac{1}{2} C_2 + C_3/R^2 ,$$

$$N_\theta = \frac{1}{2} C_2 - C_3/R^2 .$$

By substituting

$$C_1 = \frac{F}{r_{in}} D_1$$

and

$$C_2 = \frac{2F}{r_{in}} D_2$$

$$C_3 = \frac{F}{r_{in}} D_3$$

for the arbitrary constants, we can express the equations as

$$N_\phi = \frac{F}{r_{in}} \left(D_2 + \frac{D_3}{R^2}\right) , \tag{17}$$

$$N_\theta = \frac{F}{r_{in}} \left(D_2 - \frac{D_3}{R^2}\right) , \tag{18}$$

and

$$\sin \phi = \frac{R(\ln R + D_1)}{D_2 R^2 + D_3} . \tag{19}$$

From the expression for $\sin \phi$ we obtain the equation for the slope of the meridian,

$$\tan \phi = \frac{dz}{dr} = \frac{dZ}{dR} = \frac{R(\ln R + D_1)}{\sqrt{(D_2R^2 + D_3 - R \ln R - RD_1)(D_2R^2 + D_3 + R \ln R + RD_1)}} . \tag{20}$$

Solution for the Shape of the Meridian

In this section, we demonstrate how to determine the three constants, D_1, D_2, and D_3. Equation (20) may be characterized further by realizing that in order to have a smooth curve for the shape of the meridian, the following statements about the slope of the meridian must be true.

At r equal to both r_{in} and r_{out} ($R = 1.0$ and $R = r_{out}/r_{in} = R_{out}$), then the slope of the meridian must be infinity. At r equal to r_{cr}, which represents the radial distance to the crown of the shell, the slope of the meridian must be equal to zero, see Fig. 3. Stated mathematically these boundary conditions are as follows. At $R = 1.0$ and $R = r_{out}/r_{in} = R_{out}$,

$$\frac{dZ}{dR} = \infty . \tag{21}$$

At $R = R_{cr} = r_{cr}/r_{in}$,

$$\frac{dZ}{dR} = 0 . \tag{22}$$

Using condition (22),

$$D_1 = -\ln R_{cr} , \tag{23}$$

and using condition (21),

$$D_2 = \frac{R_{out}(\ln R_{cr} - \ln R_{out}) + R_{in}(\ln R_{cr} - \ln R_{in})}{(R_{in}^2 - R_{out}^2)}$$

and

$$D_3 = \frac{-R_{in}R_{out}\left[\left(R_{in} + R_{out}\right)\ln R_{cr} - R_{out}\ \ln R_{in} - R_{in}\ \ln R_{out}\right]}{(R_{in}^2 - R_{out}^2)}$$

Substituting $R_{in} = 1.0$ into the equations for D_2 and D_3 gives

$$D_2 = \frac{R_{out}(\ln R_{cr} - \ln R_{out}) + \ln R_{cr}}{(1.0 - R_{out}^2)} \tag{24}$$

and

$$D_3 = \frac{-R_{out}\left[\left(1.0 + R_{out}\right)\ln R_{cr} - \ln R_{out}\right]}{(1.0 - R_{out}^2)} \ . \tag{25}$$

Given r_{in} and r_{out}, we are left with the task of determining r_{cr}. If we choose the plane $z = 0$ to pass through the meridian at $R = 1.0$ and $R = R_{out}$, then

$$Z = \int_{1.0}^{R_{out}} \left(\frac{dZ}{dR}\right) dR = 0 \ . \tag{26}$$

Physically, this stipulates that the shell must close upon itself. Mathematically stated in a different way,

$$\int_{1.0}^{R_{cr}} \left(\frac{dZ}{dR}\right) dR = \int_{R_{cr}}^{R_{out}} \left(\frac{dZ}{dR}\right) dR \ . \tag{27}$$

Determination of the R_{cr} which satisfies Eq. (27) yields a solution to the shape of the meridian for a bending free shell of revolution. Equations (23)-(27) were solved numerically.

Table 1 presents the results of the numerical solutions to these equations. It lists for values of R_{out} from 1.2 to 5.0 the aspect ratio, radius to crown, height to crown, radial eccentricity ratio, height eccentricity ratio, and the three solution coefficients. The aspect ratio, A, is defined by

$$A = (R_{out} + 1.0)/(R_{out} - 1.0) \ .$$

The radial, λ_r, and height, λ_z, eccentricity ratios are defined by

$$\lambda_r = 2R_{cr}/(R_{out} + 1.0)$$

and

$$\lambda_z = 2Z_{cr}/(R_{out} + 1.0).$$

Tabular values for the eccentricity ratios close to 1.0 indicate a close approximation of the meridian to a circular shape. This table shows that for aspect ratios greater than 2.5, the meridian of a bending free toroidal shell is essentially circular.

A comparison of meridian shapes for four toroidal bending free shells of different aspect ratio is presented in Fig. 4. This figure shows how the solution presented in this report gradually transforms from being nearly circular for high aspect ratio into an eccentric elliptical shape for low aspect ratio. For this figure the inner toroidal radius was held constant; also, the dashed curves represent circles for the various aspect ratios.

Table 1. The results of the numerical solution for the meridian of toroidal bending free shells for tokamak fusion reactors

R_{out}	A	R_{cr}	Z_{cr}	λ_r	λ_z	D_1	D_2	D_3
1.2	11.00	1.1000	0.1001	1.0000	1.0015	-0.09530	0.02074	0.07456
1.4	6.00	1.1999	0.2020	0.9999	1.0100	-0.18220	0.03518	0.14703
1.6	4.33	1.2994	0.3065	0.9996	1.0217	-0.26192	0.04552	0.21640
1.8	3.50	1.3985	0.4143	0.9989	1.0357	-0.33538	0.05310	0.28228
2.0	3.00	1.4968	0.5257	0.9979	1.0514	-0.40335	0.05875	0.34460
2.2	2.67	1.5943	0.6412	0.9964	1.0686	-0.46645	0.06301	0.40343
2.4	2.43	1.6908	0.7609	0.9946	1.0870	-0.52519	0.06627	0.45892
2.6	2.25	1.7861	0.8853	0.9923	1.1066	-0.58002	0.06879	0.51123
2.8	2.11	1.8801	1.0145	0.9895	1.1272	-0.63131	0.07075	0.56055
3.0	2.00	1.9726	1.1490	0.9863	1.1490	-0.67936	0.07230	0.60706
3.2	1.91	2.0636	1.2890	0.9826	1.1718	-0.72443	0.07354	0.65090
3.4	1.63	2.1528	1.4350	0.9785	1.1958	-0.76676	0.07454	0.69223
3.6	1.77	2.2401	1.5873	0.9740	1.2210	-0.80653	0.07536	0.73117
3.8	1.71	2.3254	1.7466	0.9689	1.2476	-0.84390	0.07606	0.76784
4.0	1.67	2.4085	1.9134	0.9634	1.2756	-0.87901	0.07668	0.80233
4.2	1.63	2.4892	2.0885	0.9574	1.3053	-0.91196	0.07723	0.83473
4.4	1.59	2.5673	2.2725	0.9508	1.3367	-0.94285	0.07776	0.86509
4.6	1.56	2.6425	2.4664	0.9438	1.3702	-0.97174	0.07828	0.89346
4.8	1.53	2.7147	2.6715	0.9361	1.4060	-0.99868	0.07881	0.91987
5.0	1.50	2.7835	2.8889	0.9278	1.4444	-1.02371	0.07937	0.94434

Figure 5 presents the same type of comparison, except the bore diameter was held constant and the inner and outer toroidal radii were varied to obtain the desired aspect ratio. The distinguishing symbol for each aspect ratio is plotted at the crown of the shell to emphasize the eccentricity of the ellipsoidal shape.

For the shells in Fig. 5, the stress resultants N_ϕ and N_θ were tabulated and are presented graphically in Figs. 6 and 7, respectively. These figures show that for toroidal shells with high aspect ratio, the stress resultant variation with toroidal radius is small compared to toroidal shells with low aspect ratio. This should be expected because of the direct relationship of the stress resultants and the magnetic field, i.e., there is little magnetic field variation with toroidal radius in the region of the solution for high aspect ratios.

Comparison Between the Meridian Shapes of a Toroidal Bending Free Shell and the Princeton D-Shape

Figures 8 and 9 compare the meridian shape of a toroidal bending free shell and the Princeton D-shape for the same four aspect ratios plotted in Fig. 2. On each figure, a dashed curve has been plotted which represents a circle. The curve which has the straight vertical leg at the inner radius is the Princeton D-shape. These graphs show that there is a potential for a significant size, as well as superconductor cost, savings if a practical fusion reactor design can be worked out using the toroidal bending free shell concept.

Conclusion and Further Work

In this paper we have derived and solved the equations for the meridian shape of a toroidal bending free shell. Comparisons have been graphically shown between meridians calculated for various aspect ratios as well as between this shape and the Princeton D-shape. The calculated shapes for the meridian of a toroidal bending free shell have been shown to be eccentric ellipses for low aspect ratio, becoming more nearly circular with increasing aspect ratio. For aspect ratios greater than 2.5, the meridians are essentially circular.

It is argued that this new shape has a significant advantage over the Princeton D-shape. The new shape concept, if it can be made into a viable fusion reactor design, has the advantage of using less superconductor (and therefore less money) than the D-shape. Admittedly, there are several design details which must be solved for a shell structure to work for fusion reactors, yet Ref. 13 has shown that shell structures are feasible.

We are presently investigating four areas of practical interest associated with this solution:

1. How good is the solution presented in this report compared to a shell of finite thickness?
2. What effect does a more realistic approximation to the radial variation of magnetic field have upon the meridian shape?
3. How far away from reality is this solution compared to a finite element analysis where a thin shell supports lumpy thick coils of the derived shape?
4. How bad are the bending stresses in circuler coils supported by toroidal shells?

Acknowledgment

The authors would like to express their appreciation to L. Dresner for his clever suggestions on integration near infinity.

References

1. J. File, "Proposed Superconducting Coils for the Princeton Fusion Reactor," in Proc. 5th Symp. on Engineering Problems of Fusion Researcn (1974).
2. S. L. Gralnick and F. H. Tenney, "Analytical Solution of the Toroidal Constant Tension Solenoid," Proc. 6th Symp. on Engineering Problems of Fusion Research (1976).
3. J. File, R. G. Mills, and G. V. Sheffield, Large Superconducting Magnet Designs for Fusion Reactors, MATT-848, Princeton Plasma Physics Laboratory, Princeton, New Jersey (June 1971).
4. J. File and G. V. Sheffield, A Large Superconducting Magnet for Fusion Research, MATT-927, Princeton Plasma Physics Laboratory, Princeton, New Jersey (October 1972).
5. L. V. Leites, "A Coreless Toroidal Reactor for Power Systems," Elektrichestvo 11, 76 (1960).
6. V. D. Shafranov, "Optimum Shape of a Toroidal Solenoid," Sov. Phy.-Tech. Phys. 17, 1433 (1973).
7. D. W. DeMichele and J. B. Darby, Jr., "Three-Dimensional Mechanical Stresses in Toroidal Magnets for Controlled Thermonuclear Reactors," in Proc. 5th Symp. on Engineering Problems of Fusion Research (1974).
8. N. J. Diserens, Stress Analysis Studies in Optimized 'D' Shaped Tokamak Magnet Designs, RL-75-117, Rutherford Laboratory, Chilton, Didcot, Oxon, England (July 1975).
9. W. M. Stacey et al., Tokamak Experimental Power Reactor Conceptual Design, ANL/CTR-75-2, Argonne National Laboratory, Argonne, Illinois (1975).
10. R. W. Moses and W. C. Young, "Analytic Expressions for Magnetic Forces on Sectored Toroidal Coils," in Proc. 6th Symp. on Engineering Problems of Fusion Research, (1976).
11. M. Söll, Comparison of Some Analytical and Numerical Calculated Parameters for Toroidal Field Coils, IPP 4/145, Max-Plank Institute for Plasma Physics, Munich, Federal Republic of West Germany (November 1976).
12. M. Roberts (ed.), ORNL-EPR Study — Scoping Report, ORNL/TM-5038, Oak Ridge National Laboratory, Oak Ridge, Tennessee (December 1975).
13. C. A. Trachsel and B. A. Kramer, Development and Evaluation of TEPR Support Structure Concepts, SAN-76-51, McDonnell Douglas Corp., St. Louis, Missouri (May 1976).
14. S. Timoshenko and S. Woinowsky-Krieger, Theory of Plates and Shells, pp. 433-435 and 533-536, McGraw-Hill, 1970.
15. M. V. V. Murthy and J. Kiusalaas, "Toroidal-Type Shells Free of Bending Under Uniform Normal Pressure," Journal of the Franklin Institute 282, 232 (1966).

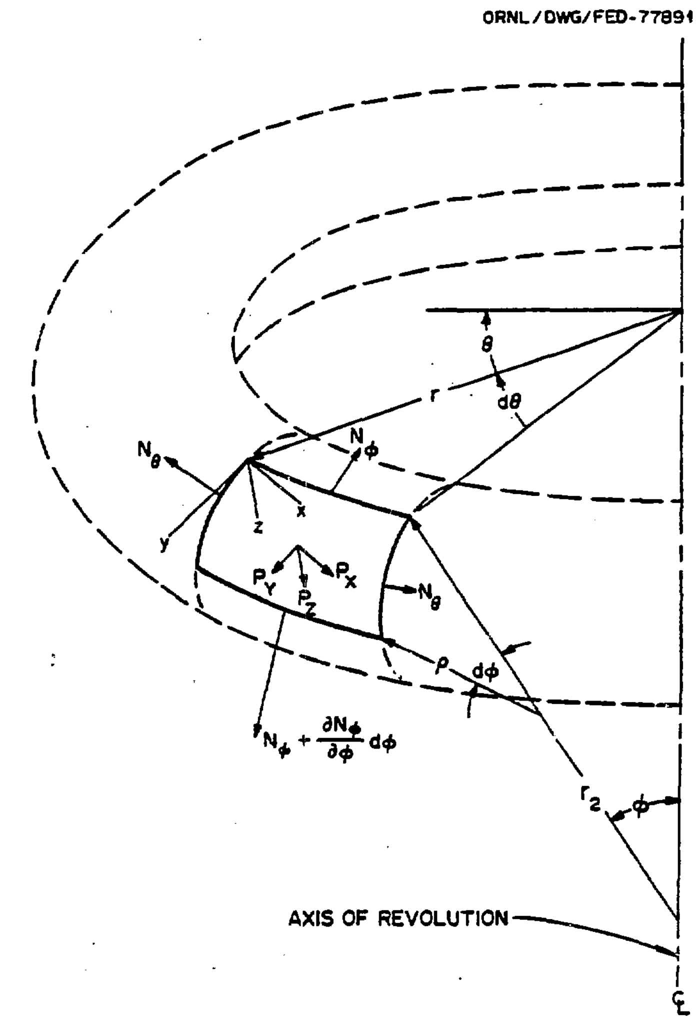

Fig. 1. Differential element of a shell of revolution under axisymmetric loading.

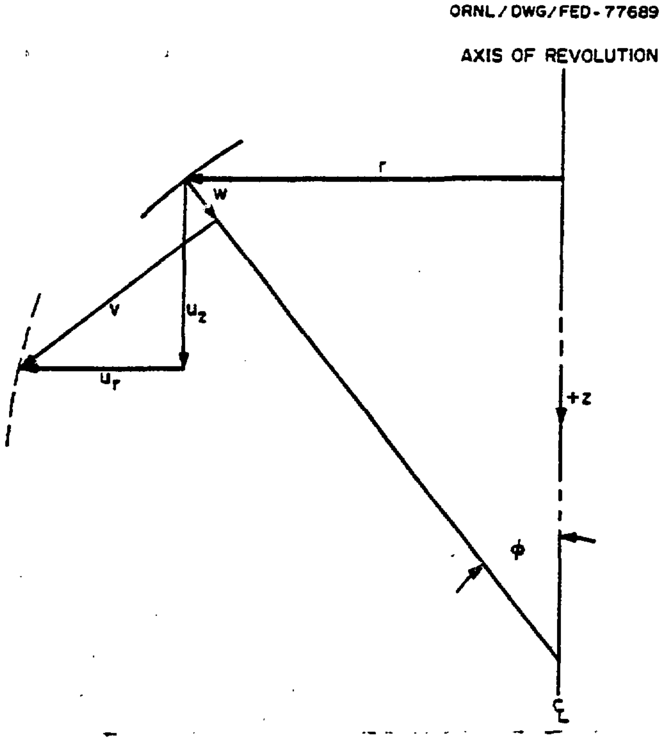

Fig. 2. Displacement components of middle surface of the toroidal shell.

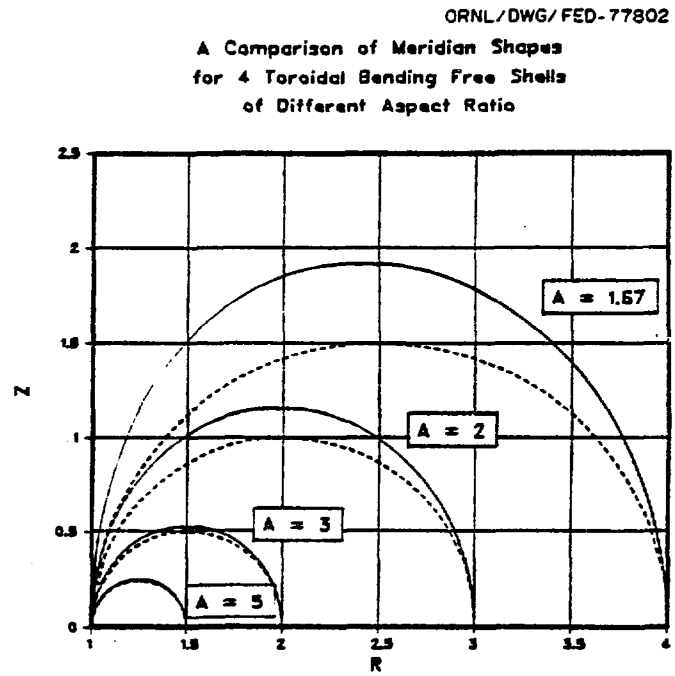

Figure 4.

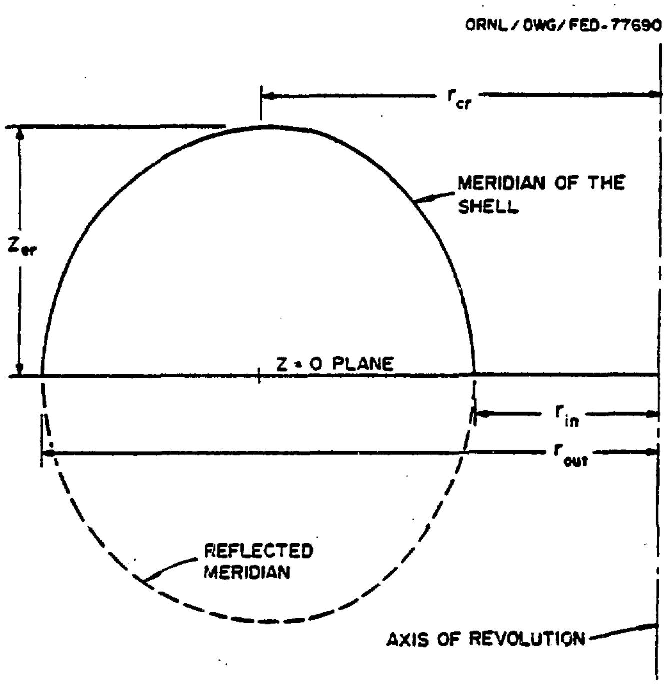

Fig. 3. The upper half of the meridian of a toroidal shell showing some significant locations.

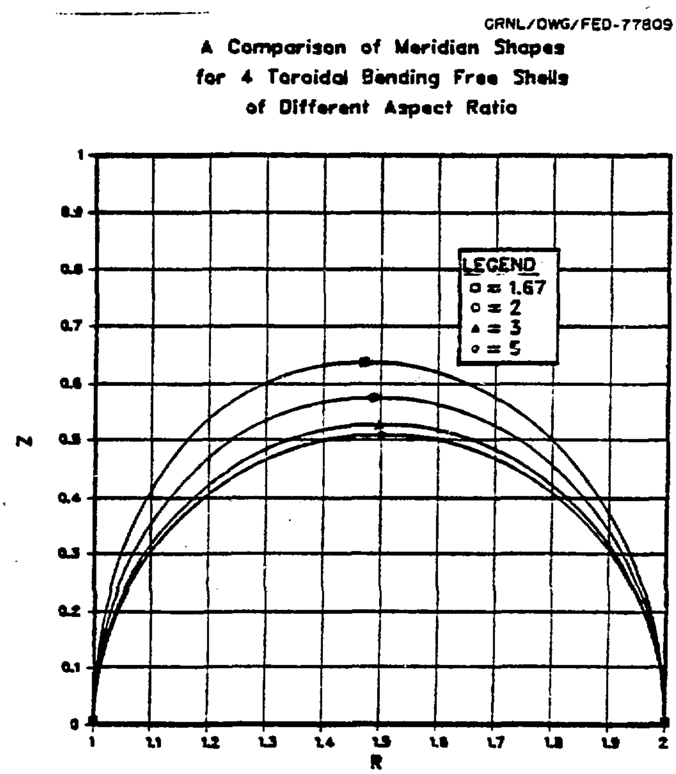

Figure 5.

ORNL/DWG/FED-77808

A Comparison of the Meridian Stress Resultants for 4 Toroidal Bending Free Shells of Different Aspect Ratio

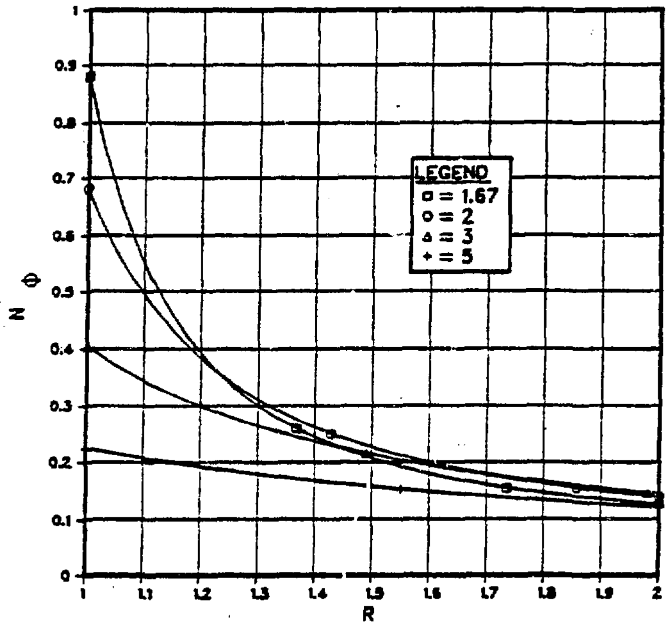

Figure 6.

ORNL/DWG/FED-77803

A Comparison Between the Meridian Shapes of a Toroidal Bending Free Shell, the Princeton D-Shape, and a Circle for an Aspect Ratio of 1.67

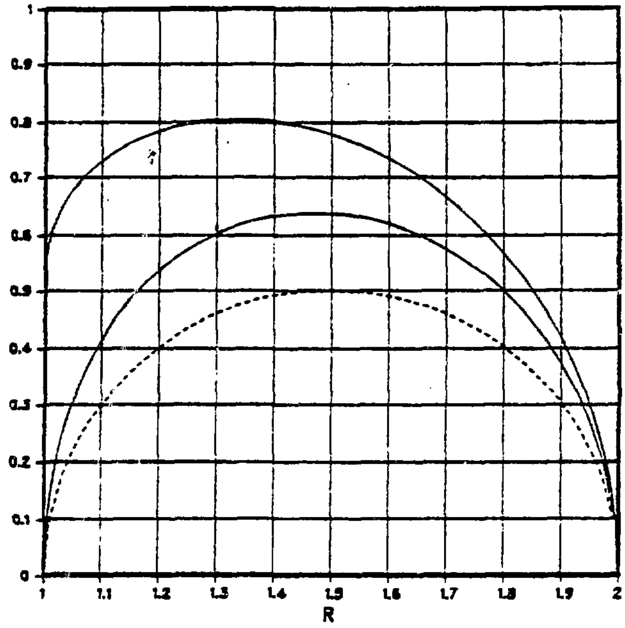

Figure 8.

ORNL/DWG/FED-77807

A Comparison of the Tangential Stress Resultants for 4 Toroidal Bending Free Shells of Different Aspect Ratio

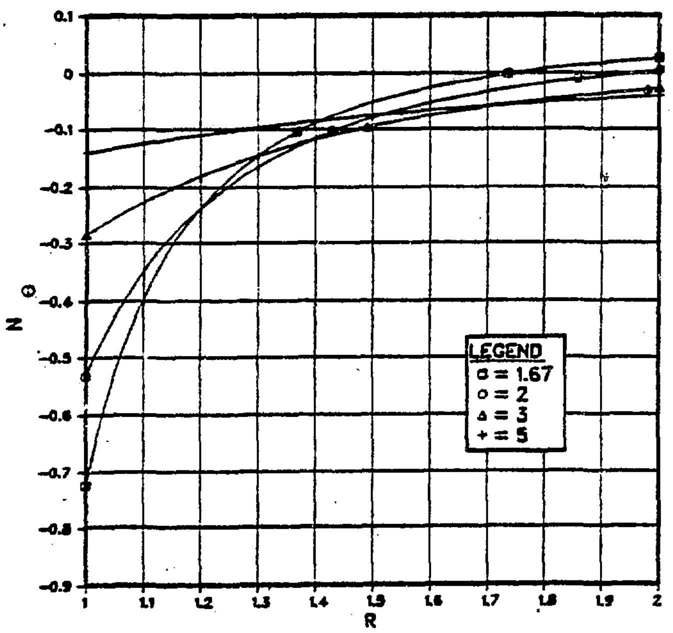

Figure 7.

ORNL/DWG/FED-77805

A Comparison Between the Meridian Shapes of a Toroidal Bending Free Shell, the Princeton D-Shape, and a Circle for an Aspect Ratio of 5.00

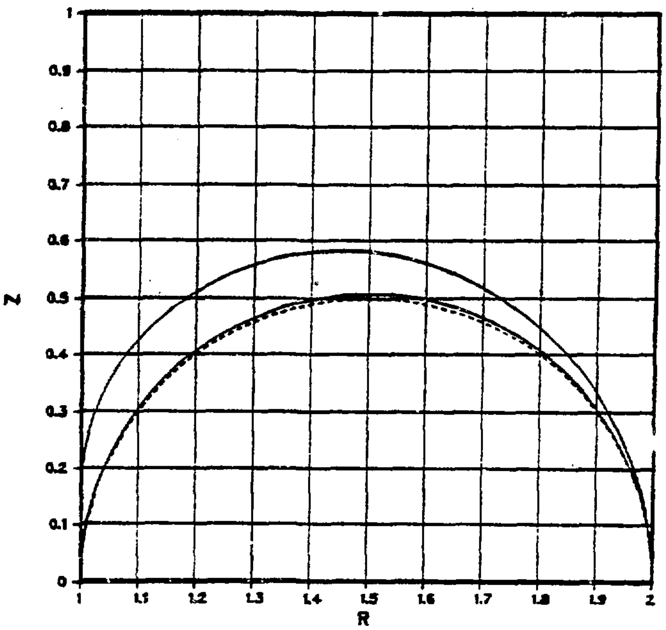

Figure 9.