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MOMENT SERIES FOR THE COEFFICIENT OF VARIATION IN WEIBULL SAMPLING*

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DISCLAIMER

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DRAFT

1. INTRODUCTION

For the 2-parameter Weibull distribution function

$$F(t) = 1 - \exp(-t/b)^{c}, \quad t > 0 \qquad (1)$$

with c and b positive, a moment estimator c* for c is the solution of the equation

$$\frac{\Gamma(1+2/c^*)}{\Gamma^2(1+1/c^*)} = 1 + v^{*2} \qquad (2)$$

where v* is the coefficient of variation in the form $\sqrt{m_2}/m_1'$, m_1' being the sample mean, m_2 the sample second central moment (it is trivial in the present context to replace m_2 by the variance).

One approach to the moments of c* (Bowman and Shenton, 1981) is to set-up moment series for the scale-free v*. We have described elsewhere the procedure for setting up series in powers of n^{-1} (n being the sample size) for the moments of statistics similar to v*; illustration for selected values of c are given in Table 1.

The series are apparently divergent and summation algorithms are essential; we consider methods due to Levin (1973) and one, (2cB), introduced ourselves (Bowman and Shenton, 1976). It is an advantage if we can find exact moments of v*. For c = 1 and n = 2, 3, and 4 these have been given by Lam (1980). However no exact results are known for general c. For n = 2 and sample values x, y we have $v^* = |x-y|/(x+y)$, and

$$Ev^{*r} = \int_0^\infty\int_0^\infty c^2(xy)^{c-1}\left(\frac{|x-y|}{x+y}\right)^r e^{-x^c-y^c}\,dx\,dy \,. \qquad (3)$$

This is readily evaluated by integration over the region x = y, y = 0 with x > 0.

For n = 3, using polar coordinates, we have

$$Ev^{*r} = 2^{3+\frac{r}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{c_1c_2s_1s_2^3\,\{\Psi(\theta,\phi\}^{\frac{r}{2}}d\theta d\phi}{\{(s_2c_1)^{\frac{2}{c}}+(s_2s_1)^{\frac{2}{c}}+c_2^{\frac{2}{c}}\}^r} \qquad (4)$$

where $s_1 = \sin\theta$, $c_1 = \cos\theta$, $s_2 = \sin\phi$, $\cos\phi$, and

$$\Psi(\theta,\phi) = (s_2c_1)^{4/c} + (s_2s_1)^{4/c} + c_2^{4/c} - (s_2^2s_1c_1)^{2/c} - (s_2c_2c_1)^{2/c} - (s_2c_2s_1)^{2/c}$$

For n = 4, a three-dimensional trigonometrical integral results from the transformation $x_1 = R\cos\theta_1\cos\theta_2\cos\theta_3$, $x_2 = R\cos\theta_1\cos\theta_2\sin\theta_3$, $x_3 = R\cos\theta_1\sin\theta_2$, $x_4 = R\sin\theta_1$, the Jacobian of the transformation being $R^3\cos^2\theta_1\cos\theta_2$.

For n = 3 a program NAGFLIB:D01DAF was used; for n = 4, we used Harwell Subroutine Library QB01AD. In view of the time involved for the latter, we confined attention to c = 1, 1.5, 2.0, 2.5, and 3.0, the first case serving as a check on the quadratures using Lam's findings.

p.2b

2. GENERAL REMARKS

Algorithms are severely tested for small sample sizes ($n = 2, 3, \ldots, 10$), but as n increases fewer terms are used and limited accuracy becomes more accessible. It is expedient at this stage to avoid questions of accuracy when n is large, for in this case one can not ignore questions of precision.

When n is small, we can evaluate moments of v^* by quadrature; however this approach is certain to run into difficulties for $n > 8$ or so (the problem is more difficult for c^*). However the use of algorithms for $n = 2, 3$, and 4 say is of considerable interest and highlights problems of error analysis for small n. For $n > 20$ or so, our experience shows that the odds are in favor of several algorithms providing acceptable approximation processes.

We must note the changes in the structure of the moment series as the shape parameter c changes. Briefly, the sign pattern changes slowly from strict alternation of positive and negative terms to a disrupted pattern as c increases beyond 1.5 or so. Simultaneously the increase in magnitude with order of coefficient decreases as c increases. This translates into a careful choice of algorithm for c small ($0.8 < c < 1.5$ approx.) with greater freedom for $c > 1.5$.

Lastly we have to cope with the first four moments (μ_1', μ_2, μ_3, μ_4). Now as far as ratios of successive terms are concerned, there is (at least in the case of v^*) little difference in structure over the four moments. **But the series do differ in the structure of the first two to four coefficients** (ignoring terms which are known to be zero, such as the n^{-1} terms in μ_3, and μ_4), and it is worth emphasizing that these differences cause fairly seroius summation problems. For some reason or other, it is always simpler to run series for the mean than for the variance; and the situation is exacerbated for the third and fourth moments. We therefore, for the most part, confine attention to means and variances.

3. THE CASE c = 1 AND (Ev^*)

The moments in this case are related, through independence, to those of $\sqrt{m_2}$; for example

$$Ev^* = E\sqrt{m_2}$$

$$\mathrm{Var}\,v^* = (n-1)/(n+1) - (E\sqrt{m_2})^2,$$

and so on for higher moments (since v^* is scale free we assume here that $b = 1$). References to this case are to be found in Bowman and Shenton (1981), Shenton _et al_ (1979), Hing-kam Lam _et al_ (1980), and Shenton and Bowman (1977).

Consider the mean, $E(v^*)$. From Table 1, consider the modified terms $e_s/(2s)!$ We have

E(v*), c = 1

s	$(-1)^s e_s/(2s)!$	Δ
0	1	
1	.750000	
		-494792
2	.255208	
		- 45052
3	.210156	
		4185
4	.214341	
		20045
5	.234386	
		26829
6	.261215	
		29959
7	.291174	
		31114
8	.322288	
		31082
9	.353370	
		30332
10	.383702	
		29184
11	.412886	
		27854
12	.440740	
		26484
13	.467224	
		25162
14	.492386	
		23932
15	.516318	
		22814
16	.539132	
		22812
17	.560944	
		20918
18	.581862	
		20122
19	.601984	
		19410
20	.621394	
		18773
21	.640167	
		18197
22	.658364	
		17676
23	.676040	
		17199
24	.693239	

The irregularities in the differences (Δ) suggest dropping the first three terms and using Borel (2cB) starting with the n^{-3} term with a = 7, n^{-4} term with a = 9, and n^{-5} term with a = 11; of course one could continue this scheme, but there is the risk that higher coefficients still would pull in loss of accuracy in the coefficients.

Exact values for n = 2, 3, 4 are due to Lam (1980). Comparisons are given in Table 2, the last two rows in each case giving the Shanks' limit using 3, and 5 terms (S_3, and S_5). There is monotonicity in the approximants in all cases and the Shanks values continue the correct trend (i.e. they do not project a value already included in the sequence). The case a = 11 and truncating five terms seems to be slightly sharper than the other two. The reason for this could be that the K's in this case are quite small. Thus

K_0 = .234386, K_1 = -.026829, K_2 = .003130,
K_3 = .001975, K_4 = .00788, K_5 = .000319,
K_6 = .000170, K_7 = .000125, K_8 = .000110.

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Notice that the errors involved in S_5 for n = 2, 3 and 4 are 0.17%, 0.07% and 0.02% respectively. It would seem reasonable to guess that errors for larger n would not be greater, and very likely in the region of 0.02% for n = 10 and much less thereafter.

Approximants to the variance using 2cB are given in Table 3. The errors for small n and S_5 are:

	n = 2	n = 3	n = 4
a = 5; n^{-2}	4.9%	0.13%	0.08%
a = 5; n^{-3}	0.78%	0.68%	0.30%
a = 9; n^{-4}	11.4%	2.9%	0.91%

This suggests the use of a = 5, n^{-3} start, for larger n. Note that for the mean and variance the sequences of approximants become tightly packed for n = 20, and n = 25; thus at n = 20, the three approaches to E(v*) yield the same result 9.33711-01. Similarly for the variance there is for n = 20 agreement in the assessment 3.30202-02.

4. The Case c > 1 and E(v*)

Comparison of the Levin 2cB, and direct sum approaches to the mean value of v* are given in Table 4. The agreement is quite remarkable (note that if two methods of summation give complete agreement, say to 12 significant digits, in both cases the less significant digits could be in errors due to loss of significance and other digital problems).

5. The Variance of v*

As mentioned in section 2, higher moments are more difficult to sum that lower moments. In Table 5 we give comparisons for the variance, where it is evident that the comparisons of different evaluations are less sharp than those occurring for the mean (Table 4).

The basic problem seems to be with the pattern of signs and the anomalous magnitude structure in the early coefficients of the series for small values of n. However, for n > 20 or so, and 1 ≤ c ≤ 4, the problem is less severe.

6. Conclusions

Basic computing was done on IBM 370/3033.

A demanding test of summation algorithms (direct sum, Padé, Levin, Borel, etc.) occurs for small sample sizes. When the moment series, such as those which arise for the mean and variance of the coefficient of variation, appear to diverge rapidly, there is a considerable element of surprise when an algorithm works extremely well for samples of 2, 3 and above. This turns out to be the case for E(v*), with a slight but important deterioration for var(v*). There is further deterioration for the higher moments, but this can be nullified by increasing the sample size.

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TABLE 1. MOMENTS SERIES FOR v*; μ_1' and μ_2

	C=1.0 μ_1'	C=1.0 μ_2	C=1.5 μ_1'	C=1.5 μ_2
0	1.000000000000000D 00		6.789686930973462D-01	
1	-1.500000000000000D 00	1.000000000000000D 00	-5.613201349285887D-01	2.677267398405026D-01
2	6.125000000000009D 00	-1.250000000000001D 01	1.655717684480837D-01	-6.252741338655030D-01
3	-1.513124999999992D 02	3.189999999999985D 02	-1.847462053672064D 00	2.805419702075228D 00
4	8.642210937499948D 03	-1.777387499999989D 04	2.192258620089601D 01	-3.174030167302986D 01
5	-8.505428320312672D 05	1.728863875000034D 06	-4.358064523222984D 02	6.169821359184447D 02
6	1.251222284423809D 08	-2.529248459374963D 08	1.214005399009297D 04	-1.698566057457510D 04
7	-2.538401905988445D 10	5.115643930187339D 10	-4.431927251015832D 05	6.156819494099406D 05
8	6.743174152794177D 12	-1.356436519339886D 13	2.020357986806791D 07	-2.793885261690395D 07
9	-2.262406892236728D 15	4.545406827420834D 15	-1.112034682681322D 09	1.532965550211385D 09
10	9.335091183857489D 17	-1.873898629258839D 18	7.209657473433931D 10	-9.916009922336329D 10
11	-4.640837914389187D 20	9.309985169622261D 20	-5.402445997147480D 12	7.417594821601848D 12
12	2.734561982239502D 23	-5.483169433576328D 23	4.609548336980600D 14	-6.320422081592837D 14
13	-1.884275122746316D 26	3.776814176008994D 26	-4.424625661605893D 16	6.060326502127352D 16
14	1.501227222116383D 29	-3.008142395573900D 29	4.730682023456777D 18	-6.473820032476090D 18
15	-1.369547748057011D 32	2.743623192876359D 32	-5.586987579397722D 20	7.640054848453776D 20
16	1.418623118359454D 35	-2.841373899324434D 35	7.236994749327359D 22	-9.890283314765727D 22
17	-1.656091270620265D 38	3.316455681025123D 38	-1.021913166334475D 25	1.395839142824942D 25
18	2.164488705901134D 41	-4.333963508456870D 41	1.564724879726098D 27	-2.136297117320892D 27
19	-3.148511719516817D 44	6.303537648520966D 44	-2.585854443253060D 29	3.529031474182933D 29
20	5.070050970385084D 47	-1.014957451947771D 48	4.593225818803173D 31	-6.266398881628273D 31
21	-8.994383654014457D 50	1.800401671674019D 51	-8.737315845800662D 33	1.191638518765838D 34
22	1.750110839792858D 54	-3.502926304845161D 54	1.773971402766864D 35	-2.418767198250694D 36
23	-3.719990760549686D 57	7.445243031033138D 57	-3.832868728219528D 38	5.224741820570659D 38
24	8.605808038387258D 60	-1.722279776947100D 61	8.788793321981414D 40	-1.197772256076727D 41

	C=2.0 μ_1'	C=2.0 μ_2	C=3.0 μ_1'	C=3.0 μ_2
0	5.227232008770633D-01		3.634465032522934D-01	
1	-3.514501786071399D-01	1.378809605122418D-01	-2.233879550000342D-01	6.649089432626640D-02
2	-1.131899247449153D-01	-8.206605877037910D-02	-1.221426137519103D-01	6.031808295146775D-03
3	-1.481401128187685D-01	7.067560464623050D-02	-6.422963219590310D-02	-1.824944669137780D-02
4	1.891390979217576D-01	-2.818299423332384D-01	3.429094884328263D-02	-6.906064755470780D-02
5	-1.792654233140374D 00	2.015483743147291D 00	3.271649589230083D-01	-2.340670989186993D-01
6	1.816207541943433D 01	-2.021080558420291D 01	5.906831568365021D-01	-2.731336705206378D-01
7	-2.010397535925470D 02	2.225408869449217D 02	-6.342038236314093D 00	4.963205901789160D 00
8	2.994498798668059D 03	-3.268492934128914D 03	-6.041232320356888D 01	4.126625571647766D 01
9	-5.139407138020691D 04	5.579499401207821D 04	-5.082457050621886D 01	8.451568039129965D 00
10	1.020734296779746D 06	-1.102639435272783D 06	4.097224071212084D 03	-3.016686680907319D 03
11	-2.314468693458788D 07	2.490339924100481D 07	4.540952849082522D 04	-3.119747965914910D 04
12	5.875950615037040D 08	-6.303538858325174D 08	1.169650449681317D 04	1.278806614773815D 04
13	-1.655577172573338D 10	1.771629316912472D 10	-6.691600472357816D 05	4.880973052914555D 06
14	5.126851704009516D 11	-5.474964817177309D 11	-9.764811436650307D 07	6.799855970278186D 07
15	-1.731361402329375D 13	1.845726375668665D 13	2.472322613108000D 08	-2.249778844764383D 08
16	6.334886536367213D 14	-6.743373944984178D 14	2.027104358903691D 10	-1.464917563981138D 10
17	-2.496865378068563D 16	2.654491102606015D 16	-2.089923510318587D 11	1.610199871604332D 11
18	1.054944395185599D 18	-1.120300310974065D 18	-1.895289334812601D 13	1.368834868575728D 13
19	-4.756681325287049D 19	5.046461930908183D 19	-3.364478962878271D 15	2.437100143629668D 15
20	2.282284973571598D 21	-2.419210256919085D 21	7.214183714503772D 16	-5.394722333857877D 16
21	-1.157752362060935D 23	1.226305801764128D 23	5.139384494613593D 19	-3.732641956261232D 19
22	6.247490007129019D 24	-6.612292357673695D 24	9.672786683089199D 20	-6.801293751028772D 20
23	-3.547774277631688D 26	3.752665384065070D 26	-2.468968021298490D 23	1.799122798719052D 23
24	1.935963660210449D 28	-2.048745669168105D 28	-2.193071595751463D 25	1.583121940489080D 25

TABLE 2. E(V*), C=1, 2CB APPROXIMANTS

	N= 2.0	N= 3.0	N= 4.0	N= 5.0	N=10.0	N=20.0	N=25.0
A= 7 START AT N-3 TERM							
10	5.31800536D-01	6.59162334D-01	7.30251964D-01	7.75925988D-01	8.76501216D-01	9.33676153D-01	9.45971687D-01
11	5.29780369D-01	6.58192289D-01	7.29932604D-01	7.75715455D-01	8.76481035D-01	9.33674821D-01	9.45971182D-01
12	5.26330882D-01	6.57404074D-01	7.29503912D-01	7.75552535D-01	8.76466173D-01	9.33673903D-01	9.45970842D-01
13	5.24211091D-01	6.56771352D-01	7.29240669D-01	7.75423759D-01	8.76454974D-01	9.33673253D-01	9.45970608D-01
14	5.22127919D-01	6.56240562D-01	7.29025214D-01	7.75320064D-01	8.76446354D-01	9.33672782D-01	9.45970442D-01
15	5.21165408D-01	6.55810252D-01	7.28847233D-01	7.75235219D-01	8.76439599D-01	9.33672434D-01	9.45970321D-01
16	5.19917698D-01	6.55437740D-01	7.28699821D-01	7.75164821D-01	8.76434224D-01	9.33672171D-01	9.45970233D-01
17	5.18827234D-01	6.55117551D-01	7.28573398D-01	7.75105694D-01	8.76429886D-01	9.33671970D-01	9.45970166D-01
18	5.17967290D-01	6.54839459D-01	7.28454895D-01	7.75055449D-01	8.76426341D-01	9.33671815D-01	9.45970116D-01
19	5.17013786D-01	6.54595620D-01	7.28370850D-01	7.75012339D-01	8.76423410D-01	9.33671692D-01	9.45970076D-01
20	5.16245618D-01	6.54379981D-01	7.28288598D-01	7.74974997D-01	8.76420962D-01	9.33671594D-01	9.45970046D-01
21	5.15557435D-01	6.54187733D-01	7.28216054D-01	7.74942374D-01	8.76418897D-01	9.33671515D-01	9.45970022D-01
22	5.14822753D-01	6.54015100D-01	7.28151615D-01	7.74913650D-01	8.76417140D-01	9.33671451D-01	9.45970002D-01
S3	5.03604979D-01	6.52495561D-01	7.27537818D-01	7.74702030D-01	8.76407110D-01	9.33671171D-01	9.45969925D-01
S5	5.04890607D-01	6.51760439D-01	7.27420158D-01	7.74620933D-01	8.76404250D-01	9.33671111D-01	9.45969909D-01
A= 9 START AT N-4 TERM							
10	4.60227343D-01	6.41775624D-01	7.24118590D-01	7.73261285D-01	8.76332220D-01	9.33668480D-01	9.45969079D-01
11	4.69514370D-01	6.43998407D-01	7.24892680D-01	7.73593756D-01	8.76352899D-01	9.33669339D-01	9.45969364D-01
12	4.76360430D-01	6.45613655D-01	7.25432275D-01	7.73829769D-01	8.76366542D-01	9.33669889D-01	9.45969542D-01
13	4.81503938D-01	6.46810520D-01	7.25855075D-01	7.74000753D-01	8.76376033D-01	9.33670250D-01	9.45969657D-01
14	4.85444355D-01	6.47715187D-01	7.26159036D-01	7.74127212D-01	8.76382773D-01	9.33670492D-01	9.45969732D-01
15	4.88513736D-01	6.48410765D-01	7.26390151D-01	7.74222419D-01	8.76387656D-01	9.33670659D-01	9.45969783D-01
16	4.90935457D-01	6.48952559D-01	7.26568267D-01	7.74295095D-01	8.76391247D-01	9.33670776D-01	9.45969817D-01
17	4.92864139D-01	6.49378952D-01	7.26705919D-01	7.74351146D-01	8.76393919D-01	9.33670858D-01	9.45969841D-01
18	4.94419784D-01	6.49716738D-01	7.26815664D-01	7.74394713D-01	8.76395925D-01	9.33670917D-01	9.45969858D-01
19	4.95657679D-01	6.49985922D-01	7.26901456D-01	7.74428734D-01	8.76397441D-01	9.33670960D-01	9.45969870D-01
20	4.96667208D-01	6.50201315D-01	7.26969464D-01	7.74455564D-01	8.76398594D-01	9.33670991D-01	9.45969879D-01
21	4.97487309D-01	6.50374382D-01	7.27023584D-01	7.74476698D-01	8.76399476D-01	9.33671014D-01	9.45969885D-01
S3	5.01037811D-01	6.51091353D-01	7.27234474D-01	7.74555820D-01	8.76402335D-01	9.33671070D-01	9.45969901D-01
S5	5.01179199D-01	6.51110664D-01	7.27243519D-01	7.74559311D-01	8.76402473D-01	9.33671079D-01	9.45969902D-01
A=11 START AT N-5 TERM							
10	7.90547151D-01	6.98782106D-01	7.40025808D-01	7.79050797D-01	8.76549499D-01	9.33674542D-01	9.45970852D-01
11	7.15362128D-01	6.86122559D-01	7.36529808D-01	7.77789771D-01	8.76504050D-01	9.33673339D-01	9.45970508D-01
12	6.54280551D-01	6.77331924D-01	7.34124277D-01	7.76930431D-01	8.76474225D-01	9.33672591D-01	9.45970299D-01
13	6.27360027D-01	6.71158273D-01	7.32451642D-01	7.76337837D-01	8.76454364D-01	9.33672117D-01	9.45970169D-01
14	6.00423912D-01	6.66716236D-01	7.31259308D-01	7.75918964D-01	8.76440799D-01	9.33671809D-01	9.45970086D-01
15	5.80325463D-01	6.63428024D-01	7.30384656D-01	7.75614211D-01	8.76431253D-01	9.33671603D-01	9.45970031D-01
16	5.64507240D-01	6.60931260D-01	7.29726395D-01	7.75386687D-01	8.76424352D-01	9.33671460D-01	9.45969994D-01
17	5.52254911D-01	6.58998638D-01	7.29221257D-01	7.75213448D-01	8.76419251D-01	9.33671359D-01	9.45969969D-01
18	5.43307199D-01	6.57482302D-01	7.28828268D-01	7.75079595D-01	8.76415449D-01	9.33671287D-01	9.45969951D-01
19	5.35670096D-01	6.56280738D-01	7.28519441D-01	7.74975368D-01	8.76412563D-01	9.33671235D-01	9.45969938D-01
20	5.29510930D-01	6.55320740D-01	7.28274675D-01	7.74893283D-01	8.76410357D-01	9.33671196D-01	9.45969929D-01
S3	5.03844379D-01	6.51503597D-01	7.27339447D-01	7.74590333D-01	8.76403206D-01	9.33671088D-01	9.45969904D-01
S5	5.00842549D-01	6.51071773D-01	7.27236231D-01	7.74557561D-01	8.76402490D-01	9.33671030D-01	9.45969902D-01
TRUE	5.00000000D-01	6.50520 D-01	7.27087 D-01		8.761 D-01*	9.338 D-01*	9.459699 D-01**

* = 10**5 simulation, ** = Levin's with no truncation, S3,S5 = Shank's 3 and 5 points formula

TABLE 3. VARIANCE OF v*, C=1, 2CB APPROXIMANTS

SERIES START AT N=4 TERM AND ADJUSTED BY A=7

```
-0.246959375000D+02  0.428795693284D+02 -0.695993071918D+02  0.106799055167D+03 -0.155593382328D+03  0.217246988891D+03
-0.292688105225D+03  0.382690911550D+03 -0.487982146200D+03  0.608723330652D+03 -0.745997854331D+03  0.899878018089D+03
-0.107119444493D+04  0.126030275001D+04 -0.146799171553D+04  0.169452976551D+04 -0.194056130281D+04  0.220659036624D+04
-0.249317512613D+04  0.280078344953D+04 -0.312099556444D+04
```

EPII

```
-0.246959375000D+02  0.181026300284D+02 -0.652810803502D+01  0.164090107679D+01  0.231331002629D+00 -0.259787655224D-01
-0.517323865497D-01 -0.241581049051D-01 -0.248786171471D-02  0.516332675625D-02  0.495840045611D-02  0.277473307939D-02
 0.121692692244D-02  0.739723770510D-03  0.911155311599D-03  0.129930145956D-02  0.162615833230D-02  0.181150323209D-02
 0.187537007446D-02  0.187144596573D-02  0.184566771497D-02
```

	N= 2.0	N= 3.0	N= 4.0	N= 5.0	N=10.0	N=20.0	N=25.0
10	5.41879975D-02	7.04705294D-02	6.94575755D-02	5.61020540D-02	5.00769367D-02	3.30195925D-02	2.82176915D-02
11	5.90581433D-02	7.16959297D-02	6.98513412D-02	5.62601621D-02	5.00844145D-02	3.30198485D-02	2.82177675D-02
12	6.28265043D-02	7.23256589D-02	7.00507243D-02	5.63391090D-02	5.00890189D-02	3.30199599D-02	2.82178004D-02
13	6.40492484D-02	7.25210590D-02	7.01302901D-02	5.63703516D-02	5.00893737D-02	3.30199902D-02	2.82178118D-02
14	6.47791002D-02	7.27226247D-02	7.01738564D-02	5.63972011D-02	5.00900741D-02	3.30200183D-02	2.82178172D-02
15	6.55510877D-02	7.28865915D-02	7.02236430D-02	5.64062406D-02	5.00903319D-02	3.30200378D-02	2.82178226D-02
16	6.66440177D-02	7.31039371D-02	7.02998019D-02	5.64303852D-02	5.00917729D-02	3.30200909D-02	2.82178239D-02
17	6.79374496D-02	7.33573302D-02	7.03639475D-02	5.64589686D-02	5.00928029D-02	3.30200846D-02	2.82178351D-02
18	6.93009372D-02	7.36206912D-02	7.04409054D-02	5.64875099D-02	5.00938100D-02	3.30201058D-02	2.82178408D-02
19	7.05397340D-02	7.39754790D-02	7.05146092D-02	5.65145383D-02	5.00947286D-02	3.30201259D-02	2.82178457D-02
20	7.19054399D-02	7.41134787D-02	7.05926914D-02	5.65390655D-02	5.00955391D-02	3.30201422D-02	2.82178498D-02
21	7.30927247D-02	7.43336351D-02	7.06449893D-02	5.65516809D-02	5.00962488D-02	3.30201557D-02	2.82178531D-02
S3	9.09421857D-02	7.70499355D-02	7.13159501D-02	5.67799181D-02	5.01012220D-02	3.30202246D-02	2.82178684D-02
S5	8.97464805D-02	7.69186308D-02	7.12290167D-02	5.67715314D-02	5.01011227D-02	3.30202239D-02	2.82178683D-02

SERIES START AT N=5 TERM AND ADJUSTED BY A=9

```
 0.428795693284D+02 -0.695993071918D+02  0.106799055167D+03 -0.155593382328D+03  0.217246988891D+03 -0.292688105225D+03
 0.382690911550D+03 -0.487982146200D+03  0.608723330652D+03 -0.745997854331D+03  0.899878018089D+03 -0.107119444493D+04
 0.126030275001D+04 -0.146799171553D+04  0.169452976551D+04 -0.194056130281D+04  0.220659036624D+04 -0.249317512613D+04
 0.280078344953D+04 -0.312099556444D+04
```

K SERIES

```
 0.428795693284D+02 -0.260207308634D+02  0.102780091118D+02 -0.141857007416D+01 -0.257160796152D+00 -0.258996210263D-01
 0.275702815536D-01  0.216902432300D-01  0.765118847221D-02 -0.194926299685D-03 -0.219366830314D-02 -0.155720516309D-02
-0.487203152480D-03  0.191431579961D-03  0.398146256910D-03  0.327314338904D-03  0.194827496687D-03  0.638659223569D-04
-0.392211944700D-05 -0.247717569026D-04
```

	N= 2.0	N= 3.0	N= 4.0	N= 5.0	N=10.0	N=20.0	N=25.0
10	1.88563785D-01	9.22324441D-02	7.52152613D-02	5.81032056D-02	5.01400682D-02	3.30210390D-02	2.82180332D-02
11	1.31737146D-01	8.31659787D-02	7.22466615D-02	5.72884575D-02	5.01154957D-02	3.30205134D-02	2.82179438D-02
12	8.30499594D-02	7.70797258D-02	7.12770828D-02	5.67559582D-02	5.01000595D-02	3.30202026D-02	2.82178833D-02
13	8.14377711D-02	7.52776612D-02	7.08179619D-02	5.66015785D-02	5.00957874D-02	3.30201212D-02	2.82178427D-02
14	8.55908035D-02	7.59141589D-02	7.09781061D-02	5.66548793D-02	5.00972079D-02	3.30201467D-02	2.82178490D-02
15	9.41275822D-02	7.72089936D-02	7.12995025D-02	5.67607978D-02	5.00999238D-02	3.30201931D-02	2.82178603D-02
16	1.01051134D-01	7.83674290D-02	7.15541138D-02	5.68439978D-02	5.01019773D-02	3.30202255D-02	2.82178682D-02
17	1.04814954D-01	7.87613525D-02	7.16994041D-02	5.68876616D-02	5.01030201D-02	3.30202426D-02	2.82178720D-02
18	1.06067038D-01	7.89736481D-02	7.17342089D-02	5.69017693D-02	5.01033449D-02	3.30202474D-02	2.82178731D-02
19	1.05992843D-01	7.89529858D-02	7.17307719D-02	5.69009594D-02	5.01033060D-02	3.30202471D-02	2.82178730D-02
20	1.05541738D-01	7.88879440D-02	7.17156738D-02	5.68961718D-02	5.01032237D-02	3.30202457D-02	2.82178727D-02
S3	1.06021596D-01	7.89757719D-02	7.17339345D-02	5.69019341D-02	5.01033487D-02	3.30202474D-02	2.82178731D-02
S5	1.05010820D-01	7.88250450D-02	7.16985665D-02	5.68907566D-02	5.01031105D-02	3.30202443D-02	2.82178724D-02

TABLE 4. THREE ALGORITHMS ON E(v*)

C	n=2	n=3	n=4	n=5	n=10	n=20	n=25
1.5	.382657 T .381964 L_{12}	.488905 T .488857 L_{12}	.538535 T .538555 L_{12}	.57333 S .56739 L_{12} .56794 $*_{10}$	.624492 S .623496 L_{12} .623529 $*_{10}$	.651087 S .651160 L_{12} .651161 $*_{10}$	.656674 S .6566959 L_{12} .6566963 $*_{10}$
2.0	.306853 T .306971 L_{12}	.388807 T .388759 L_{12}	.425863 T .425847 L_{12}	.44702 S .44687 L_{12} .44691 $*_{15}$	.486299 S .486308 L_{12} .486309 $*_{15}$	.504850 S .504850 L_{12} .504850 $*_{12}$	.5084750 S .5084750 L_{12} .5084750 $*_{15}$
2.5	.254794 T .251590 L_8 .259754 L_{13}	.321713 T .321278 L_8 .321980 L_{13}	.351650 T .351575 L_8 .351670 L_{13}	.36857 S .36848 L_8 .36850 L_{13}	.399732 S .399732 L_8 .399732 L_{13}	.41417907 S .41417907 L_8 .41417907 L_{13}	.41698042 S .41698042 L_8 .41698042 L_{13}
3.0	.217203 T .217386 L_8	.273919 T .274103 L_8	.299245 T .299321 L_8	.31330 S .31351 L_8	.339829 S .339829 L_8	.35196404 S .35196404 L_8	.35431157 S .35431157 L_8
3.5	.188945 T .187048 L_5	.238265 T .238745 L_9	.260330 T .260414 L_9	.27260 S .27277 L_9	.295781 S .295781 L_9	.30640516 S .30640516 L_9	.30846072 S .30846072 L_9

(T refers to true, S refers to the direct sum of the series to the smallest terms up to that of n^{-24}. $*ij$ refers to 2CB with a=i and starting at n^{-j} term, followed by 5-points Shanks formula applied to last 5 approximants. L_s refers to Lenin's s^{th} approximants.)

TABLE 5. THREE ALGORITHMS ON $100\mu_2(v^*)$

c	n=2	n=3	n=4	n=5	n=10	n=20	n=25
1.0	8.3333 T 8.3984 $*_{53}$ 8.0546 L'_{17}	7.6694 T 7.6170 $*_{53}$ 7.6065 L'_{17}	7.1345 T 7.1128 $*_{53}$ 7.1216 L'_{17}	6.6725 $*_{53}$ 6.6792 L'_{17}	5.0100 $*_{53}$ 5.0106 L'_{17}	3.30202 $*_{53}$ 3.30204 L'_{17}	2.821787 $*_{53}$ 2.821793 L'_{17}
1.5	6.3600 T 6.2758 L_{17} 6.3686 +	5.2842 T 5.3024 L_{17} 5.2338 +	4.4154 T 4.2033 L_{17} 4.3925 +	5.0978 S 3.7838 L_{18} 3.7740 +	2.3325 S 2.20697 L_{18} 2.20682 +	1.2168 S 1.206448 L_{18} 1.206461 +	0.98701 S 0.983935 L_{18} 0.983947 +
2.0	4.7434 T 4.7562 L_{10} 5.1878 $*_{11}$	3.7399 T 3.7496 L_{10} 3.8169 $*_{11}$	2.9785 T 2.9815 L_{10} 2.9975 $*_{11}$	2.4408 S 2.4596 L_{10} 2.4645 $*_{11}$	1.3030 S 1.3020 L_{10} 1.3021 $*_{11}$	.669632 S .669637 L_{10} .669638 $*_{11}$	.5387887 S .5387882 L_{10} .5387884 $*_{11}$
2.5	3.5842 T 4.0323 L_6 4.5132 L_{13}	2.7578 T 2.8445 L_6 2.8464 L_{13}	2.1538 T 2.1779 L_6 2.1690 L_{13}	1.7464 S 1.7603 L_6 1.7554 L_{13}	.89439 S .89461 L_6 .89444 L_{13}	.450004 S .450034 L_6 .450004 L_{13}	.36037700 S .36037737 L_6 .36037700 L_{13}
3.0	2.7667 T 3.2971 L_9	2.1085 T 2.1751 L_9	1.6362 T 1.6491 L_9	1.3364 S 1.3278 L_9	.668256 S .668244 L_9	.3336839 S .3336839 L_9	.26679179 S .26679179 L_9

(T, S, $*_{ij}$, L_S, refer as same as in Table 4. + refers to 2CB a=1 and starting n^{-1} term and 17th approximants signaled from comparison at n=2. L'_{17} is derived from $\mu_2(v^*) = (n-1)/(n+1) - (E\sqrt{m_2})^2$, with L_{17} used on $E(\sqrt{m_2})$).