An Algorithm for Noisy Image Segmentation*

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DISCLAIMER

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(Extended Abstract)

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Abstract

This paper presents a segmentation algorithm for gray-level images and addresses issues related to its performance on noisy images. It formulates an image segmentation problem as a partition of an image into (arbitrarily-shaped) connected regions to minimize the sum of gray-level variations over all partitioned regions, under the constraints that (1) each partitioned region has at least a specified number of pixels, and (2) two adjacent regions have significantly different "average" gray-levels. To overcome the computational difficulty of directly solving this problem, a minimum spanning tree representation of a gray-level image has been developed. With this tree representation, an image segmentation problem is effectively reduced to a tree partitioning problem, which can be solved efficiently. To evaluate the algorithm, we have studied how noise affects the performance of the algorithm. Two types of noise, transmission noise and Gaussian additive noise, are considered, and their effects on both phases of the algorithm, construction of a tree representation and partition of a tree, are studied. Evaluation results have shown that the algorithm is stable and robust in the presence of these types of noise.

1 Introduction

Image segmentation is one of the most fundamental problems in low-level image processing. The problem is to partition (segment) an image into connected regions of similar textures or similar colors/gray-levels, with adjacent regions having significant dissimilarity. Many algorithms have been proposed to solve this problem (see surveys [1, 2]). Most of these algorithms fit into two categories: (1) boundary detection-based approaches, which partition an image by discovering closed boundary contours, and (2) region clustering-based approaches, which group "similar" neighboring pixels into clusters. Rigorous mathematical solutions to the image segmentation problems are generally difficult to achieve due to their (intrinsic) computational complexity. Hence many researchers have exploited either probabilistic/stochastic methods, which guarantee only asymptotic results, or heuristic methods while sacrificing the mathematical rigor.

In this paper, we present an efficient region-based segmentation algorithm. We formulate an image segmentation problem as a partition of an image into a number (not predetermined) of arbitrarily-shaped connected regions to minimize the sum of gray-level variations over all partitioned regions under the constraints that (1) each partitioned region has at least a specified number of pixels, and (2) two adjacent regions have significantly different "average" gray-levels. To overcome the computational difficulty of directly solving this problem, we have developed a minimum spanning tree representation of an image. The minimum spanning tree representation, though simple, captures the essential information of an image for the purpose of segmentation, and it facilitates a fast segmentation algorithm. The technical contribution of our approach includes (1) a new spanning tree representation of an image that captures all the key information for the purpose of segmentation, and (2) a fast and mathematically rigorous tree partitioning algorithm.

To evaluate the algorithm, we have studied how two types of noise, transmission noise and Gaussian additive noise, affect the performance of the algorithm. We have shown, both analytically and experimentally, that (1) both types of noise have very little effect on the minimum spanning tree construction algorithm, i.e., the property that an originally homogeneous region corresponds to one subtree of the spanning tree will generally not be affected by noise; (2) transmission noise, in general, has less effect on the performance of our tree-partitioning algorithm than Gaussian additive noise does.

2 Image Segmentation: the problem formulation

Consider a gray-level image I. Each pixel x of I has a gray level $\mathcal{G}(x) \in [0, \mathcal{K}]$. An image *segmentation problem* can be naturally formulated as follows: find a partition $\{I_1, ..., I_k\}$ of I with each I_i being a connected region of I, such that

$$\text{minimize} \qquad \sum_{i=1}^{k} \sum_{x_i^j \in I_i} (average(I_i) - \mathcal{G}(x_i^j))^2$$

$$\text{subject to:} \quad \begin{array}{ll} (1) & \|I_i\| \geq L, \text{ for each } I_i, \\ (2) & |average(I_i) - average(I_{i'})| \geq D, \text{ for all adjacent } I_i \text{ and } I_{i'}. \end{array}$$

where $\|\cdot\|$ denotes the cardinality of a set, $average(I_i)$ denotes the average gray-level of region I_i, and L and D are two (application-dependent) parameters.

Though this formulation captures the intuition of segmenting an image it is computationally difficult to solve due to two reasons: (1) segmenting a 2-D object to optimize some non-trivial function is always difficult, and (2) explicit calculation of averages implicitly requires to consider all the possible partitions. Two strategies have been developed to overcome these difficulties: a tree representation of an image, and an approximation scheme to avoid explicit calculation of averages.

2.1 Spanning tree representation of an image

For a given image I, we define a weighted (undirected) planar graph $G(I) = (V, E)$ as follows: The vertex set $V = \{$ all pixels of I $\}$ and the edge set $E = \{(u, v) | u, v \in V$ and $distance(u, v) \leq \sqrt{2}$ $\}$, with $distance(u, v)$ representing the Euclidean distance in terms of the coordinates of the image array; Each edge $(u, v) \in E$ has a weight $w(u, v) = |\mathcal{G}(u) - \mathcal{G}(v)|$.

A *spanning tree* T of a connected graph $G(I)$ (note that $G(I)$ is connected) is a connected subgraph of $G(I)$ such that (1) T contains every vertex of $G(I)$, and (2) T does not contain cycles. A *minimum* spanning tree is a spanning tree with a minimum total weight.

A minimum spanning tree of a weighted graph can be found using greedy methods, like in the classical Kruskal's algorithm [3]: the initial solution is a singleton set containing an edge with the smallest weight, and then the current partial solution is repeatedly expanded by adding the next smallest weighted edge (from the unconsidered edges) under the constraint that no cycles are formed until no more edges can be added. For the above defined planar graph $G(I)$, a minimum spanning tree can be constructed in $O(\|V\| \log(\|V\|))$ time and in $O(\|V\|)$ space.

A key property of a minimum spanning tree representation obtained by Kruskal's algorithm is that *pixels of a homogeneous region are connected in the tree structure only through pixels of this region*, i.e., pixels of a homogeneous region form a (connected) subtree of the minimum spanning tree. The following theorem formalizes this statement.

Consider an object A in a given image I. Let $G(I)$ be the planar graph representation of I and T be its minimum spanning tree obtained by Kruskal's algorithm. A is called *T-connected* if every pair of pixels of A are connected in T only through pixels of A. We use $G(A)$ to denote the subgraph of $G(I)$ induced by the pixels of A. A set of edges C of $G(A)$ is called a *cutset* if deleting C divides $G(A)$ into two unconnected parts.

Theorem 1 *A is not T-connected if and only if there exists a cutset C of $G(A)$ and a path P in $G(I)$ that has its two end vertices on two sides of the cut of $G(A)$ and has its remaining vertices outside of $G(A)$ such that every edge of P has smaller[1] weight than every edge of C.* □

2.2 An approximation scheme

To formulate the image segmentation problem in a natural and intuitive way, we have explicitly used the average gray-levels of a region in the problem formulation, which makes the computation difficult. This subsection presents an approximation scheme to avoid the explicit calculation of averages.

Consider the following formulation of an image segmentation problem. Given an image I and two parameters D and L, find a partition $\{I_1, ..., I_k\}$ of I with each I_i being a connected region of I, and a $g_i \in \mathcal{R}$ (real value) for each I_i, such that

$$\text{minimize} \quad \sum_{i=1}^{k} \sum_{x_i^j \in I_i} (g_i - \mathcal{G}(x_i^j))^2$$

$$\text{subject to:} \quad (1) \quad \|I_i\| \geq L, \text{ for each } I_i,$$
$$(2) \quad |g_i - g_{i'}| \geq D, \text{ for all adjacent } I_i \text{ and } I_{i'}.$$

The relationship between this formulation, which does not involve explicit calculation of averages, and the original one can be intuitively described as follows: if a solution to this formulation is stable around the given parameter D, then the two formulations are equivalent. This can be stated more rigorously as in the following theorem. Let

$$F(k, I, g) = \sum_{i=1}^{k} \sum_{x_i^j \in I_i} (g_i - \mathcal{G}(x_i^j))^2,$$

and

$$R(D, L) = \{(k, I, g) | \text{ which satisfies constraints (1) and (2)}\},$$

[1]We ignore the case of equality for the simplicity of discussion.

where $I = \bigcup_{i=1}^{k} I_i$ and $g = (g_1, ..., g_k)$. Hence the above formulation can be rewritten as

$$\min_{k,I,g}\{F(k,I,g)|(k,I,g) \in R(D,L)\}.$$

Theorem 2 *For the given parameters D and L, if there is an $\epsilon > 0$ such that*

$$\min_{k,I,g}\{F(k,I,g)|(k,I,g) \in R(d,L)\} = F_0 \tag{1}$$

for some constant F_0, for all $d \in [D, D+\epsilon]$, then any minimum solution $I^* = \{I_1^*, ..., I_k^*\}$ and $g^* = \{g_1^*, ..., g_k^*\}$ to $\min_{k,I,g}\{F(k,I,g)|(k,I,g) \in R(D+\epsilon, L)\}$ has $g_i^* = average(I_i^*)$, for all $i \in [1,k]$. □

Note that g_i's, as defined above, are real values $\in [0, \mathcal{K}]$. To facilitate a fast algorithm, we restrict g_i's to integer values $\in [0, \mathcal{K}]$. Now we can give the tree-based image segmentation problem as follows. Given a minimum spanning tree representation T of an image and two parameters D and L, find a partition $\{T_1, ..., T_k\}$ of T with each T_i being a connected subtree of T, and an integer $g_i \in [0, \mathcal{K}]$ for each T_i, such that

$$\begin{array}{llll} \text{minimize} & \sum_{i=1}^{k} \sum_{x_i^j \in T_i} (g_i - \mathcal{G}(x_i^j))^2 & & \\ & & & (P) \\ \text{subject to:} & (1) & \|T_i\| \geq L, \text{ for each } T_i, & \\ & (2) & |g_i - g_{i'}| \geq D, \text{ for all adjacent } T_i \text{ and } T_{i'}. & \end{array}$$

To estimate how close the approximation problem is to the original problem, we have the following result:

$$\frac{E(\sum_{i=1}^{k} \|T_i\| (average(T_i) - g_i)^2)}{E(\sum_{i=1}^{k} \sum_{x_i^j \in T_i} (average(x_i^j) - \mathcal{G}(x_i^j))^2)} \leq k/N, \tag{2}$$

which indicates the minimum value of the approximation problem is fairly close to the minimum value of the original optimization problem, where $E()$ represents the mathematical expectation.

3 A Tree-based Image Segmentation Algorithm

A dynamic programming algorithm is developed to solve the optimization problem (P). The algorithm first converts the given tree into a *rooted* tree by selecting an arbitrary vertex as the root. Hence the *parent-children* relation is defined. We assume that the vertices of T are labeled consecutively from 1 to $\|T\|$ with the tree root labeled as 1. We use T^i to denote the subtree rooted at vertex i. For each tree vertex i, the dynamic programming algorithm solves a number of constraint version of the problem (P) on T^i by combining solutions to the "corresponding" problems on T^j's, with j's being i's children. It does this repeatedly in such a bottom-up fashion and stops when it reaches the tree root.

Let $score(i,k,g)$ denote the minimum value of (P) on T^i, under the additional constraint that the partitioned subtree of T^i containing i has at least k vertices and is mapped to a fixed value g, for $k \in [0, L]$ and $g \in [0, \mathcal{K}]$. These quantities can be efficiently calculated using the following lemma and can be used to construct an optimum partition of T.

Lemma 1 *(a) If $i_1, i_2, ..., i_n$ are the children of vertex i, $n \leq 8$ and $1 \leq k \leq L$, we have*

$$\begin{array}{l} score(i,k,g) = \min \sum_{j=1}^{n} score(i_j,k_j,g) + (g - \mathcal{G}(i))^2, \\ \quad for\ k = \sum_{j=1}^{n} k_j, k_j \geq 0, \quad when\ \|T^i\| \geq L \\ scores(i,k,g) = \begin{cases} \sum_{p \in \mathcal{D}(i)} (g - \mathcal{G}(p))^2, & \|T^i\| = k \\ +\infty, & \|T^i\| \neq k \end{cases} \\ \quad when\ \|T^i\| < L \end{array}$$

where $\mathcal{D}(i)$ is the set of all i's descendants, i is defined to be $\in \mathcal{D}(i)$ and $score(i_j, 0, g)$ is defined to be

$$\min_{|g'-g| \geq D} score(i_j, L, g').$$

(b) $\min_g score(1, L, g)$ is a minimum solution of (P), where 1 represents the tree root. □

Based on Lemma 1, we can solve the optimization problem (P) by calculating $score()$ for each tree vertex in a bottom-up fashion using the recurrence from Lemma 1(a), and stopping at the tree root.

Theorem 3 $\min_g score(1, L, g)$ *can be correctly calculated by the above algorithm in* $O(\max\{(\|T\| - L), 1\}\mathcal{K}(\log(\mathcal{K}) + L^2))$ *time and in* $O(\|T\|L\mathcal{K})$ *space.* □

4 Algorithm Evaluation on Noisy Images

Potentially noise affects the algorithm's performance in both stages of the algorithm: spanning tree construction and tree partitioning. We will show that noise has greater effects on the performance in the tree partitioning stage than in the spanning tree construction stage. In this study, we consider two types of noise: transmission noise and Gaussian additive noise.

The model for generating *transmission noise* is defined as follows: each pixel of the image has a probability $\mathcal{P}$ to keep its original gray level during transmission and the probability $1 - \mathcal{P}$ to randomly change to arbitrary gray level $\in [0, \mathcal{K}]$. *Gaussian additive noise* adds to each pixel independently a random normal value (using the floor function for real-to-integer conversion) according to a normal distribution $N(0, \sigma^2)$ censored to $[-\mathcal{K}/2, \mathcal{K}/2]$.

4.1 Effect of noise on tree representation

One basic premise for our image segmentation algorithm to be effective is that each object, given as a homogeneous region in an image, is represented as one subtree of the spanning tree representation. In the following, we show how noise affects this property. Theorem 1 provides the basic framework for such a study.

To estimate how probable the if-and-only-if condition in Theorem 1 is we have conducted the following computer simulation. The experiment is done on a 256-gray-level image I having one object A in the center of the image. I is a 256×256 image and A is a 30×30 square. The background has a uniform gray level 100 and A has a uniform gray level 150. We add transmission noise and Gaussian additive noise, respectively, to I as follows. When adding transmission noise, each pixel of I has a probability 0.3 to keep its original gray level and the probability 0.7 to randomly and uniformly change to arbitrary gray level $\in [0, 255]$. When adding Gaussian additive noise, each pixel of I is added by a value $\lfloor \delta + 1/2 \rfloor$ (modulo 256), where δ is random number generated according to the normal distribution $N(0, \sigma^2)$ censored to [-128, 128] with $\sigma = 50$.

For each type of noise, we estimated the probability that there exist a path P connecting two pixels a and b, and a cutset C of A separating a and b such that every edge of P has smaller weight than every edge of C, where a and b are two randomly chosen pixels both of which are 5-pixels from the left boundary of A and are at least 5 pixels from the lower and upper boundaries of A, and P has at least 20 edges.

We have observed, for this particular experiment, that the probability that there exist such a P and a cutset C is very small ($< 10^{-3}$), for both types of noise. This experiment suggested that both types of noise have very little effect on the property that a homogeneous region corresponds to one subtree of the minimum spanning tree constructed by Kruskal's algorithm.

4.2 Effect of noise on tree partitioning

Though both types of noise have little effect on the property that a homogeneous region corresponds to one subtree of the spanning tree representation they could affect the tree partitioning result in a form we call *corrosions*. Consider an object A in a given image and its representing subtree T_A. With noise, T_A may contain a subtree (or subtrees) that has a (significantly) different average gray level than the rest of T_A, and contains more than enough vertices ($\geq L$) to be partitioned into a separate region. This subsection presents a comparative study on how the two types of noise affect the formation of corrosions.

Let $g(A)$ be the (uniform) gray level of A before noise is added. We compare the probabilities, P_1 and P_2, that a connected region A' of A will have its gray level changed to the same value $g(A)+k$, for any $k \neq 0$, when transmission noise and Gaussian additive noise are added, respectively. Let p_k denote the probability that one pixel of A' changes its gray level from $g(A)$ to $g(A)+k$ when Gaussian additive noise is added. For the simplicity of discussion, we assume that $g(A) = 0$, hence $k \in [1, \mathcal{K}]$. Recall $\mathcal{P}$ denotes the probability that a pixel keeps its original gray level in the presence of transmission noise. It can be shown by a simple calculation that

$$P_1 = \left(\frac{1-\mathcal{P}}{\mathcal{K}-1}\right)^n (\mathcal{K}-1), \quad \text{and} \quad P_2 = \sum_{k=1}^{\mathcal{K}} p_k^n,$$

where $n = \|A'\|$ (note that $P_2 = \sum_{k=1}^{\mathcal{K}} p_k^n$ is true for any type of independent noise). Theorem 4 shows the relationship between P_1 and P_2, which can be proved using Jensen's inequality [5] (page 433).

Theorem 4 *For any $\mathcal{N} \in [2, \mathcal{K}]$ and $n > 0$, when $\sum_{k=0}^{\mathcal{N}} p_k = 1$ and $p_0 = \mathcal{P}$,*

$$\sum_{k=1}^{\mathcal{N}} p_k^n \geq \left(\frac{1-\mathcal{P}}{\mathcal{N}-1}\right)^n (\mathcal{N}-1).$$

□

Theorem 4 implies that transmission noise is the least possible to form corrosions among all possible forms of noises (including Gaussian additive noise) when $\mathcal{P}$ or p_0 is fixed.

4.3 Tests on noisy images

This subsection presents a case-study on an aerial image of 202x503 pixels and with 256 gray levels, and on how noise of different types affects the performance of the segmentation algorithm. Throughout this study, the same set of parameters D and L are used. Segmentation on each image takes less than 1 CPU minute on a SPARC-20 workstation. Figure 1 gives the test examples on the image with noise added. For each figure, the image on the left is the original image with added noise and the one on the right represents the segmentation results.

Table 1 summarizes the performance of algorithm and the effect of the averaging operation on the two types of noise. Each entry of the first row represents the correlational coefficient between the original image and the image with noise, and each entry of the second row represents the correlational coefficient between the segmentation result of the original image and the segmentation of the noisy image.

Table 1: Performance summary of segmentations

	Transmission noise				Gaussian additive noise			
	$\mathcal{P} = 0.1$	$\mathcal{P} = 0.3$	$\mathcal{P} = 0.5$	$\mathcal{P} = 0.7$	$\sigma = 40$	$\sigma = 60$	$\sigma = 80$	$\sigma = 100$
noisy image	0.86	0.62	0.41	0.24	0.82	0.69	0.57	0.47
segmentation	0.95	0.89	0.80	0.70	0.87	0.84	0.81	0.76

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References

[1] N. Pal and S. Pal, "A review on image segmentation techniques", *Pattern Recognition*, Vol. 26, No. 9, pp. 1277 - 1294, 1993.

[2] R. M. Haralick and L. G. Shapiro, "Image segmentation techniques", *Computer Vision, Graphics, and Image Processing*, Vol. 29, pp. 100 - 132, 1982.

[3] J. B. Jr. Kruskal, "On the shortest spanning subtree of a graph and the traveling salesman problem", *Proc. Amer. Math Soc*, Vol. 7, No. 1, pp. 48 - 50, 1956.

[4] Y. Xu and E. C. Uberbacher, "2-D Image Segmentation Using Minimum Spanning Trees", *Image and Vision Computing*, Vol. 15 pp. 47 - 57, 1997.

[5] P. J. Bickel and K. A. Doksum, *Mathematical Statistics: Basic Ideas and Selected Topics*, Holden-Day Inc., 1977.

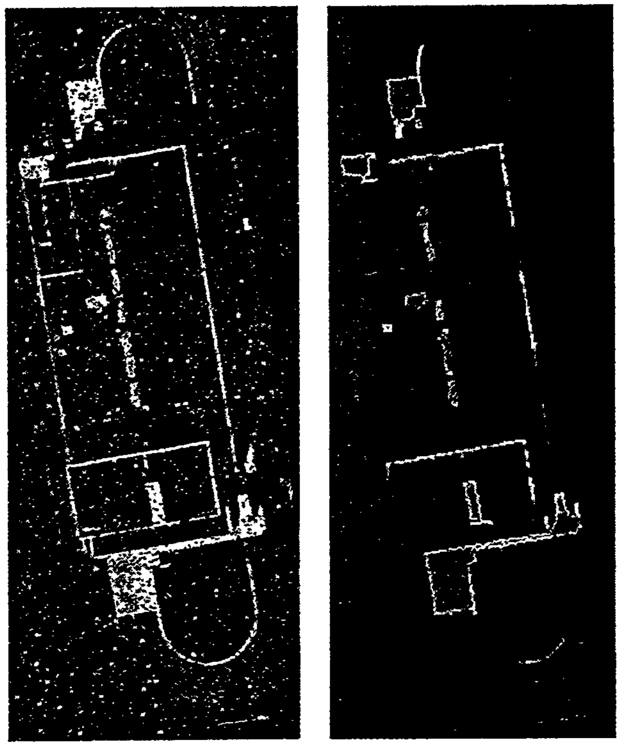

Figure 6: (a) Aerial image with added transmission noise and $\mathcal{P} = 0.1$.

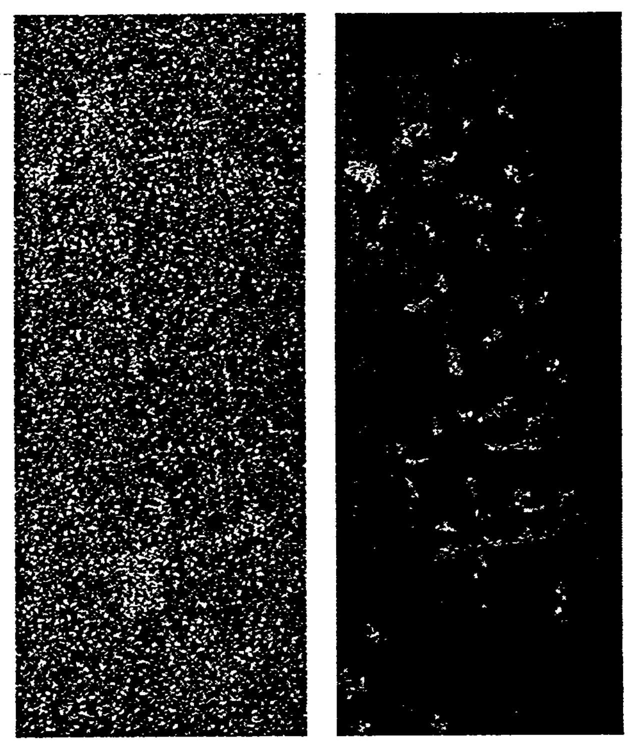

Figure 6: (d) Aerial image with added transmission noise and $\mathcal{P} = 0.7$.